Introduction to Astrophysics and Cosmology

Observational cosmology

Based on: Chapter 29 of An Introduction to Modern Astrophysics

Helga Dénes 2025 S1 Yachay Tech

hdenes@yachaytech.edu.ec

Observational cosmology

Most of the key parameters of cosmology encountered in the previous section, such as H_0, q_0, and the various Ω_0's, are not quantities that can be directly measured by astronomers. Observers are primarily limited to **measuring the spectrum, redshift, radiant flux, and polarization of the starlight from a distant object.** We now proceed to link these observables to the theoretical framework we have erected.

Let's begin by uncovering the origin of **the cosmological redshift**. We start with the Robertson–Walker metric, Eq. (106), with ds = 0 for a light ray, and $d\theta = d\phi = 0$ for a **radial path traveled** from the point of the light's emission at comoving coordinate ϖ_e to its arrival at Earth at $\varpi = 0$. Taking the negative square root (so ϖ decreases with increasing time) gives

$$\frac{-c\,dt}{R(t)} = \frac{d\varpi}{\sqrt{1-k\varpi^2}}.$$

Integrating this from a larger ϖ_{far} at an initial time t_i to a smaller ϖ_{near} at time t_f , we have

$$\int_{t_i}^{t_f} \frac{c\,dt}{R(t)} = -\int_{\varpi_{\text{far}}}^{\varpi_{\text{near}}} \frac{d\varpi}{\sqrt{1-k\varpi^2}} = \int_{\varpi_{\text{near}}}^{\varpi_{\text{far}}} \frac{d\varpi}{\sqrt{1-k\varpi^2}}. \qquad (137)$$

Cosmological redshift

A moment's thought reveals that the same result describes an **outwardly moving light ray**. Suppose that one crest of the light wave was emitted at time t_e and received at t_0, and the next wave crest was emitted at $t_e + \Delta t_e$ and received at $t_0 + \Delta t_0$. These times, which describe how long it takes for the successive crests of the light wave to travel to Earth, satisfy the relations

$$\int_{t_e}^{t_0} \frac{c\,dt}{R(t)} = \int_0^{\varpi_e} \frac{d\varpi}{\sqrt{1 - k\varpi^2}} \tag{138}$$

for the first crest and

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c\,dt}{R(t)} = \int_0^{\varpi_e} \frac{d\varpi}{\sqrt{1 - k\varpi^2}} \tag{139}$$

for the next. The right-hand sides are the same, since the comoving coordinate of an object does not change as the universe expands (assuming its peculiar velocity is negligible).

Cosmological redshift

Subtracting Eq. (138) from Eq. (139) produces

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{R(t)} - \int_{t_e}^{t_0} \frac{dt}{R(t)} = 0. \tag{140}$$

But

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{R(t)} = \int_{t_e+\Delta t_e}^{t_e} \frac{dt}{R(t)} + \int_{t_e}^{t_0} \frac{dt}{R(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{R(t)},$$

So

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{R(t)} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{R(t)} = 0.$$

Cosmological redshift

Any change in $R(t)$ during the time intervals Δt_e and Δt_0 between the emission of the two successive wave crests can safely be neglected. This allows us to treat $R(t)$ as a constant with respect to the time integration, so that, using $R(t_0) = 1$,

$$\Delta t_0 = \frac{\Delta t_e}{R(t_e)}. \tag{141}$$

The times Δt_e and Δt_0 are just the periods of the emitted and received light waves and are related to their wavelengths by $\lambda = c\Delta t$.

Making this substitution into Eq. (141) and using the definition of the redshift z results in the **expression for the cosmological redshift,**

$$\boxed{\frac{1}{R(t_e)} = \frac{\lambda_0}{\lambda_e} = 1 + z. \tag{142}}$$

Cosmological redshift

This derivation shows that the **cosmological redshift is due to the wavelength of a photon expanding along with the space** through which the photon moves during its journey to Earth. Equation (142) is just Eq. (4) discussed previously. Combining Eqs. (141) and (142) results in the formula for **cosmological time dilation**,

$$\boxed{\frac{\Delta t_0}{\Delta t_e} = 1 + z.} \tag{143}$$

Note that these relations for the cosmological redshift and time dilation hold regardless of the functional form of the scale factor, R(t). Experimental confirmation of cosmological time dilation has been frustrated by the lack of a reliable natural clock located at cosmological distances. However, a **measurement of cosmological time dilation has been made using the temporal changes in the spectrum of a moderate-redshift (z = 0.361) Type Ia supernova.** The results were consistent with Eq. (143).

Distance to very far objects

How to measure the distance to objects in the most remote regions of the universe? The proper distance of an object from Earth can be found from the Robertson–Walker metric. Recall that the differential proper distance is just $\sqrt{-(ds)^2}$ with dt = 0. Furthermore, if the comoving coordinate of the object is ϖ (with Earth at $\varpi = 0$), then $d\theta = d\phi = 0$ along a radial line from Earth to the object. Inserting these into the Robertson–Walker metric (Eq. 106), we can find the proper distance, $d_p(t)$, to the object at time t by integrating

$$d_p(t) = R(t) \int_0^{\varpi} \frac{d\varpi'}{\sqrt{1 - k\varpi'^2}}. \tag{144}$$

Using Eq. (138) this becomes

$$d_p(t) = R(t) \int_{t_e}^{t_0} \frac{c\,dt'}{R(t')}. \tag{145}$$

Distance to very far objects

The physical meaning of this is: as the photon moves in from ϖ_e, in each interval of time dt it travels a small distance of cdt . These intervals cannot simply be added up, because **the universe expands as the photon travels.**

Dividing cdt by the scale factor at the time, R(t), converts this small distance to what it would be at the present time, t_0. Integrating then yields the proper distance from ϖ_e to $\varpi = 0$ today, at time t_0.

Multiplying by the scale factor R(t) then converts this to the distance at some other time t. It is worth emphasizing that the current value of the proper distance, $d_{p,0} \equiv d_p(t_0)$, to an object is how far away it is *today*, and not its distance when its light was emitted. As long as the object has zero peculiar velocity (constant comoving coordinate ϖ), it suffices to find $d_{p,0}$ because the proper distance at any other time can be obtained from

$$d_p(t) = R(t) d_{p,0}. \tag{146}$$

In particular, if the object's redshift is z, then its distance at time t_e, when its light was emitted, is

$$d_p(t_e) = d_{p,0} R(t_e) = \frac{d_{p,0}}{1+z}. \tag{147}$$

Distance to very far objects

Integrating Eq. (144) and using $R(t_0) = 1$ shows that the expression for the present proper distance in a flat universe is

$$d_{p,0} = \varpi \qquad \text{(for } k = 0\text{)}, \tag{148}$$

the expression for a **closed universe** is

$$d_{p,0} = \frac{1}{\sqrt{k}} \sin^{-1}(\varpi\sqrt{k}) \qquad \text{(for } k > 0\text{)}, \tag{149}$$

and the expression for an **open universe** is

$$d_{p,0} = \frac{1}{\sqrt{|k|}} \sinh^{-1}(\varpi\sqrt{|k|}) \qquad \text{(for } k < 0\text{)}. \tag{150}$$

Distance to very far objects

In a flat universe, the present proper distance to an object is just its coordinate distance $d_{c,0} = \varpi$ (cf. Eq. 3). However, the coordinate distance will not agree with the proper distance if $k \neq 0$. Because $\sin^{-1}(x) \geq x$, in a **closed universe (k > 0) the proper distance to an object is *greater* than its coordinate distance.**

Similarly, $\sinh^{-1}(x) \leq x$, so in an **open universe (k < 0) the proper distance to an object is *less* than its coordinate distance.**

We can examine the above equations for $d_{p,0}$ because measuring distances in an expanding universe has some interesting aspects.

Distances in a closed universe ($k > 0$) are especially interesting. Solving Eq. (149) for ϖ gives

$$\varpi = \frac{1}{\sqrt{k}} \sin(d_{p,0}\sqrt{k}). \qquad (151)$$

Note that in a closed universe there is a maximum value of the comoving coordinate of $\varpi_{max} = 1/\sqrt{k}$.

Distance to very far objects

Also, there are an infinite number of distances along a radial line to the same point X in space, located, say, at ϖ_X. If $d_{p,0}$ is one value of the present proper distance to X at time t, then for any integer n,$d_{p,0} + 2\pi\, n/\sqrt{k}$ will also bring us back to X, with the same value of ϖ_X. **The extra multiples of $2\pi/\sqrt{k}$ correspond to traversing the circumference of the universe n times before stopping at X**.

However, such a journey, running circles around the universe, would not be physically possible. As it happens, **a photon emitted at t = 0 in a closed, matter-dominated universe with $\Lambda = 0$ would return to its starting point just as the universe ended in a Big Crunch.**

This illustrates that although **there is no boundary to a closed universe, it contains only a finite amount of space,** like the unbounded surface of a sphere. Furthermore, a closed universe **curves back on itself;** moving outward from Earth (or from any other choice of origin), the farthest you can get from home is a point where $\varpi = 1/\sqrt{k}$. From that point, a step in *any* direction brings you closer to where you started.

Distance to very far objects

Note that suggestive phrases like "the circumference of the universe" and "curves back on itself" do not imply a curved path through three-dimensional space, since there is no deviation from a radial line as ϖ increases. Despite these caveats, we can define the circumference of a closed universe (including the time-dependence) as

$$C_{\text{univ}}(t) = \frac{2\pi R(t)}{\sqrt{k}} \tag{152}$$

which is **the proper distance along a radial line that brings you back to your starting point.** This expression for the circumference is consistent with our definition of curvature since, from Eq. (105), the radius of curvature of a closed universe at time t is $1/\sqrt{K(t)}$ = R(t)/$\sqrt{k}$. Nevertheless, the radius of curvature must not be thought of as the radius of an actual circular path.

The Horizon

As the universe expands and ages, photons from increasingly distant objects have more time to complete their journey to Earth. This means that as time increases, we might expect that more of the universe will come into causal contact with the observer. **The proper distance to the farthest observable point** (called the **particle horizon**) at time t is the **horizon distance**, $d_h(t)$. Note that **two points separated by a distance greater than d_h are not in causal contact**. Thus d_h may be thought of as the *diameter* of the largest causally connected region.

We will now derive an expression for $d_h(t)$, the size of the observable universe as a function of time. (It is important to note that because the farthest observable point moves outward through increasingly larger values of ϖ,$d_h(t)$ is *not* proportional to R(t).)

Consider an observer at the origin ($\varpi = 0$), and let the particle horizon for this observer be located at ϖ_e at time t. This means that a photon emitted at ϖ_e at $t = 0$ would reach the origin at time t. With an appropriate change of limits in Eq. (145), the horizon distance at time t is found to be

$$d_h(t) = R(t) \int_0^t \frac{c\,dt'}{R(t')}. \qquad (153)$$

The Horizon

First we consider **distances in the early universe**, when the effect of dark energy was negligible. During the radiation era, the universe was essentially flat and the scale factor was of the form $R(t) = Ct^{1/2}$, where C is a constant (see Eq. 87). Inserting this into Eq. (153) gives

$$d_h(t) = 2ct \qquad \text{(radiation era).} \tag{154}$$

After the radiation era, the expansion of the universe was governed by the effects of matter and, later, dark energy.

For the **matter era**, assuming a flat universe, the scale factor is given by Eq. (91), which is of the form $R(t) = Ct^{2/3}$, where again C is a constant. (Since the radiation era lasted only 55,000 years following the Big Bang, for the purposes of the following calculations we will ignore radiation and set the lower limit to $t = 0$.) Substituting this into Eq. (153) results in

$$d_h(t) = 3ct \qquad (\text{for } k = 0). \tag{155}$$

The Horizon

Using Eqs. (4) and (90), this expression **can be rewritten in terms of the redshift** as

$$d_h(z) = \frac{2c}{H_0\sqrt{\Omega_{m,0}}} \frac{1}{(1+z)^{3/2}} \qquad \text{(for } k = 0\text{).} \qquad (156)$$

We can obtain a rough estimate for the **present horizon distance** by setting z = 0 to obtain

$$\boxed{d_{h,0} \approx \frac{2c}{H_0\sqrt{\Omega_{m,0}}} = 5.02 \times 10^{26}\ \mathrm{m} = 16{,}300\ \mathrm{Mpc} = 16.3\ \mathrm{Gpc} \qquad (157)}$$

using WMAP values.

The Horizon

Finally, in the Λ **era**, we substitute Eq. (131) for the scale factor into Eq. (153) for the horizon distance to get, for k = 0,

$$d_h(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0 t\sqrt{\Omega_{\Lambda,0}}\right)\int_0^t \frac{c\,dt'}{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}\sinh^{2/3}\left(\frac{3}{2}H_0 t'\sqrt{\Omega_{\Lambda,0}}\right)}. \tag{158}$$

This has no simple analytic solution and must be integrated numerically. Using t_0 =13.7 billion years, we calculate that at **the present time, the distance to the particle horizon in a flat universe** is

$$d_{h,0} = 4.50 \times 10^{26}\ \mathrm{m} = 14{,}600\ \mathrm{Mpc} = 14.6\ \mathrm{Gpc} \tag{159}$$

The Horizon

Figure 22 uses WMAP values and shows $\mathbf{d_h}$**, the size of the observable universe, as a function of time**.

Of course, when viewing an object near the particle horizon, astronomers see it as it was when the light was emitted, not as it would appear in today's universe.

