Introduction to Astrophysics and Cosmology

Relativistic cosmology

Based on: Chapter 29 of An Introduction to Modern Astrophysics

Helga Dénes 2025 S1 Yachay Tech

hdenes@yachaytech.edu.ec

Relativistic cosmology

The appearance of objects at truly cosmological distances is affected by the curvature of the spacetime through which the light travels on its way to Earth. The geometrical properties of the universe as a whole will be best understood by starting with simple analogies before proceeding to the description that Einstein's general theory of relativity can provide.

The foundations of plane geometry (appropriate for a flat universe) were laid by Euclid sometime around 300 b.c. These theorems, are derived from just five postulates:

1. It is possible to draw a straight line from any point to any point.
2. It is possible to produce a finite straight line continuously in a straight line.
3. It is possible to describe a circle with any center and distance (radius).
4. All right angles are equal to one another.

5a. Given, in a plane, a line L and a point P not on L, then through P there exists one and only one line parallel to L.

Relativistic cosmology

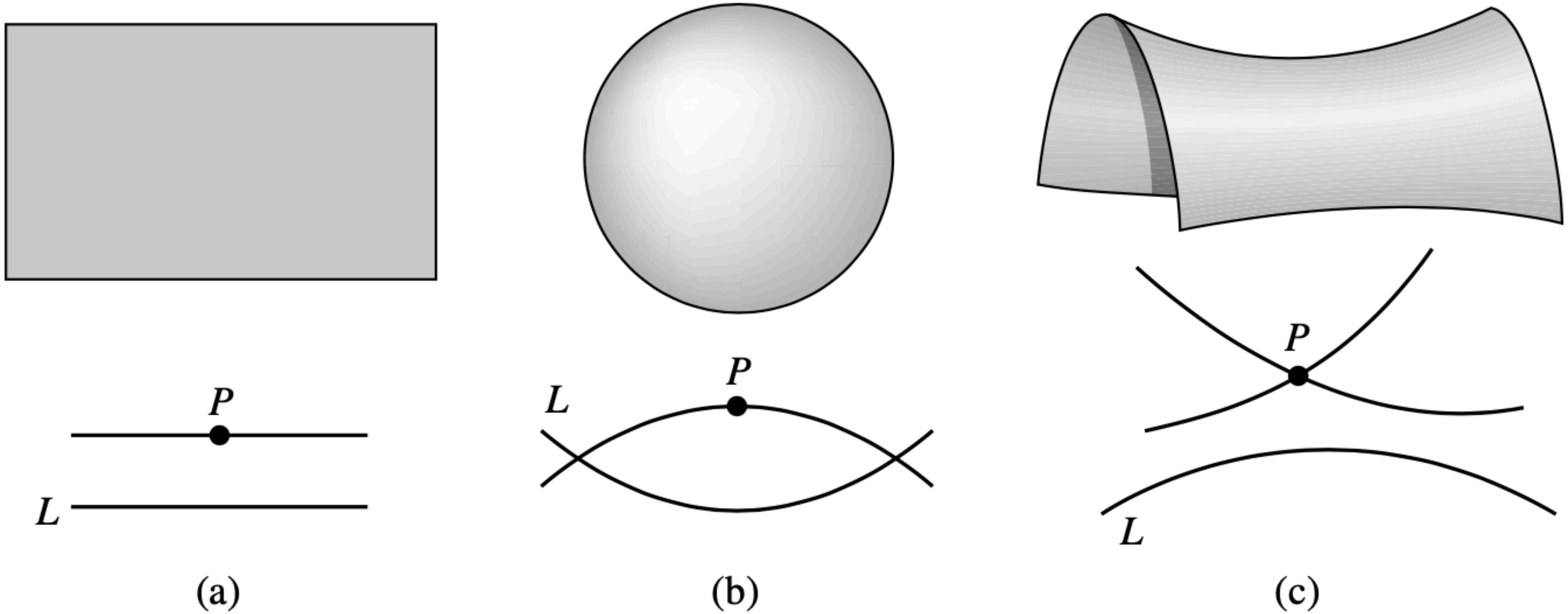


FIGURE 15 The parallel postulate, illustrated for three alternative geometries: (a) Euclidean, (b) elliptic, and (c) hyperbolic.

Relativistic cosmology

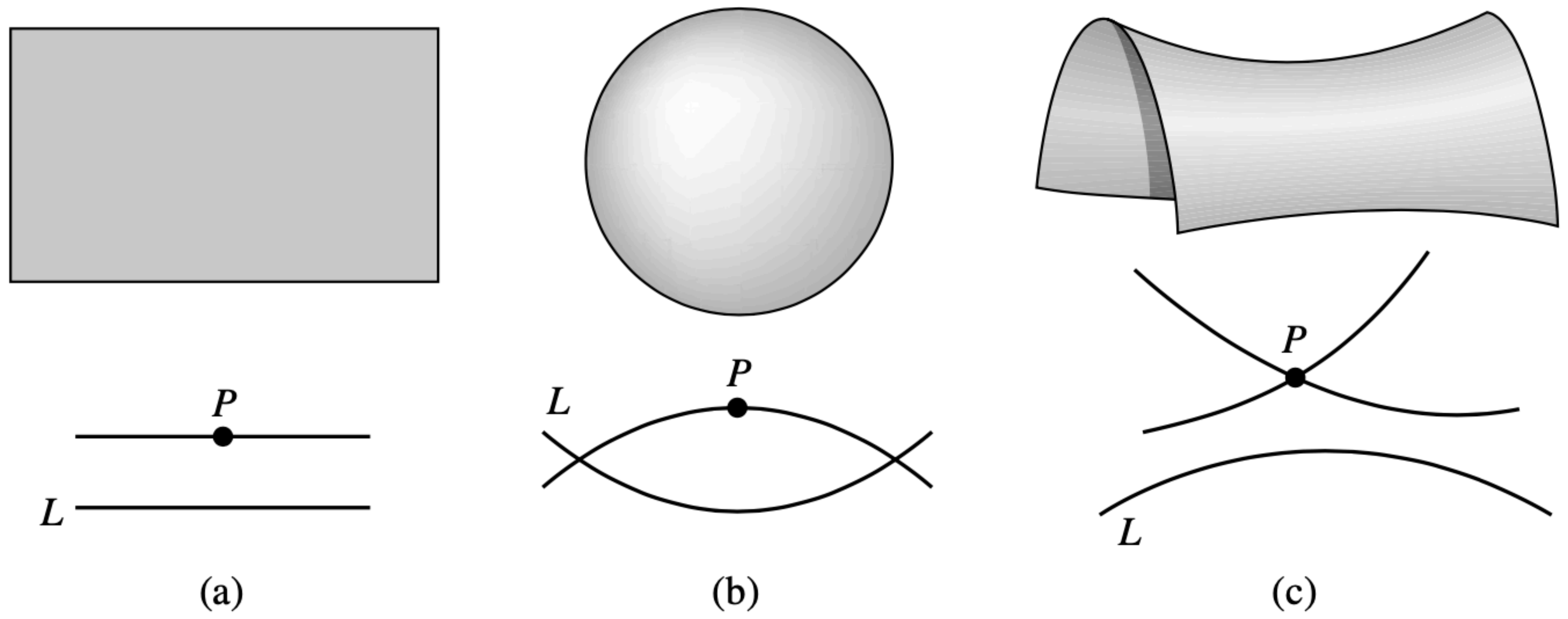


FIGURE 15 The parallel postulate, illustrated for three alternative geometries: (a) Euclidean, (b) elliptic, and (c) hyperbolic.

There are *three* different fifth postulates, each of which leads to a different geometry. In 1868 it was proved that the two additional, non-Euclidean geometries are as logically consistent as Euclid's original version. In addition to the Euclidean geometry, there is the fifth postulate of the **elliptic geometry developed by Riemann**:

5b. Given, in a plane, a line L and a point P not on L, then through P there exists *no line* parallel to L.

As Fig. 15(b) shows, this describes the geometry of the **surface of a sphere,** where two lines that both start out perpendicular to the sphere's equator meet at its poles. In elliptic geometry, the angles of a triangle add up to *more* than 180°, and the circumference of a circle is less than $2\pi r$.

On the other hand, the fifth postulate of the **hyperbolic geometry,** developed by Gauss, Bolyai, and Lobachevski, is:

5c. Given, in a plane, a line L and a point P not on L, then through P there exist *at least two lines* parallel to L.

Relativistic cosmology

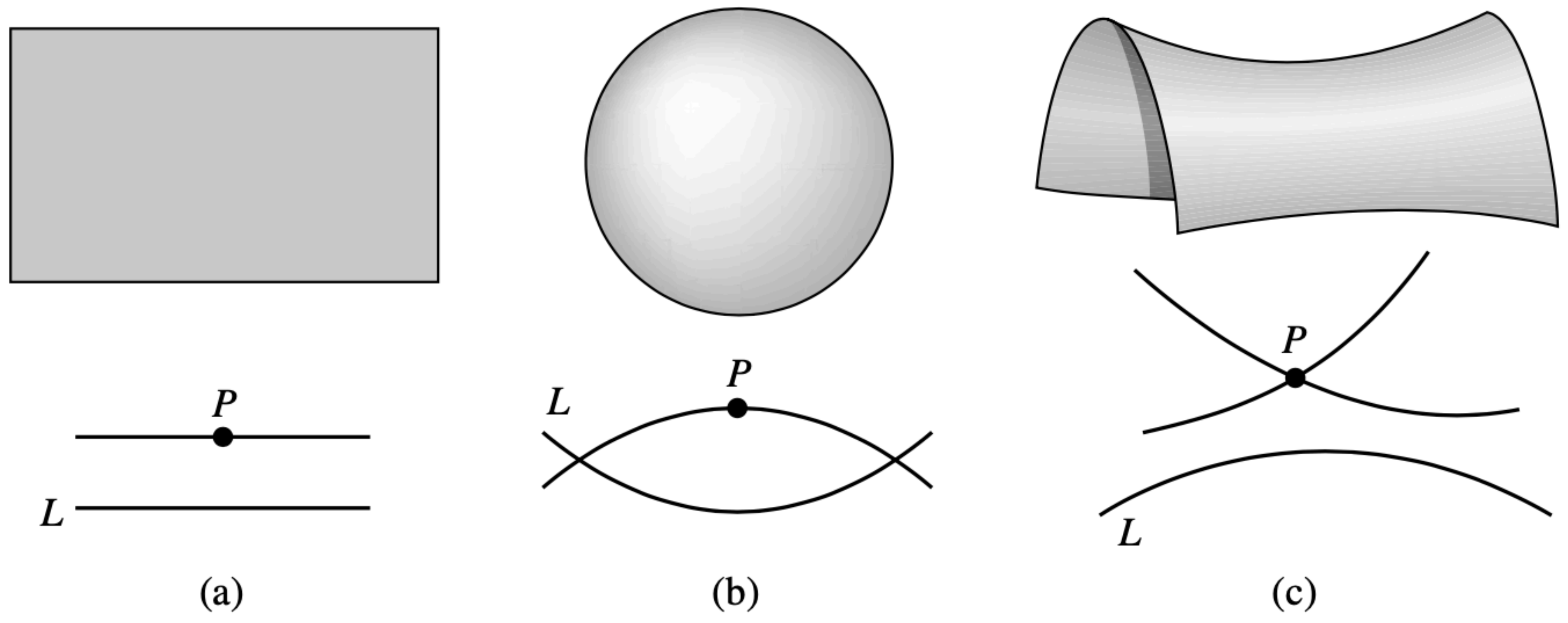


FIGURE 15 The parallel postulate, illustrated for three alternative geometries: (a) Euclidean, (b) elliptic, and (c) hyperbolic.

Figure 15(c) shows this geometry applied to a **saddle-shaped hyperboloid**. Neither of the two lines shown passing through point P intersects the line L, and it is possible to draw (in this example) infinitely many more. In hyperbolic geometry, the angles of a triangle add up to *less* than 180°, and the circumference of a circle exceeds $2\pi r$.

Which geometry to adopt is an arbitrary choice, since all three are equally valid from a mathematical viewpoint. **Which of these three geometries describes the spatial structure of the physical universe is a question that must be answered *empirically*, by observation**.

