Introduction to Astrophysics and Cosmology

Newtonian Cosmology

Based on: Chapter 29 of An Introduction to Modern Astrophysics

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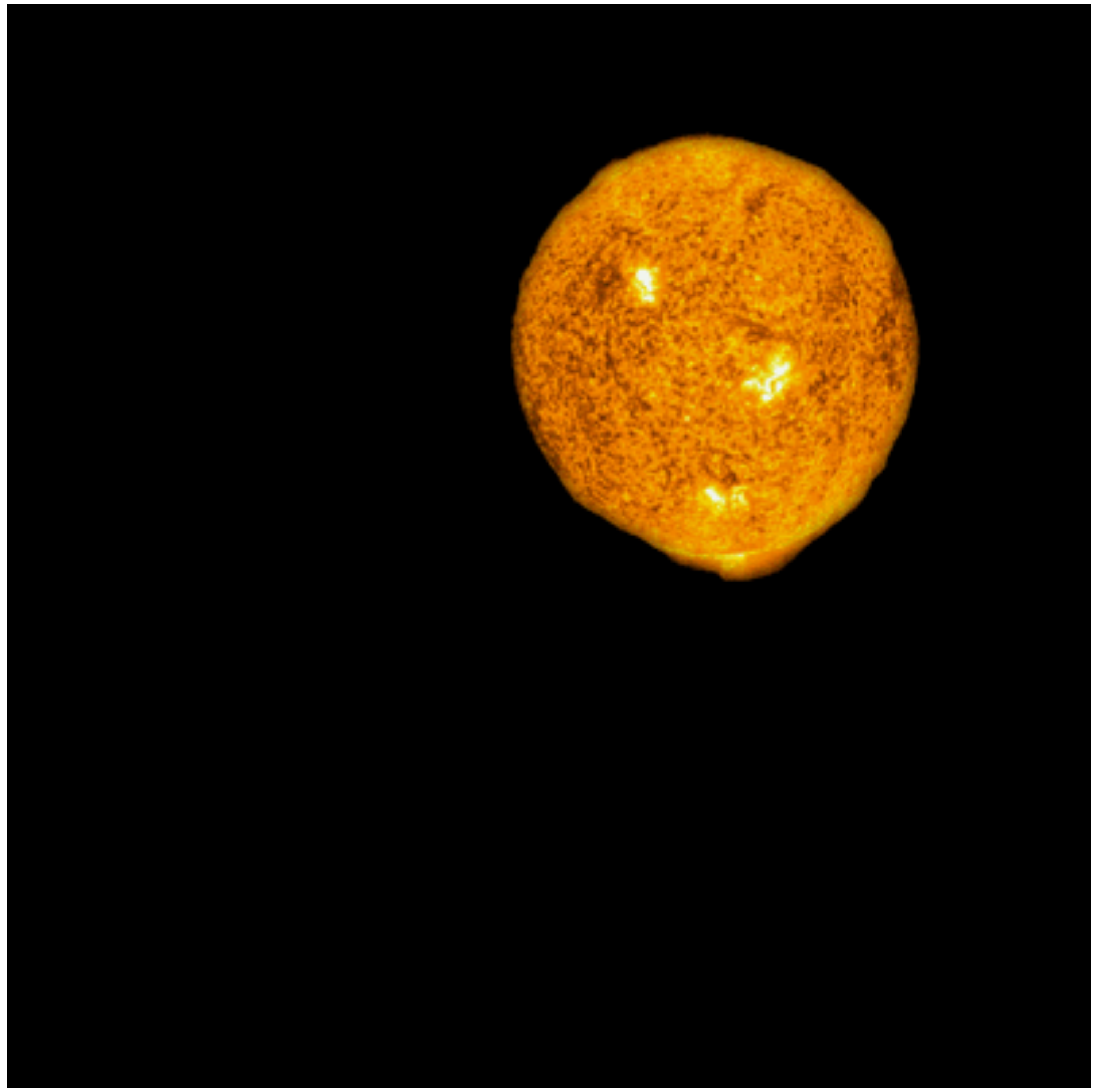
Newtonian cosmology



Cosmology is the study of the origin and evolution of the universe.

Although general relativity is required for a complete understanding of the structure and evolution of the universe, it is useful to develop some intuition by first considering the expansion of the universe from a **Newtonian point of view**.

Newton believed in an infinite static universe filled with a uniform scattering of stars. If the distribution of matter did not extend forever, he realized, then it would collapse inward due to its own self-gravity. However, Newton's contemporary, Edmund Halley, worried about a sky filled with an infinite number of stars. **Why then, asked Halley, is the sky dark at night?**

Olbers's Paradox

Why then, asked Halley, is the sky dark at night?

This question was put in its strongest form Heinrich Olbers. Olbers argued in 1823 that if we live in an infinite, transparent universe filled with stars, then in any direction one looks in the night sky, one's line of sight will fall on the surface of a star. This conclusion is valid regardless of whether the stars are uniformly distributed, as Newton believed, or grouped in galaxies. Olbers's argument was so strong that its disagreement with the obvious fact that the night sky is indeed dark became known as **Olbers's paradox**.

Olbers believed that the answer to this paradox was that space is not transparent. The ideas of thermodynamics were still being developed at that time, and Olbers could not appreciate that his explanation was incorrect. The flaw was that any obscuring matter hiding the stars beyond would be heated up by the starlight until it glowed as brightly as a stellar surface. Surprisingly, the first correct answer came from Edgar Allan Poe. Poe proposed that **because light has a finite speed and the universe is not infinitely old, the light from the most distant sources has not yet arrived.** This solution was independently put on a firm scientific foundation by William Thomson. In more modern terms, **the solution to Olbers's paradox is that our universe is simply too young for it to be filled with light.**

The cosmological principle

The assumption of an isotropic and homogeneous universe is called the **cosmological principle**. To show that the expansion of the universe appears the same to all observers at all locations, let Earth be at the origin of the coordinate system shown in Fig. 1, and consider two galaxies, A and B, located at positions r_A and r_B, respectively. According to the Hubble law, the recessional velocities of the two galaxies are described by the vectors

$$\mathbf{v}_A = H_0\mathbf{r}_A$$

$$\mathbf{v}_B = H_0\mathbf{r}_B.$$

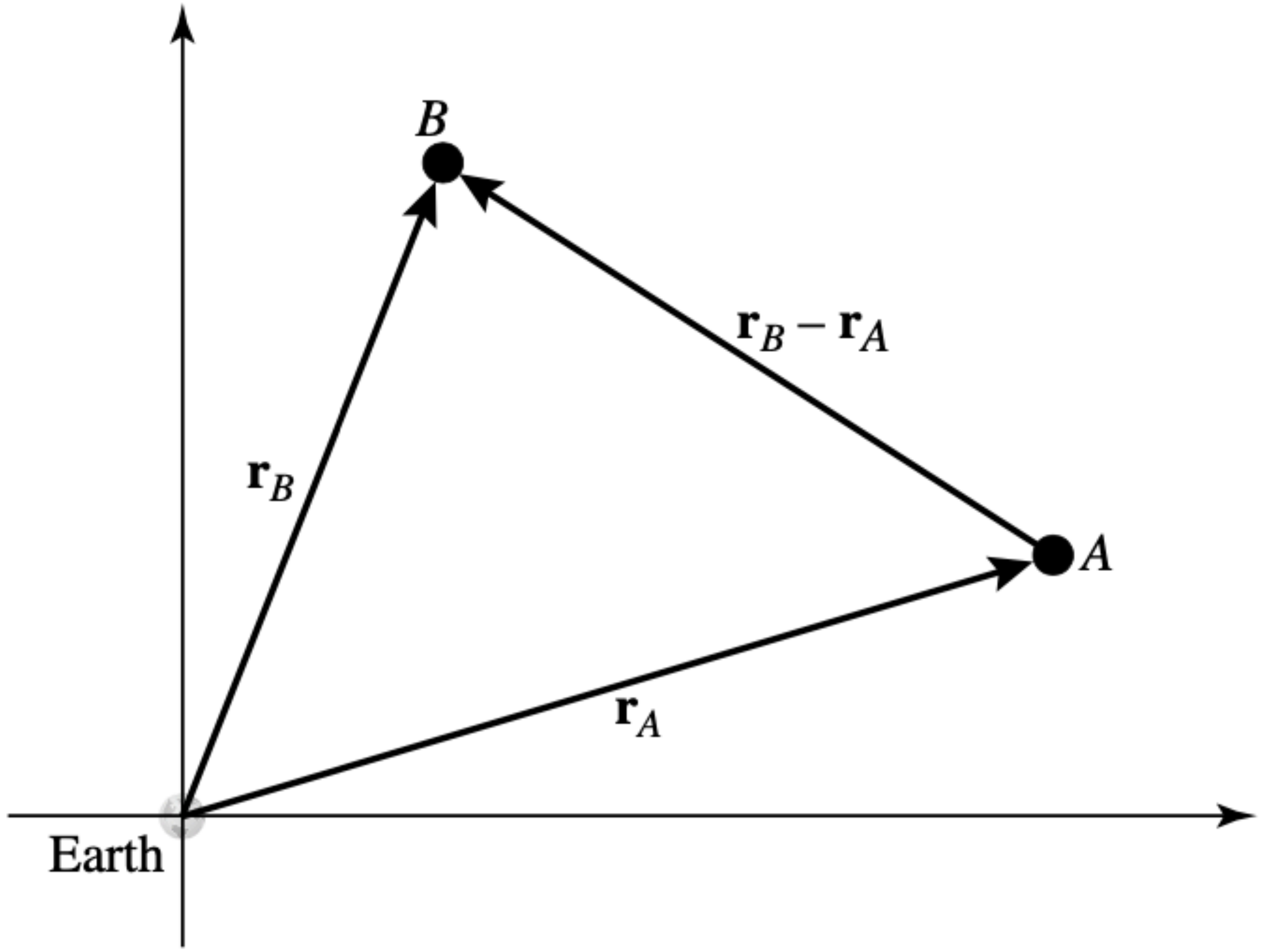


FIGURE 1 The expansion of the universe, with Earth at the origin.

The cosmological principle

The recessional velocity of Galaxy B as seen by an observer in Galaxy A is therefore

$$\mathbf{v}_B - \mathbf{v}_A = H_0\mathbf{r}_B - H_0\mathbf{r}_A = H_0(\mathbf{r}_B - \mathbf{r}_A),$$

so the observer in Galaxy A sees all of the other galaxies in the universe moving away with recessional velocities described by the *same* Hubble law as on Earth.

Although the value of the Hubble "constant," H_0, is assumed to be the same for all observers, it is actually a function of time, $H(t)$. If the present time is t_0, then $H_0 \equiv H(t_0)$.

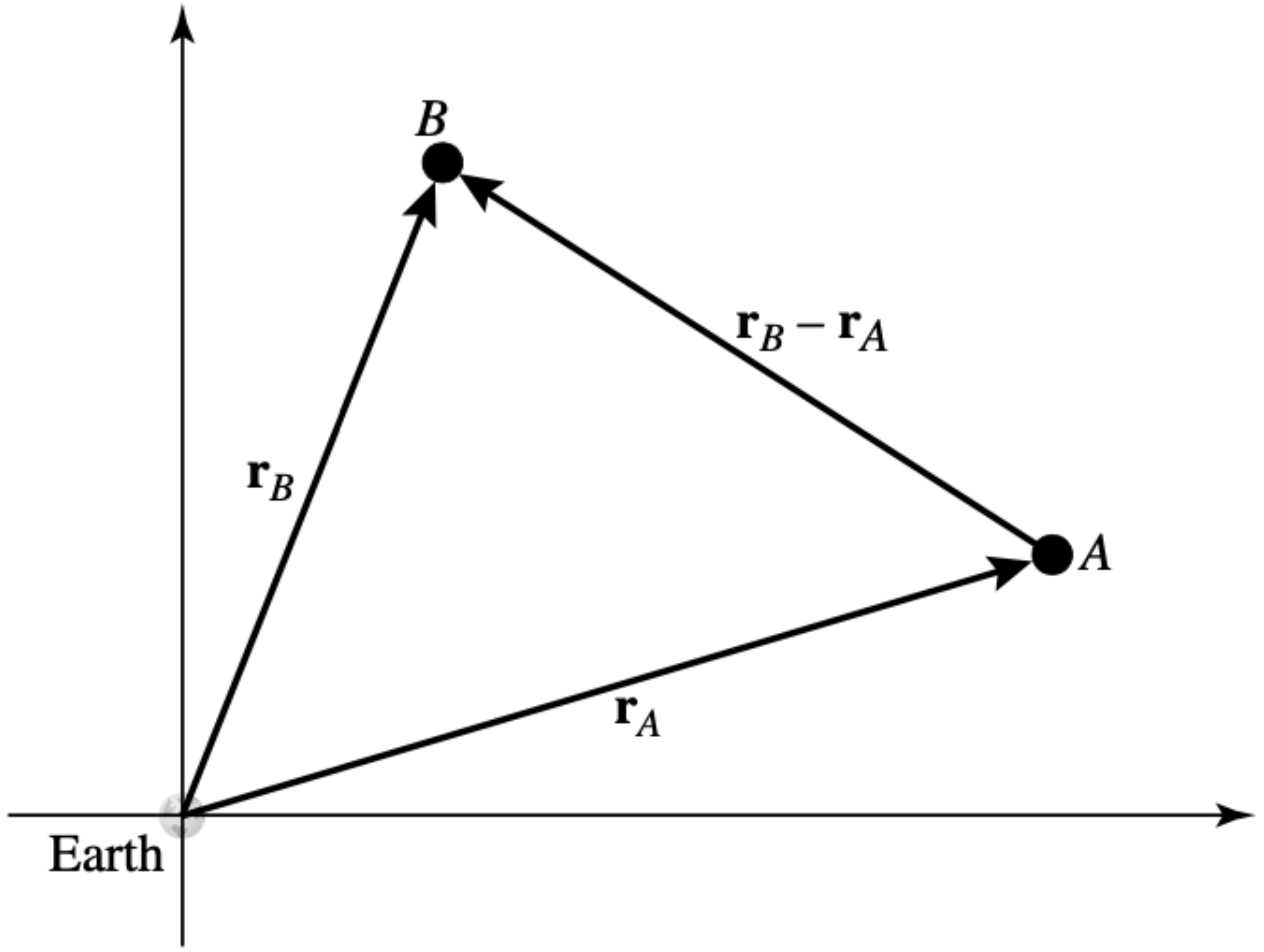


FIGURE 1 The expansion of the universe, with Earth at the origin.