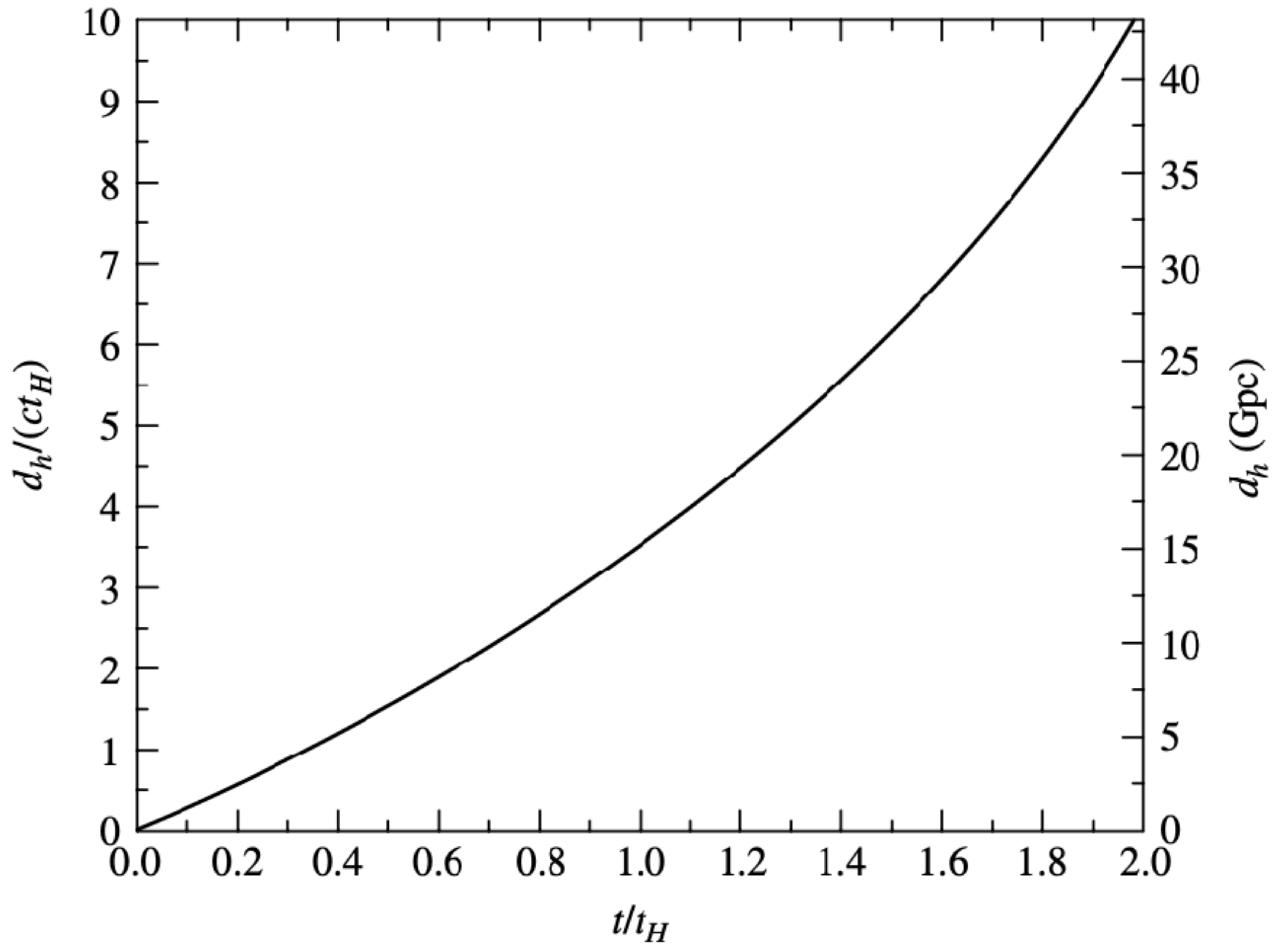


FIGURE 22 The proper distance from Earth to the particle horizon as a function of time, using WMAP values. The horizon distance is expressed as a fraction of ct_H. The right axis shows d_h in billions of parsecs.

The Horizon

Note that the distance to the particle **horizon** in Eqs. (154) and (155) is **proportional to t**, while **the scale factor in the radiation and matter eras is proportional to $t^{1/2}$ and $t^{2/3}$**, respectively. This means that **during those eras the size of the observable universe increased more rapidly than the universe expanded**, so the universe became increasingly causally connected as it aged. However, the integral in Eq. (158)—without the term in front—is just the present distance to the point that will be at the particle horizon at time t, as we can see by comparing Eq. (153) with Eq. (145), evaluated at t_0 with $t_e = 0$. As $t \to \infty$, this integral converges to 19.3 Gpc. **This means that the proper distance *today* to the farthest object that will *ever* be observable in the future is 19.3 Gpc**. **Everything within a sphere, centered on Earth, of radius 19.3 Gpc will eventually become visible, while everything beyond will be forever hidden.** In the future, the particle horizon and the scale factor will both grow exponentially as $e^{H_0\sqrt{\Omega_{\Lambda,0}}}$ (Eq. 133).

Ultimately, an object located at the particle horizon will remain at the particle horizon as the universe expands. The particle horizon will never catch up to any object that is presently more than 19.3 Gpc away, and so its light will *never* reach us.

The Horizon

What will we observe when looking at an object at the ultimate ($t \to \infty$) particle horizon?

Although photons from the object will continue to arrive, they will be increasingly deeply redshifted, and their **arrival rate will decline toward zero due to cosmological time dilation** (Eq. 143). Thus the object will fade from view, apparently frozen in time, as its redshift diverges to infinity. This bears a striking similarity to the description of how we would view an astronaut falling into a black hole, although the physics of the two situations is completely different.

The Horizon

Example 4.1. Helium nuclei were being formed when the temperature was roughly 10^9 K and t=178s. From Eq.(58), the scale factor at that time was $R=2.73\times10^{-9}$. From Eq. (154), the horizon distance was then about

$$d_h(t) = 2ct = 1.07 \times 10^{11} \text{ m} = 0.7 \text{ AU}.$$

This is the diameter of a causally connected region, which we call C, when the universe was 178 s old.

The region C (which has a comoving boundary, so it always contains the same mass) has been expanding along with the rest of the universe since t = 178 s. **How large is C today?** Assuming a flat universe, Eq. (146) shows that C has expanded by a factor of $1/R = 3.66 \times 10^8$, with a present diameter of

$$\frac{d_h(t)}{R(t)} = 3.92 \times 10^{19} \text{ m},$$

about 1.3 kpc. In other words, the largest region that was causally connected at t = 178 s is now just over a kiloparsec across, roughly 8.7×10^{-8} of the present horizon distance, $d_{h,0}$.

The Horizon

This illustrates that **as the early universe aged, the amount of material that is in causal contact increased dramatically**. Today's causally connected universe extends far beyond C because, since t = 178 s, light from more distant regions has had time to arrive and causally connect those regions with C. The comoving boundary of C cannot keep up with the more rapid recession of the particle horizon.

$$\int_0^t \frac{c\,dt'}{R(t')} = \int_{\varpi}^{\varpi_e} d\varpi'. \qquad (160)$$

This illustrates that **as the early universe aged, the amount of material that is in causal contact increased dramatically**. Today's causally connected universe extends far beyond C because, since t = 178 s, light from more distant regions has had time to arrive and causally connect those regions with C. The comoving boundary of C cannot keep up with the more rapid recession of the particle horizon.

The arrival of photons

You may be wondering, if the scale factor R was zero at the Big Bang and everything was right next to everything else, then why has it taken the age of the universe for a Big Bang photon to reach Earth? What is the path followed by the photon?

In the following discussion, the actual complications of the Big Bang will be neglected. Instead, we will consider a perfectly transparent, expanding, flat universe where a single photon is emitted at comoving coordinate ϖ_e at time $t = 0$.

What, then, is the proper distance of that photon from our position ($\varpi = 0$) at a later time t ? The coordinate, ϖ , of the photon at time t may be found from Eq. (137) with $k = 0$,

$$\int_0^t \frac{c\,dt'}{R(t')} = \int_{\varpi}^{\varpi_e} d\varpi'. \tag{160}$$

To simplify the calculation, we will ignore the Λ era and adopt a flat, matter-dominated universe with a scale factor given by Eq. (132),

$$R(t) = \left(\frac{3}{2} H_0 t \sqrt{\Omega_{m,0}}\right)^{2/3}. \tag{161}$$

The arrival of photons

Setting R = 1, the age of this model universe is found to be

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_{m,0}}}, \tag{162}$$

so the scale factor is simply

$$R(t) = \left(\frac{t}{t_0}\right)^{2/3} .$$

Integrating Eq. (160) yields

$$\varpi = \varpi_e - 3ct_0 \left(\frac{t}{t_0}\right)^{1/3} . \tag{163}$$

The arrival of photons

We can evaluate ϖ_e by noting that at $t=t_0$, $\varpi = 0$; thus

$$\varpi_e = 3ct_0, \tag{164}$$

the present horizon distance for this model (Eq. 155). Inserting this value for the photon's starting point into Eq. (163) and multiplying both sides by the scale factor R(t) shows that **the proper distance of the photon from Earth is, as a function of time,**

$$d_p(t) = 3ct_0\left[\left(\frac{t}{t_0}\right)^{2/3} - \left(\frac{t}{t_0}\right)\right] \tag{165}$$

for our model flat universe.

The arrival of photons

Since the Big Bang, the entire system of comoving coordinates has been stretching out.

As shown in Fig. 23, **the initial expansion of the universe actually carried the photon *away* from Earth.** Although the photon's comoving coordinate was always decreasing from an initial value of ϖ_e toward Earth's position at $\varpi = 0$, the scale factor R(t) increased so rapidly that at first the proper distance between the photon and Earth increased with time. This means that a photon emitted from the present particle horizon at t = 0 is only now reaching Earth. Photons emitted from a greater ϖ, beyond the present particle horizon, have yet to arrive—and in fact may *never* arrive if ϖ is sufficiently large that the exponential expansion of the universe ultimately carries the photon away from Earth.

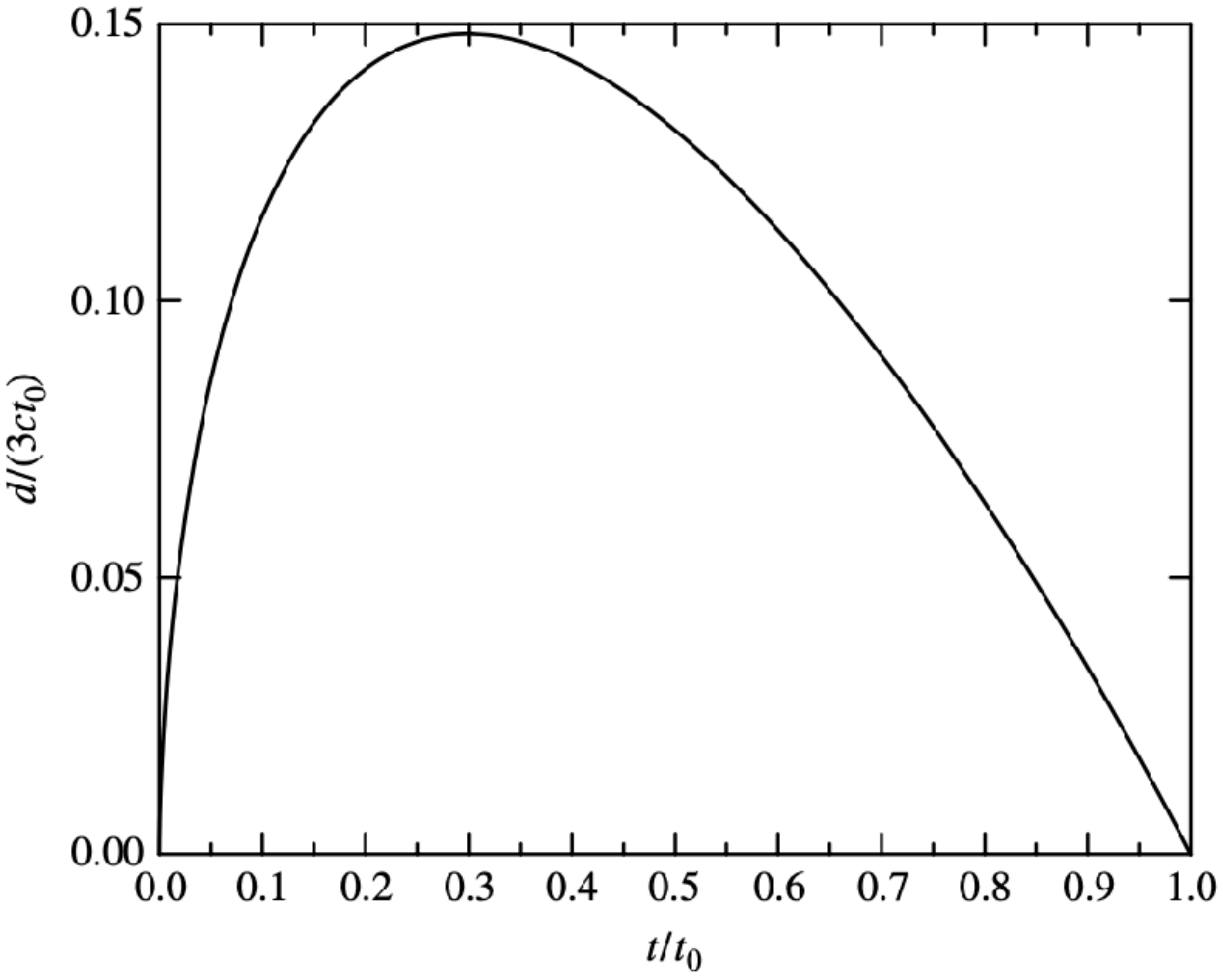


FIGURE 23 The proper distance from Earth of a photon emitted from the present particle horizon at the time of the Big Bang. The photon's proper distance is expressed as a fraction of $3ct_0$.