The Roberston-Walker metric

The spacetime surrounding any massive object is curved.

What evidence do we have for this?

The Curvature of spacetime

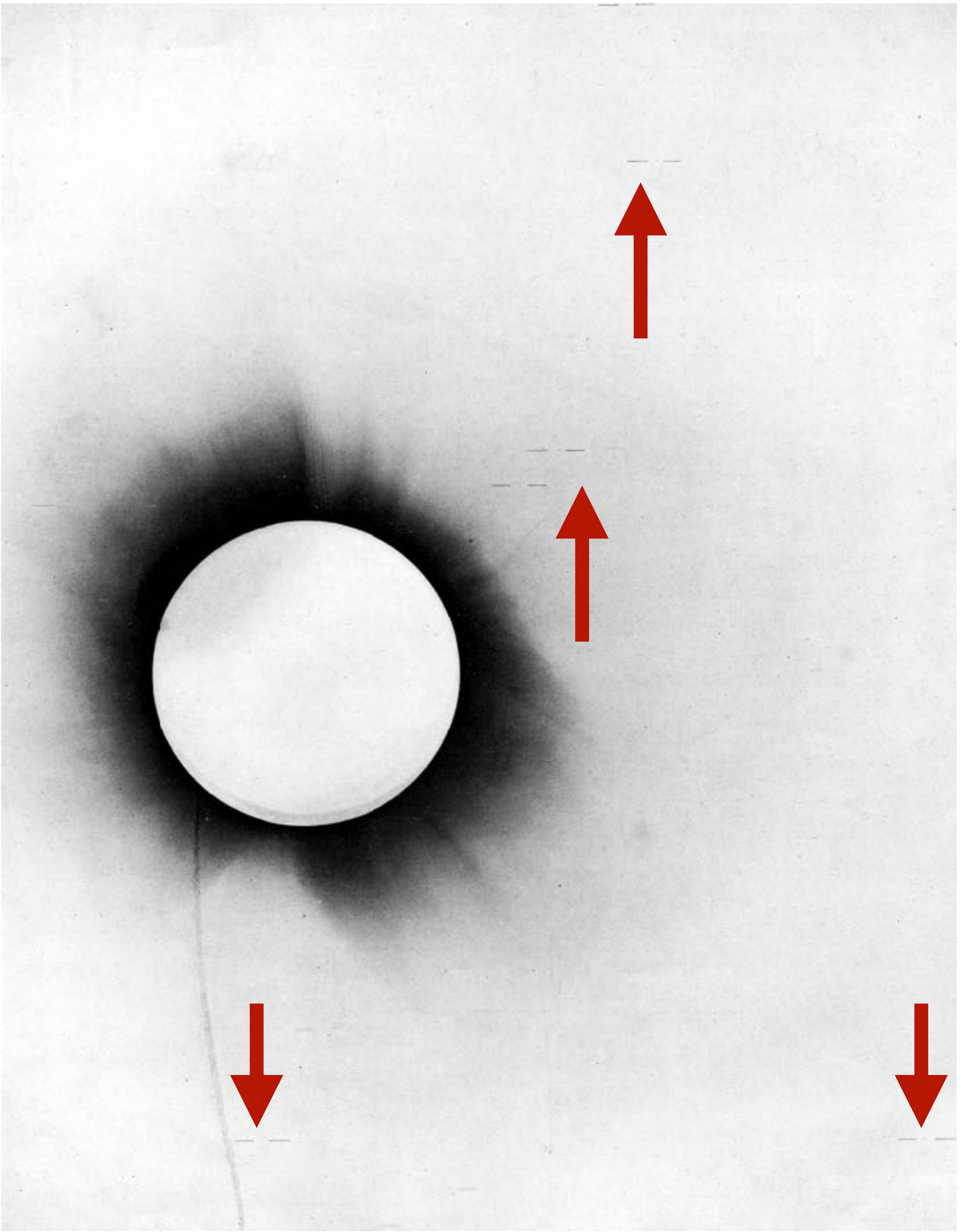
The passage of a ray of light near the Sun -> the bend of the photon's trajectory is small because the photon's speed carries it quickly through the curved space.

Can this effect be observed?

- Prediction: light bending 1.75 arcseconds for light that grazes the Sun.
- Observations: change in position of stars as they passed near the Sun, confirmed in 1919 during a solar eclipse

In general relativity, **gravity is the result of objects moving through curved spacetime, and everything that passes through, even massless particles such as photons, is affected.**

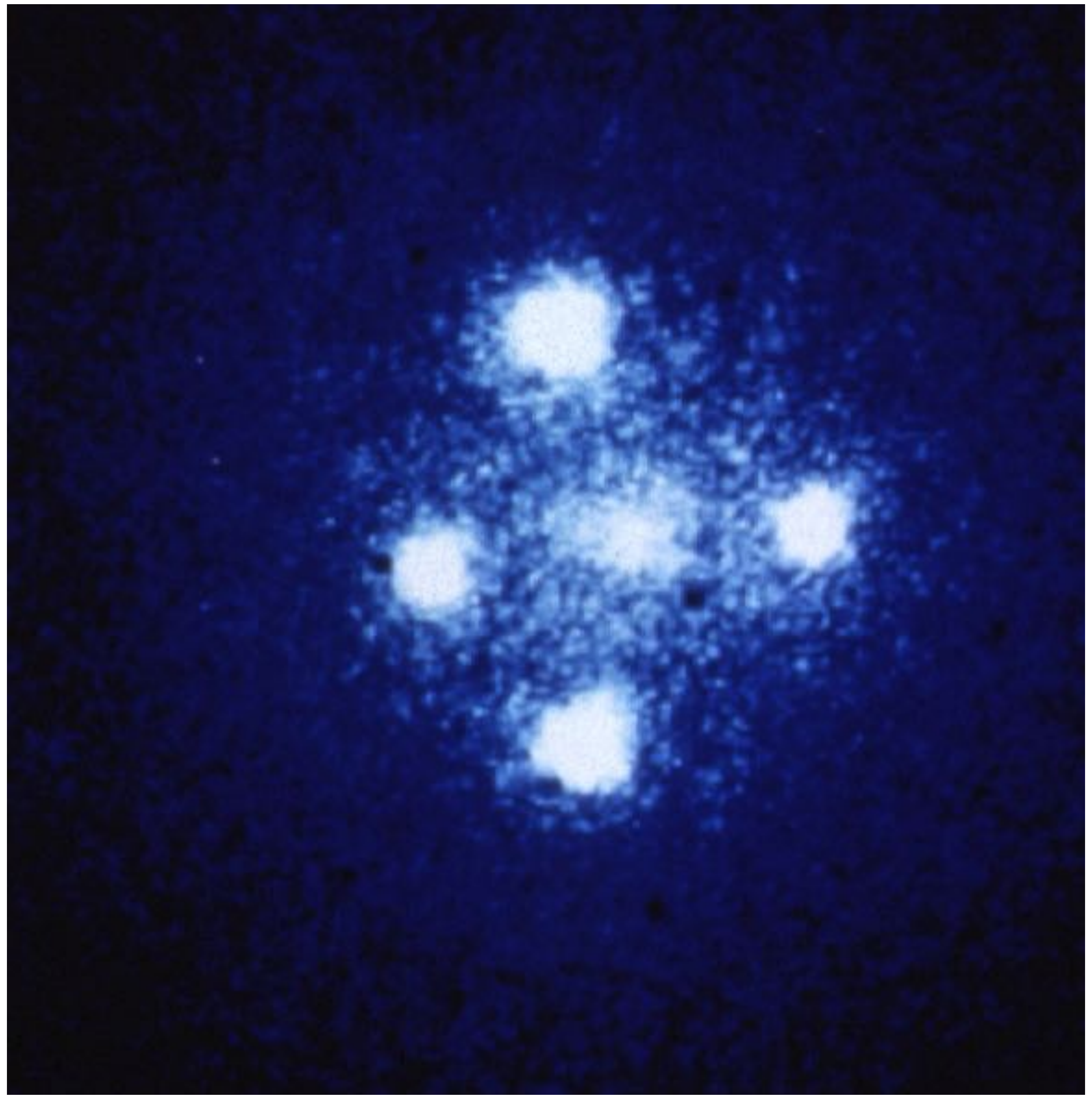
Image of the Solar eclipse in 1919 - shows the position of the stars



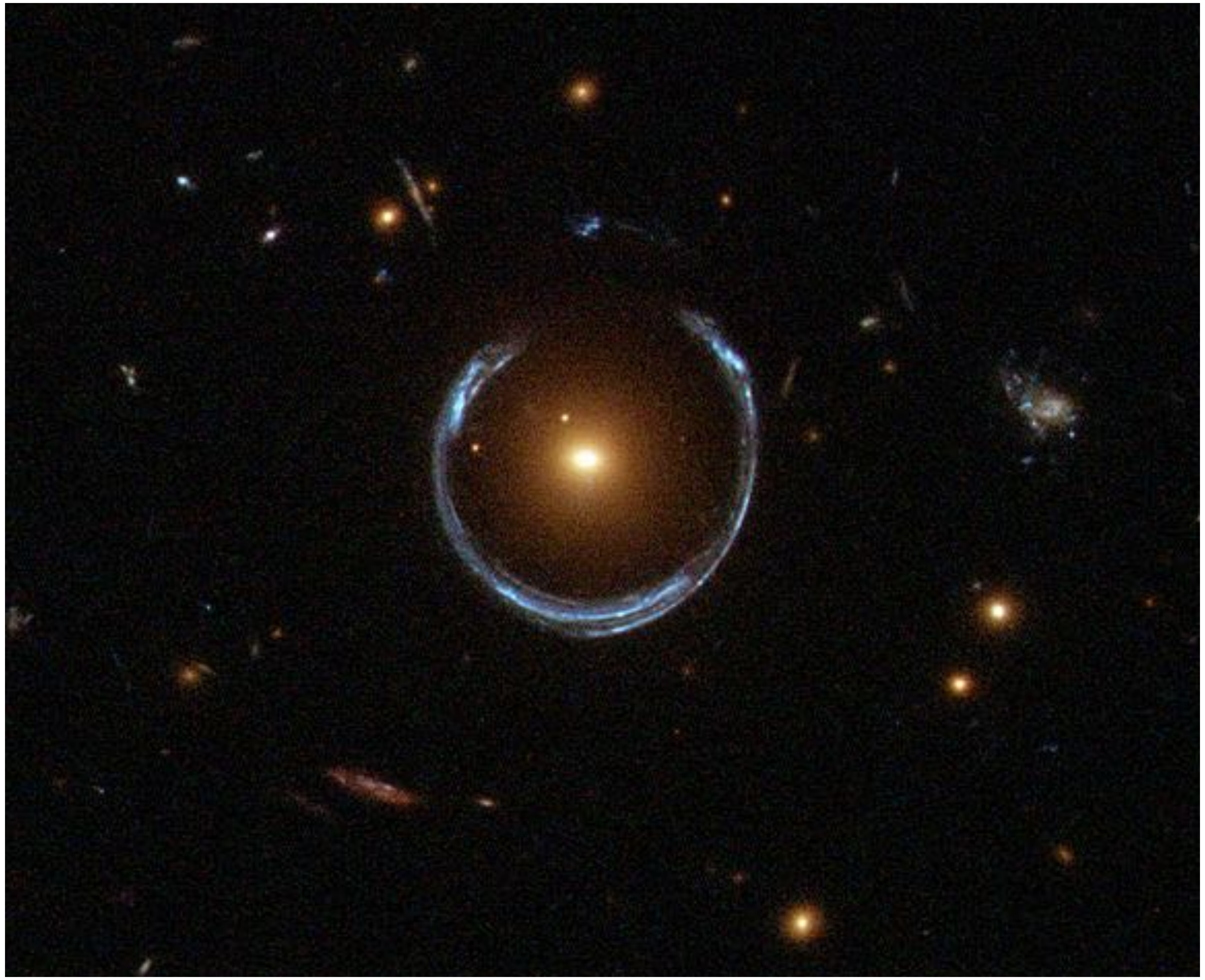
The Curvature of spacetime

More modern observations of the bending of light:
gravitational lensing

Multiple images like this are called Einstein crosses.