The cosmological principle

To develop an understanding of how the expansion of the universe varies with time, **imagine a universe filled with a pressureless "dust" of uniform density, $\rho(t)$, and choose an arbitrary point for the origin.** Unlike the actual universe, this model universe is both perfectly isotropic and homogeneous at all scales. The pressureless dust represents all of the matter in the universe after being homogenized and uniformly dispersed. There are no photons or neutrinos in this single-component model of the universe.

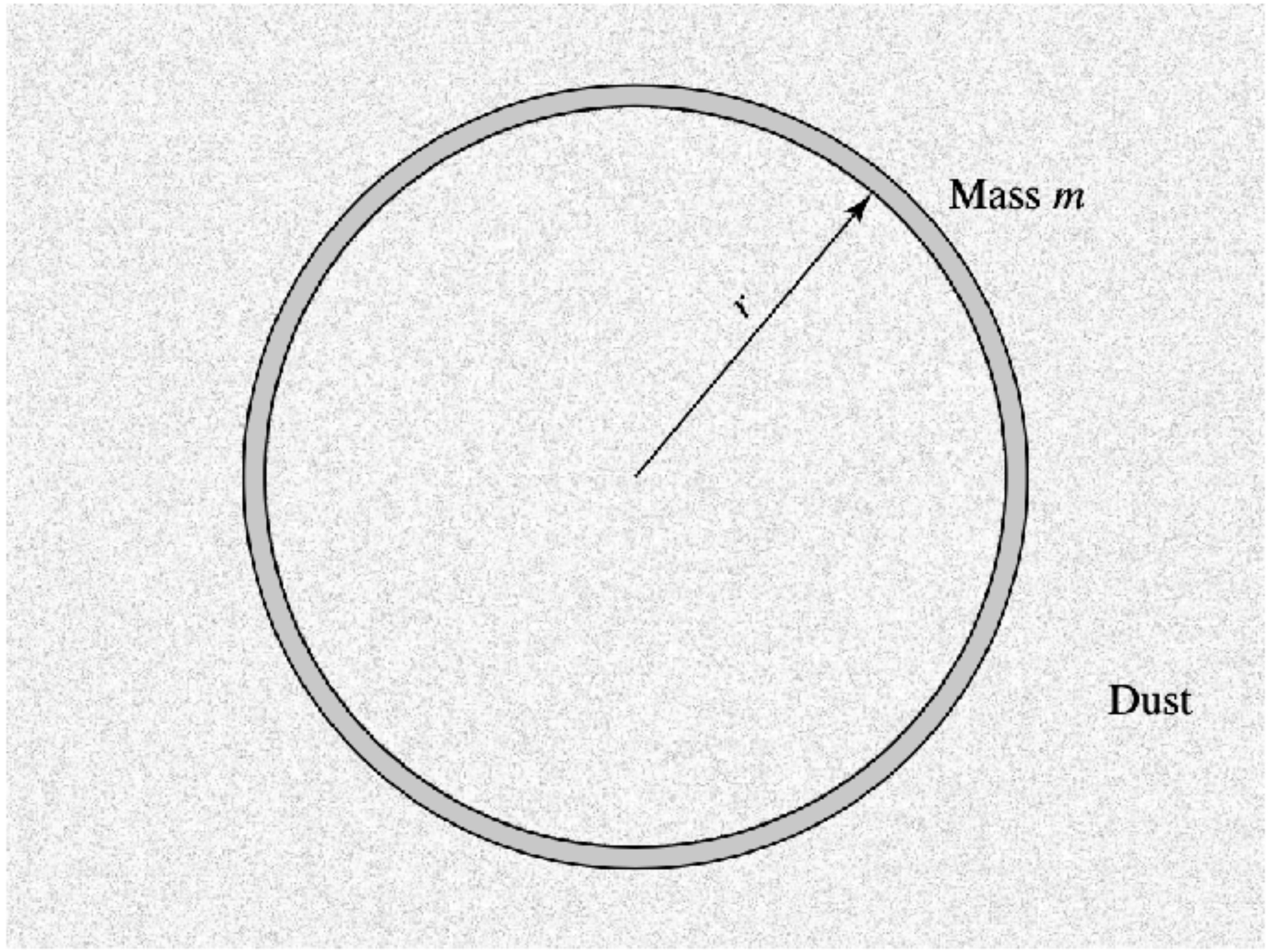


FIGURE 2 Spherical mass shell in a dust-filled universe.

As the universe expands, the dust is carried radially outward from the origin. Let r(t) be the radius of a thin spherical shell of mass m at time t; see Fig. 2. This shell of mass expands along with the universe with recessional velocity v(t) = dr(t)/dt, so it always contains the same dust particles. Then the mechanical energy E of the shell is

$$K(t) + U(t) = E.$$

The cosmological principle

As the shell expands, the gravitational pull from the mass inside causes the kinetic energy, K, to decrease while the gravitational potential energy, U, increases. However, by conservation of energy, the total energy, E, of the shell does not change as the shell moves outward.

For future convenience, the total energy of the shell is written in terms of two constants, k and ϖ, such that $E = -\frac{1}{2}mkc^2\varpi^2$. The constant k has units of (length)$^{-2}$. The other constant, ϖ ("varpi"), labels this particular mass shell and may be thought of as the *present* radius of the shell, so $r(t_0) = \varpi$. The statement of the conservation of the mass shell's energy is then

$$\frac{1}{2}mv^2(t) - G\frac{M_r m}{r(t)} = -\frac{1}{2}mkc^2\varpi^2. \tag{1}$$

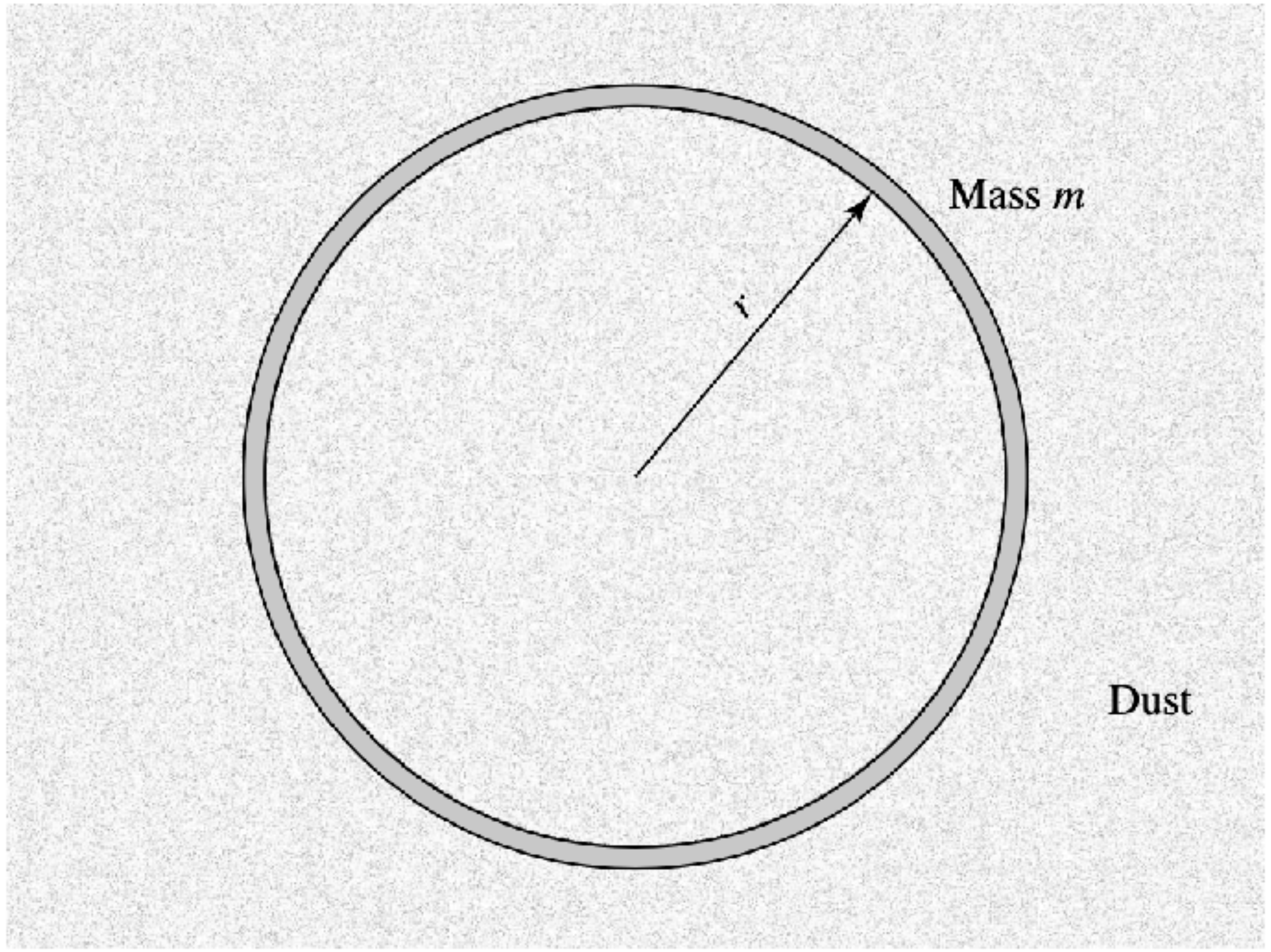


FIGURE 2 Spherical mass shell in a dust-filled universe.

Mr is the mass interior to the shell,

$$M_r = \frac{4}{3}\pi r^3(t)\rho(t).$$

The cosmological principle

Although the radius of the shell and the density of the dust are continually changing, the combination $r^3(t)\rho(t)$ does not vary because the mass interior to a specific shell remains constant as the universe expands. Canceling m and substituting for M_r in Eq. (1) gives

$$v^2 - \frac{8}{3}\pi G\rho r^2 = -kc^2\varpi^2. \qquad (2)$$

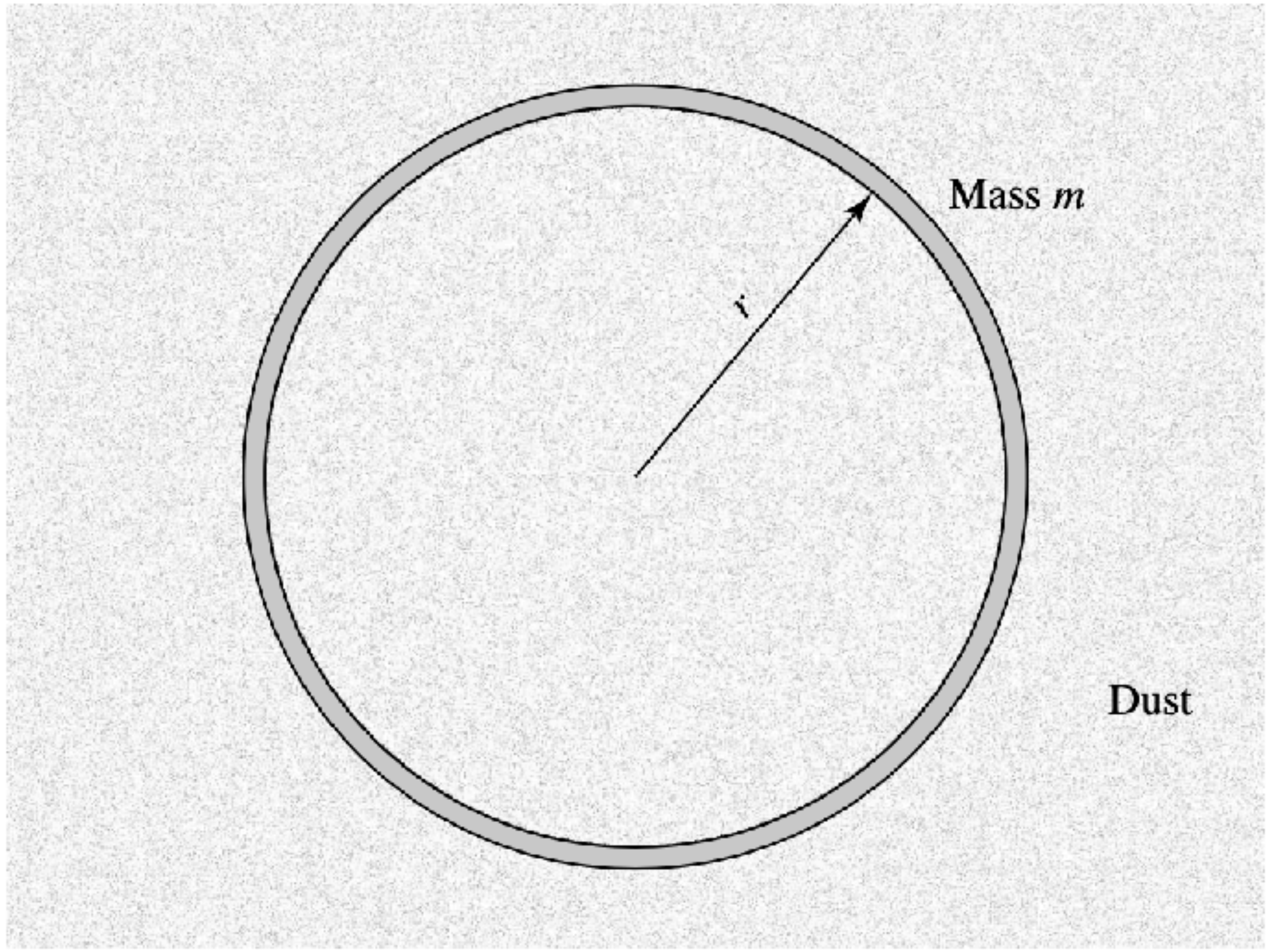


FIGURE 2 Spherical mass shell in a dust-filled universe.