The maximum visible age

The previous calculation assumes that the photon was emitted at $t = 0$. **Is it possible that the exponential expansion of space could carry a presently visible object away from Earth so fast that the object would never again be seen in the sky from some future time forward?**

To answer this, consider an object (say, a galaxy) that is now visible, meaning that its light was emitted at an earlier time t_e and it arrives today at t_0. Assuming that the galaxy is still visible at some time in the future, we will let the time of the emission of its future photons be t_i and their arrival time here be t_f , where $t_e < t_i$ and $t_0 < t_f$. Applying these conditions to Eq. (140) gives

$$\int_{t_e}^{t_0} \frac{dt}{R(t)} = \int_{t_i}^{t_f} \frac{dt}{R(t)}, \tag{166}$$

where the scale factor R is given by Eq. (131) for a flat universe ($k = 0$). **Because the scale factor increases monotonically, it may be, for sufficiently large t_i, that *no* value of t_f can satisfy this equality.** In that case, a photon emitted at time t_i will never reach Earth. The latest time of emission, t_{mva} , for photons to eventually reach us (the **maximum visible age** of the source) may be found by setting $t_f = \infty$.

The maximum visible age

Just like an object located at the ultimate particle horizon, **as photons from this galaxy continue to arrive, they will be increasingly deeply redshifted and their flux will drop toward zero**. The galaxy will fade from view. The farther a source is, the sooner it will fade away. **This places a fundamental limit on extragalactic astronomy.**

Figure 24 shows that if the redshift of an object is roughly larger than 1.8, then $t_{mva} < t_H$, and we will never see it even as it appears *today*. That is, the light emitted by the object today will never reach Earth because those photons will eventually be carried away from us by the accelerating Hubble flow.

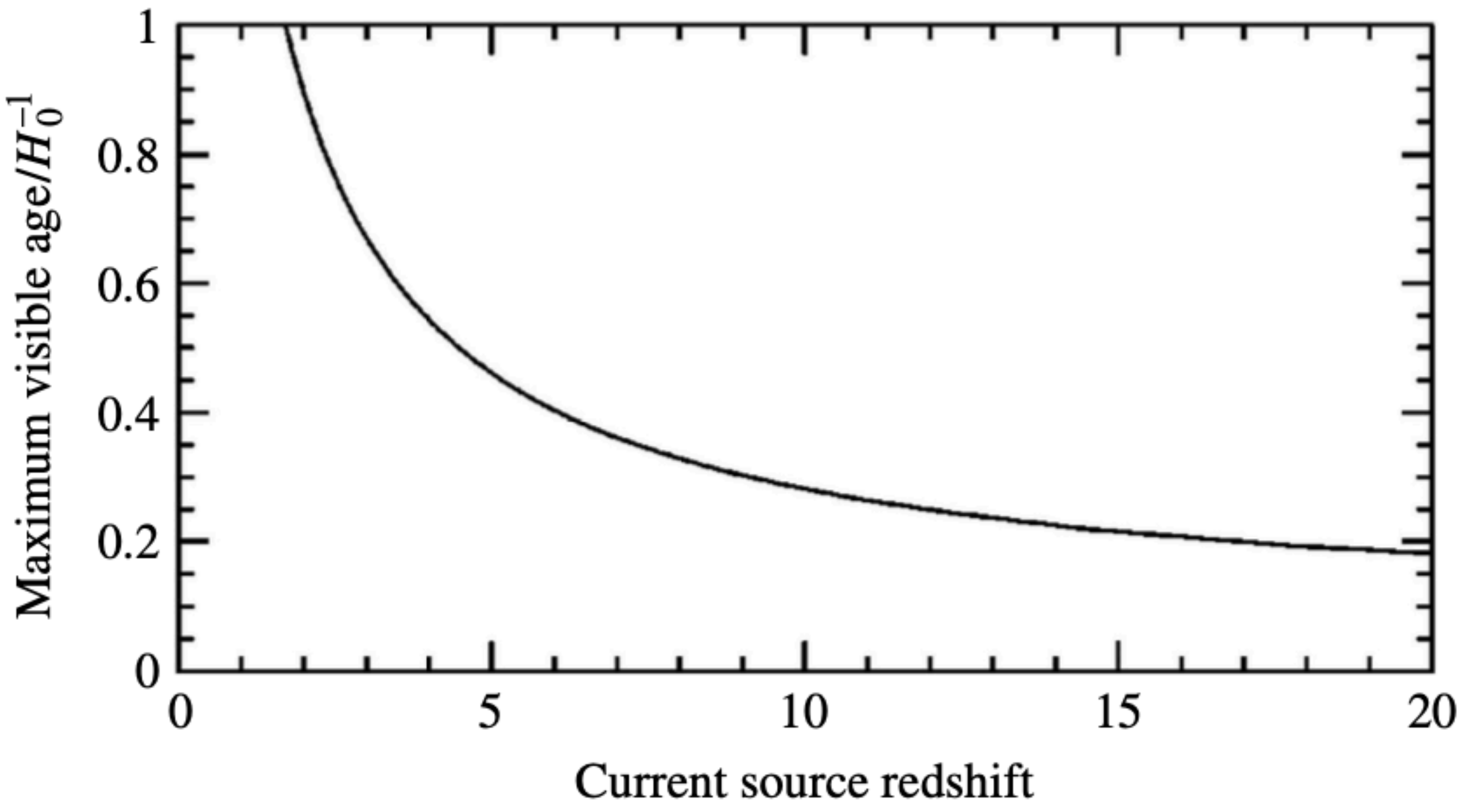


FIGURE 24 The maximum visible age of a source, in units of $t_H = 1/H_0$, as a function of its current redshift. (Figure adapted from Loeb, *Phys. Rev. D*, *65*, 047301, 2002.)

The maximum visible age

Gravitationally bound systems do not participate in the expansion of the universe, so our Solar System and Galaxy will not become causally fragmented.

Objects with a redshift in the range 5–10 can be observed only as they appeared when the universe was approximately 4 to 6 billion years old.

As more light is carried away from Earth by the expansion of the universe, the observable sky will become increasingly empty. Similarly, any signal we send toward a galaxy with $z \approx 1.8$ or more will never arrive. Because there can be no contact between that galaxy and Earth anytime in the future, we are no longer in causal contact with it. As the universe ages, it is becoming causally fragmented, with one region no longer capable of influencing another.

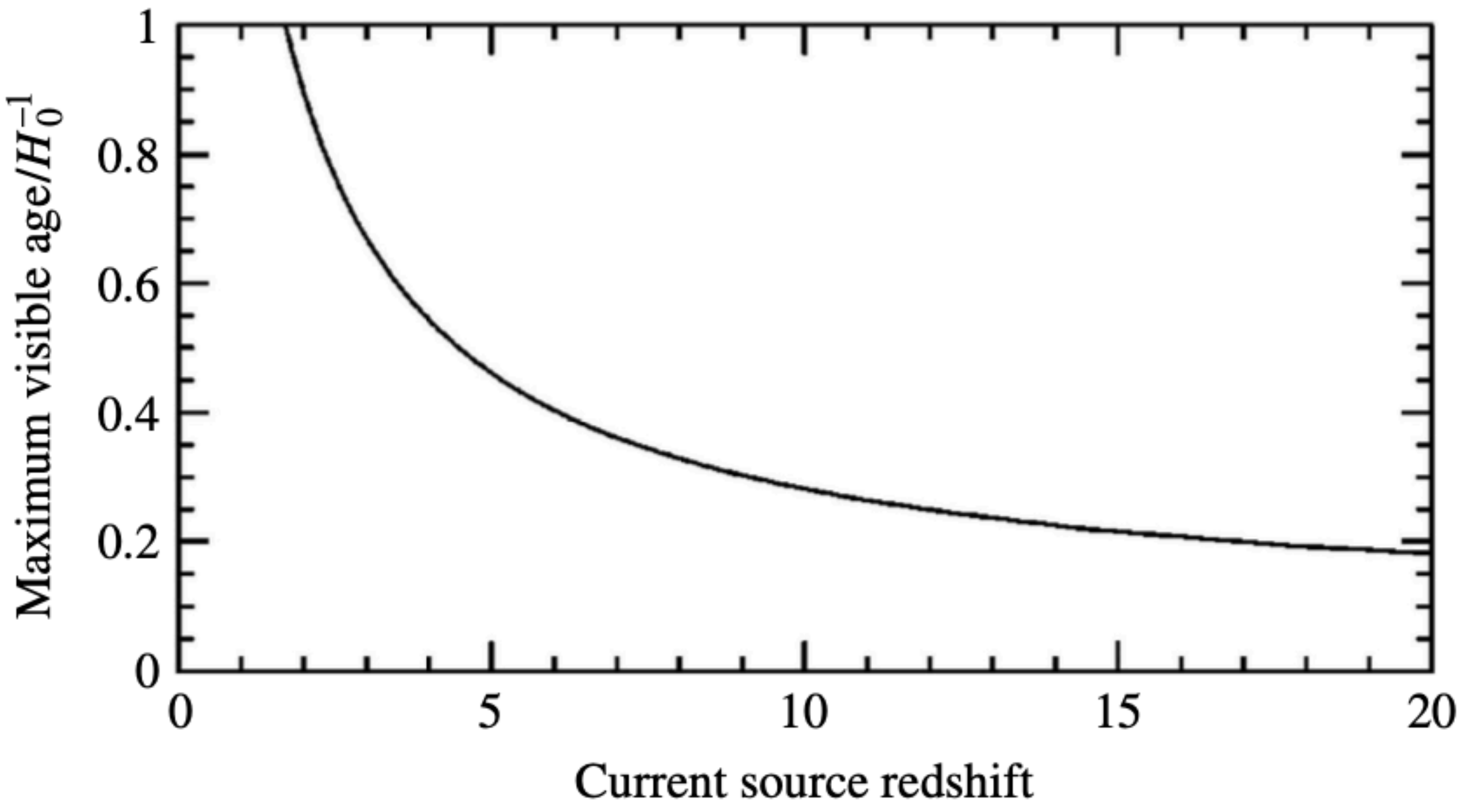


FIGURE 24 The maximum visible age of a source, in units of $t_H = 1/H_0$, as a function of its current redshift. (Figure adapted from Loeb, *Phys. Rev. D*, *65*, 047301, 2002.)

The comoving coordinate

Returning to Eqs. (148–150), we want to express the comoving coordinate ϖ as a function of the redshift z. We begin by using Eq. (145) to find another expression for the present proper distance $d_{p,0}$. Writing $dt = dR/(dR/dt)$, Eq. (145) may be written as

$$d_{p,0} = \int_{R(t_e)}^{R(t_0)} \frac{c\,dR}{R(dR/dt)}.$$

It is useful to use $R(t_0) = 1$, $R(t_e) = 1/(1 + z)$, $dR = -R^2\,dz$ [from differentiating Eq. (4), $R = 1/(1 + z)$] and Eq. (8) to define the dimensionless integral

$$I(z) = H_0 \int_{\frac{1}{1+z}}^{1} \frac{dR}{R(dR/dt)} = H_0 \int_0^z \frac{dz'}{H(z')}. \tag{167}$$

Using Eq. (122), we obtain

$$I(z) \equiv \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\mathrm{rel},0}(1+z')^4 + \Omega_{\Lambda,0} + (1-\Omega_0)(1+z')^2}}. \tag{168}$$

The comoving coordinate

With this definition of the integral I(z), **the present proper distance is**

$$d_{p,0}(z) = \frac{c}{H_0} I(z). \tag{169}$$

Comparing this with Eqs. (148–150) and using Eq. (121) for k, we find our expressions for the comoving coordinate ϖ(z):

$$\varpi(z) = \frac{c}{H_0} I(z) \qquad (\Omega_0 = 1) \tag{170}$$

$$= \frac{c}{H_0\sqrt{\Omega_0 - 1}} \sin\left[I(z)\sqrt{\Omega_0 - 1}\right] \qquad (\Omega_0 > 1) \tag{171}$$

$$= \frac{c}{H_0\sqrt{1 - \Omega_0}} \sinh\left[I(z)\sqrt{1 - \Omega_0}\right] \qquad (\Omega_0 < 1). \tag{172}$$

The comoving coordinate

These exact expressions must be evaluated numerically. For later reference, we define

$$S(z) \equiv I(z) \qquad (\Omega_0 = 1) \tag{173}$$

$$\equiv \frac{1}{\sqrt{\Omega_0 - 1}} \sin\left[I(z)\sqrt{\Omega_0 - 1}\right] \qquad (\Omega_0 > 1) \tag{174}$$

$$\equiv \frac{1}{\sqrt{1 - \Omega_0}} \sinh\left[I(z)\sqrt{1 - \Omega_0}\right] \qquad (\Omega_0 < 1), \tag{2175}$$

so we may simply write

$$\varpi(z) = \frac{c}{H_0} S(z). \tag{176}$$

We can approximate (to second order in z) that S(z) ≃ I (z). Thus

$$\varpi(z) \simeq \frac{c}{H_0} I(z). \qquad (\text{for } z \ll 1) \tag{177}$$

The comoving coordinate

Because the comoving coordinate ϖ is so important in observational cosmology, it will be helpful to find an approximate expression for the integral I(z). (Again we will ignore the brief radiation era, so $\Omega_{rel,0} = 0$ and $\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0}$.) The integrand may be expressed as a Taylor series about z = 0 as

$$I(z) = \int_0^z \left\{1 - (1+q_0)z' + \left[\frac{1}{2} + 2q_0 + \frac{3}{2}q_0^2 + \frac{1}{2}(1-\Omega_0)\right]z'^2 + \cdots\right\} dz' \quad (178)$$

where we have used (Eq. 124) for the deceleration parameter, $q_0 = \frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0}$. Integrating gives our result,

$$I(z) = z - \frac{1}{2}(1+q_0)z^2 + \left[\frac{1}{6} + \frac{2}{3}q_0 + \frac{1}{2}q_0^2 + \frac{1}{6}(1-\Omega_0)\right]z^3 + \cdots . \quad (179)$$

Eqs. (170–172) then provide series expressions for the comoving coordinate ϖ of an object observed at a redshift z.