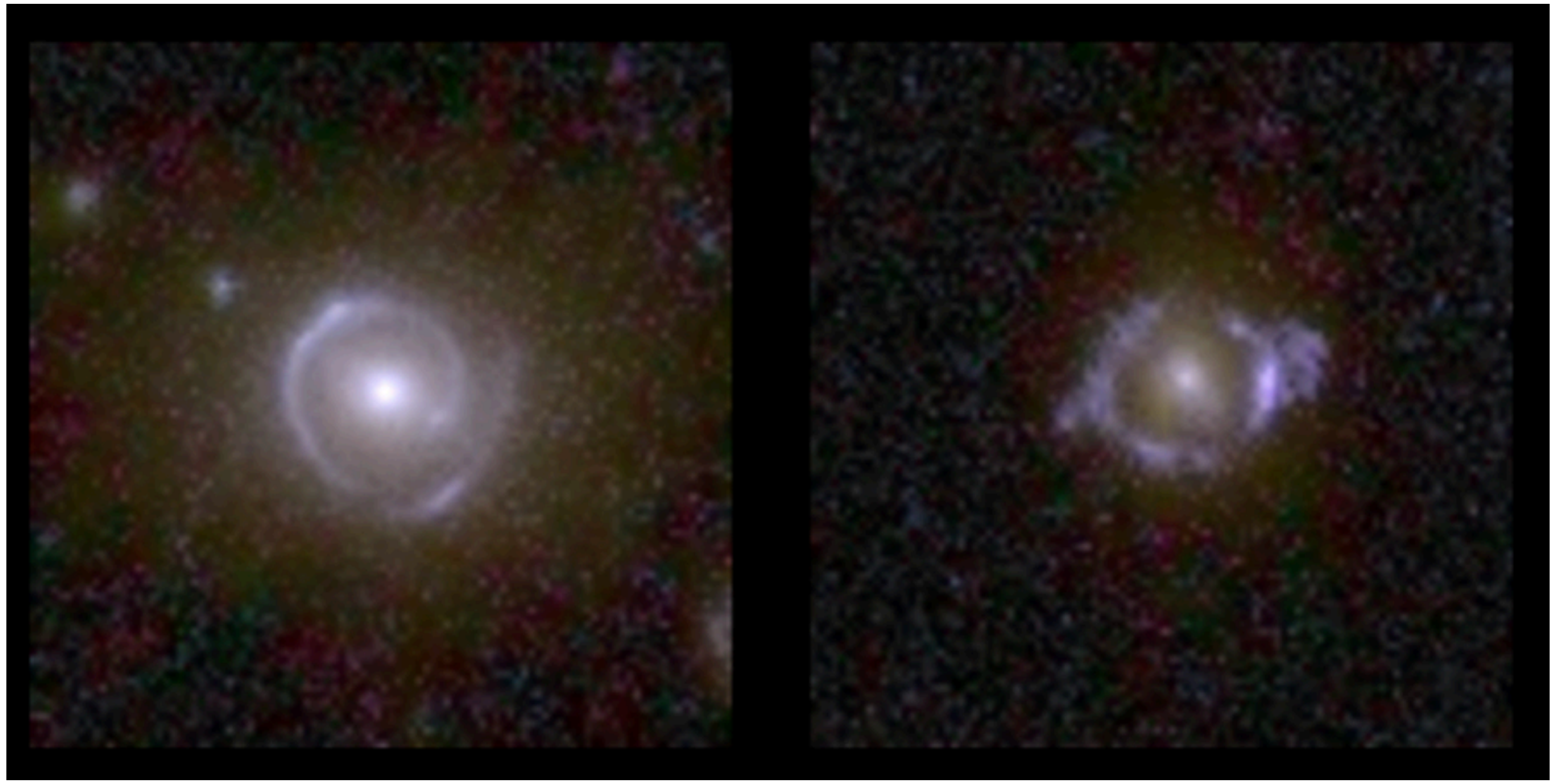


Rings like this are now known as Einstein Rings



The Curvature of spacetime

Rings like this are now known as Einstein Rings



https://www.esa.int/Science_Exploration/Space_Science/Euclid/Euclid_opens_data_treasure_trove_offers_glimpse_of_deep_fields#msdynttrid=lKBalrIGoY9X63i8Rr1-XgruRxmJ5gBiczqP1zn5Vf4

The Roberston-Walker metric

The spacetime surrounding any massive object is curved. The spatial curvature is revealed in the radial term of the Schwarzschild metric.

However, the Schwarzschild metric is valid only outside matter. We will have to find another metric to describe the spacetime of the dust-filled universe of Section 1. Although a derivation of this metric is not presented, the following argument should help elucidate some of its properties. Our search for the metric is made somewhat easier by the cosmological principle.

In a homogeneous and isotropic universe, although the curvature of space may change with time, the curvature must have the same value everywhere at a given time since the Big Bang.

Let's begin by considering the curvature of a two-dimensional surface. "Curvature" has a precise mathematical meaning; for example, a sphere of radius R has a constant curvature, K, that is defined to be $K \equiv 1/R^2$. Gauss realized that **the curvature of any surface can be determined *locally***, on an arbitrarily small patch of its surface. It is instructive to imagine how the curvature of a sphere of radius R might be measured by an ant on the sphere's surface. It is free to roam about the sphere's surface but is unable to gain an outside perspective and view the sphere as a whole. **How could the inquisitive ant determine the sphere's curvature?**

The Roberston-Walker metric

How could the inquisitive ant determine the sphere's curvature?

Starting at the north pole of the sphere (call this point P ; see Fig. 16), the ant could mark a series of other points all of which are a distance D from P . When connected, these points form a circle with P at its center. The ant now **measures the circumference**, C_{meas}, of the circle and compares it with the expected value of $C_{exp} = 2\pi D$. However, the two values do not agree; since $D = R\theta$,

$$C_{\exp} = 2\pi R\theta$$

while

$$C_{\text{meas}} = 2\pi R \sin\theta = 2\pi R \sin(D/R).$$

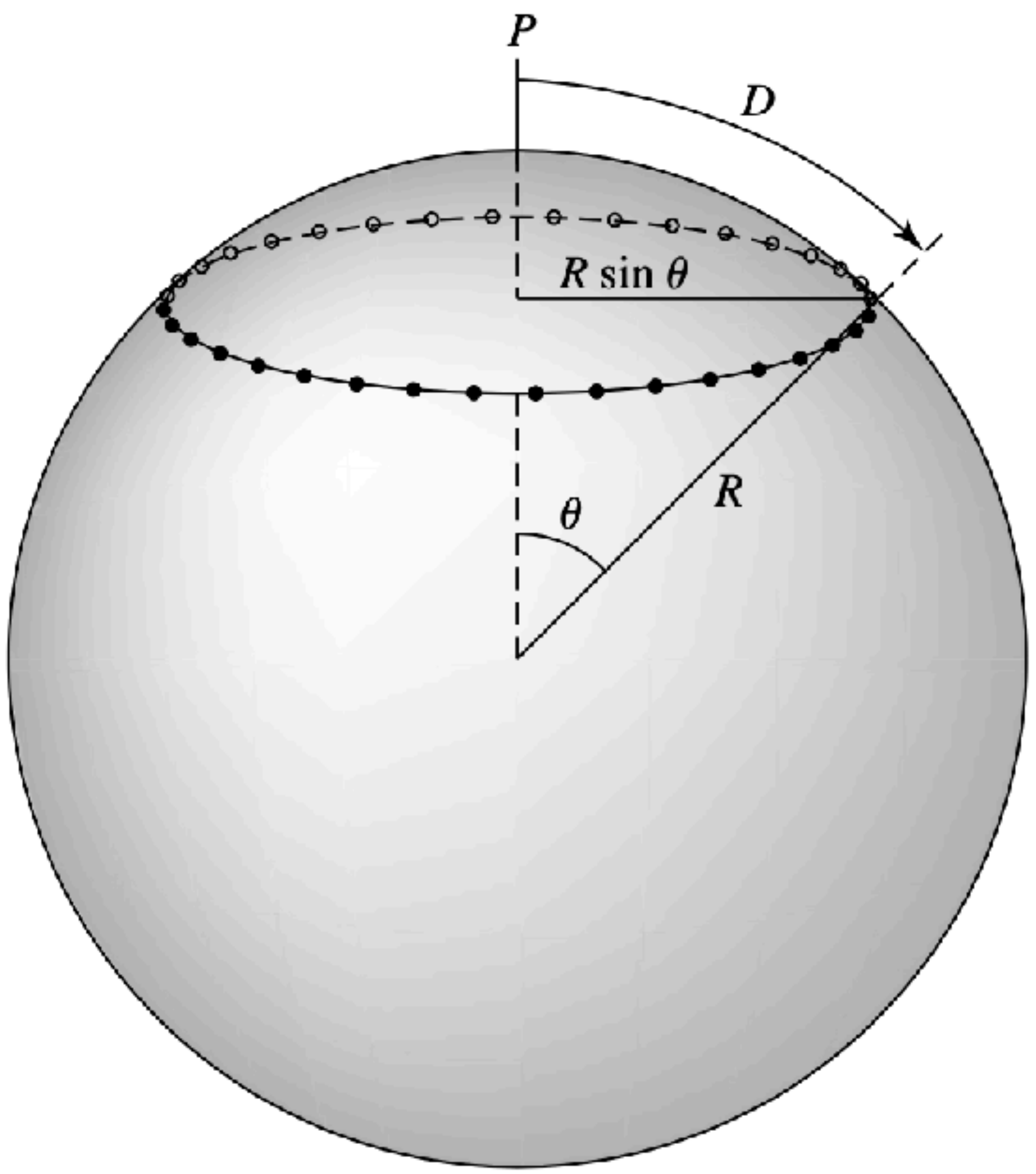


FIGURE 16 A local measurement of the curvature of a sphere.

The Roberston-Walker metric

When the ant divides the fractional discrepancy between these two values of the circumference by the expected area of the circle, $A_{exp} = \pi D^2$, and then multiplies by 6π, the result (in the limit $D \to 0$) is **the curvature of the sphere**. That is, the ant starts with

$$6\pi \cdot \frac{C_{\text{exp}} - C_{\text{meas}}}{C_{\text{exp}} A_{\text{exp}}} = 6\pi \cdot \frac{2\pi D - C_{\text{meas}}}{(2\pi D)(\pi D^2)} = \frac{3}{\pi} \frac{2\pi D - C_{\text{meas}}}{D^3}.$$

Substituting for C_{meas}, the clever ant then uses a Taylor series for $\sin(D/R)$,

$$\frac{3}{\pi} \frac{2\pi D - 2\pi R \sin(D/R)}{D^3} = \frac{6}{D^3} \left\{ D - R \left[\frac{D}{R} - \frac{1}{3!} \left(\frac{D}{R} \right)^3 + \frac{1}{5!} \left(\frac{D}{R} \right)^5 + \cdots \right] \right\}$$
$$= \frac{1}{R^2} - \frac{1}{20} \frac{D^2}{R^4} + \cdots .$$