The **constant k** determines the ultimate fate of the universe:

• If **k > 0,** the total energy of the shell is negative, and the universe is *finite*, or **closed**. In this case, the **expansion will someday halt and reverse itself.**

• If **k < 0**, the total energy of the shell is positive, and the universe is *infinite*, or **open**. In this case, the **expansion will continue forever.**

• If **k = 0**, the total energy of the shell is zero, and the universe is **flat**, neither open nor closed. In this case, the **expansion will continue to slow down**, coming to a halt only as $t \to \infty$ and the universe is infinitely dispersed.

Newtonian cosmology

The Newtonian cosmology always takes place in a flat spacetime.
The terms *closed*, *open*, and *flat* above should be understood as describing the *dynamics* of the universal expansion. This can also be interpreted to describe the geometry of spacetime.

The cosmological principle requires that the expansion proceed in the same way for all shells; for example, the time required for every shell to double its distance from the origin is assumed to be the same. This means that the radius of a particular shell (identified by ϖ) at any time can be written as

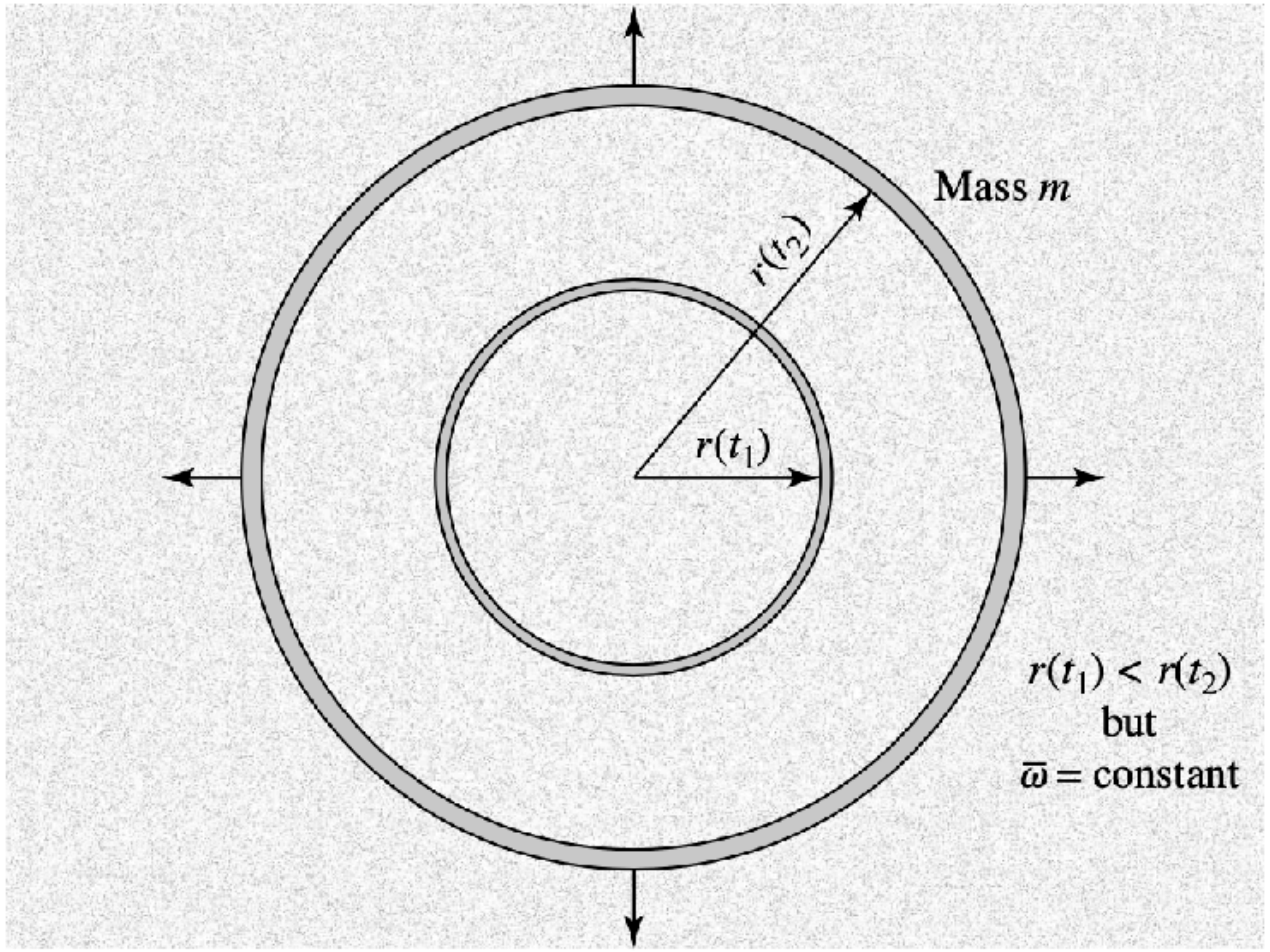


FIGURE 3 An expanding mass shell seen at two different times, $t_1 < t_2$. As the mass shell expands, its comoving coordinate, ϖ, is the same at times t_1 and t_2, while $r(t_1) < r(t_2)$.

$$r(t) = R(t)\varpi. \qquad (3)$$

Newtonian cosmology

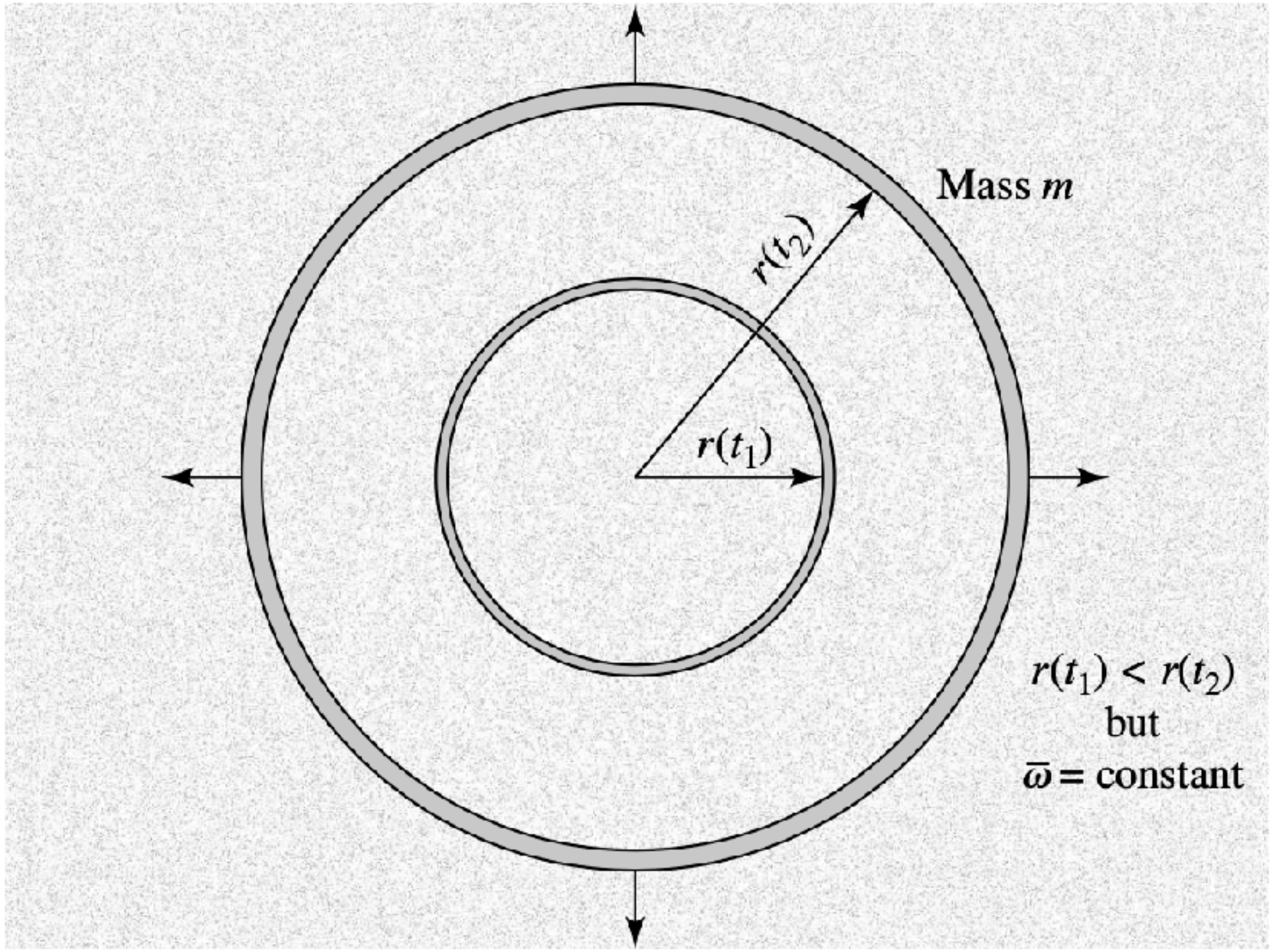
$$r(t) = R(t)\varpi. \tag{3}$$

In this expression, r(t) is called the **coordinate distance**.

Because ϖ labels a shell and follows it as it expands, ϖ is referred to as a **comoving coordinate**; see Fig. 3.

R(t) is a dimensionless **scale factor** (the same for all shells) that describes the expansion; $R(t_0) = 1$ corresponds to $r(t_0) = \varpi$. The scale factor R is equal to $R_{emitted}/R_{obs}$. Thus R and the redshift z are related by

$$R = \frac{1}{1+z}, \tag{4}$$



Newtonian cosmology

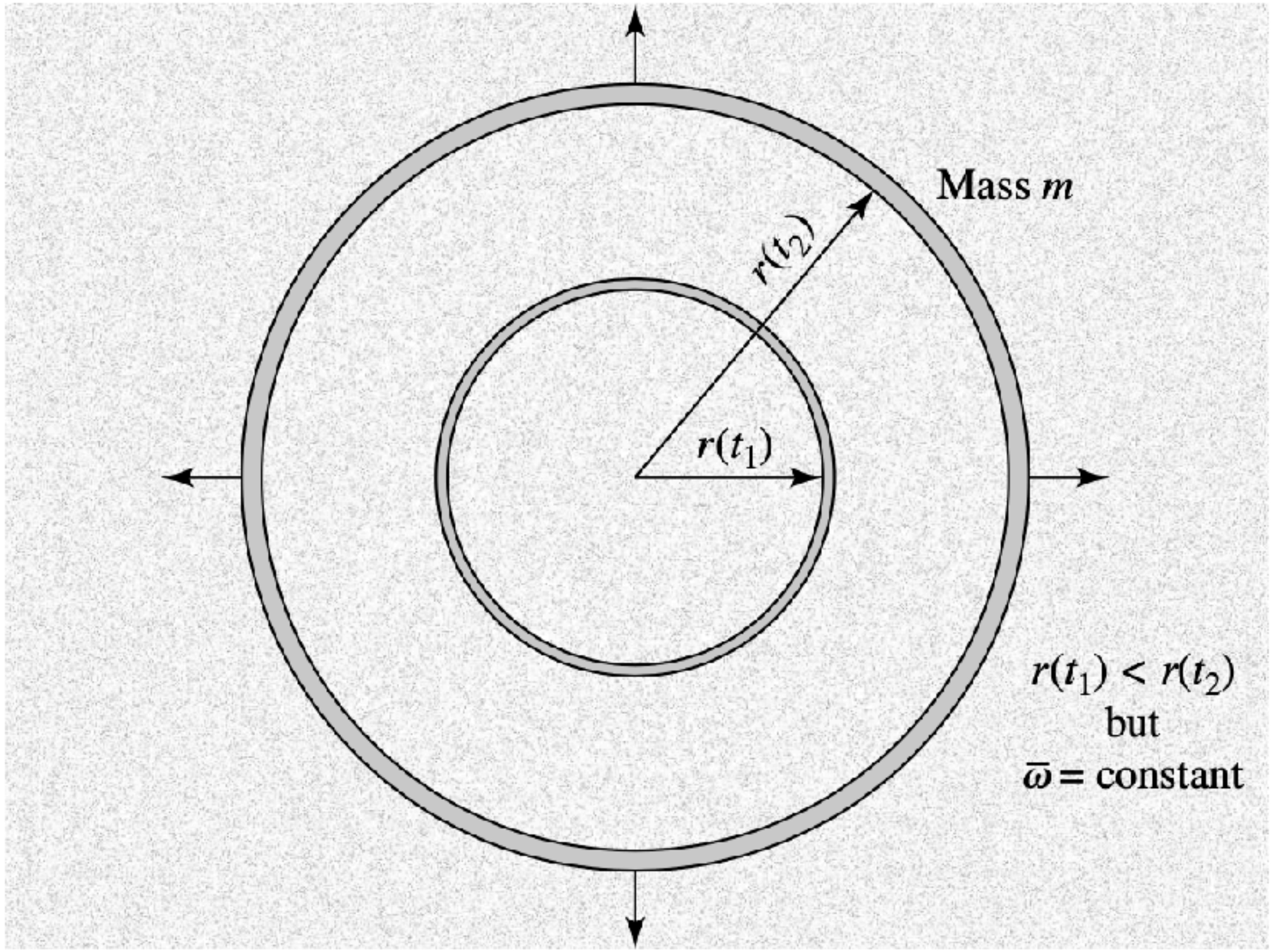
For example, looking back to a redshift of z = 3 implies a universe for which the scale factor was R = 1/4.