The comoving coordinate

Note that the squared term in Eq. (179) involves only q_0 and thus depends only on the dynamics of the expanding universe, while the cubed term involves both q_0 and k (through Eq. 121) and so depends on both the dynamics and the geometry of the cosmos. A further simplification comes from using only the first two terms of the series expression for I(z) along with Eq (177) to obtain, to second order in z,

$$\varpi \simeq \frac{cz}{H_0}\left[1 - \frac{1}{2}(1+q_0)z\right] \qquad (\text{for } z \ll 1). \tag{180}$$

Equation (180) is valid regardless of whether or not the universe is flat and whether or not the cosmological constant, Λ, is nonzero. In fact, Eq. (180) can be derived very generally, without reference to the Friedmann equation or any specific model of the universe. The procedure uses the fact that the deceleration parameter is defined as a second time derivative of the scale factor; recall Eq. (54).

The proper distance

An **approximate expression for the proper distance of an object *at the present time***. According to Eq. (169), this is

$$d_{p,0} \simeq \frac{cz}{H_0}\left[1 - \frac{1}{2}(1+q_0)z\right] \qquad (\text{for } z \ll 1). \qquad (181)$$

The first term is just the Hubble law, after applying for the redshift.

Since $q_0 = \Omega_{m,0}/2 - \Omega_{\Lambda,0}$, we see from the second term that larger values of $\Omega_{m,0}$ imply smaller distances (more mass to slow down the expansion of the universe), as do smaller values of $\Omega_{\Lambda,0}$ (less dark energy pressure to speed up the expansion).

The second term involves a departure from the linearity of the Hubble law that can be used to determine the deceleration parameter, q_0. For $q_0 = -0.6$, the second term is 10% of the first when $z = 0.13$.

How can we measure the distance to distant objects in a way that is not relying on redshift?

The Luminosity Distance

We associate the source of the emitted photons with the rate at which energy arrives at a telescope's detectors. Suppose that a radiant **flux F is measured for a source** of light **with a known luminosity L**. (For now, we will assume that F is a bolometric flux, measured over all wavelengths.) Then the inverse square law can be used to define the **luminosity distance**, d_L, of the star by

$$\boxed{d_L^2 \equiv \frac{L}{4\pi F}.} \qquad (182)$$

Let a source of light be located at the origin ($\varpi = 0$) of a comoving coordinate system. The source emits photons that arrive at a spherical surface around the origin for which ϖ = constant > 0. From the Robertson–Walker metric, Eq. (106), the surface area of the sphere at the present time (R = 1) is $4\pi\varpi^2$; see Fig. 25.

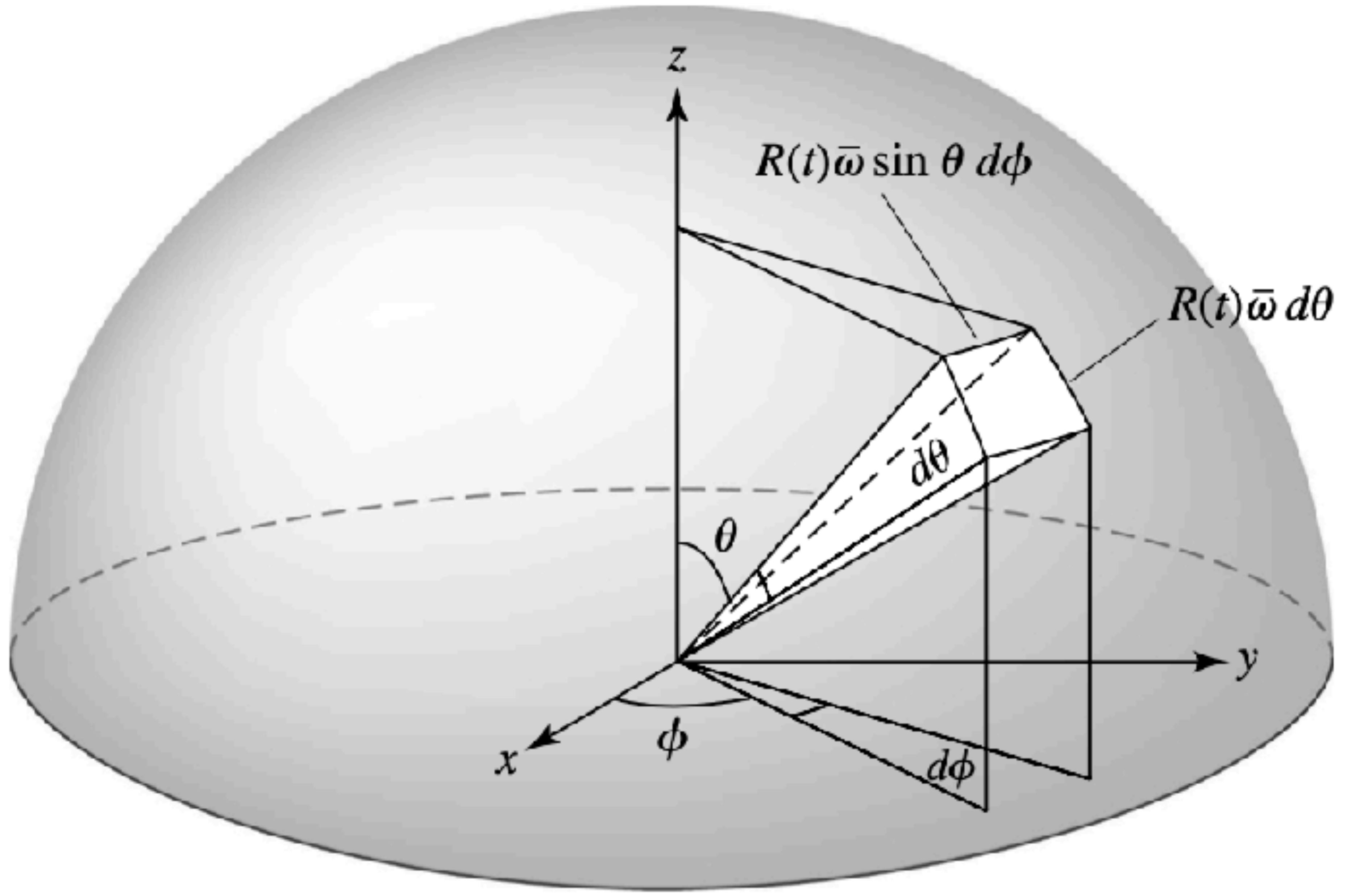


FIGURE 25 An element of area on the surface of a sphere centered at $\varpi = 0$. Integrating over the angles θ and ϕ shows that the surface area of the sphere is $4\pi[R(t)\varpi]^2$.

The Luminosity Distance

After traveling out to ϖ from the source, the photons will be spread over this surface area, and so the radiant **flux will diminish as $1/\varpi^2$**. Two effects, in addition to the inverse square law, act to further reduce the value of the radiant flux measured at this sphere.

The **cosmological redshift**, Eq. (142), shows that the energy of each photon, $E_{photon} = hc/\lambda$, is reduced by a factor of 1 + z.

Also, **cosmological time dilation,** Eq. (143), affects the average time interval *between* photons emitted by the source.

This means that **the rate at which the photons arrive at the sphere is less than the rate at which they leave the source** by another factor of 1 + z. Combining these effects, the radiant flux at the sphere's surface is

$$F = \frac{L}{4\pi\varpi^2(1+z)^2}.$$

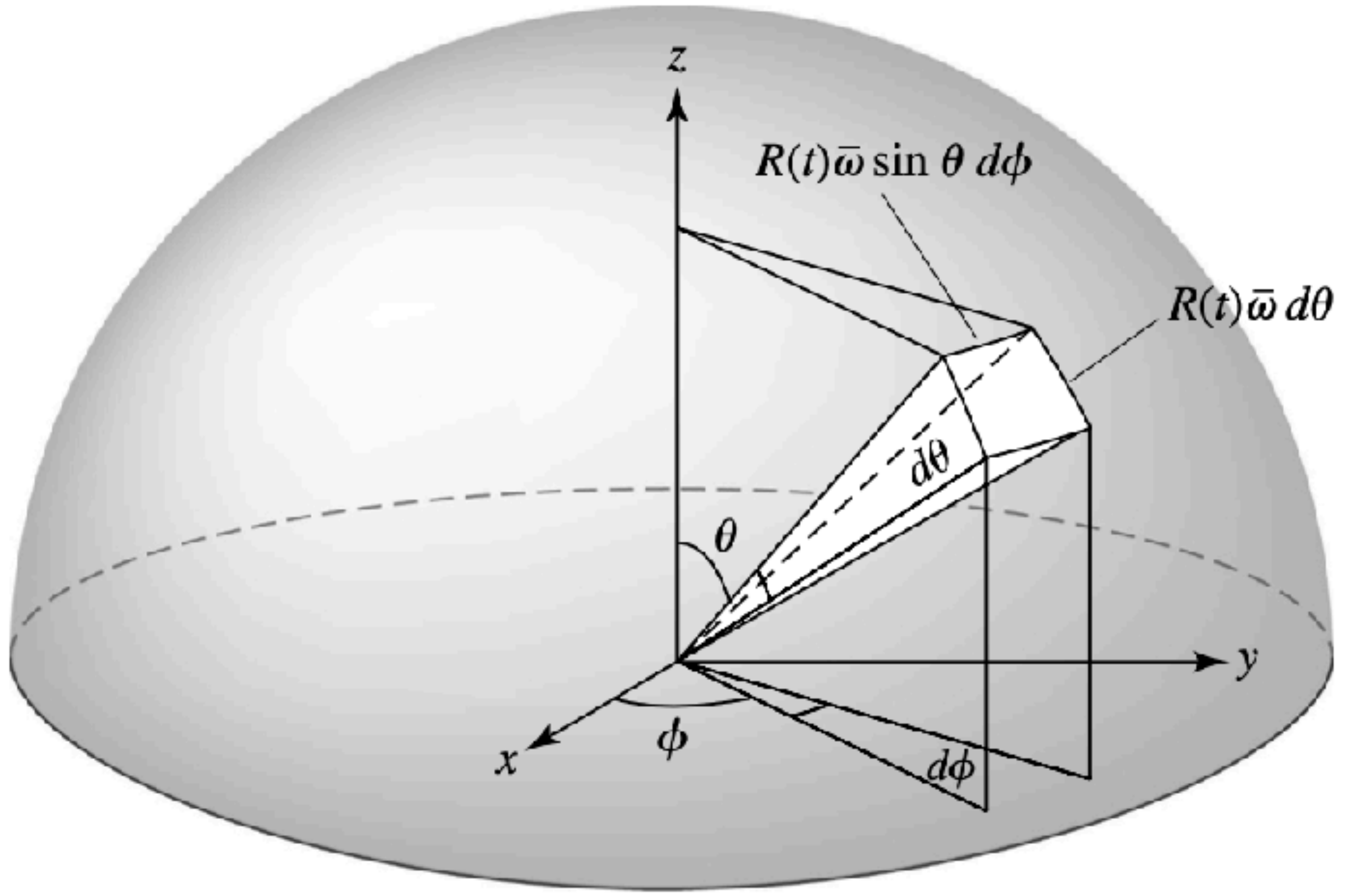


FIGURE 25 An element of area on the surface of a sphere centered at $\varpi = 0$. Integrating over the angles θ and ϕ shows that the surface area of the sphere is $4\pi[R(t)\varpi]^2$.