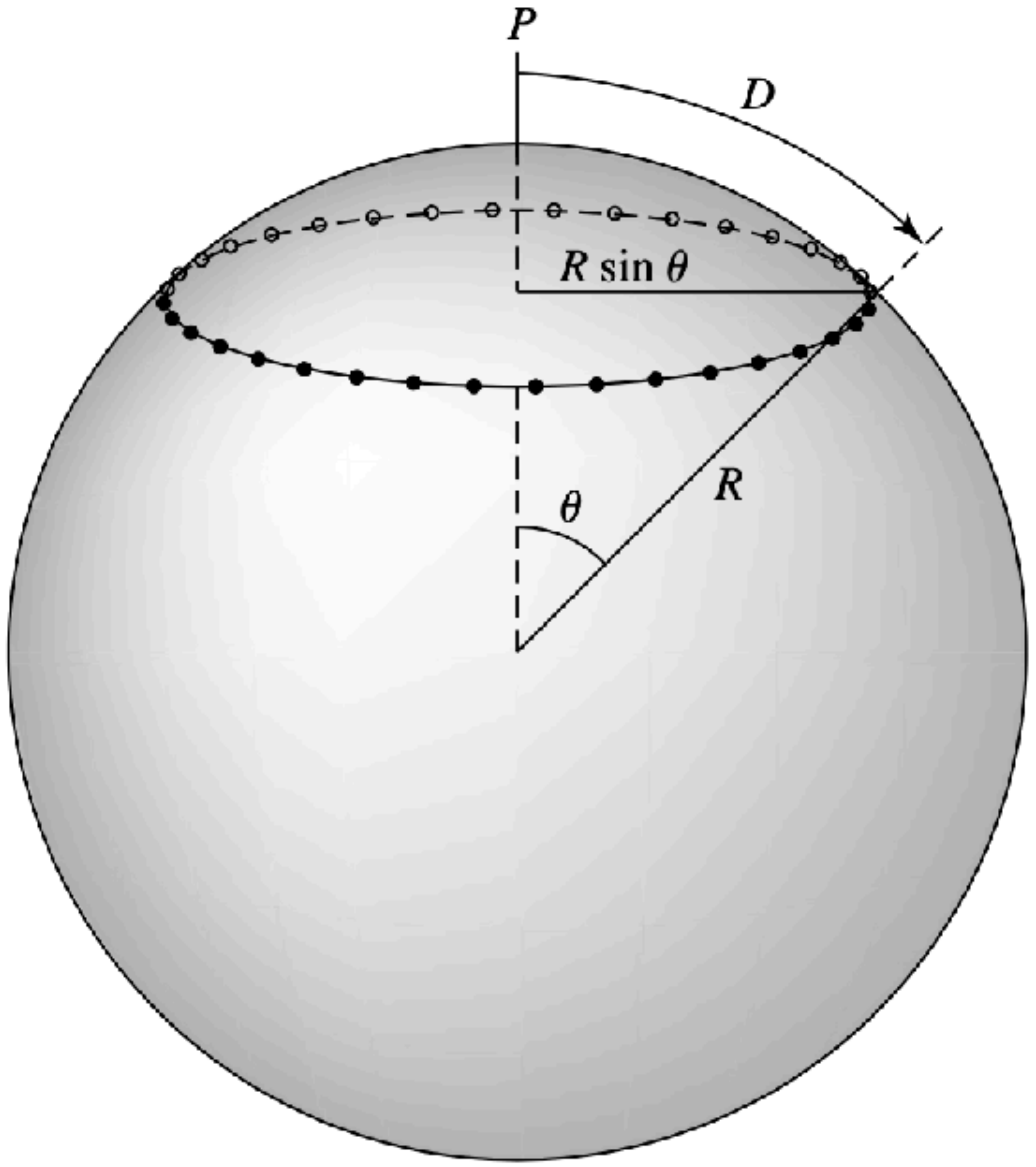


FIGURE 16 A local measurement of the curvature of a sphere.

The Roberston-Walker metric

In the limit D → 0, this is $1/R^2$, the curvature of the sphere. In fact, the prescription

$$\boxed{K = \frac{3}{\pi} \lim_{D \to 0} \frac{2\pi D - C_{\text{meas}}}{D^3} \qquad (103)}$$

can be used to calculate the curvature at any point on a two-dimensional surface; see Fig. 17.

For a **flat plane, K = 0,** while for a **saddle-shaped hyperboloid, K is *negative*** because the measured circumference exceeds 2πD.

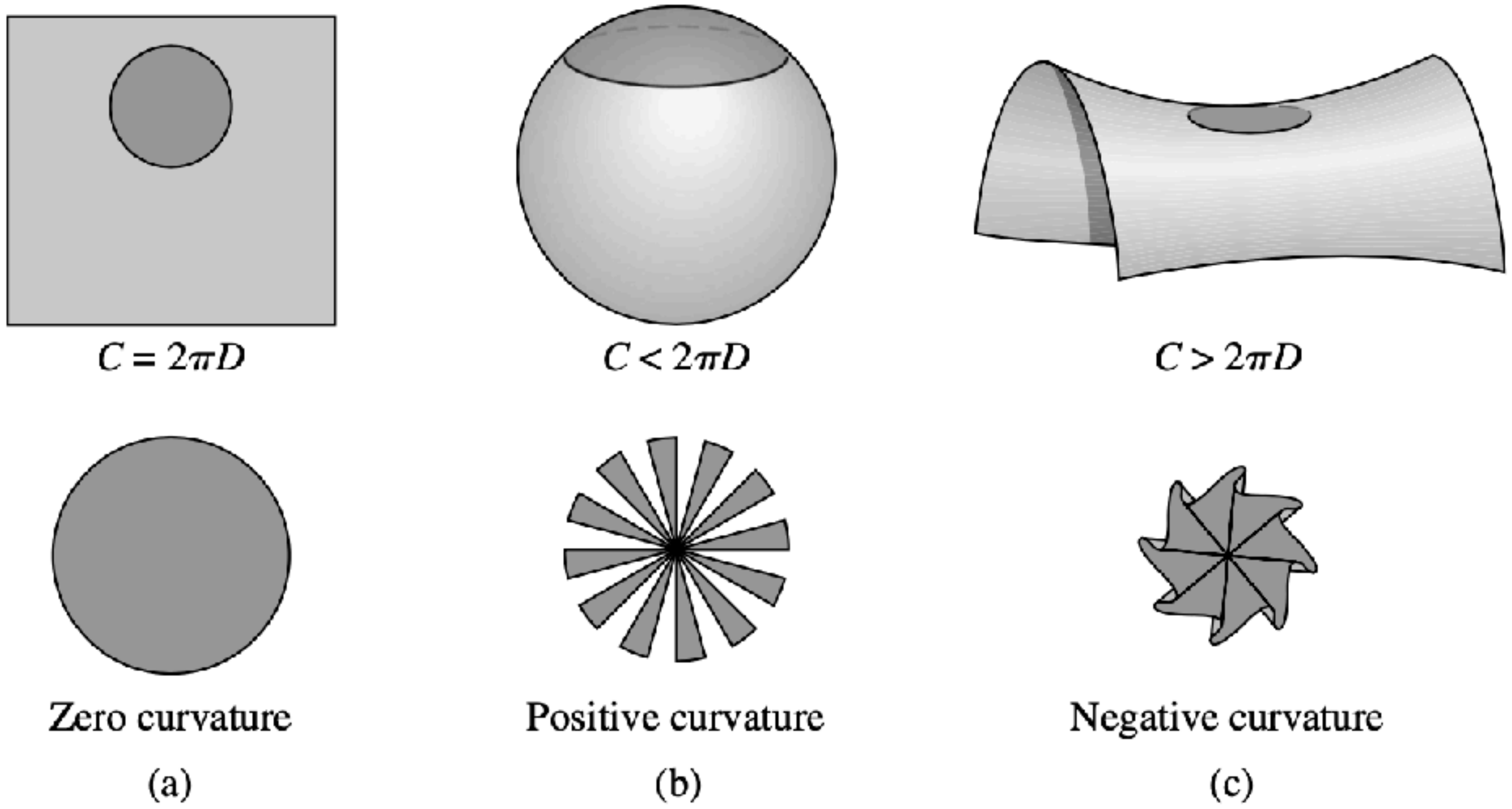


FIGURE 17 Calculating the curvature of a surface in three geometries: (a) a flat plane, (b) the surface of a sphere, and (c) the surface of a hyperboloid.

The Roberston-Walker metric

The next step toward the spacetime metric that describes a uniform dust-filled universe comes from considering **how a small distance is measured in two dimensions**. For a flat plane, polar coordinates are an appropriate choice of variables, and **the differential distance dl between two nearby points P1 and P2 on the plane** [see Fig. 18(a)] is given by

$$(d\ell)^2 = (dr)^2 + (r\,d\phi)^2.$$

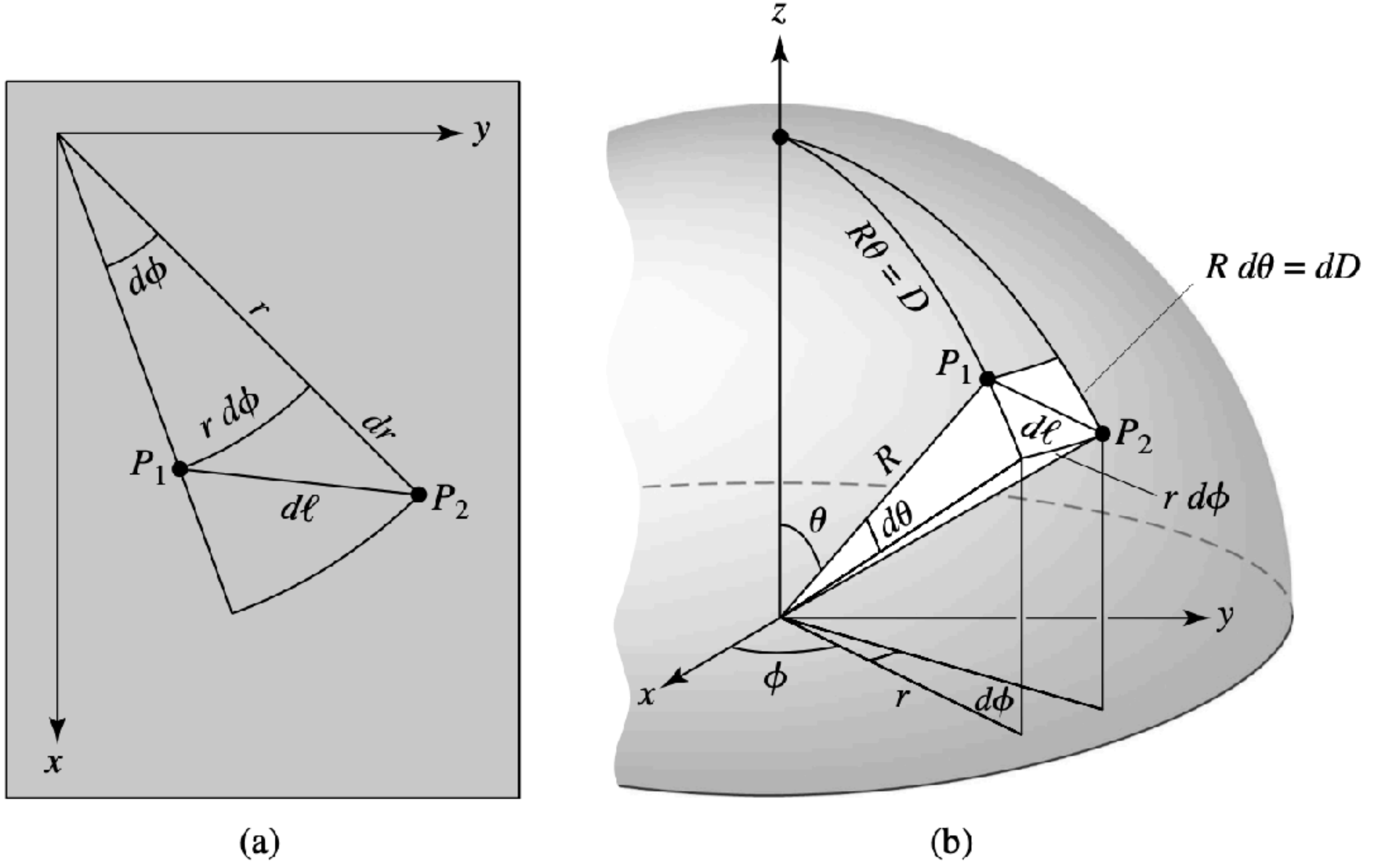


FIGURE 18 $d\ell$ as measured for (a) a flat plane and (b) the surface of a sphere.