The previous statement that $r^3\rho$ does not vary for a specific shell means that $R^3\rho$ remains constant for *all* shells. That is, since $R(t_0) = 1$,

$$R^3(t)\rho(t) = R^3(t_0)\rho(t_0) = \rho_0, \quad (5)$$

where ρ_0 is the density of the dust-filled universe at the present time. Using Eq. (4), this can also be written as

$$\boxed{\rho(z) = \rho_0(1+z)^3,} \quad (6)$$



which gives the **average density of the universe as observed at redshift z.**

Note that Eqs. (5) and (6) are valid only for a universe consisting of pressureless dust. The more general counterpart of Eq. (5) will be derived later.

The Evolution of the Universe

The evolution of our Newtonian universe, which can be described by the time behavior of the scale factor R(t). The first step is to write the Hubble parameter, H (t), in terms of the scale factor.

The Hubble law is

$$v(t) = H(t)r(t) = H(t)R(t)\varpi. \tag{7}$$

Because v(t) is the time derivative of r(t), Eq. (3) gives

$$v(t) = \frac{dR(t)}{dt}\varpi.$$

Comparing this with Eq. (7) shows that

$$H(t) = \frac{1}{R(t)}\frac{dR(t)}{dt}. \tag{8}$$

The Evolution of the Universe

Inserting Eqs. (3) and (7) into (2) and canceling the ϖ^2 results in

$$\left(H^2 - \frac{8}{3}\pi G\rho\right) R^2 = -kc^2, \tag{9}$$

Substituting Eq. (8)

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -kc^2. \tag{10}$$

The left-hand sides of Eqs. (9) and (10) apply to *all shells* and involve the functions of time H(t), ρ(t), and R(t), while the right-hand sides are constant.

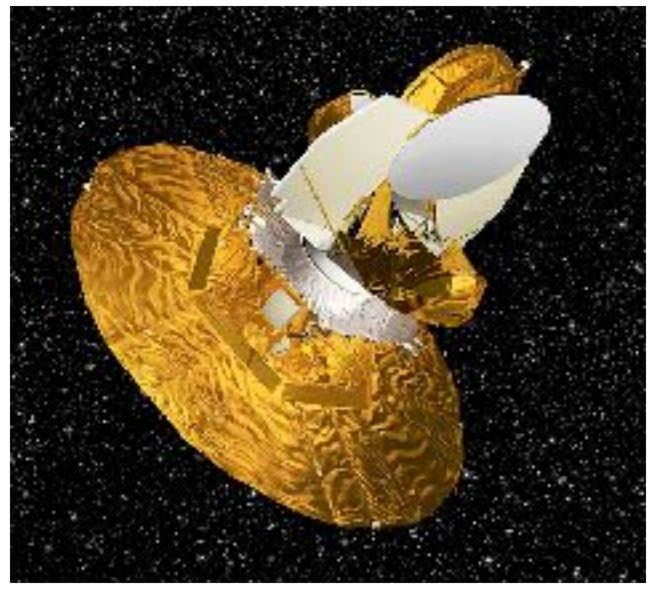
The Evolution of the Universe

Using Eq. (5), Eq. (10) can be written in terms of R and t only:

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho_0}{3R} = -kc^2. \tag{11}$$

This result, along with Eqs. (9) and (10), will be used to **describe the expansion of the universe.**

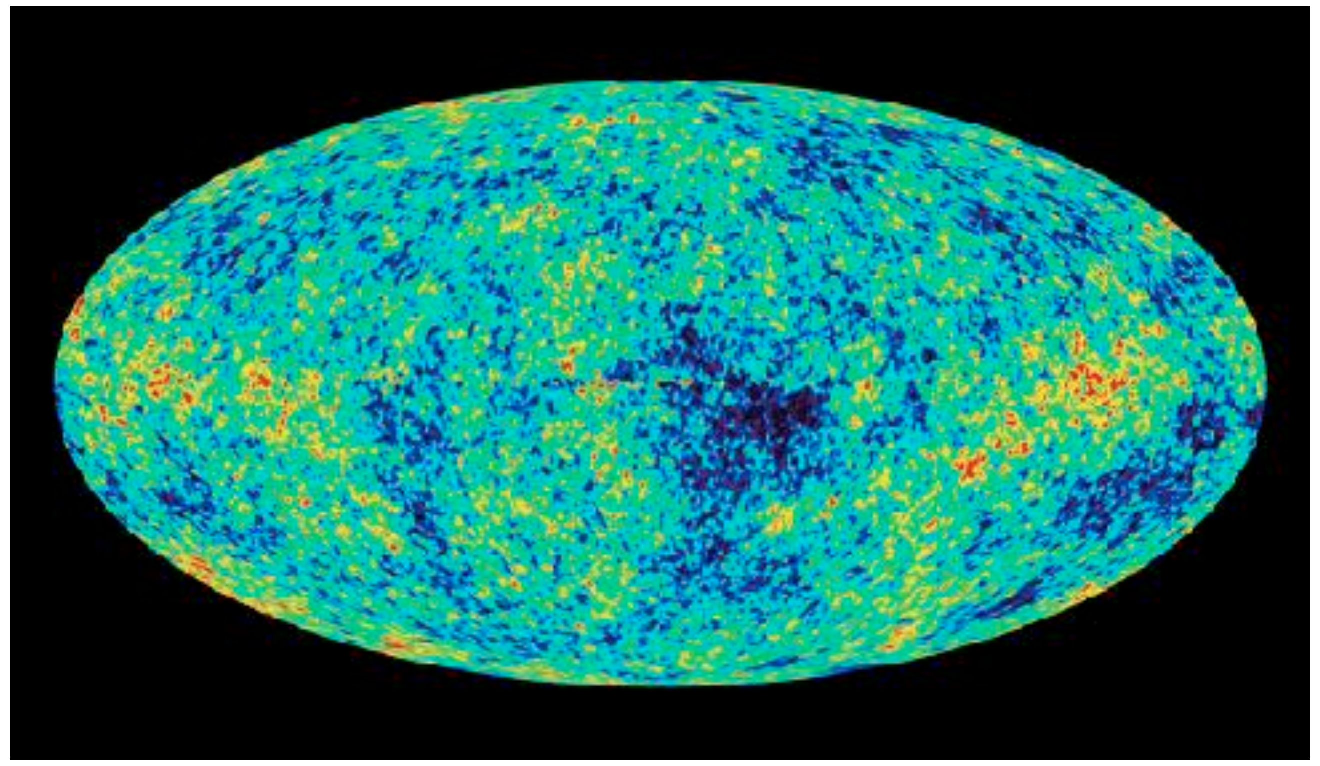
The Evolution of the Universe



Now we are ready to examine the motion of mass shells in the three cases of a flat, closed, or open universe. First, consider the case of a **flat universe (k = 0),** corresponding to each shell expanding at exactly its escape velocity. **The value of the density that will result in a value of k = 0 is known as the critical density**, ρ_c(t).

From Eq. (9),

$$\boxed{\rho_c(t) = \frac{3H^2(t)}{8\pi G}.} \tag{12}$$



To evaluate this at the present time, it is useful to know that the Hubble constant in conventional units is

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} = 3.24 \times 10^{-18} h \text{ s}^{-1} \tag{13}$$

which, using the WMAP value of

$$[h]_{\text{WMAP}} = 0.71,$$

$$[H_0]_{\text{WMAP}} = 71 \text{ km s}^{-1} \text{ Mpc}^{-1} = 2.30 \times 10^{-18} \text{ s}^{-1}. \tag{14}$$

The Evolution of the Universe

The present value of the critical density, $\rho_{c,0}$ is then

$$\boxed{\rho_{c,0} = \frac{3H_0^2}{8\pi G}} \tag{15}$$
$$= 1.88 \times 10^{-26} h^2 \text{ kg m}^{-3},$$

with a WMAP value of

$$\rho_{c,0} = 9.47 \times 10^{-27} \text{ kg m}^{-3}. \tag{16}$$

This is equivalent to **about six hydrogen atoms per cubic meter.**

However, the WMAP value of the average density of **baryonic matter** in the universe is about 4% of the critical density,

$$\rho_{b,0} = 4.17 \times 10^{-28} \text{ kg m}^{-3} \qquad (\text{for } h = 0.71), \tag{17}$$

The Evolution of the Universe

By “baryonic matter,” we mean matter made of *baryons* (e.g., protons and neutrons); hence the “b” subscript designating baryonic matter in Eq. (17).

This value is consistent with that obtained from comparing the theoretical and observed abundances of light elements, such as ^3He and ^7Li, that were formed in the early universe. The **density of nonbaryonic dark matter**, is not included in the value of $\rho_{b,0}$. Dark matter is revealed only by its gravitational influence on baryonic matter. Presumably it interacts very weakly (if at all) with photons and charged particles via the electromagnetic force, so it does not absorb, emit, or scatter appreciable amounts of light. Our model universe of pressureless dust includes both types of matter, baryonic and nonbaryonic, luminous and dark.

The Evolution of the Universe

The ratio of a measured density to the critical density is an important parameter in cosmology. Accordingly, it is useful to define the **density parameter**,

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)}, \tag{18}$$

which has a present value of

$$\boxed{\Omega_0 = \frac{\rho_0}{\rho_{c,0}} = \frac{8\pi G\rho_0}{3H_0^2}.} \tag{19}$$

The Evolution of the Universe

TABLE 1 Mass-to-Light Ratios and Density Parameters, Measured for a Variety of Systems. The complicated dependence on h for the values from the X-ray halo of M87 and Local Group timing is not shown. (Adapted from Binney and Tremaine, *Galactic Dynamics*, Princeton University Press, Princeton, NJ, 1987, and Schramm, *Physica Scripta*, *T36*, 22, 1991.)

Method	M/L $(M_\odot/L_\odot)$	Ω_0
Solar neighborhood	3	$0.002h^{-1}$
Elliptical galaxy cores	$12h$	0.007
Local escape speed	30	$0.018h^{-1}$
Satellite galaxies	30	$0.018h^{-1}$
Magellanic Stream	> 80	$> 0.05h^{-1}$
X-ray halo of M87	> 750	$> 0.46h^{-1}$
Local Group timing	100	$0.06h^{-1}$
Groups of galaxies	$260h$	0.16
Clusters of galaxies	$400h$	0.25
Gravitational lenses	—	0.1 – 0.3
Big Bang nucleosynthesis	—	0.065 ± 0.045

The Evolution of the Universe

Table 1 shows the mass-to-light ratios of a variety of astronomical systems and the density parameters derived for them. With the exception of Big Bang nucleosynthesis, these values were obtained by studying gravitational effects and thus include both baryonic and dark matter.

There is a significant trend that **more large scale systems have larger mass-to-light ratios and density parameters**, but, as shown in Fig. 4, for the largest systems the density parameters seem to reach a “ceiling” at a maximum value of $\Omega_0 \simeq 0.3$. This is consistent with the WMAP result for the value of the average density of all types of matter, baryonic and dark:

$$[\Omega_{m,0}]_{\mathrm{WMAP}} = (0.135\ ^{+0.008}_{-0.009})h^{-2} = 0.27 \pm 0.04 \qquad (\text{for } h = 0.71). \qquad (20)$$

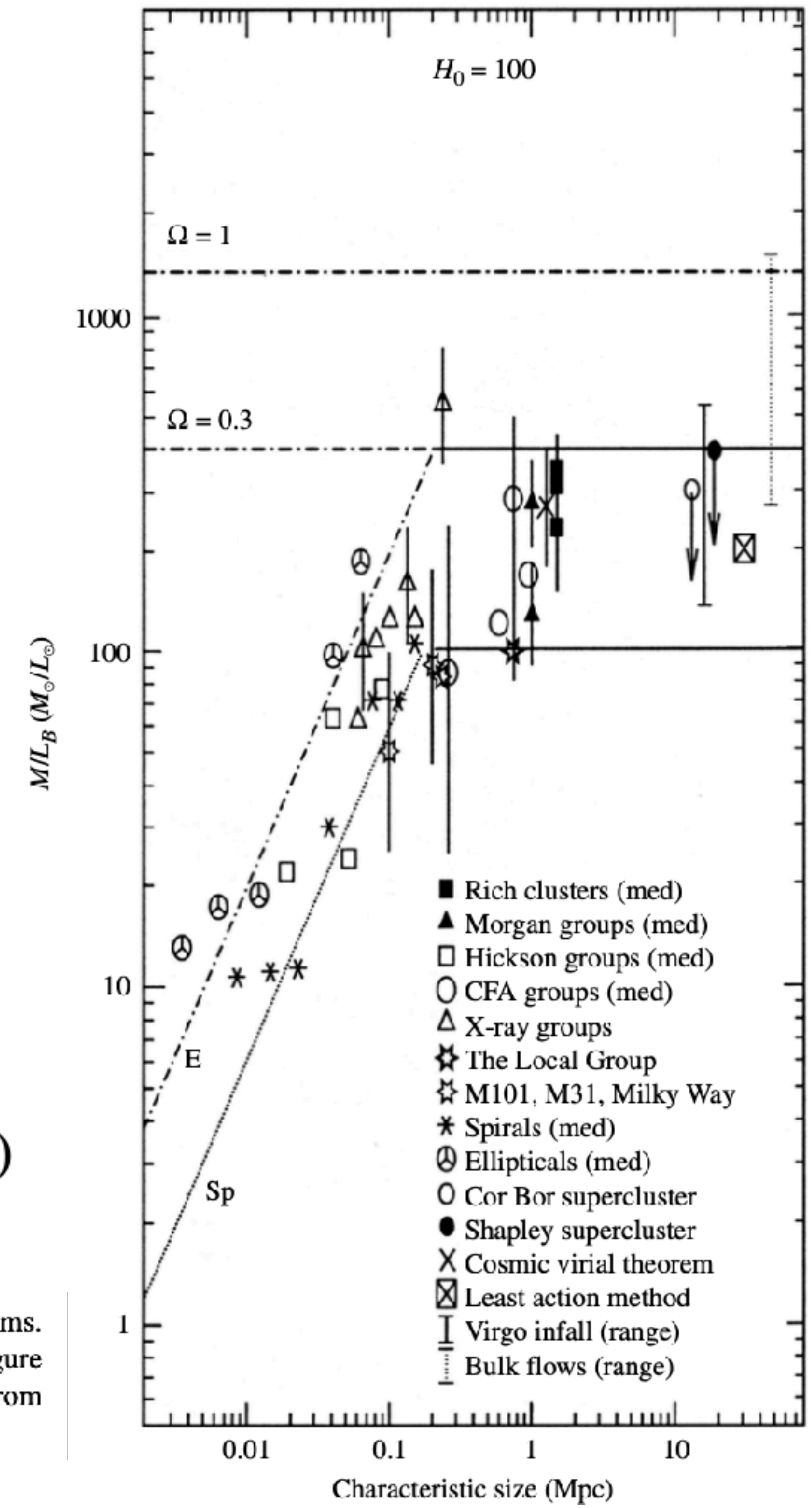


FIGURE 4 The mass-to-light ratio as a function of the characteristic size of a variety of systems. H_0 was taken to be 100 km s^{-1} Mpc^{-1} for this figure prior to publication of the WMAP results. (Figure adapted from Dodelson, *Modern Cosmology*, Academic Press, New York, 2003, with permission from Elsevier. Data from Bahcall et al., *Ap. J.*, 541, 1, 2000.)