The redshift-magnitude relation

Substituting this into Eq. (182), we find that

$$\boxed{d_L = \varpi(1+z), \qquad (183)}$$

where ϖ must be evaluated numerically using Eq. (168) and Eqs. (170–172). **The luminosity distance, d_L, is the distance actually measured by the distance modulus m – M.**

Although **the luminosity distance is *not* the same as either the present proper distance (Eqs. 148–150) or the coordinate distance (Eq. 3), the three distances do agree for z ≪ 1.**

Equation (176) shows that the luminosity distance is exactly given by

$$d_L(z) = \frac{c}{H_0}(1+z)S(z). \qquad (184)$$

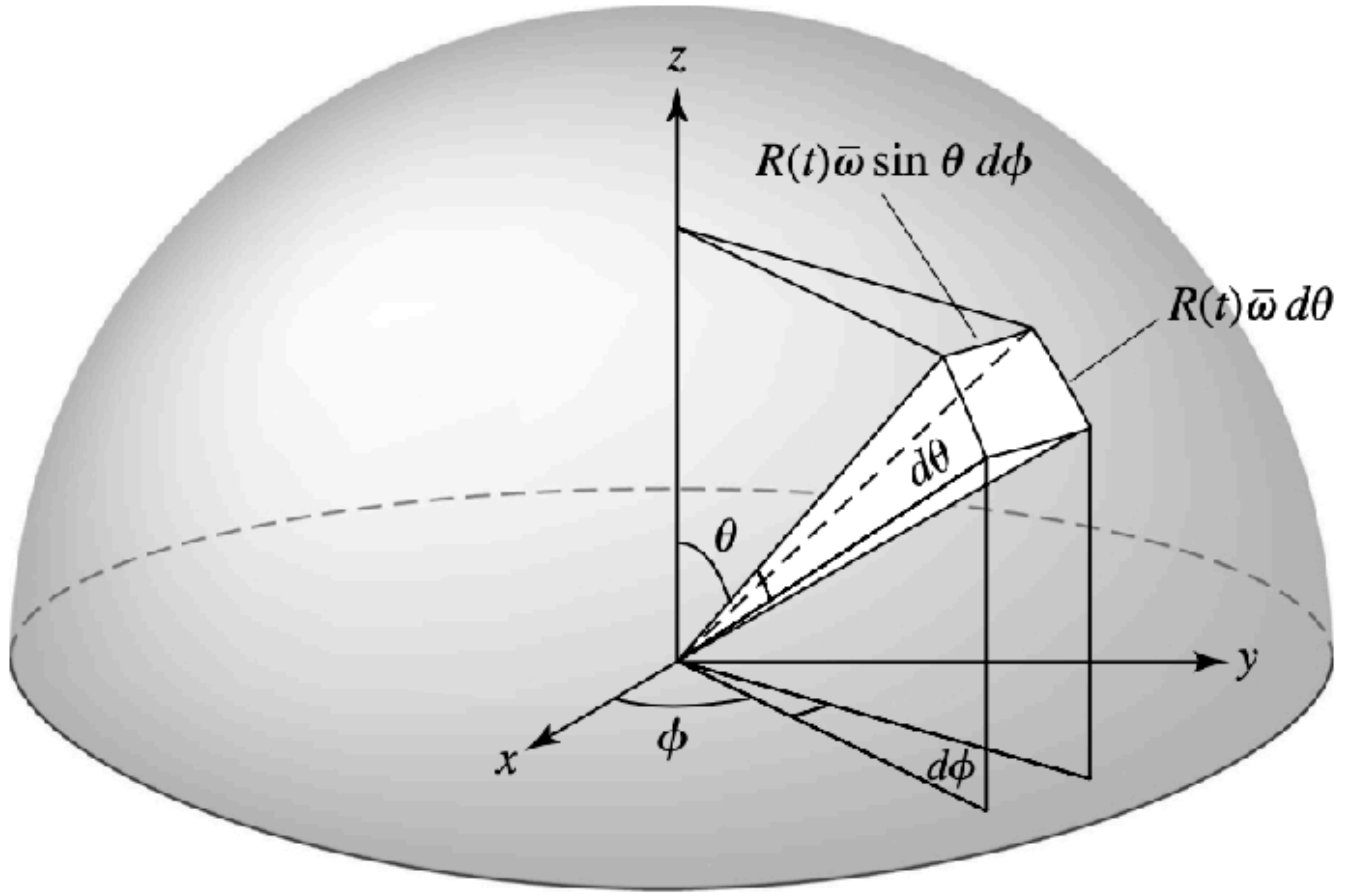


FIGURE 25 An element of area on the surface of a sphere centered at $\varpi = 0$. Integrating over the angles θ and ϕ shows that the surface area of the sphere is $4\pi[R(t)\varpi]^2$.

The Luminosity Distance

Using our approximation to second order in z, Eq. (180),

$$d_L(z) \simeq \frac{cz}{H_0}\left[1 + \frac{1}{2}(1 - q_0)z\right] \qquad \text{(for } z \ll 1\text{)}. \tag{185}$$

Comparing this with Eq. (181), we see that the luminosity distance is approximately equal to the proper distance only when z is very small and the first term in each expansion dominates. For larger values of z, $d_p(z) < d_L(z)$.

Ultimately, the exact expressions for both $d_p(z)$ and $d_L(z)$ are the most useful: Eqs. (169) and (184), respectively, evaluated by numerical integration.

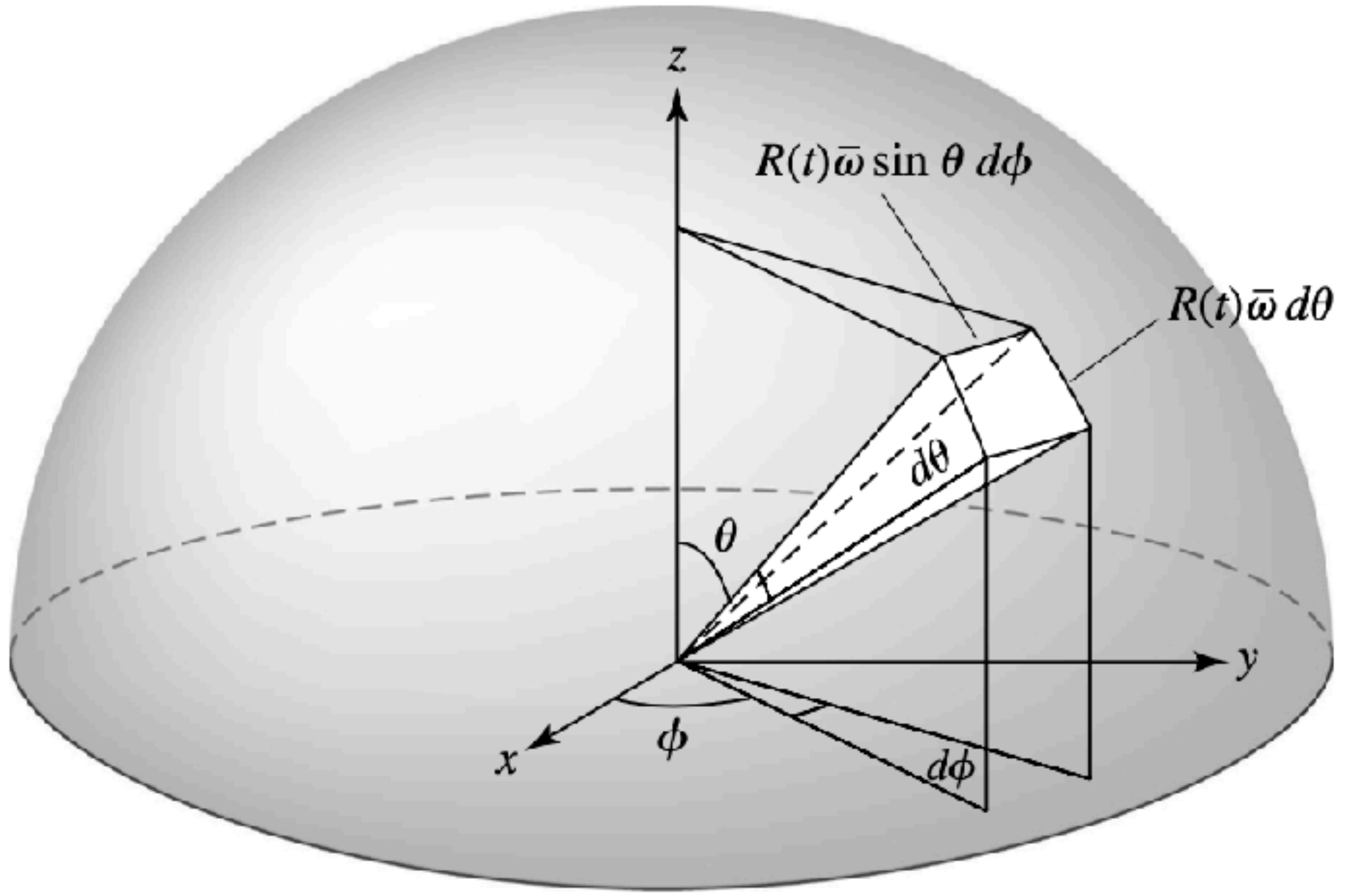


FIGURE 25 An element of area on the surface of a sphere centered at $\varpi = 0$. Integrating over the angles θ and ϕ shows that the surface area of the sphere is $4\pi[R(t)\varpi]^2$.

The Luminosity Distance

The **redshift–magnitude relation** comes from using the luminosity distance for the distance modulus,

$$m - M = 5\log_{10}(d_L/10\ \text{pc}) \tag{186}$$

Equation (184), along with $H_0 = 100h$ km s^{-1} Mpc^{-1} (Eq. 13), quickly produces

$$\begin{aligned} m - M &= 5\log_{10}\left[\frac{c}{(100\ \text{km s}^{-1}\ \text{Mpc}^{-1})(10\ \text{pc})}\right] - 5\log_{10}(h) \\ &\quad + 5\log_{10}(1+z) + 5\log_{10}[S(z)] \\ &= 42.38 - 5\log_{10}(h) + 5\log_{10}(1+z) + 5\log_{10}[S(z)]. \end{aligned} \tag{187}$$

The Luminosity Distance

In the same manner, using the approximate Eq. (185) for the luminosity distance with Eq. (186) leads to, for $z \ll 1$,

$$m - M \simeq 5\log_{10}\left[\frac{c}{(100\text{ km s}^{-1}\text{ Mpc}^{-1})(10\text{ pc})}\right] - 5\log_{10}(h) + 5\log_{10}(z) + 5\log_{10}\left[1 + \frac{1}{2}(1-q_0)z\right] \qquad (\text{for } z \ll 1).$$

Expanding the last term on the right in a Taylor series about $z = 0$ and keeping only the first-order terms in z results in

$$\boxed{m - M \simeq 42.38 - 5\log_{10}(h) + 5\log_{10}(z) + 1.086(1-q_0)z \qquad (\text{for } z \ll 1). \qquad (188)}$$

We can measure q_0 using this relation

The Luminosity Distance

Figure 26 shows the redshift z plotted on a logarithmic scale as a function of m – M.

For z ≪ 1, the redshift–magnitude relation is linear. Observations confirm the linearity of the log10(z) term (which is just the Hubble law) for small z.

At larger z, **the fourth term** on the right-hand side of Eq. (188), 1.086 (1 – q_0)z, **will cause the line to curve upward.** **Accurately measuring this departure from a straight line allows the value of the deceleration parameter to be determined.** At still larger z, the curve is sensitive to the individual values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

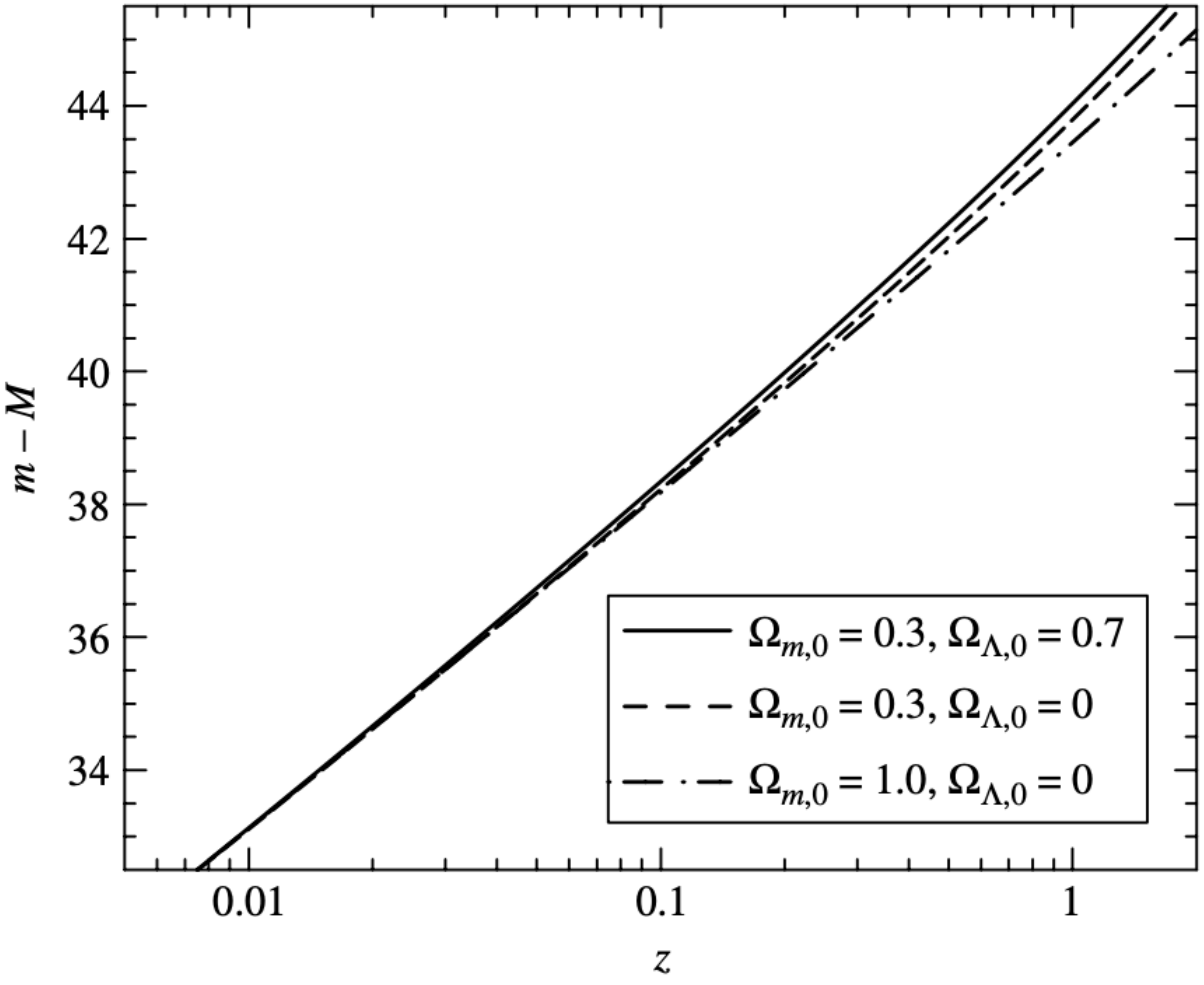


FIGURE 26 The redshift–magnitude relation for $h = 0.71$ and several values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

The Luminosity Distance

The cosmological redshift affects the measurement of an object's spectrum because these observations are usually made within a specific wavelength region. For example, observations made in the V-band at 550 nm can be affected as the **cosmological redshift brings shorter-wavelength radiation into the V band**. This effect can be corrected for by adding a compensating term called the **K-*correction*** to Eq. (188) if the spectrum, I_λ, of the object is known.