The Roberston-Walker metric

Polar coordinates can also be used to measure the **differential distance between two nearby points** on the surface of a **sphere**. As an example, we will return to the surface of the sphere of radius R and curvature K = $1/R^2$ considered previously. Then, as shown in

Fig. 18(b), **the distance dl between two points P1 and P2 on the sphere** is now given by

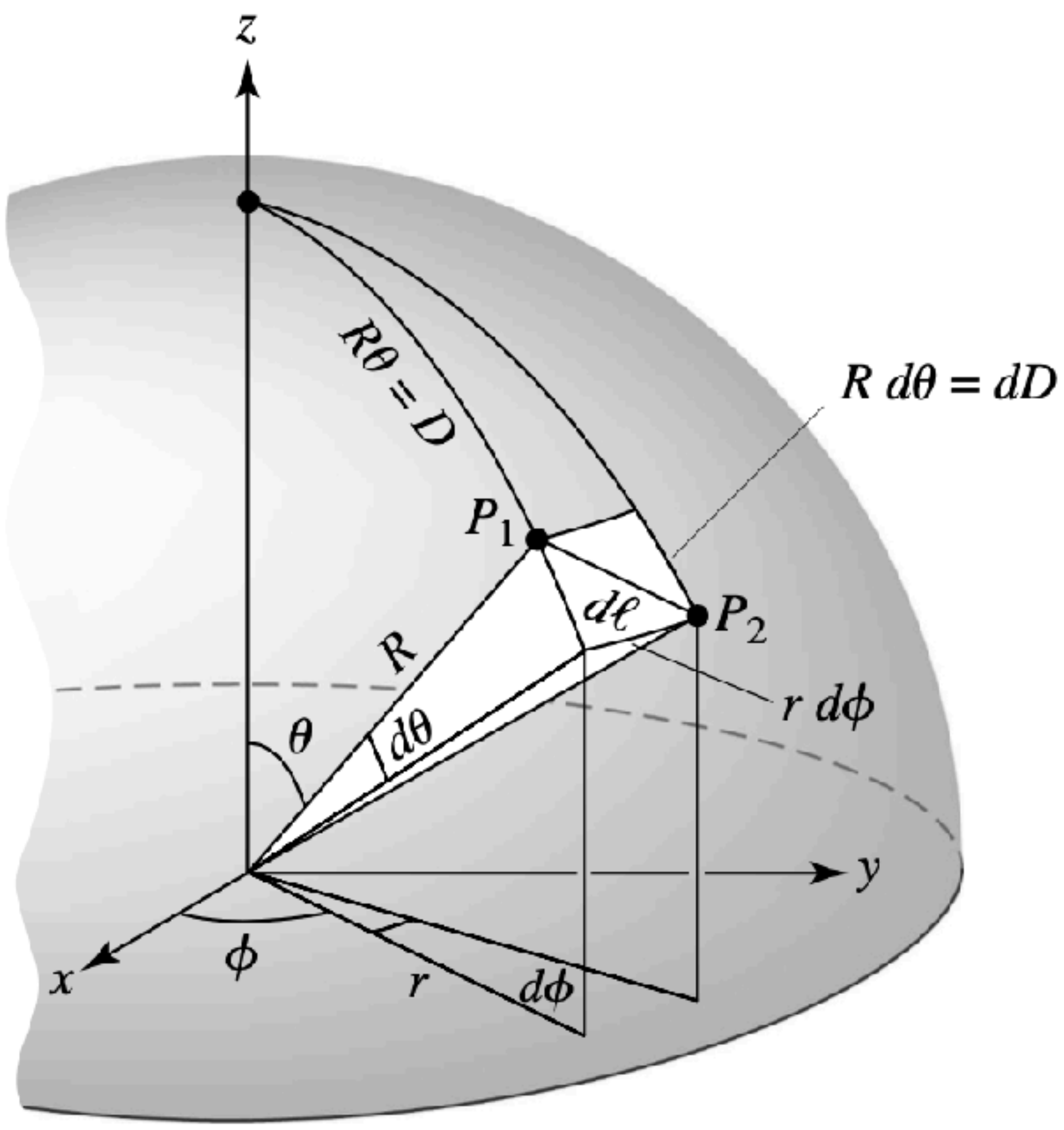
$$(d\ell)^2 = (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2.$$

But r = R sinθ, so dr =R cosθ dθ and

$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}}.$$

The differential distance on the sphere's surface may therefore be written as

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2,$$



(b)

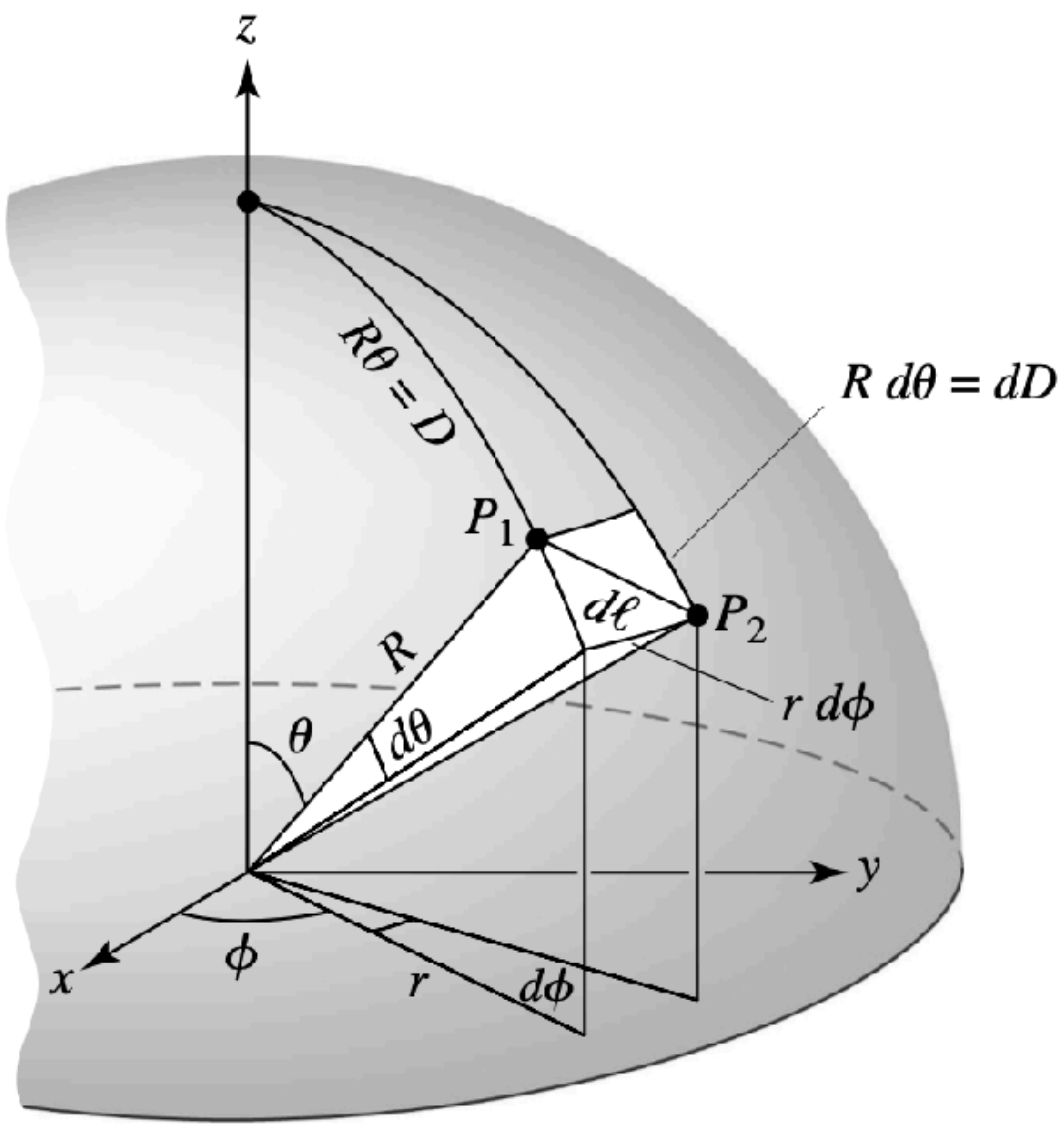
The Roberston-Walker metric

in terms of the plane polar coordinates r and ϕ. **More generally, in terms of the curvature K of a two-dimensional surface,**

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\phi)^2.$$

The extension to three dimensions is accomplished simply by making a transition from **polar to spherical coordinates,**

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2, \qquad (104)$$



(b)

where r now measures the radial distance from the origin.

Equation (104) **shows the effect of the curvature of our three-dimensional universe on spatial distances.**

The Roberston-Walker metric

The final step toward the spacetime metric comes from recalling that by "distance," we mean the ***proper distance*** between two **spacetime events** that occur simultaneously according to an observer. In an expanding universe, **the positions of two points must be recorded at the same time if their separation is to have any meaning**. In an isotropic, homogeneous universe, there is no reason why time should pass at different rates at different locations; consequently, **the temporal term should simply be cdt.** If we take

$$\boxed{(ds)^2 = (c\,dt)^2 - \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2}$$

as the metric for an isotropic, homogeneous universe, then for the proper distance agrees with Eq. (104). That is, the differential proper distance is just $dL = \sqrt{-(ds)^2}$ with dt = 0.

All that remains is **to express this metric in terms of the dimensionless scale factor, R(t)**, defined by r(t) = R(t)ϖ (Eq. 3). Because the expansion of the universe affects all of its geometric properties, including its curvature, it is useful to **define the time-dependent curvature in terms of a time-independent constant, k,** as

$$K(t) \equiv \frac{k}{R^2(t)}. \tag{105}$$

The Roberston-Walker metric

These substitutions for r and K result in

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\,\sin\theta\,d\phi)^2\right], \qquad (106)$$

which is known as the **Robertson–Walker metric**. The Robertson–Walker metric **determines the spacetime interval between two events in an isotropic, homogeneous universe**.

In fact, we will use the same technique to define ϖ for a curved spacetime to specify the radial coordinate r. From the Robertson–Walker metric, the area today ($t = t_0$ so $R(t_0) = 1$) of a spherical surface centered on the point $\varpi = 0$ is $4\pi\varpi^2$. By definition, this surface is located at the coordinate ϖ. It is important to remember that the ϖ in Eq. (106) is a ***comoving* coordinate that follows a given object as the universe expands.** Furthermore, the **time, t, is a *universal time* that essentially measures the time that has elapsed since the Big Bang.** This is not an absolute time, but merely reflects a choice of how the clocks of distant observers are to be synchronized.

The Friedman equation

Solving Einstein's field equations for an isotropic, homogeneous universe leads to a description of the dynamic evolution of the universe in the form of a differential equation for the scale factor, R(t). This is known as the **Friedmann equation**,

$$\boxed{\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right]R^2 = -kc^2,} \qquad (107)$$

In 1922 Friedmann solved Einstein's field equations for an isotropic, homogeneous universe to obtain this equation for a nonstatic universe. The same equation was derived independently in 1927 by the Belgian cleric Abbé Georges Lemaître.

The cosmological constant

Einstein realized that, as originally conceived, his field equations could not produce a static universe. Hubble's discovery of the expanding universe had not yet been made, so in 1917 Einstein modified his equations by adding an ad hoc term that contained the **cosmological constant**, Λ. With this addition, the general solution of Einstein's field equations is

$$\boxed{\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2. \qquad (108)}$$

Einstein's field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

Mass-energy represents curvature in spacetime.

Einstein tensor

Represents curvature + metric

Metric tensor

represents geometry

Stress-energy tensor

Represents energy/mass

Einstein gravitational constant

https://en.wikipedia.org/wiki/Einstein_field_equations

The cosmological constant

Except for the cosmological constant, this is the same as the Friedmann equation. The additional term containing Λ would result from the Newtonian cosmology of Section 1 if a potential energy term,

$$U_\Lambda \equiv -\frac{1}{6}\Lambda m c^2 r^2,$$

were added to the left-hand side of Eq. (1). The conservation of mechanical energy applied to an expanding shell of mass m then becomes

$$\frac{1}{2}mv^2 - G\frac{M_r m}{r} - \frac{1}{6}\Lambda m c^2 r^2 = -\frac{1}{2}mkc^2\varpi^2.$$

The force due to this new potential is

$$\mathbf{F}_\Lambda = -\frac{\partial U_\Lambda}{\partial r}\,\hat{\mathbf{r}} = \frac{1}{3}\Lambda m c^2 r\,\hat{\mathbf{r}} \tag{109}$$

which is radially outward for $\Lambda > 0$.

The cosmological constant

In effect, a **positive cosmological constant produces a repulsive force on the mass shell**. This allowed Einstein to achieve his goal of **balancing his static, closed universe against a gravitational collapse** in an (unstable) equilibrium. Later, after the expansion of the universe had been discovered, Einstein expressed his regret at including the Λ-term in his field equations.