The Evolution of the Universe

This corresponds to a mass density of

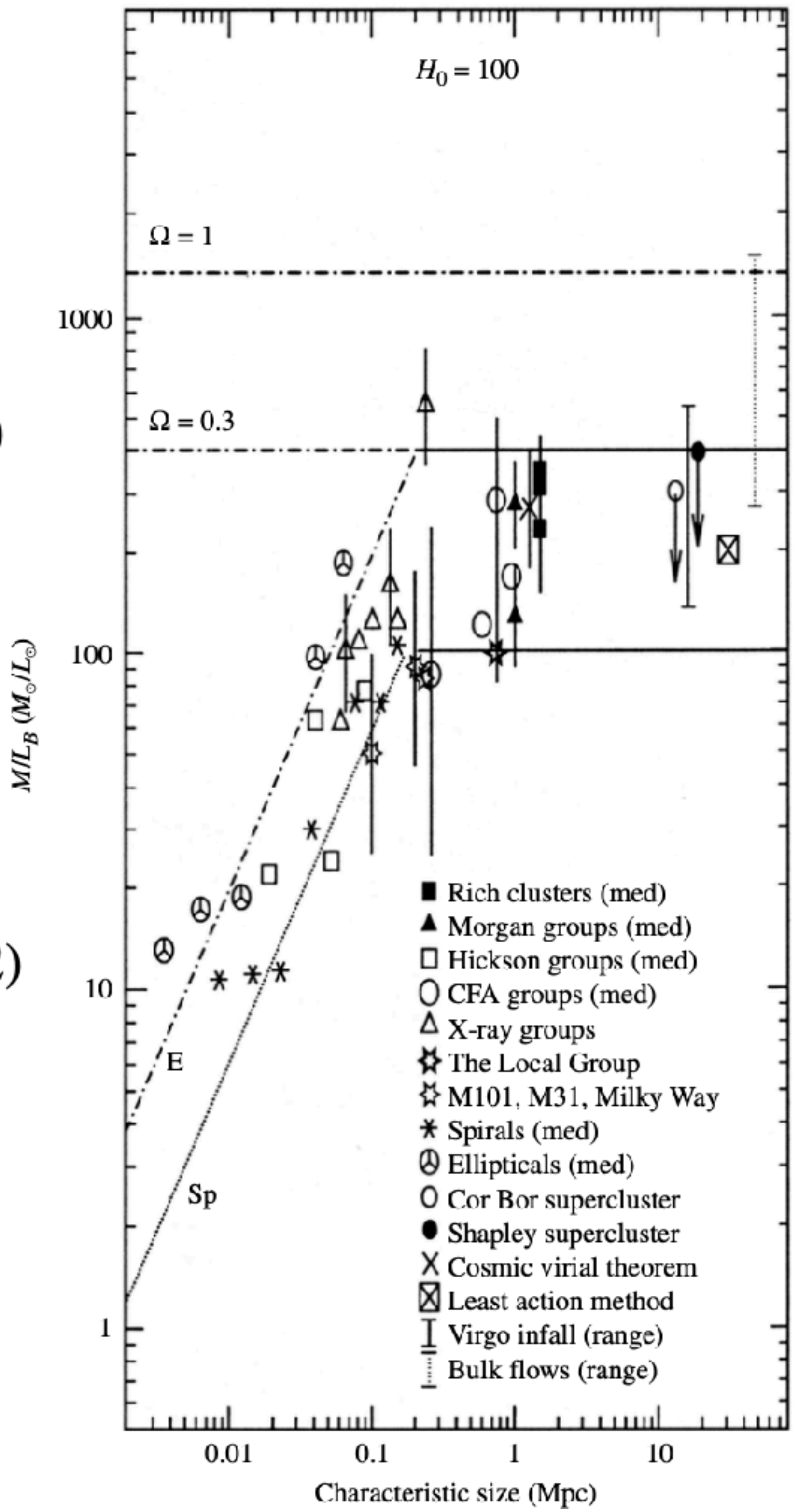
$$\rho_{m,0} = 2.56 \times 10^{-27}\ \mathrm{kg\ m^{-3}} \qquad (\text{for } h = 0.71) \tag{21}$$

The "m" subscript, which stands for "mass," anticipates models of the universe with more than one component. This subscript will be suppressed for the present one-component model.

The WMAP value of the density parameter for baryonic matter is

$$[\Omega_{b,0}]_{\mathrm{WMAP}} = (0.0224 \pm 0.0009)h^{-2} = 0.044 \pm 0.004 \qquad (\text{for } h = 0.71). \tag{22}$$

Thus, according to the WMAP results, **baryonic matter accounts for only about 16% of the matter in the universe; the other 84% is some sort of nonbaryonic dark matter.**



The Evolution of the Universe

The general characteristics of the expansion of our model universe composed of pressureless dust can now be determined. First note that, from Eqs. (6) and (19)

$$\frac{\Omega}{\Omega_0} = \frac{\rho}{\rho_0}\frac{H_0^2}{H^2} = (1+z)^3\,\frac{H_0^2}{H^2},$$

So

$$\Omega H^2 = (1+z)^3 \Omega_0 H_0^2. \tag{23}$$

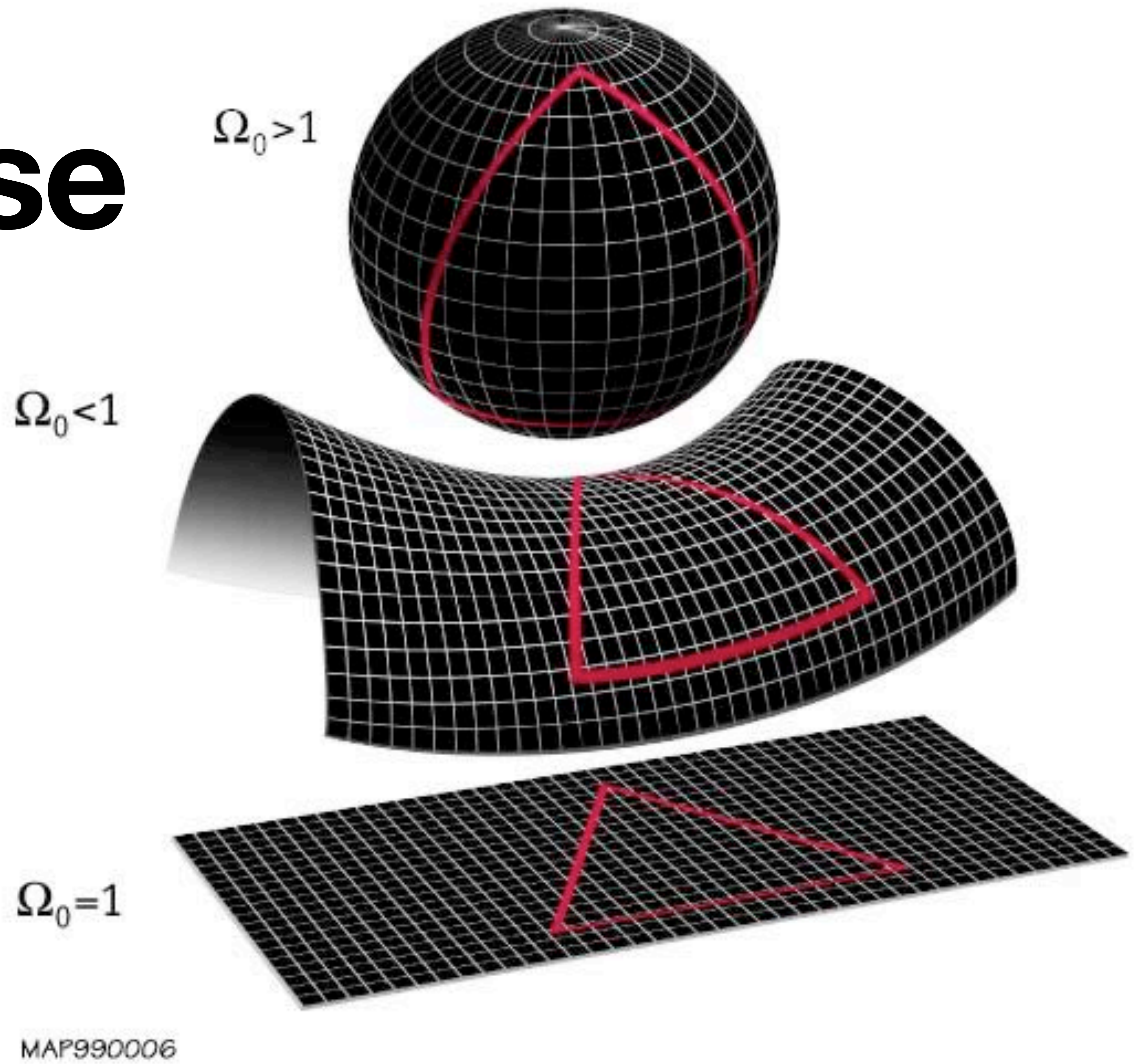
Another relation between Ω and H comes from combining the density parameter, Eq. (18), with Eq. (9):

$$H^2(1-\Omega)R^2 = -kc^2 \tag{24}$$

As a special case at $t = t_0$,

$$H_0^2(1-\Omega_0) = -kc^2. \tag{25}$$

The Evolution of the Universe



This confirms that:

- **If Ω_0 >1, then k > 0 and the universe is closed.**
- **If Ω_0 <1, then k < 0 and the universe is open.**
- **If Ω_0 =1, then k = 0 and the universe is flat.**

Remember that we are now dealing with a simple model of a one-component universe of pressureless dust. Later we will study more realistic multicomponent models, which will show that a measurement of the mass density parameter alone is not enough for us to draw any conclusions about the ultimate fate of our physical universe.

Equating Eqs. (24) and (5), and using (4), we find

$$H^2(1-\Omega) = H_0^2(1-\Omega_0)(1+z)^2. \tag{26}$$

The Evolution of the Universe

Thus we have two equations, Eqs. (23) and (26), with the two unknowns Ω and H . These may be easily solved to find

$$H = H_0(1+z)\,(1+\Omega_0 z)^{1/2} \tag{27}$$

$$\Omega = \left(\frac{1+z}{1+\Omega_0 z}\right)\Omega_0 = 1 + \frac{\Omega_0 - 1}{1+\Omega_0 z}. \tag{28}$$

Equation (27) implies that **at very early times, as R → 0 and z → ∞,** the Hubble parameter H → ∞.

Equation (28) shows that the sign of Ω – 1 does not change, and in particular that **if Ω = 1 at any time, then Ω = 1 at all times**. **The character of the universe does not change as the universe evolves**.

Equation (28) also shows that **at very early times**, as z → ∞, the density parameter **Ω → 1 regardless of today's value of Ω_0**. *Therefore, **the early universe was essentially flat***; see Fig. 5.

The assumption of a flat early universe will greatly simplify the description of the first few minutes of the universe.