In the mid-1990's two rival teams of astronomers made **observations of Type Ia supernovae at cosmological distances**. Both teams were astonished to discover that supernovae observed with a redshift of $z \approx 0.5$ were about 0.25 magnitude *dimmer* than expected for a universe with $\Omega_{m,0} \simeq 0.3$ and $\Lambda = 0$. The supernovae were farther away than they would be in this canonical decelerating universe. The possibility of an accelerating universe and a nonzero cosmological constant immediately leapt to their minds, but it took nearly a year of intense work to eliminate several plausible alternative explanations.

The Luminosity Distance

The redshift–magnitude diagram in Fig. 27 shows a more recent results of these two teams.

Both groups found that their analyses **ruled out a flat universe with $\Omega_{m,0}$= 1 and Λ = 0** (the scenario championed by most theorists at the time) and were **also incompatible with an open universe** having $\Omega_{m,0} \simeq 0.3$ and $\Lambda = 0$.

Instead, their findings **favored a universe with $\Omega_{m,0} \simeq 0.3$ and $\Omega_{\Lambda,0} = 0.7$.**

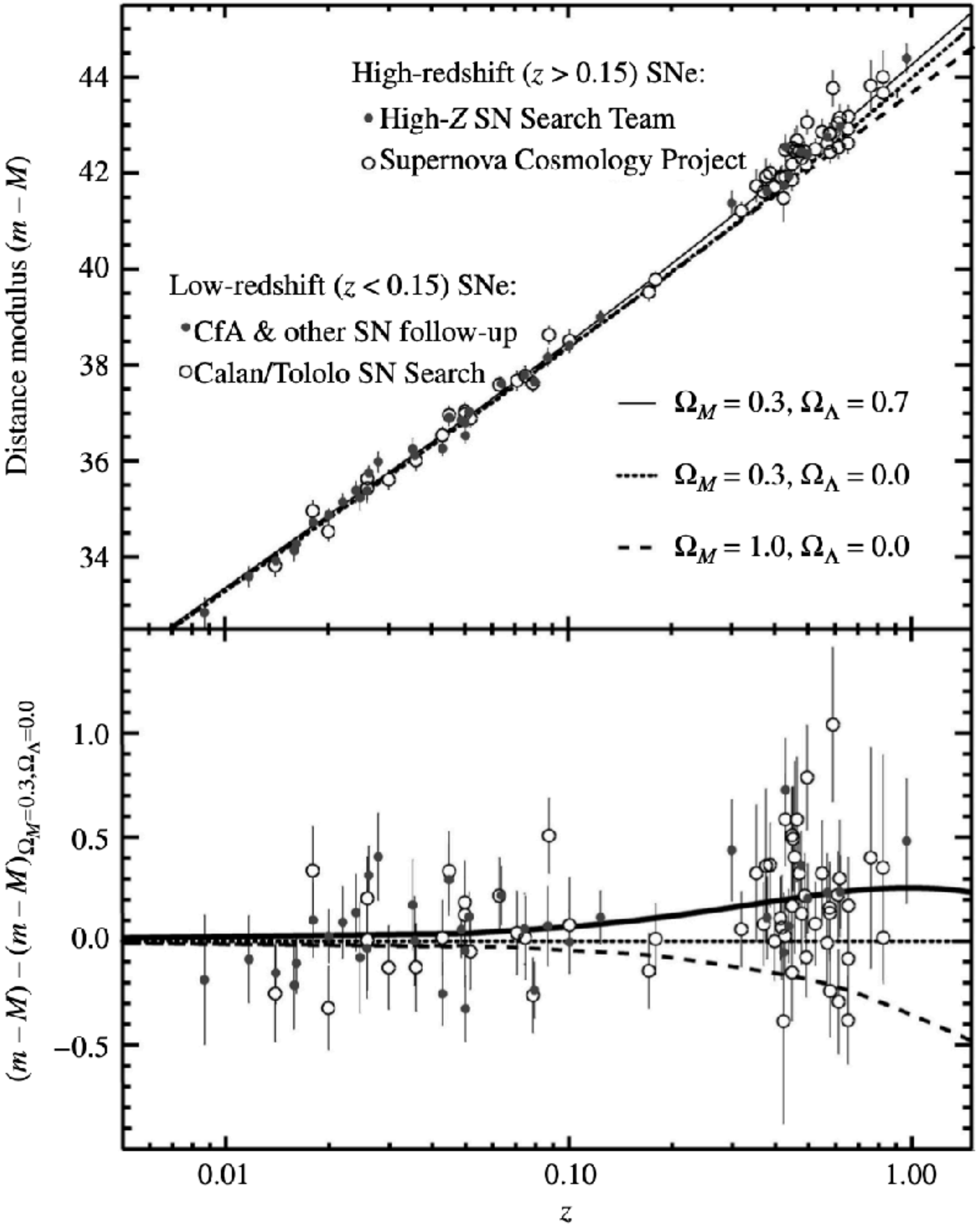


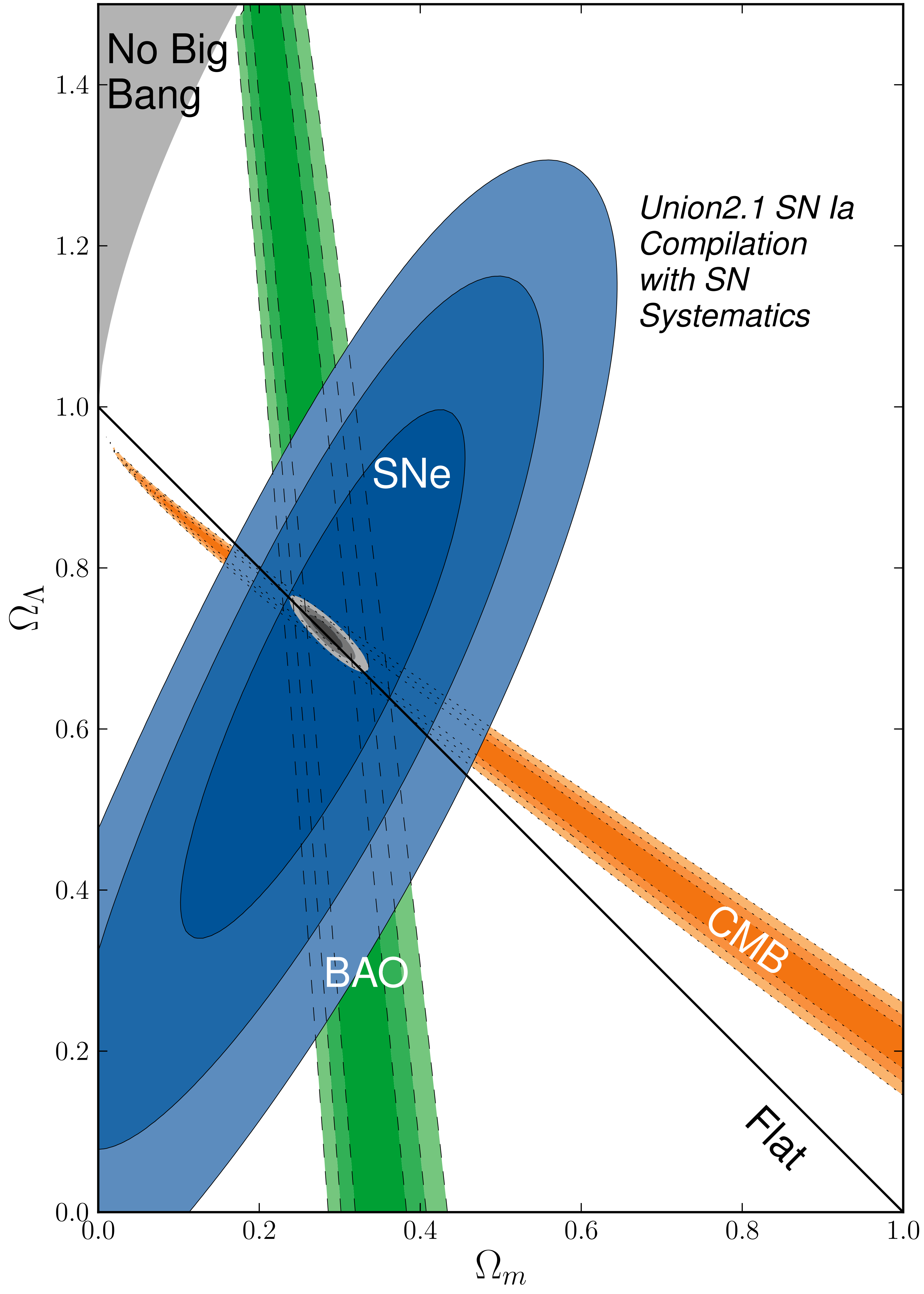
FIGURE 27 The redshift–magnitude relation measured for high-z supernovae. The K-correction has been applied to the apparent magnitudes. The lower graph shows the data after subtracting the theoretical curve for $\Omega_{m,0} = 0.3, \Omega_{\lambda,0} = 0$. [Figure adapted from Perlmutter and Schmidt, *Supernovae and Gamma-Ray Bursters*, K. Weiler (ed.), *Lecture Notes in Physics*, *598*, 195, 2003. Data from Perlmutter et al, *Ap. J.*, *517*, 565, 1999 (SCP) and Riess et al, *A. J.*, *116*, 1009, 1998 (HZSNS).]

The Luminosity Distance

The Figure shows the location on the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane of the **most likely set of values that are consistent with the high-z supernovae results.**

Regions with different colours represent different methods to measure these parameters.

Supernova Cosmology Project
Suzuki, et al., *Ap.J.* (2011)



https://supernova.lbl.gov/union/

The Luminosity Distance

If we look at supernovae beyond $z_{accel} = 0.76$, when the universe started accelerating, we should find the signature of a decelerating universe.

Figure 29 shows further results of observations of high-z supernovae. **Supernovae in this high-z sample appear *brighter* than they would if the universe had expanded at a constant rate (i.e., with $\Lambda = 0$),** as expected for the deceleration phase of the early universe.