A nonzero cosmological constant implies that space would be curved even in an empty universe that is devoid of matter, an idea that Einstein disliked because it conflicted with his ideas concerning mass as the cause of spacetime curvature.

Ironically, Willem de Sitter used Einstein's field equations with the Λ-term to describe an expanding, empty universe, with the expansion powered by the cosmological constant, and Hubble viewed the distance-dependent redshift of de Sitter's universe as theoretical support for an expanding universe in his 1929 paper.

The effects of dark energy

In the late 1990s, observations forced astronomers to recognize a nonzero cosmological constant also called **dark energy**.

We begin by rewriting the **Friedmann equation (including the Λ-term)** in a form that makes it explicit that we are dealing with a three-component universe of mass (baryonic and dark), relativistic particles (photons and neutrinos), and dark energy,

$$\boxed{\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{\mathrm{rel}}) - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2. \qquad (110)}$$

The **fluid equation** (Eq. 50) also emerges from solving Einstein's field equations with the cosmological constant included, as

$$\frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}, \qquad (111)$$

where ρ and P are the density and pressure due to every component of the universe (including, as we shall see, dark energy).

The effects of dark energy

Note that although Λ was included in the field equations, it does not appear in the fluid equation. The Friedmann and fluid equations can be combined to produce the **acceleration equation**,

$$\frac{d^2R}{dt^2} = \left\{ -\frac{4}{3}\pi G \left[\rho_m + \rho_{\text{rel}} + \frac{3(P_m + P_{\text{rel}})}{c^2} \right] + \frac{1}{3}\Lambda c^2 \right\} R. \tag{112}$$

(Although $P_m = 0$ for a pressure less dust universe, it is included in the acceleration equation for the sake of completeness.)

If we define **the equivalent mass density of the dark energy** to be

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0}, \tag{113}$$

then the Friedmann equation becomes, in parallel with Eq. (65),

$$\left[\left(\frac{1}{R}\frac{dR}{dt} \right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{\text{rel}} + \rho_\Lambda) \right] R^2 = -kc^2. \tag{114}$$

The effects of dark energy

Because ρ_Λ remains constant as the universe expands, more and more dark energy must continually appear to fill the increasing volume.

The pressure due to dark energy, as calculated from Eq. (111), is

$$P_\Lambda = -\rho_\Lambda c^2. \tag{115}$$

Thus $w_\Lambda = -1$ in the general equation of state P = $w\rho c^2$ (Eq. 52). This equation of state is unlike any other we have encountered. **A positive cosmological constant corresponds to a positive mass density and a negative pressure!**

Substituting the expressions for ρ_Λ and P_Λ into Eq. (112), the **acceleration equation**, yields

$$\frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\mathrm{rel}} + \rho_\Lambda + \frac{3(P_m + P_{\mathrm{rel}} + P_\Lambda)}{c^2}\right]\right\} R. \tag{116}$$

With the inclusion of ρ_Λ and P_Λ, these equations have the same form as their Newtonian counterparts, Eqs. (10), (50), and (51).

The effects of dark energy

However, the **interpretation of the constant k has changed**. In Section 1, k was related to the mechanical energy of an expanding mass shell by Eq. (1). Here, it is seen to be the **present value of the curvature of the universe** [Eq. (105) with R = 1].

Using Eq. (8) and recalling from Eq. (12) that $3H^3/8\pi G = \rho c$, the **Friedmann equation can be written as**

$$\boxed{H^2\left[1-(\Omega_m+\Omega_{\mathrm{rel}}+\Omega_\Lambda)\right]R^2=-kc^2, \qquad (117)}$$

Where

$$\Omega_\Lambda=\frac{\rho_\Lambda}{\rho_c}=\frac{\Lambda c^2}{3H^2}. \qquad (118)$$

We define the **total density parameter** as

$$\boxed{\Omega\equiv\Omega_m+\Omega_{\mathrm{rel}}+\Omega_\Lambda. \qquad (119)}$$

The effects of dark energy

The Friedmann equation then becomes

$$H^2(1-\Omega)R^2 = -kc^2, \tag{120}$$

so for a **flat universe (k = 0)**, we must have $\Omega(t) = 1$. It is useful to note that, as a special case at $t = t_0$,

$$H_0^2(1-\Omega_0) = -kc^2. \tag{121}$$

Using Eqs. (120) and (121) along with Eqs. (79), (80) and (118) the **Hubble parameter as a function of the redshift z** is found to be

$$H = H_0(1+z)\left[\Omega_{m,0}(1+z) + \Omega_{\mathrm{rel},0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0\right]^{1/2} \tag{122}$$

The effects of dark energy

$$\Omega_0 = \Omega_{m,0} + \Omega_{\text{rel},0} + \Omega_{\Lambda,0} = 1;$$

The WMAP values for $\Omega_{m,0}$, $\Omega_{rel,0}$, and $\Omega_{\Lambda,0}$ are

$$[\Omega_{m,0}]_{\text{WMAP}} = 0.27 \pm 0.04,$$
$$\Omega_{\text{rel},0} = 8.24 \times 10^{-5},$$
$$[\Omega_{\Lambda,0}]_{\text{WMAP}} = 0.73 \pm 0.04,$$

Current measurements:

$\Omega_{mass} \approx 0.315 \pm 0.018$

$\Omega_{relativistic} \approx 9.24 \times 10^{-5}$

$\Omega_{\Lambda} \approx 0.6817 \pm 0.0018$

$\Omega_{total} = \Omega_{mass} + \Omega_{relativistic} + \Omega_{\Lambda} = 1.00 \pm 0.02$

The actual value for critical density value is measured as $\rho_c = 9.47 \times 10^{-27}$ kg m^{-3}. From these values, within experimental error, **the universe seems to be spatially flat**. And dark energy dominates the current expansion.

https://en.wikipedia.org/wiki/Shape_of_the_universe

The effects of dark energy

The **deceleration parameter** (Eq. 54) may be written as

$$q(t) = \frac{1}{2} \sum_i (1 + 3w_i)\Omega_i(t), \tag{123}$$

where w is the coefficient from the equation of state $P_i = w_i \rho_i c^2$ and the "i" subscript identifies one of the components of the universe (i.e., pressureless dust, relativistic particles, or dark energy). Using $w_m = 0$, $w_{rel} = 1/3$, and $w_\Lambda = -1$, we obtain

$$q(t) = \frac{1}{2}\Omega_m(t) + \Omega_{\mathrm{rel}}(t) - \Omega_\Lambda(t). \tag{124}$$

With WMAP values we find that the current value of the deceleration parameter is

$$q_0 = -0.60.$$

The minus sign indicates that **expansion of the universe is now accelerating!**

The effects of dark energy

Assuming $\Lambda > 0$, the equivalent mass density of dark energy ρ_Λ adds to the effect of the other densities in the Friedmann equation (Eq. 114) on the curvature of the universe, k. However, in the acceleration equation, the negative pressure P_Λ opposes the gravitational effect of a positive ρ_Λ and acts to *increase* the acceleration of the universe, as seen by the positive term $\Lambda c^2/3$ added to the acceleration equation, Eq (112).

The presence of a cosmological constant therefore decouples the geometry of the universe (open, closed, flat), which is described by k, from the dynamics of the universe, which are governed by the interplay of ρ_m, ρ_{rel}, and ρ_Λ.

The dark energy era

The Λ-terms in Eqs. (114) and (116) became dominant as the scale factor R increased because ρ_Λ is constant while the mass density $\rho_m \propto R^{-3}$ and $\rho_{rel} \propto R^{-4}$. Just as the radiation era yielded to the mass era when the universe was about 55,000 years old (Eq. 85), **the mass era has changed into the Λ era**. **Dark energy now governs the expansion of the universe.**

The transition from the matter era to the present Λ era occurred when the scale factor satisfied $\rho_m = \rho_\Lambda$. Since ρ_Λ is constant, Eq. (82) can be used to show that the transition value of R is

$$R_{m,\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}. \tag{125}$$

Inserting WMAP values gives

$$R_{m,\Lambda} = 0.72, \tag{126}$$

The dark energy era

which corresponds to a redshift of

$$z_{m,\Lambda} = \frac{1}{R_{m,\Lambda}} - 1 = 0.39. \tag{127}$$

The acceleration equation with WMAP values shows that the acceleration of the universe changed sign (from negative to positive) when the scale factor was

$$R_{\text{accel}} = 2^{-1/3} R_{m,\Lambda} = 0.57,$$

corresponding to

$$z_{\text{accel}} = 0.76,$$

meaning that the acceleration became positive ***before*** **the Λ-term dominated the Friedmann equation**.

As R → 0 we can deduce that the effects of the cosmological constant are **negligible in the early universe** because $\rho_m \propto R^{-3}$ and $\rho_{rel} \propto R^{-4}$ while ρ_Λ remains constant. *All of the results for the early universe obtained earlier are valid for the present relativistic cosmology as well.*

The dark energy era

The behavior of the scale factor R for a flat universe can be found by setting k = 0 in the Friedmann equation (Eq. 114). A little algebra and Eqs. (79) and (118) lead to

$$t = \sqrt{\frac{3}{8\pi G}} \int_0^R \frac{R'\,dR'}{\sqrt{\rho_{m,0}R' + \rho_{\mathrm{rel},0} + \rho_{\Lambda,0}R'^4}}. \qquad (128)$$

Although this can be integrated numerically, it has no simple analytic solution. Figure 19 displays a numerical solution of Eq. (128) using WMAP values, showing the different behaviors of the scale factor in the radiation, matter, and Λ eras.

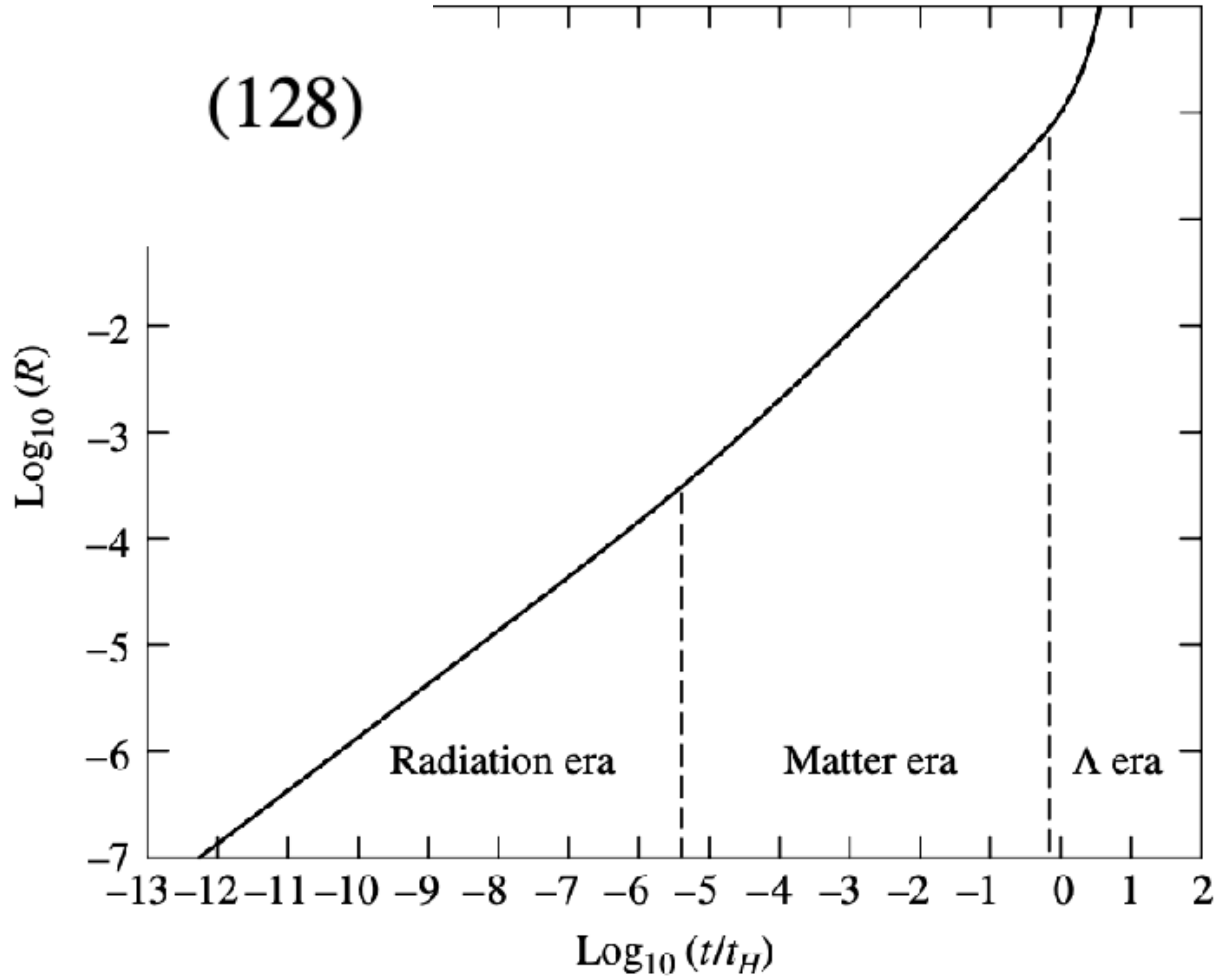


FIGURE 19 A logarithmic graph of the scale factor R as a function of time. During the radiation era, $R \propto t^{1/2}$; during the matter era, $R \propto t^{2/3}$; and during the Λ era, R grows exponentially.