The Evolution of the Universe

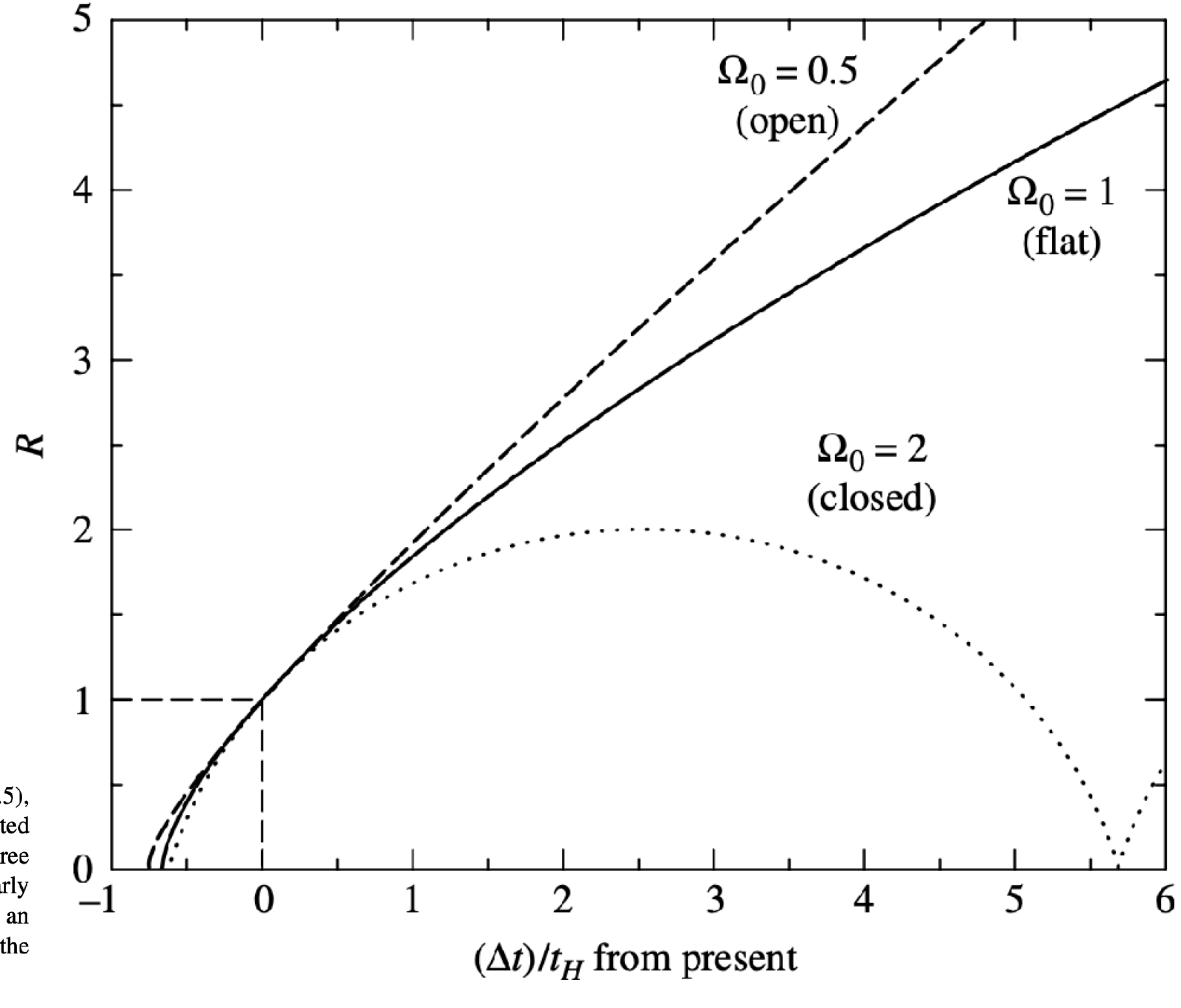


FIGURE 5 The evolution of the scale factor, R, for three model universes—open ($\Omega_0 = 0.5$), flat ($\Omega_0 = 1$), and closed ($\Omega_0 = 2$)—as a function of time, measured from the present. The dotted lines locate the position of today's universe on the three curves. At the present time ($R = 1$) all three universes have the same value of H_0, as exhibited by the curves having the same slope. For the early universe ($R < 1$) there is little difference among the kinematic behaviors of a flat, a closed, and an open universe because the early universe was essentially flat. The elapsed time Δt is in units of the Hubble time, t_H.

The Evolution of the Universe

Example 1.1. We will find that when the universe was about 3 minutes old, protons and neutrons combined to form helium nuclei. This occurred at a redshift of z = 3.68×10^8.

Using the WMAP value of $[\Omega_{m,0}]$WMAP = 0.27 for Ω_0, we find that at the time of helium formation, the value of Ω was

$$\Omega = 1 + \frac{\Omega_0 - 1}{1 + \Omega_0 z} = 1 + \frac{0.27 - 1}{1 + (0.27)(3.68 \times 10^8)} = 0.99999999265. \tag{29}$$

At even earlier times the value of Ω contains a much longer string of nines. During the late twentieth century, it appeared absurd to theoreticians that a mechanism would exist to fine-tune Ω to a value so very close to unity without having an exactly flat universe with $\Omega = 1$. And yet, observational measurements of the value of the density parameter continued to hover around $\Omega_0 \simeq 0.3$. The solution to this puzzle will be described later.

The Evolution of the Universe

The expansion of a flat, one-component universe of pressureless dust as a function of time may be found by solving Eq.(11) with k = 0 (so $\rho_0 = \rho_{c,0}$ and $\Omega_0 = 1$):

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G\rho_{c,0}}{3R}.$$

Taking the square root of each side and integrating (with R = 0 at t = 0) gives

$$\int_0^R \sqrt{R'}\,dR' = \sqrt{\frac{8\pi G\rho_{c,0}}{3}}\int_0^t dt'$$

$$R_{\text{flat}} = (6\pi G\rho_{c,0})^{1/3} t^{2/3} \tag{30}$$

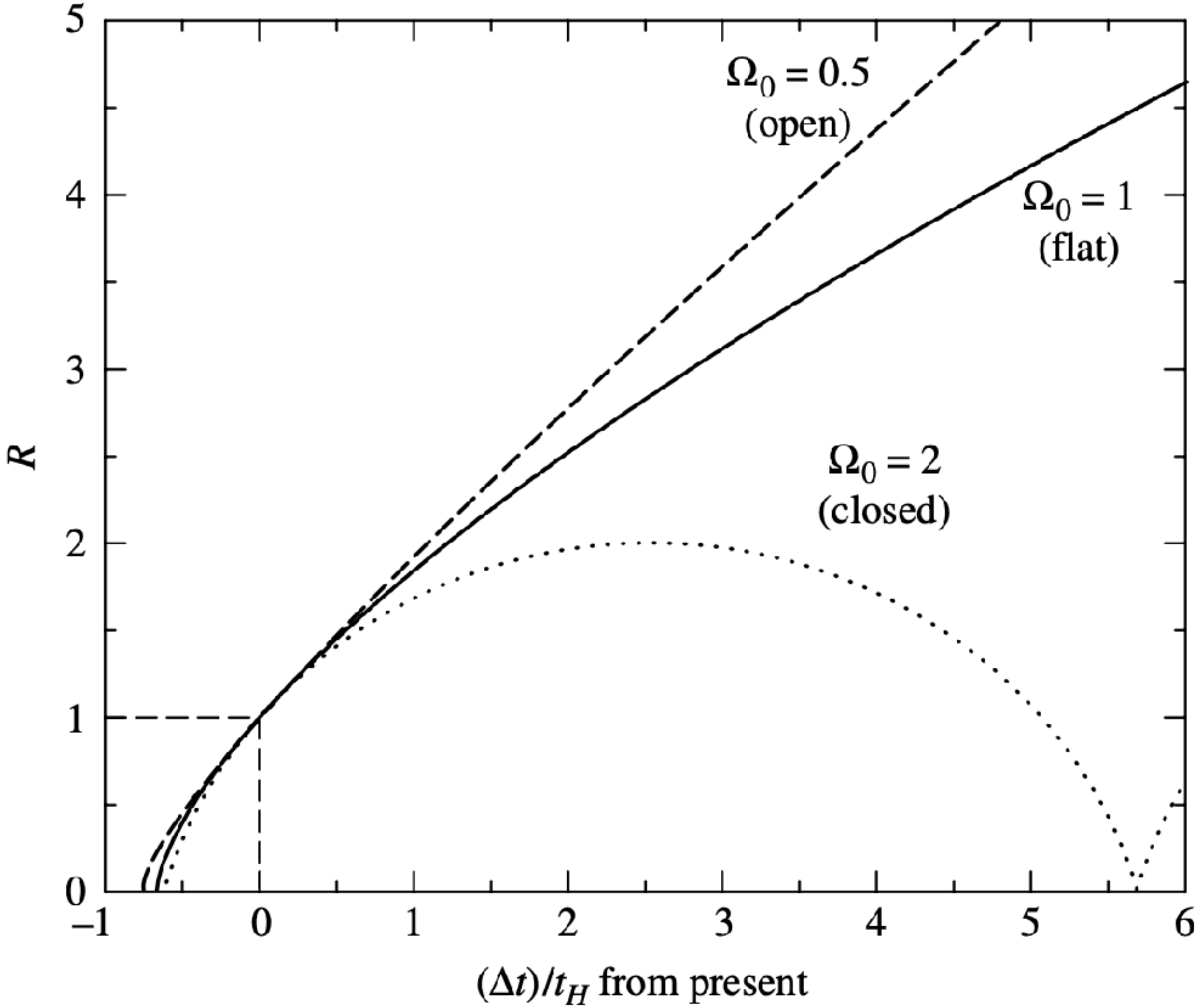
$$= \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3} \qquad (\text{for } \Omega_0 = 1), \tag{31}$$

The Evolution of the Universe

$$R_{\text{flat}} = (6\pi G\rho_{c,0})^{1/3} t^{2/3}$$
$$= \left(\frac{3}{2}\right)^{2/3} \left(\frac{t}{t_H}\right)^{2/3} \qquad \text{(for } \Omega_0 = 1\text{)},$$

where the last expression was obtained by using Eq.(15) and $t_H = 1/H_0$ for the Hubble time.

The increase in R for $\Omega_0 = 1$ is shown in Fig. 5, with time in units of the Hubble time.



The Evolution of the Universe

If $\Omega_0 \neq 1$, the density is not equal to the critical density and Eq. (11) is more difficult to solve.

If Ω_0 **> 1,** the universe is closed and the solution can be expressed in parametric form as

$$R_{\text{closed}} = \frac{4\pi G\rho_0}{3kc^2}\,[1-\cos(x)] \tag{32}$$

$$= \frac{1}{2}\,\frac{\Omega_0}{\Omega_0-1}\,[1-\cos(x)] \tag{33}$$

$$t_{\text{closed}} = \frac{4\pi G\rho_0}{3k^{3/2}c^3}\,[x-\sin(x)] \tag{34}$$

$$= \frac{1}{2H_0}\,\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\,[x-\sin(x)]\,, \tag{35}$$

where the variable $x \geq 0$ merely parameterizes the solution.

The behavior of this solution with $\Omega_0 = 2$ is shown in Fig. 5. The “bounce” that occurs after the **contraction of the universe** is a mathematical artifact and does not imply an endless sequence of oscillating universes.

The Evolution of the Universe

On the other hand, if Ω_0 **<1**, the universe is open and the parametric form of the solution of Eq. (11) is:

$$R_{\text{open}} = \frac{4\pi G\rho_0}{3|k|c^2}\,[\cosh(x) - 1] \tag{36}$$

$$= \frac{1}{2}\,\frac{\Omega_0}{1-\Omega_0}\,[\cosh(x) - 1] \tag{37}$$

$$t_{\text{open}} = \frac{4\pi G\rho_0}{3|k|^{3/2}c^3}\,[\sinh(x) - x] \tag{38}$$

$$= \frac{1}{2H_0}\,\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\,[\sinh(x) - x]\,. \tag{39}$$

Recall that the hyperbolic cosine is defined as $\cosh(x) \equiv (e^{x} + e^{-x})/2 \geq 1$. Similarly, the hyperbolic sine is given by $\sinh(x) \equiv (e^{x} - e^{-x})/2 \geq x$, so R_{open} increases monotonically with t. See Fig. 5 for the appearance of the solution with $\Omega_0 = 0.5$. If $\Omega_0 \leq 1$, then **the universe will continue to expand forever.**

The age of the Universe

We are now ready to calculate the age of the universe as a function of the redshift z. Before continuing, a note of caution should be sounded about referring to any time t as the "age of the universe." The laws of physics, as we presently understand them, cannot remain valid under the extreme conditions that must prevail as $t \to 0$. In using t as a measure of the time since the Big Bang, we must always keep in mind that this is an *extrapolated time* and cannot be taken literally at the earliest instants ($t < 10^{-43}$ s).

Keeping this in mind, we now proceed by using Eq. (4) to replace R by 1/(1 + z) in Eq. (31) for a flat universe. **The age of a flat universe (in units of the Hubble time)** that is observed at redshift z is then found to be

$$\frac{t_{\text{flat}}(z)}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}} \qquad (\text{for } \Omega_0 = 1). \tag{40}$$

Replacing R by 1/(1 + z) in Eq. (33) for a closed universe and using Eq. (35) to eliminate x leads to

$$\frac{t_{\text{closed}}(z)}{t_H} = \frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0}\right) - \frac{2\sqrt{(\Omega_0-1)(\Omega_0 z+1)}}{\Omega_0(1+z)}\right]$$
$$(\text{for } \Omega_0 > 1). \tag{41}$$

The age of the Universe

Following a similar procedure using Eq. (37) for an open universe and using Eq. (39) to eliminate x results in

$$\frac{t_{\text{open}}(z)}{t_H} = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\left[-\cosh^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0}\right) + \frac{2\sqrt{(1-\Omega_0)(\Omega_0 z + 1)}}{\Omega_0(1+z)}\right]$$

$$(\text{for } \Omega_0 < 1). \tag{42}$$

In the limit of large redshift, Eqs. (40) through (42) reduce to:

$$\frac{t(z)}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}\Omega_0^{1/2}}, \tag{43}$$

where the higher-order terms may be neglected for $\Omega_0 \neq 1$. Because the early universe was flat to a very good approximation, precise **observations are required to determine whether the universe is flat, closed, or open.**

The age of the Universe

The current age of the universe, t_0, may be easily found by setting z = 0 in Eqs. (40– 42) to find

for a flat universe:

$$\frac{t_{\text{flat},0}}{t_H} = \frac{2}{3} \qquad (\text{for } \Omega_0 = 1) \qquad (44)$$

for a closed universe:

$$\frac{t_{\text{closed},0}}{t_H} = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \left[\cos^{-1}\left(\frac{2}{\Omega_0} - 1\right) - \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0} \right] \qquad (\text{for } \Omega_0 > 1) \qquad (45)$$

for an open universe:

$$\frac{t_{\text{open},0}}{t_H} = \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}} \left[-\cosh^{-1}\left(\frac{2}{\Omega_0} - 1\right) + \frac{2\sqrt{1 - \Omega_0}}{\Omega_0} \right] \qquad (\text{for } \Omega_0 < 1) \qquad (46)$$

The age of the Universe

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Newtonian cosmology

The age of the universe for these models, expressed as a fraction of the Hubble time, is shown in Fig. 6.