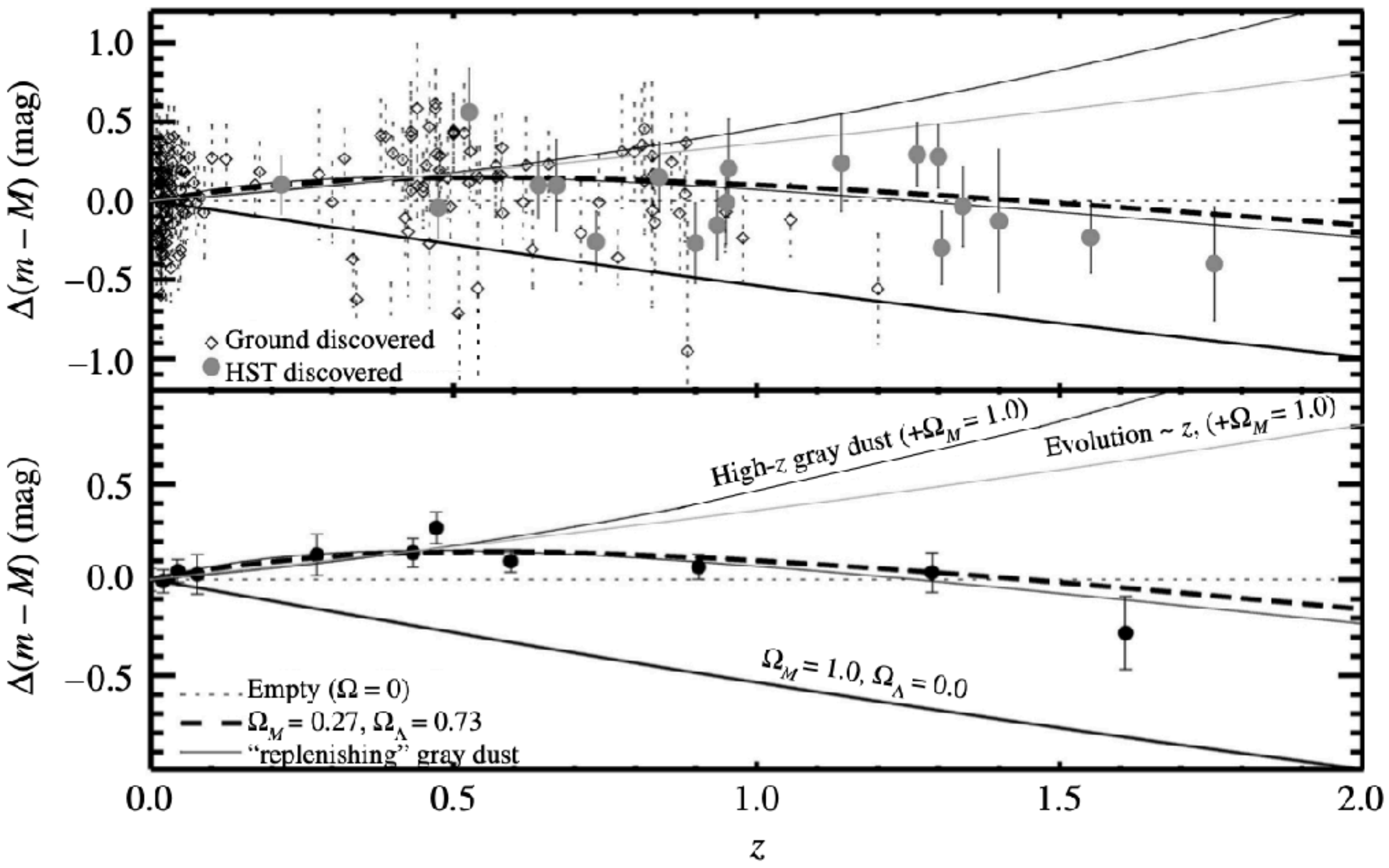


FIGURE 29 The redshift–magnitude relation measured for very high-z supernovae. The K-correction has been applied to the apparent magnitudes, and the theoretical curve for $\Omega = 0$ (a "coasting universe") has been subtracted from the data. The lower graph illustrates the averages of binned data (grouped according to redshift) and compares them to curves of alternative models incorporating "gray dust" or evolutionary effects for supernovae. (Figure adapted from Riess et al, *Ap. J.*, *607*, 665, 2004.)

The Luminosity Distance

It is perhaps surprising that the values of the Hubble constant obtained from the redshift–magnitude diagrams (**$\mathbf{H_0}$ ~ 70** ± 10 km s^{-1} Mpc^{-1}) are not better determined. The spread in the values of H_0 obtained by various groups using supernovae is due to their different calibrations of **Cepheid distances.**

This systematic uncertainty does not affect the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ because these are determined by the *departures from linearity* in the redshift–magnitude diagram.

Angular diameter distance

Another measure of an object's distance may be found by comparing its linear diameter D (assumed known) with its observed angular diameter θ (assumed small). The **angular diameter distance**, d_A is then defined to be

$$\boxed{d_A \equiv \frac{D}{\theta}.} \qquad (189)$$

To place this into context with our previous results, consider a galaxy of redshift z located at comoving coordinate ϖ.

We can use $d_L = \sqrt{-(ds)^2}$ to find an expression for D, the proper distance from one side of the galaxy to the other. Integrating the Robertson– Walker metric (Eq. 106) across the galaxy in the plane of the sky with dt = d ϖ = dϕ = 0, we obtain

$$D = R(t_e)\varpi\,\theta = \frac{\varpi\,\theta}{1+z}.$$

Angular diameter distance

D is the galaxy's diameter at the time t_e, when the light we observe was emitted. Since the light from the galaxy traveled a radial path to Earth, θ is the angular size of the galaxy as measured by astronomers. Equation (176) can be used to express the **diameter**

$$D = \frac{c}{H_0} \frac{S(z)\theta}{1+z}. \tag{190}$$

Thus the angular diameter distance is

$$d_A = \frac{c}{H_0} \frac{S(z)}{1+z}. \tag{191}$$

From Eq. (123) we find that the angular diameter distance and the luminosity distance are related by

$$d_A = \frac{d_L}{(1+z)^2}. \tag{192}$$

Angular diameter distance

Figure 30 shows a graph of θ in units of H_0D/c,

$$\frac{c\theta}{H_0 D} = \frac{(1+z)}{S(z)}, \qquad (193)$$

as a function of the redshift z for several model universes.

It is surprising that **the angular diameter of a galaxy does *not* continue to decrease with distance**. In fact, beyond a certain redshift, the angular size actually *increases* with distance. This is due to **the universe acting as a sort of gravitational lens, enlarging the appearance of a galaxy beyond what would be expected in a static Euclidean universe**.

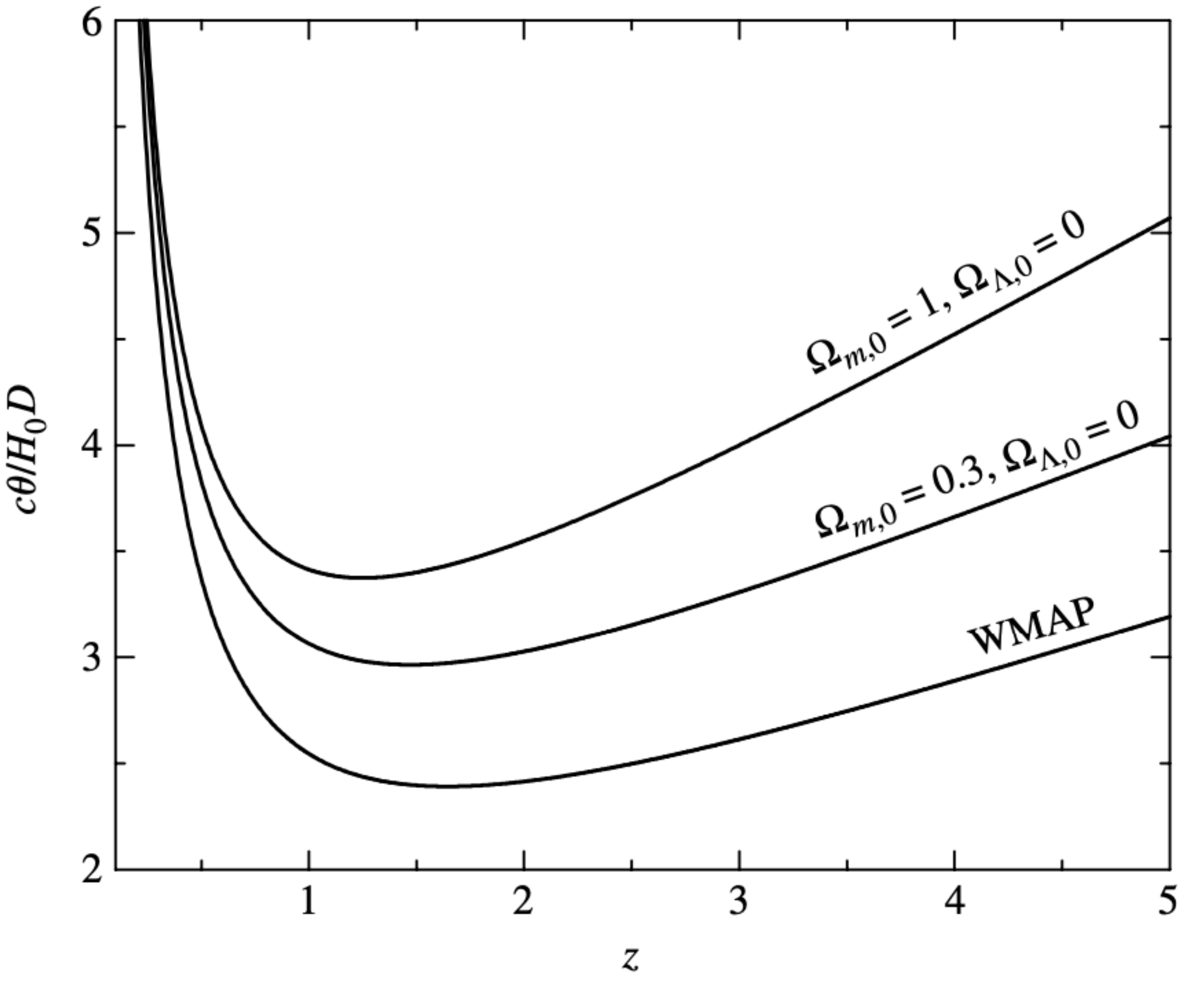


FIGURE 30 The angular diameter θ of a galaxy in units of H_0D/c for several values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

Angular diameter distance

In principle, observations of galaxies of a known linear diameter D would allow observers to determine which values of cosmological parameters are preferred. In practice, however, galaxies do not have sharp boundaries, and they evolve as the universe ages.

The most productive use of the angular diameter distance has been coupled with observations of the Sunyaev–Zel'dovich effect.