The dark energy era

To make further progress, we will neglect the reign of relativistic particles during the first 55,000 years or so of the universe by setting $\rho_{rel} = 0$. Integrating eventually yields, for k = 0,

$$t(R) = \frac{2}{3}\frac{1}{H_0\sqrt{\Omega_{\Lambda,0}}}\ln\left[\sqrt{\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)R^3} + \sqrt{1+\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)R^3}\,\right]. \qquad (129)$$

The present age of the universe may be obtained by substituting R = 1 into Eq. (129).

Using the WMAP values for $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ we obtain

$$t_0 = 4.32 \times 10^{17}\ \mathrm{s} = 1.37 \times 10^{10}\ \mathrm{yr}.$$

This is in good agreement with the best determination of the age of the universe currently available, the published WMAP value of

$$\boxed{[t_0]_{\mathrm{WMAP}} = 13.7 \pm 0.2\ \mathrm{Gyr}.}$$

The dark energy era

The acceleration of the universe changed sign (from negative to positive) when $R = R_{accel} = 0.57$. According to Eq. (129), **the expansion of the universe began to speed up when its age was**

$$t_{\text{accel}} = 2.23 \times 10^{17}\ \text{s} = 7.08\ \text{Gyr}.$$

Thus the expansion of the universe has been accelerating for approximately the second half of its existence. For this reason, **t_0 is very nearly equal to the Hubble time**:

$$t_0 = 0.993\, t_H\, .$$

At the present epoch, the effects of deceleration during the radiation and matter eras and acceleration during the Λ era nearly cancel, so the age of the universe is what we would have calculated for a constant rate of expansion.

The dark energy era

Equation (129) can be inverted to obtain, for $k = 0$,

$$R(t) = \left(\frac{\Omega_{m,0}}{4\,\Omega_{\Lambda,0}}\right)^{1/3} \left(e^{3H_0t\sqrt{\Omega_{\Lambda,0}}/2} - e^{-3H_0t\sqrt{\Omega_{\Lambda,0}}/2}\right)^{2/3} \tag{130}$$

$$= \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0t\sqrt{\Omega_{\Lambda,0}}\right) \tag{131}$$

The dark energy era

Figure 20 shows the evolution of the scale factor as a function of time.

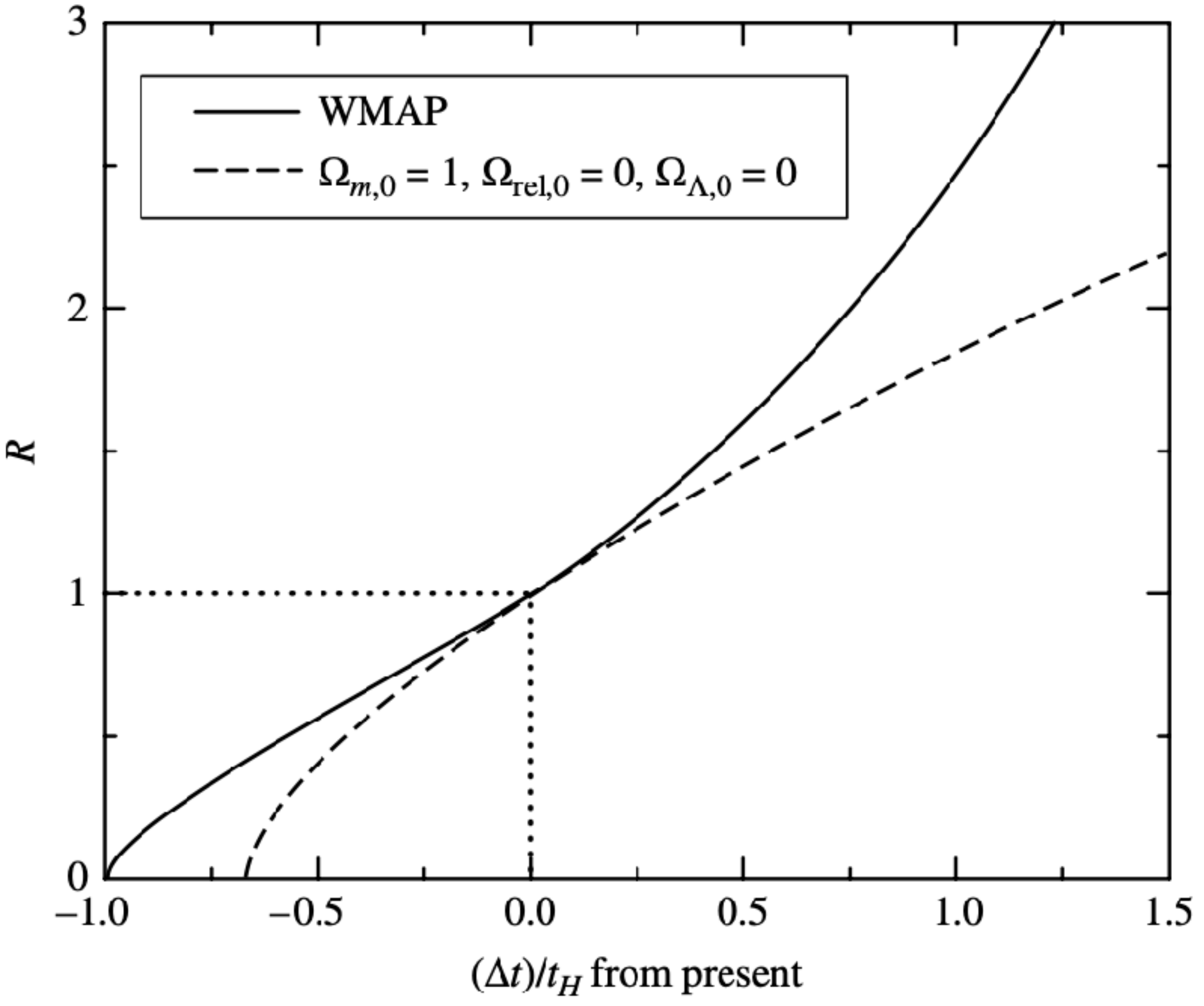


FIGURE 20 The scale factor R as a function of time, measured from the present, for a WMAP universe with $t_0 \simeq t_H$, and a flat, one-component universe of pressureless dust with $t_0 = 2t_H/3$ (Eq. 44). The dotted lines locate the position of today's universe on the two curves.

The dark energy era

In the limit of $3H_0 t\sqrt{\Omega_{\Lambda,0}}/2 \ll 1$ (essentially when t ≪ t_H), Eq. (131) reduces to Eq. (91),

$$R(t) \simeq \left(\frac{3}{2} H_0 t \sqrt{\Omega_{m,0}}\right)^{2/3} = \left(\frac{3\sqrt{\Omega_{m,0}}}{2}\right)^{2/3} \left(\frac{t}{t_H}\right)^{2/3}, \tag{132}$$

as appropriate for a **matter-dominated universe**. But when t ≫ t_H , the second exponential in Eq. (131) can be neglected, leaving

$$R(t) \simeq \left(\frac{\Omega_{m,0}}{4\,\Omega_{\Lambda,0}}\right)^{1/3} e^{H_0 t \sqrt{\Omega_{\Lambda,0}}}. \tag{133}$$

When the **cosmological constant dominates the Friedmann equation,** the scale factor **R grows exponentially** with a characteristic time of $t_H/\sqrt{\Omega_{\Lambda,0}}$.

Model universes

Every model of a three-component universe (matter, relativistic particles, and dark energy) is specified by the values of the three density parameters $\Omega_{m,0}$, $\Omega_{rel,0}$, and $\Omega_{\Lambda,0}$. At the present epoch $\Omega_{rel,0}$ is negligible, so we can **consider a two-dimensional plot of $\Omega_{m,0}$ vs. $\Omega_{\Lambda,0}$,** as shown in Fig. 21.

The $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane is divided into several regions. The Friedmann equation (Eq. 117) shows that the line $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$ determines the sign of k and so divides the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane into open and closed universes. The sign of the deceleration parameter (Eq. 123) is determined by the sign of the quantity $\Omega_{m,0} - 2\Omega_{\Lambda,0}$, so the line $\Omega_{m,0} - 2\Omega_{\Lambda,0} = 0$ divides the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane into accelerating and decelerating universes.

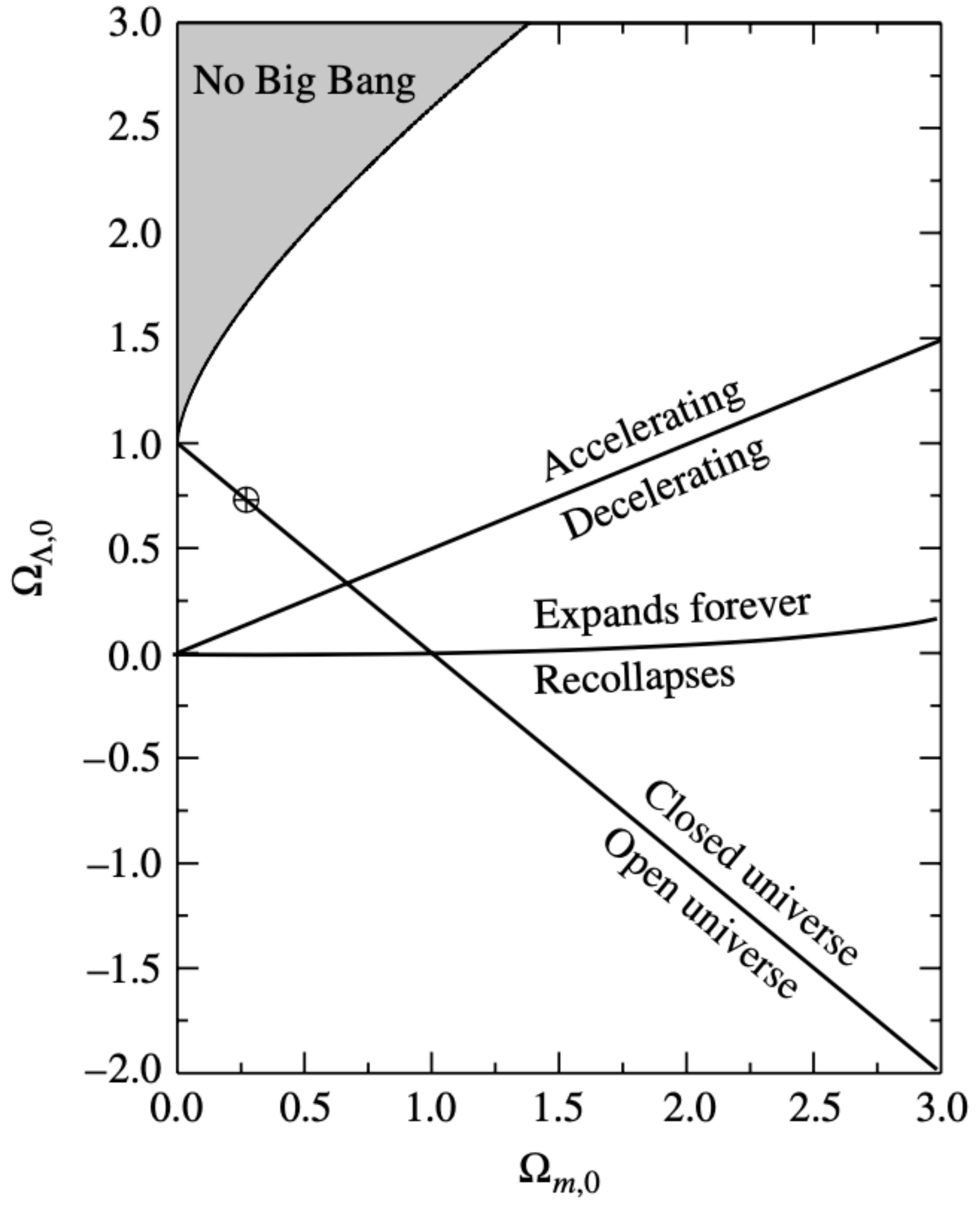


FIGURE 21 Model universes on the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane. Every point on this plane represents a possible universe. The point ($\Omega_{m,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$) is indicated by the circle.