According to the inflation scenario **the universe should be essentially flat**, a scenario supported by recent observations. If the average density of the universe is equal to the critical density, then **the age of the universe is two-thirds of the Hubble time.**

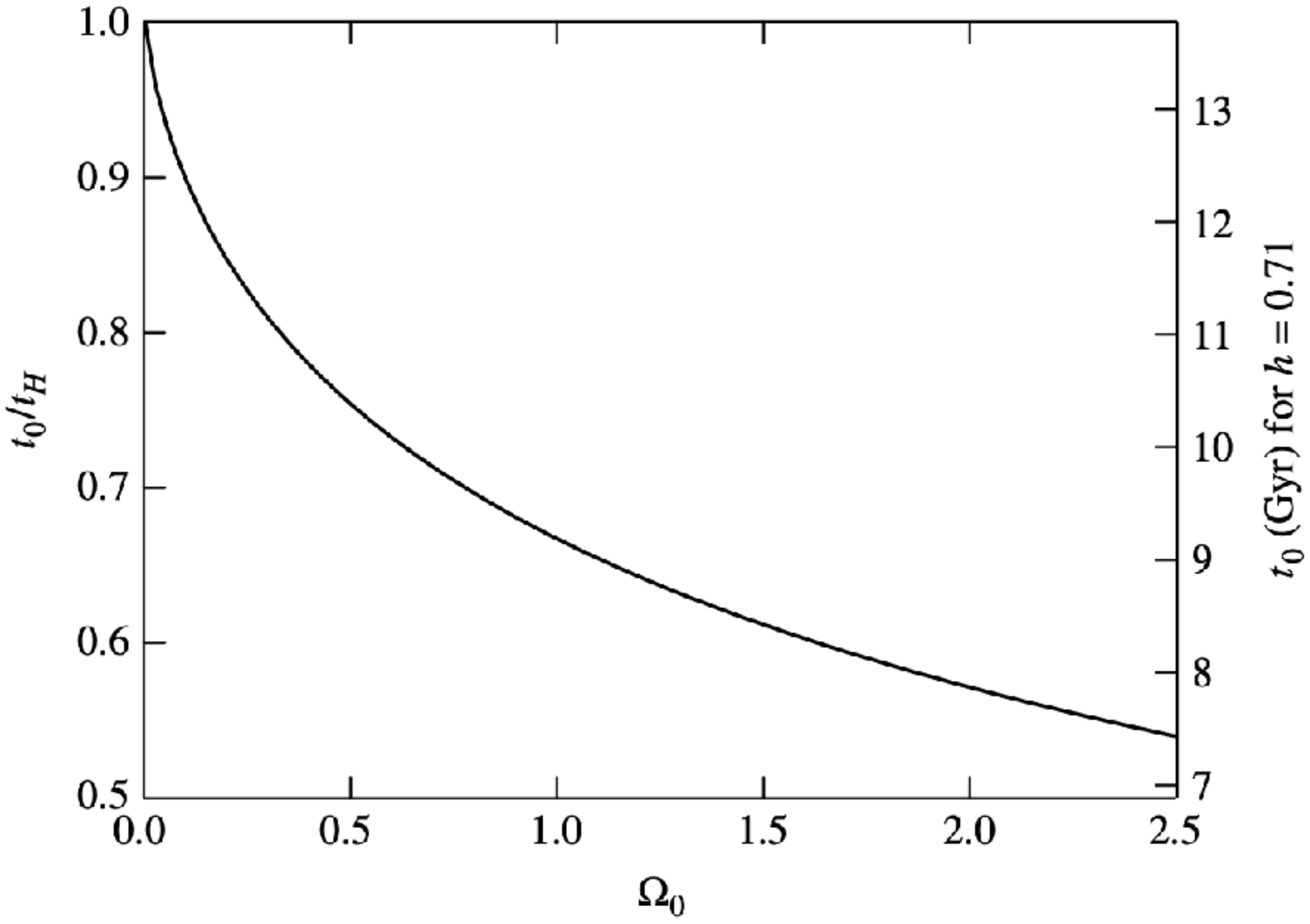


FIGURE 6 The age of the universe as a function of the density parameter, Ω_0. The age is expressed as a fraction of the Hubble time, $t_H \simeq 10^{10}h^{-1}$ yr. The right axis shows the age in billions of years for $h = 0.71$.

Newtonian cosmology

Using the WMAP value of [h]WMAP = 0.71 gives an age of about 9.2 Gyr.

Although this result is less than the **currently accepted value of 13.7 Gyr,** it is remarkable that this simple model of an expanding universe of pressureless dust produces ages that are in rough accordance with **the mean age of the oldest globular clusters, 11.5 billion years.**

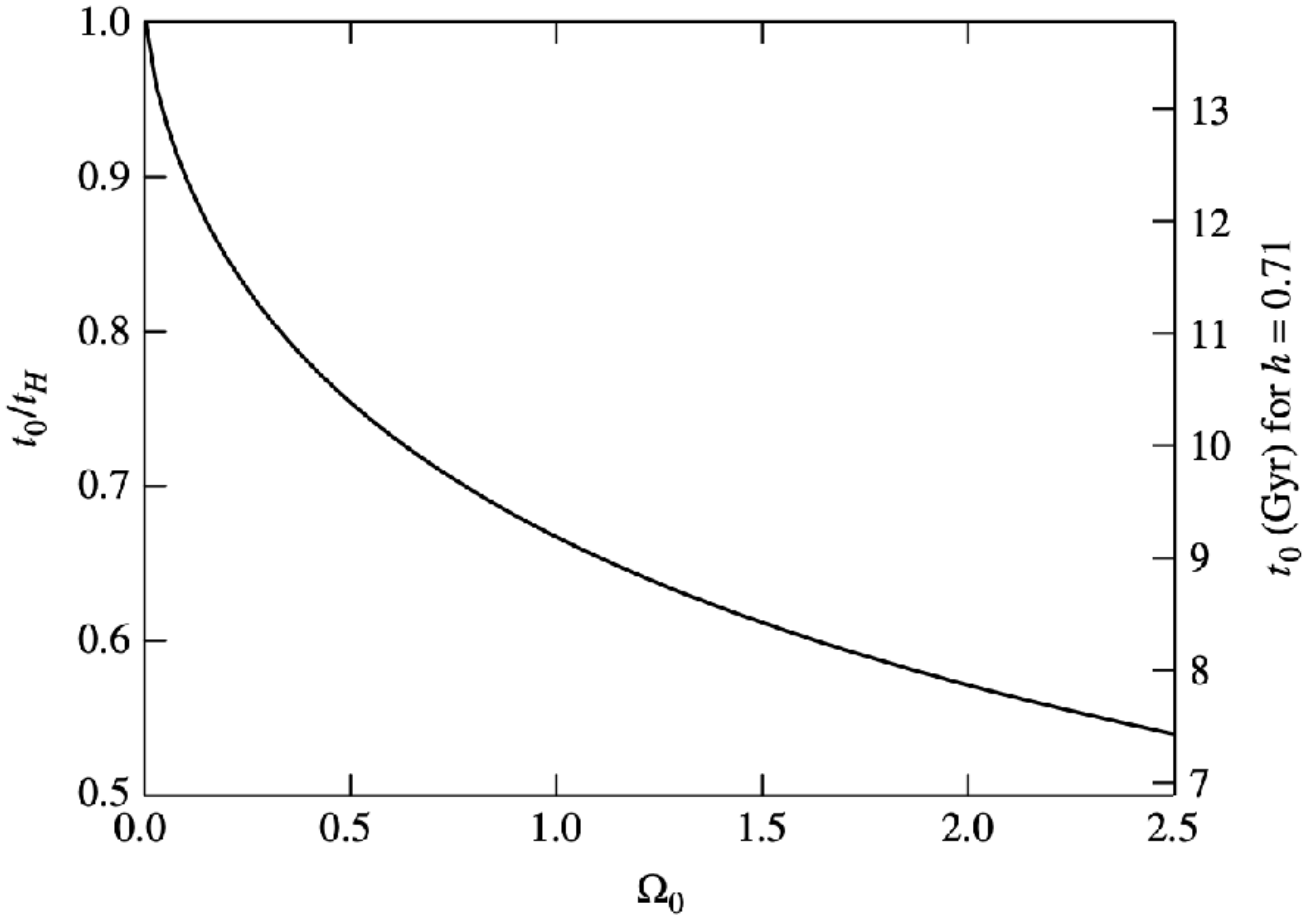


FIGURE 6 The age of the universe as a function of the density parameter, Ω_0. The age is expressed as a fraction of the Hubble time, $t_H \simeq 10^{10}h^{-1}$ yr. The right axis shows the age in billions of years for $h = 0.71$.

The Lookback time

The **lookback time**, t_L, **is defined as how far back in time we are looking when we view an object with redshift z.** This is just the difference between the present age of the universe and its age at time t(z),

$$t_L = t_0 - t(z). \quad (47)$$

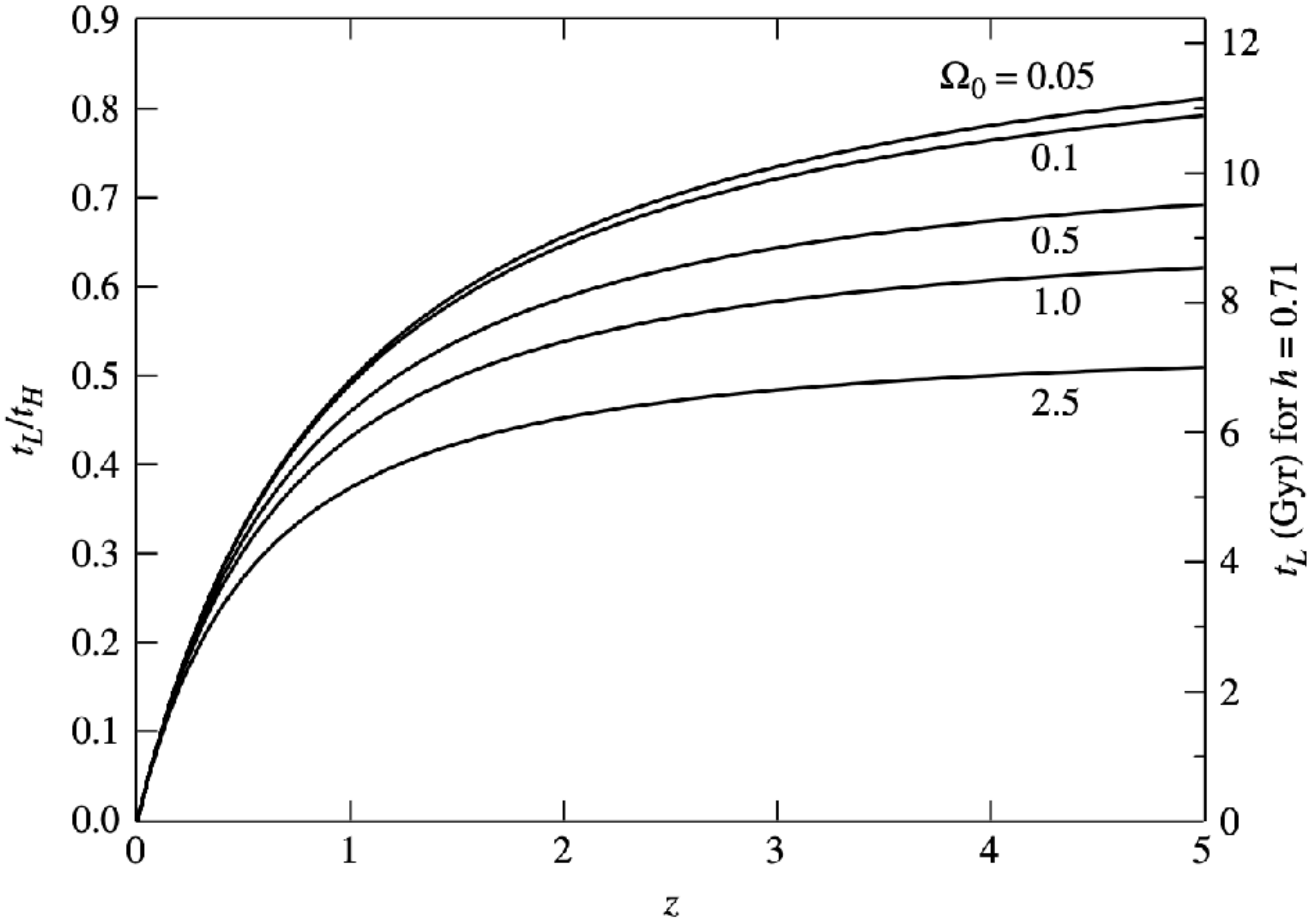


FIGURE 7 The lookback time as a function of the redshift, z, for a range of values of the density parameter, Ω_0. The lookback time is expressed as a fraction of the Hubble time, $t_H \simeq 10^{10}h^{-1}$ yr. The right axis shows the lookback time in billions of years if $h = 0.71$.