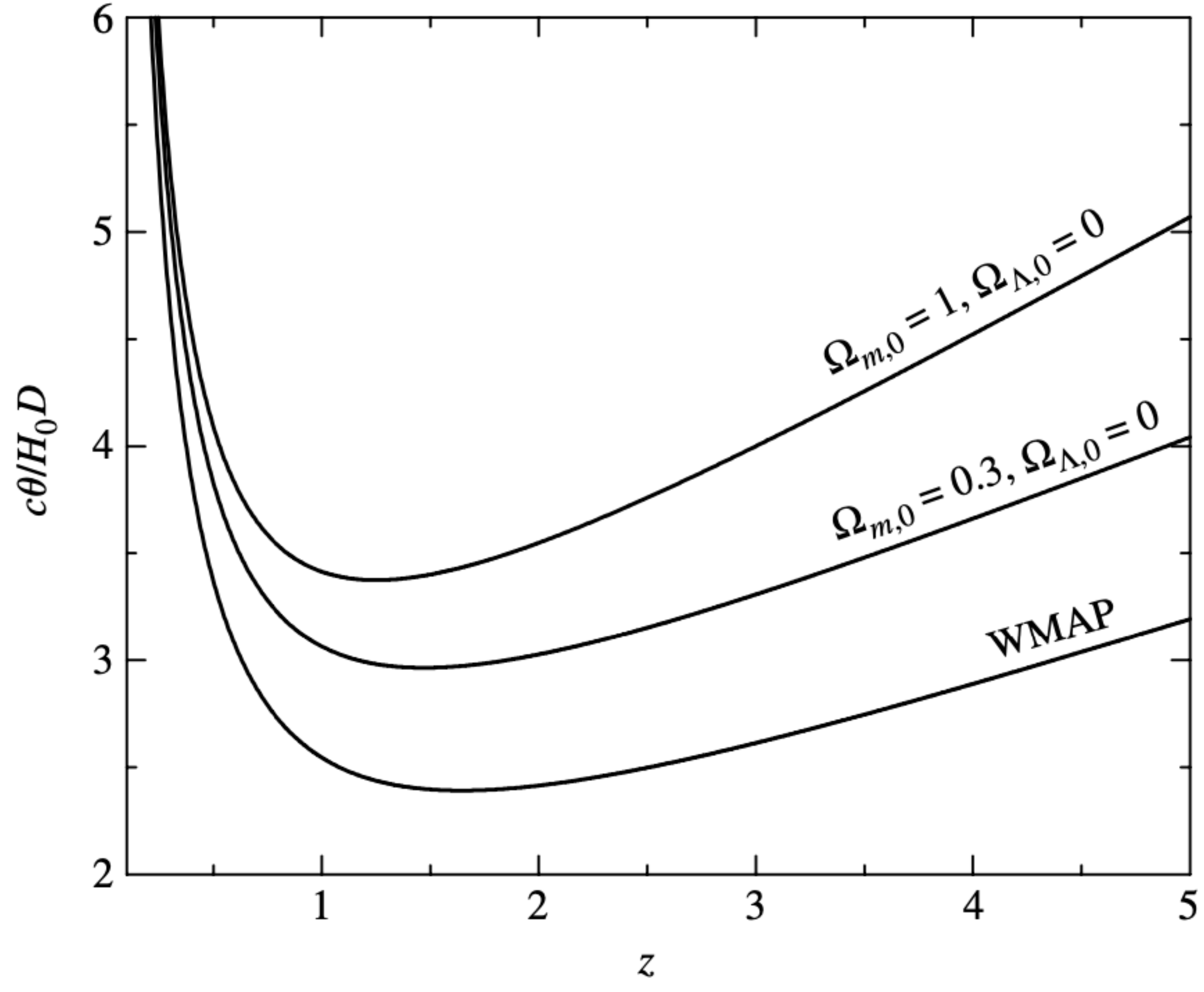


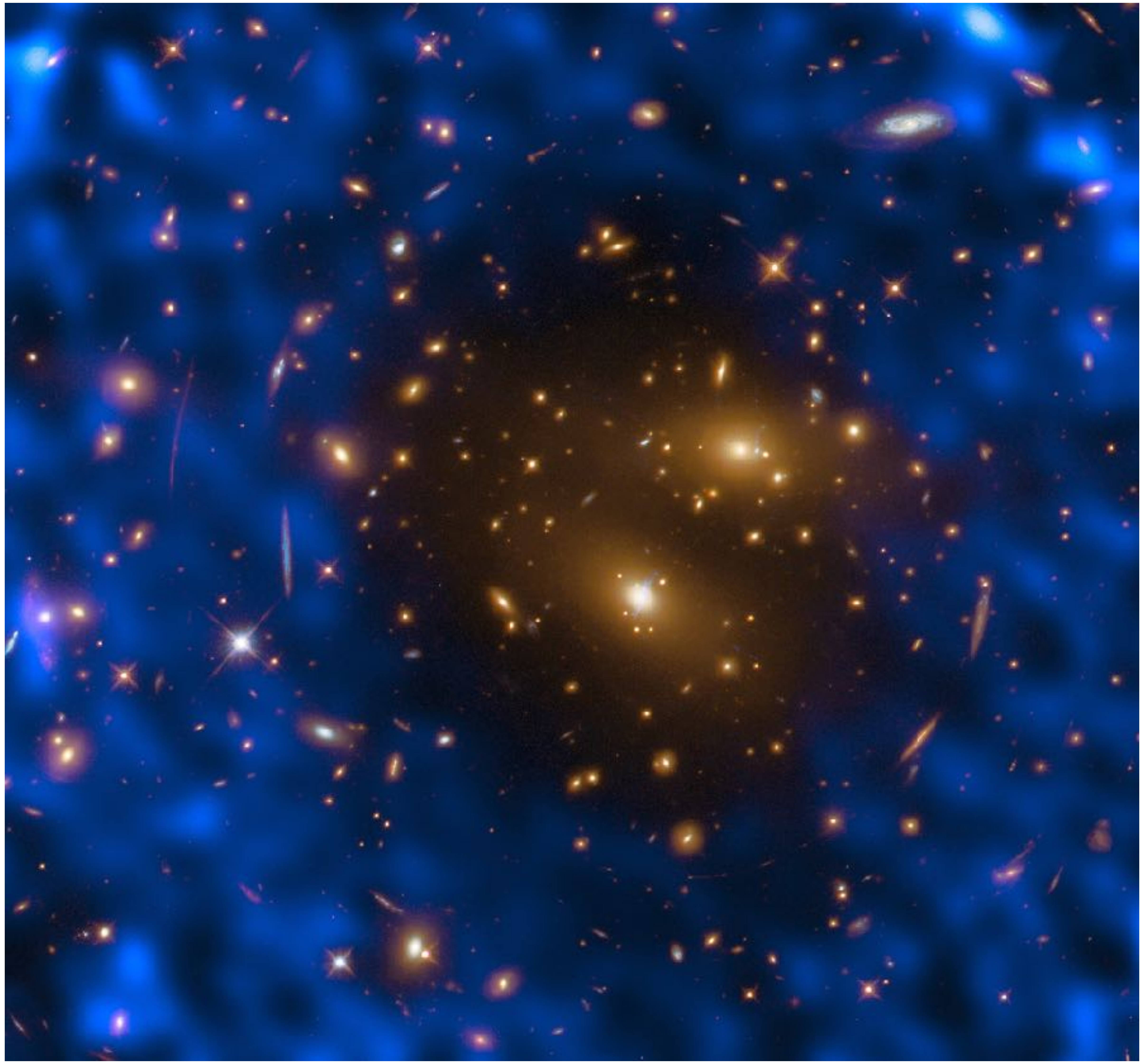
FIGURE 30 The angular diameter θ of a galaxy in units of $H_0 D/c$ for several values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$.

Angular diameter distance

The Sunyaev–Zel'dovich effect provides an independent determination of the Hubble constant. Measurements of $\Delta T/T_0$ (Eq. 64) along with the X-ray flux F_X and temperature T_e of the intracluster gas in rich clusters of galaxies can be used to model the physical properties of the cluster.

Comparing the calculated diameter D of the cluster with its measured angular diameter θ yields d_A, the cluster's angular diameter distance.

On the other hand, the **measured X-ray flux** from the cluster and for the X-ray luminosity of the intracluster gas **determine the cluster's luminosity distance**.



https://en.wikipedia.org/wiki/RX_J1347.5%E2%88%921145

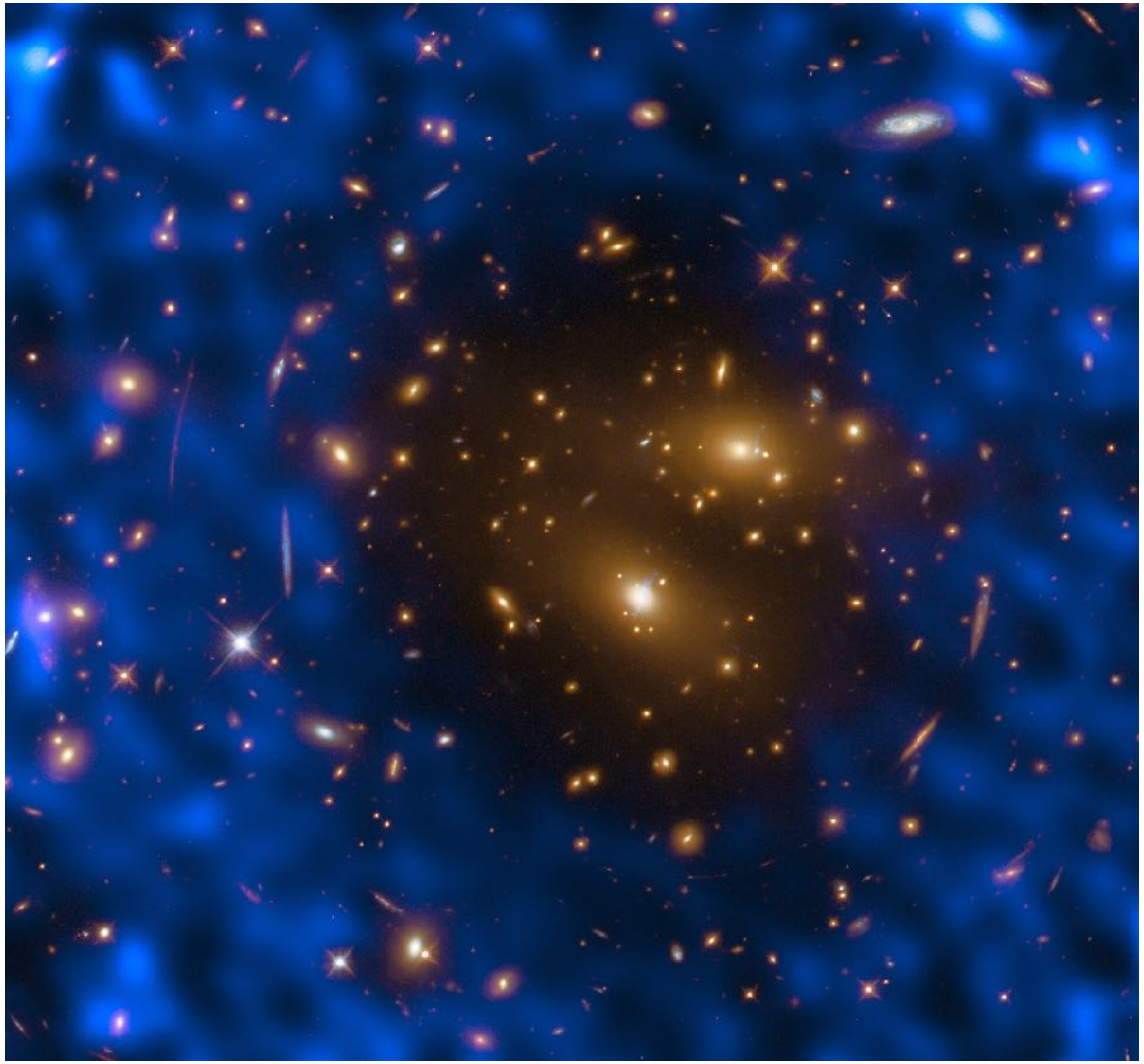
Angular diameter distance

The connection between these two distances, Eq. (192), can then be used to **calculate the Hubble constant.**

$$H_0 = Cf(z)\frac{F_X T_e^{3/2}}{\theta(\Delta T/T_0)^2},$$

where f (z) is a function of the redshift z of the cluster and C is a constant.

A team of astronomers measured the value of the Hubble constant for five clusters and obtained an average of $H_0 = 65$ km s^{-1} Mpc^{-1}, consistent with the values obtained from other recent measurements.

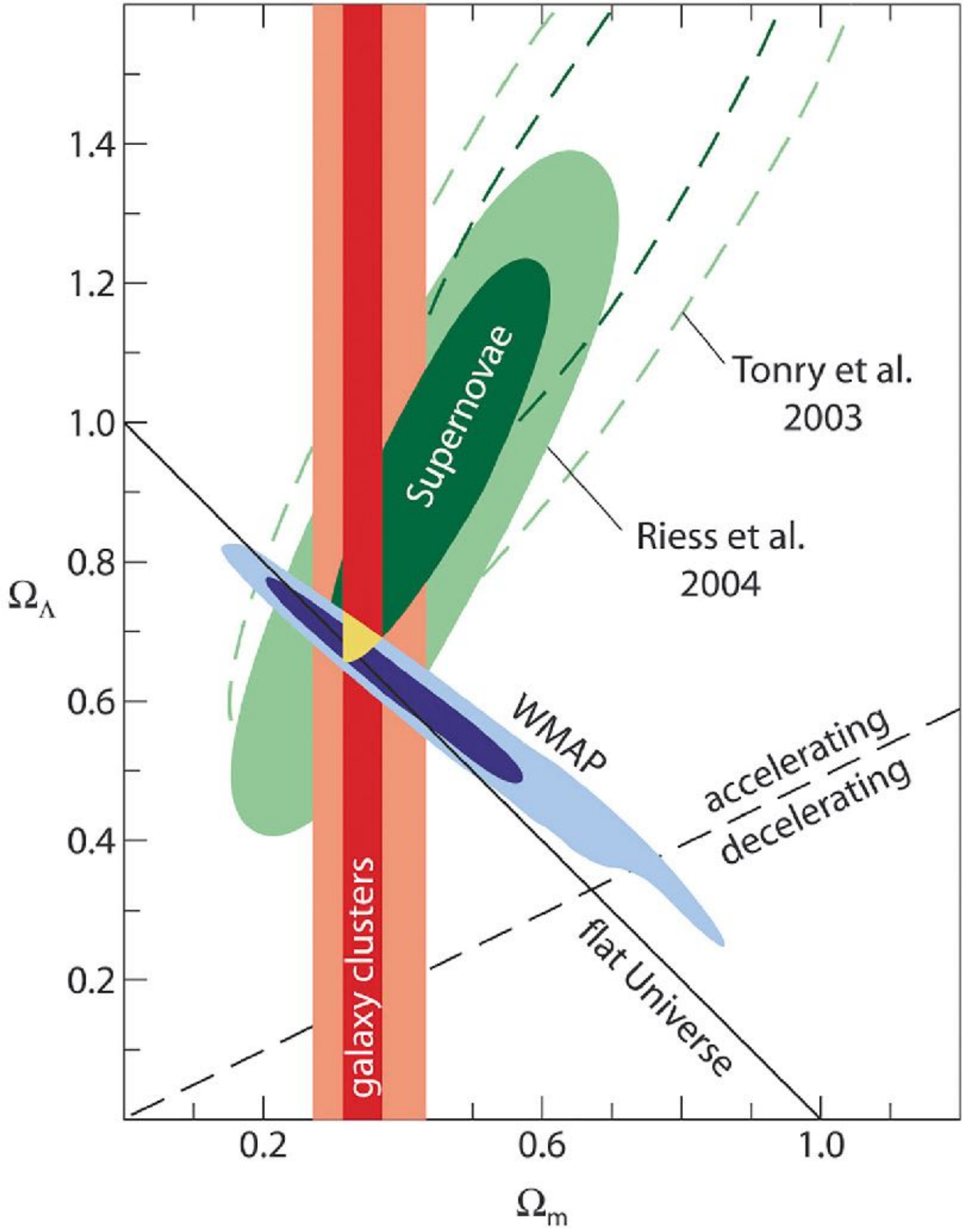


https://en.wikipedia.org/wiki/RX_J1347.5%E2%88%921145

Combined constrains

Current observational constraints on the cosmic density of all matter including dark matter (Ω_M) and the dark energy (Ω_Λ) relative to the density of a critical-density Universe.

All three observational tests by means of supernovae (green), the cosmic microwave background (blue) and galaxy clusters converge at a Universe around $\Omega_M \sim 0.3$ and $\Omega_\Lambda \sim 0.7$. The dark red region for the galaxy cluster determination corresponds to 95% certainty (2-sigma statistical deviation) when assuming good knowledge of all other cosmological parameters, and the light red region assumes a minimum knowledge. For the supernovae and WMAP results, the inner and outer regions corespond to 68% (1-sigma) and 95% certainty, respectively. References: Schuecker et al. 2003; Tonry et al. 2003; Riess et al. 2004



https://www.eso.org/public/spain/images/eso0419d/?lang

Evidence for cosmology models

- The redshift-magnitude relation
- The Luminosity distance (Supernova Ia)
- Angular size distance (Sunyaev–Zel'dovich effect)
- The properties of the CMB