The Lookback time

For example, for a flat universe, Eqs. (40) and (44) show that the lookback time is, in units of the Hubble time,

$$\frac{t_L}{t_H} = \frac{2}{3}\left[1 - \frac{1}{(1+z)^{3/2}}\right] \qquad \text{(for } \Omega_0 = 1\text{)}. \tag{48}$$

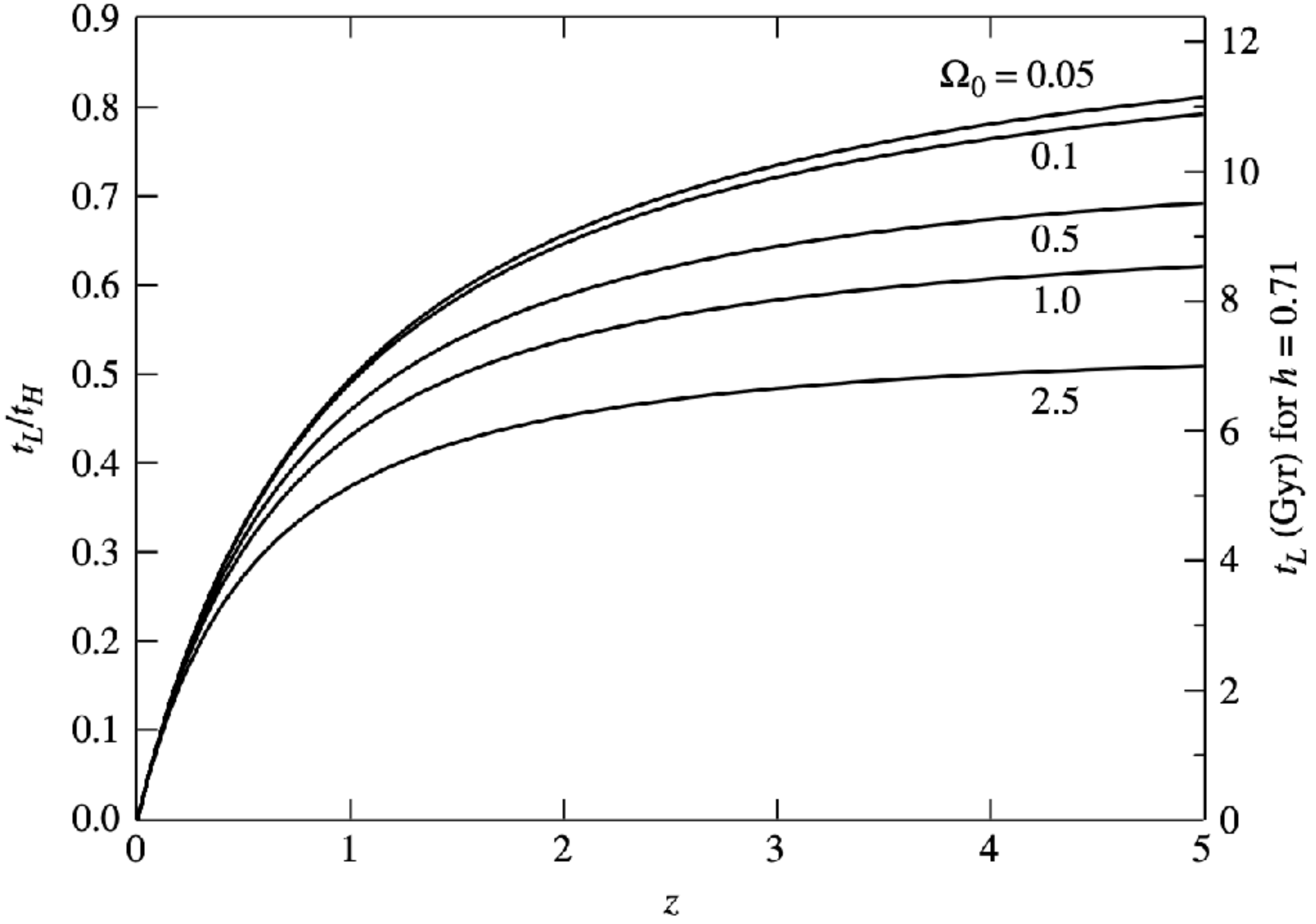
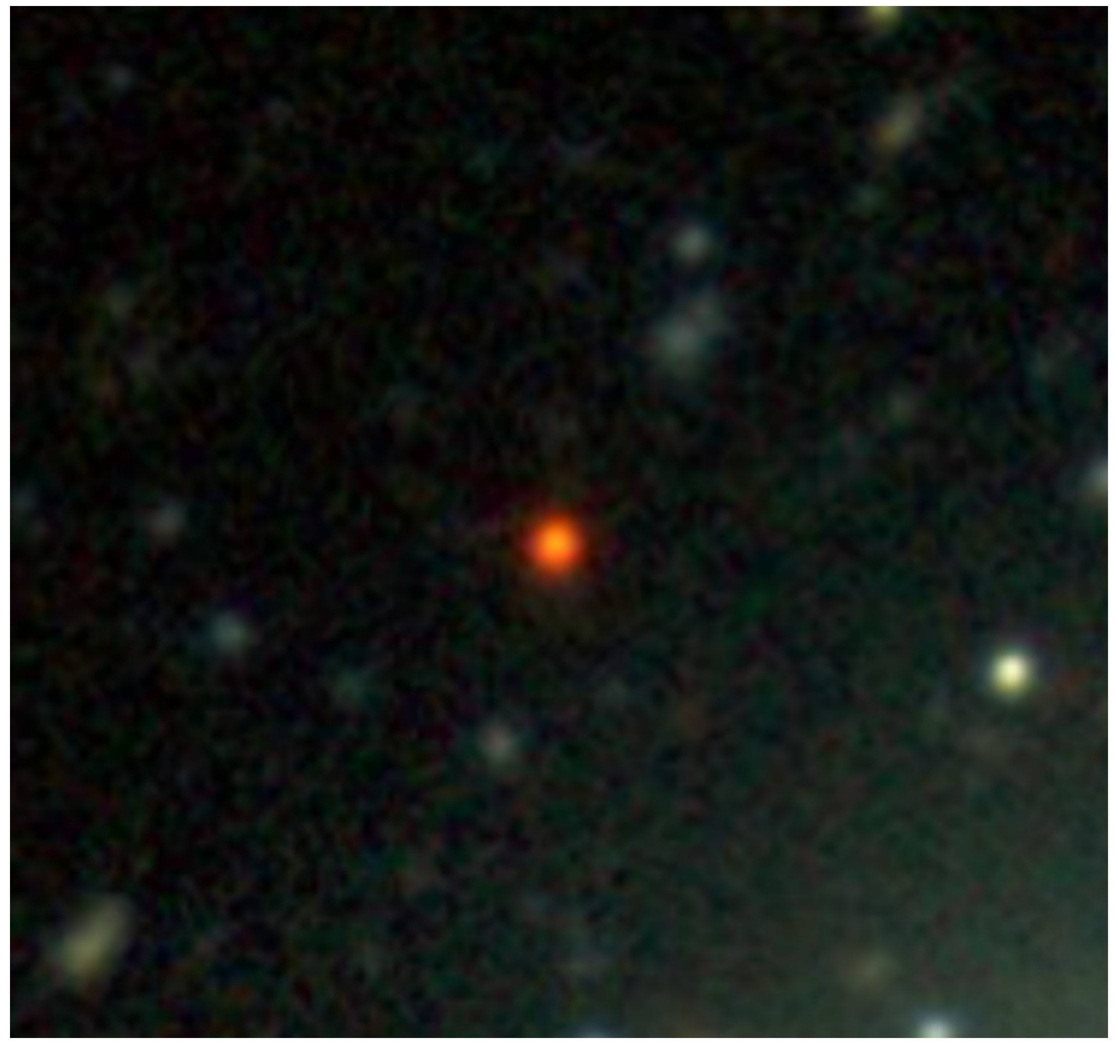


Figure 7 shows the lookback times for flat, closed, and open models of the universe.

The Lookback time

Example 1.2. The redshift of the quasar SDSS 1030+0524 was found to be z = 6.28. Assuming a flat universe of pressureless dust, Eq. (48) shows that the lookback time to this quasar is

$$\frac{t_L}{t_H} = \frac{2}{3}(1 - 0.0509) = 0.633.$$



quasar SDSS 1030+0524

Since the age of a flat universe is $t_0 = 2t_H/3$,

$$\frac{t_L}{t_0} = 0.949.$$

This means that **only 5% of the history of the universe had unfolded when the light left this quasar. At that time the universe was smaller by about a factor of 7** when, according to Eq. (4), the scale factor was

$$R = \frac{1}{1+z} = 0.137.$$

Extending the model - adding pressure

Let’s take stock of the basic equations we have derived so far and generalize them a bit, anticipating some features of the equations of general relativity we will encounter later. We start with Eq. (10),

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right]R^2 = -kc^2.$$

Considering Einstein’s $E_{rest} = mc^2$, we broaden the meaning of the density ρ to include matter in all of its forms. For nonrelativistic particles, ρ is the usual mass density. For relativistic particles, such as photons and neutrinos, ρ is the **equivalent mass density**— that is, the energy density divided by c^2.

Equation (5), $R^3\rho = \rho_0$, describes the conservation of mass within the expanding shell. Again acknowledging the equivalence of mass and energy, this equation is also a statement of the **conservation of energy for a pressureless dust universe.**

Extending the model - adding pressure

A thermodynamic argument supplies the generalization of Eq. (5) for models of the universe that incorporate pressure-producing components. Imagine a universe filled with a fluid (dust, photons, etc.) of uniform density ρ, pressure P , and temperature T , and choose an arbitrary point for the origin. Let r be the radius of a comoving spherical surface, centered on the origin. We will employ the **first law of thermodynamics, which applies the law of conservation of internal energy, U , to the fluid within the expanding sphere:**

$$dU = đQ - đW. \tag{49}$$

First note that the entire **universe has the same temperature, so there can be no heat flow**: dQ = 0. That is, the **expansion of the universe is adiabatic**. Any change in internal energy must be produced by work done by the fluid. Writing the result as a time derivative,

$$\frac{dU}{dt} = -\frac{dW}{dt} = -P\frac{dV}{dt}.$$

Extending the model - adding pressure

and substituting $V = \frac{4}{3}\pi r^3$, we obtain

$$\frac{dU}{dt} = -\frac{4}{3}\pi P \frac{d(r^3)}{dt}.$$

If we define the internal energy per unit volume u as

$$u = \frac{U}{\frac{4}{3}\pi r^3},$$

then we find

$$\frac{d(r^3 u)}{dt} = -P \frac{d(r^3)}{dt}.$$

Extending the model - adding pressure

Writing u in terms of the equivalent mass density ρ,

$$\rho = \frac{u}{c^2},$$

gives

$$\frac{d(r^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(r^3)}{dt}.$$

Finally, using r = Rϖ (Eq 3) we obtain the **fluid equation,**

$$\frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}. \tag{50}$$

For a universe of pressureless dust, P = 0 so $R^3\rho$ = constant, in agreement with Eq. (5). An equation describing the acceleration of the universal expansion can be obtained by multiplying Eq. (10) by R and then taking a time derivative. Using Eq. (50) to replace d(ρR^3)/dt and using Eq. (10) to eliminate the $-$kc^2, we arrive at the **acceleration equation**

$$\frac{d^2R}{dt^2} = -\frac{4}{3}\pi G\left(\rho + \frac{3P}{c^2}\right)R. \tag{51}$$

Extending the model - adding pressure

Note that **the effect of the pressure P is to slow down the expansion (assuming $P > 0$).** If this seems counterintuitive, recall that because the pressure is the same everywhere in the universe, both inside and outside the shell, there is no pressure gradient to exert a net force on the expanding sphere. The answer lies in the motion of the particles that creates the fluid's pressure. **The equivalent mass of the particles' kinetic energy creates a gravitational attraction that slows down the expansion just as their actual mass does.** In fact, the assumption that $P = 0$ is valid for much of the history of the universe. For instance, you will find that $\rho \gg P/c^2$ in today's universe.

Equation (51) is an illustration of **Birkhoff's theorem**. In 1923 the mathematician G. D. Birkhoff proved quite **generally that for a spherically symmetric distribution of matter, Einstein's field equations have a unique solution.**

As a corollary, the acceleration of an expanding shell in our fluid universe is determined solely by the fluid lying within the shell.

Equation (51) shows that the acceleration does not depend on any factors other than ρ, P, and R. Because Birkhoff's theorem **holds even when general relativity is included**, it is quite important in the study of cosmology.

Extending the model - adding pressure

Equations (10), (50), and (51) have three unknowns: R, ρ, and P. However, the equations are not independent; any two may be used to derive the third. To solve these two equations for R, ρ, and P , we need a third relation, an ***equation of state*, that links the variables**. Such an equation of state can be written generally as

$$P = wu = w\rho c^2, \tag{52}$$

where w is a constant.

The pressure is proportional to the energy density of the fluid.

For example, for mass in the form of pressureless dust, $w_m = 0$, and for blackbody radiation, with the equation of state $P_{rad} = u_{rad}/3$, we have $w_{rad} = 1/3$. Inserting the general equation of state, Eq. (52), into the fluid equation, Eq. (50), quickly produces the relation

$$R^{3(1+w)}\rho = \text{constant} = \rho_0, \tag{53}$$

where ρ_0 is the present value of the mass density. For pressureless dust ($w_m = 0$), we recover Eq. (5), $R^3\rho_m = \rho_{m,0}$.

The deceleration parameter

Finally, we introduce a useful dimensionless quantity that **describes the acceleration of the universal expansion**: the **deceleration parameter**, q(t), which is defined as

$$\boxed{q(t) \equiv -\frac{R(t)\,[d^2R(t)/dt^2]}{[dR(t)/dt]^2}.} \tag{54}$$

Both the name and the minus sign (to ensure that $q > 0$ for a deceleration) betray the certainty of twentieth-century astronomers that the expansion of the universe must be slowing down with time (we currently think that **the expansion is accelerating**). For a pressureless dust universe,

$$q(t) = \frac{1}{2}\,\Omega(t), \tag{55}$$

and so at the present time,

$$q_0 = \frac{1}{2}\,\Omega_0. \tag{56}$$

Thus, for a pressureless dust universe, $q_0 = 0.5$ for a flat universe, while $q_0 > 0.5$ and $q_0 < 0.5$ correspond to a closed and an open universe, respectively.