Model universes

Although it seems that our universe will expand forever, driven by dark energy, it is easy to conceive of other model universes that will eventually collapse. This includes any universe with $\Omega_{\Lambda,0} < 0$, as well as universes with $\Omega_{\Lambda,0} > 0$ but containing enough matter to bring the expansion to a halt before dark energy dominates. When the expansion stops, dR/dt = 0. Equation (114), together with Eqs. (81), (82), (113), and (15), can be used to express dR/dt as

$$\left(\frac{dR}{dt}\right)^2 = H_0^2\left(\frac{\Omega_{m,0}}{R} + \frac{\Omega_{\mathrm{rel},0}}{R^2} + \Omega_{\Lambda,0}R^2 + 1 - \Omega_{m,0} - \Omega_{\mathrm{rel},0} - \Omega_{\Lambda,0}\right). \quad (134)$$

Setting the left-hand side equal to zero, canceling the H_0^2, and neglecting the radiation era, we have a cubic equation for the scale factor R.

$$\frac{\Omega_{m,0}}{R} + \Omega_{\Lambda,0}R^2 + 1 - \Omega_{m,0} - \Omega_{\Lambda,0} = 0. \quad (135)$$

Model universes

We want to know when this cubic equation for R has a positive, real root. It turns out that if $\Omega_{m,0} < 1$ and $\Omega_{\Lambda,0} > 0$, Eq. (135) has no positive, real root, and we conclude that these universes will expand forever. But a universe with $\Omega_{m,0} > 1$ will expand forever only if

$$\Omega_{\Lambda,0} > 4\Omega_{m,0}\left\{\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{1}{\Omega_{m,0}} - 1\right) + \frac{4\pi}{3}\right]\right\}^{3}$$

Model universes

On this diagram, lines of constant age t_0 are roughly diagonal from the lower left to the upper right, and t_0 increases from lower right to upper left. In fact, t_0 becomes *infinite* as we approach a line across the upper-left corner of the diagram where there is an (unstable) equilibrium between the inward pull of gravity and the outward push of dark energy. Models on this line are infinitely old, meaning that they never unfolded from a hot, dense Big Bang. This line is given by

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\left\{\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{1}{\Omega_{m,0}} - 1\right)\right]\right\}^3$$

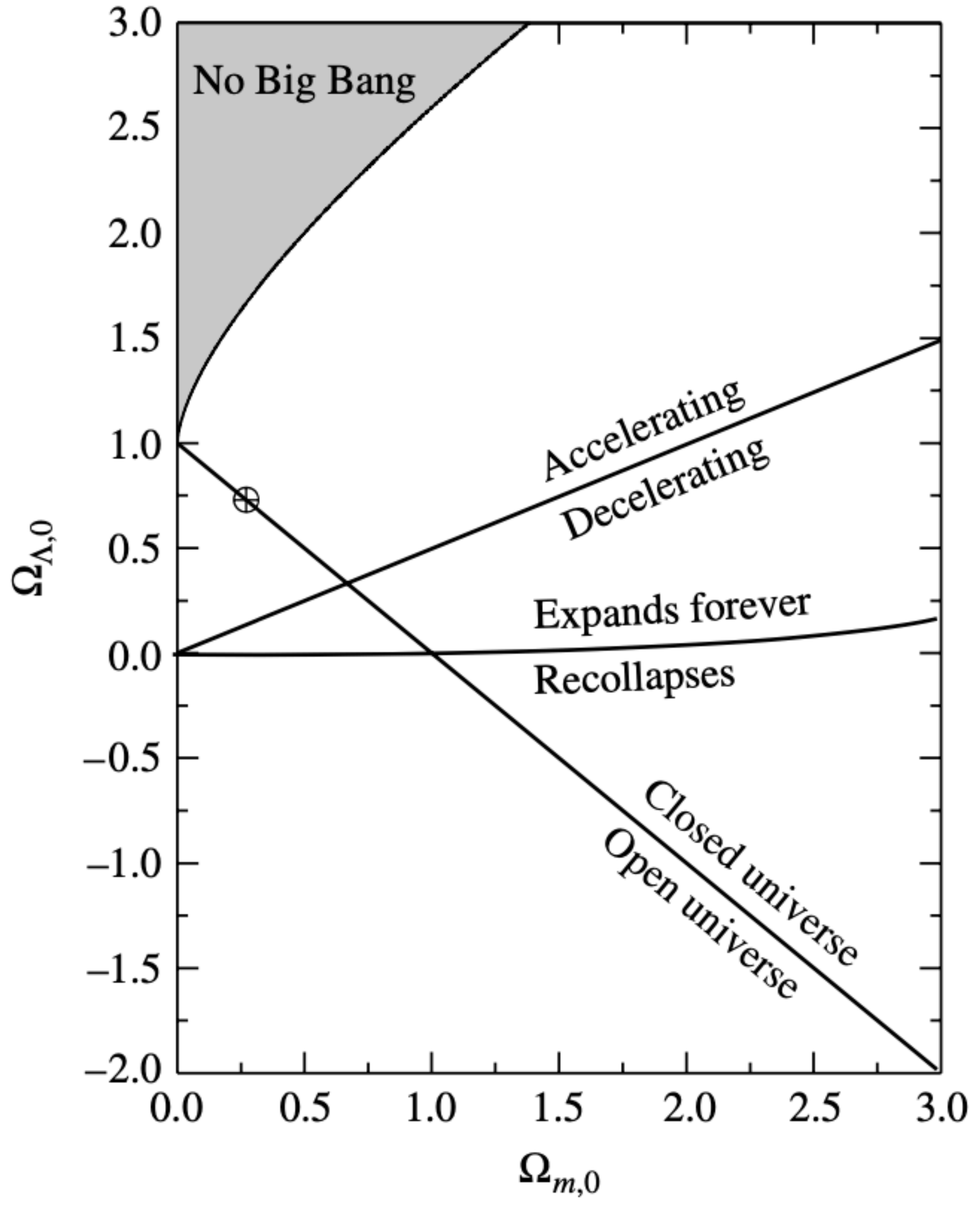


FIGURE 21 Model universes on the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane. Every point on this plane represents a possible universe. The point ($\Omega_{m,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$) is indicated by the circle.

Model universes

if $\Omega_{m,0} > 0.5$;

Models beyond this line represent "bounce" universes that are now on the rebound from an earlier collapse. We merely state the result that these "bounce" models have a maximum redshift (at the bounce) that satisfies

$$z_{\text{bounce}} \le 2\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)\right] - 1 \qquad (136)$$

if $\Omega_{m,0} > 0.5$;

Because objects are observed beyond this maximum redshift, these "bounce" models may be rejected.

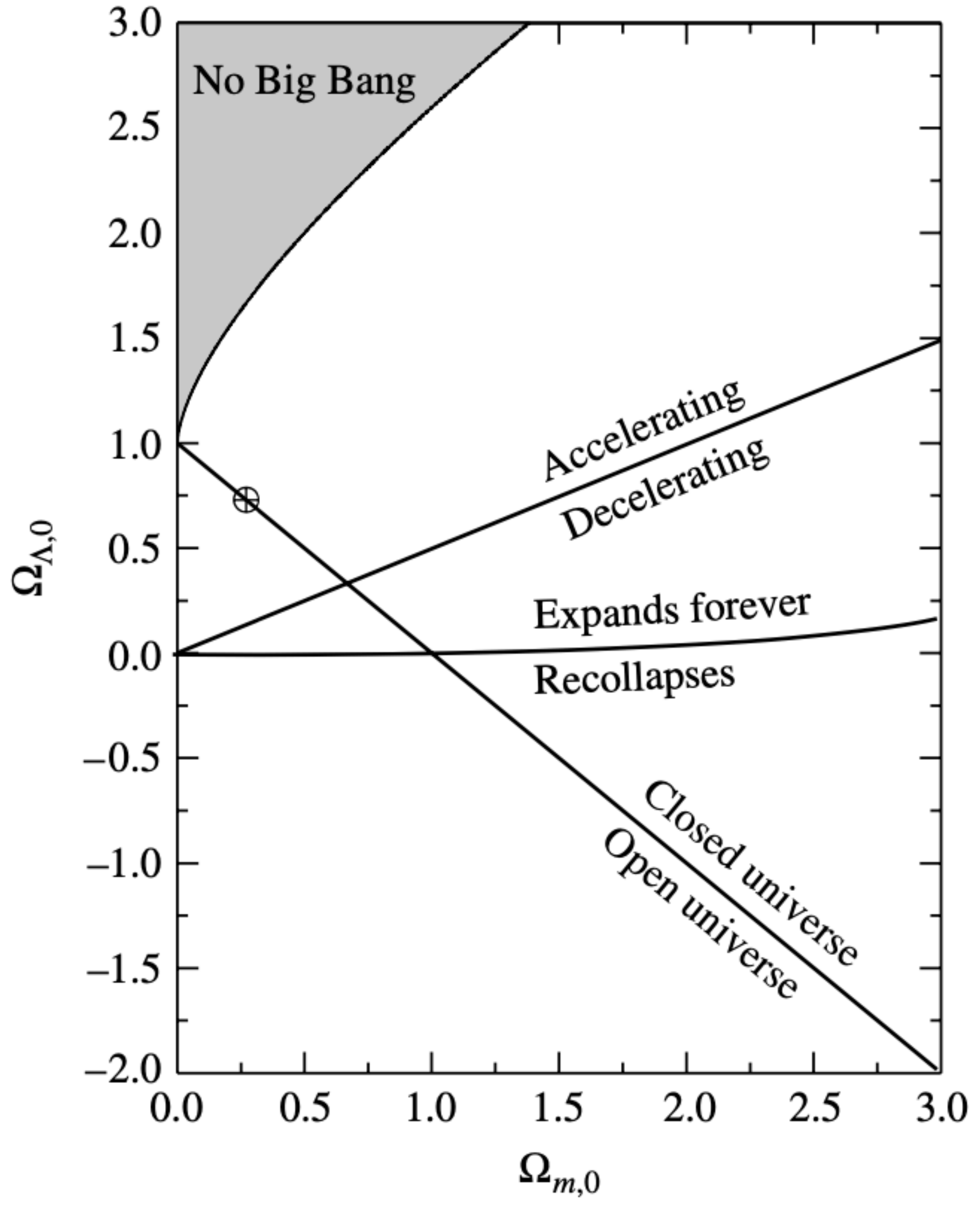


FIGURE 21 Model universes on the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane. Every point on this plane represents a possible universe. The point ($\Omega_{m,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$) is indicated by the circle.

Model universes

Astronomers have the task of determining which point on Fig. 21 represents *our* universe. This task is possible because the dynamics of the expansion of the universe determines $q_0 = \Omega_{m,0}/2 - \Omega_{\Lambda,0}$, and the geometry of the universe determines $\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0}$.

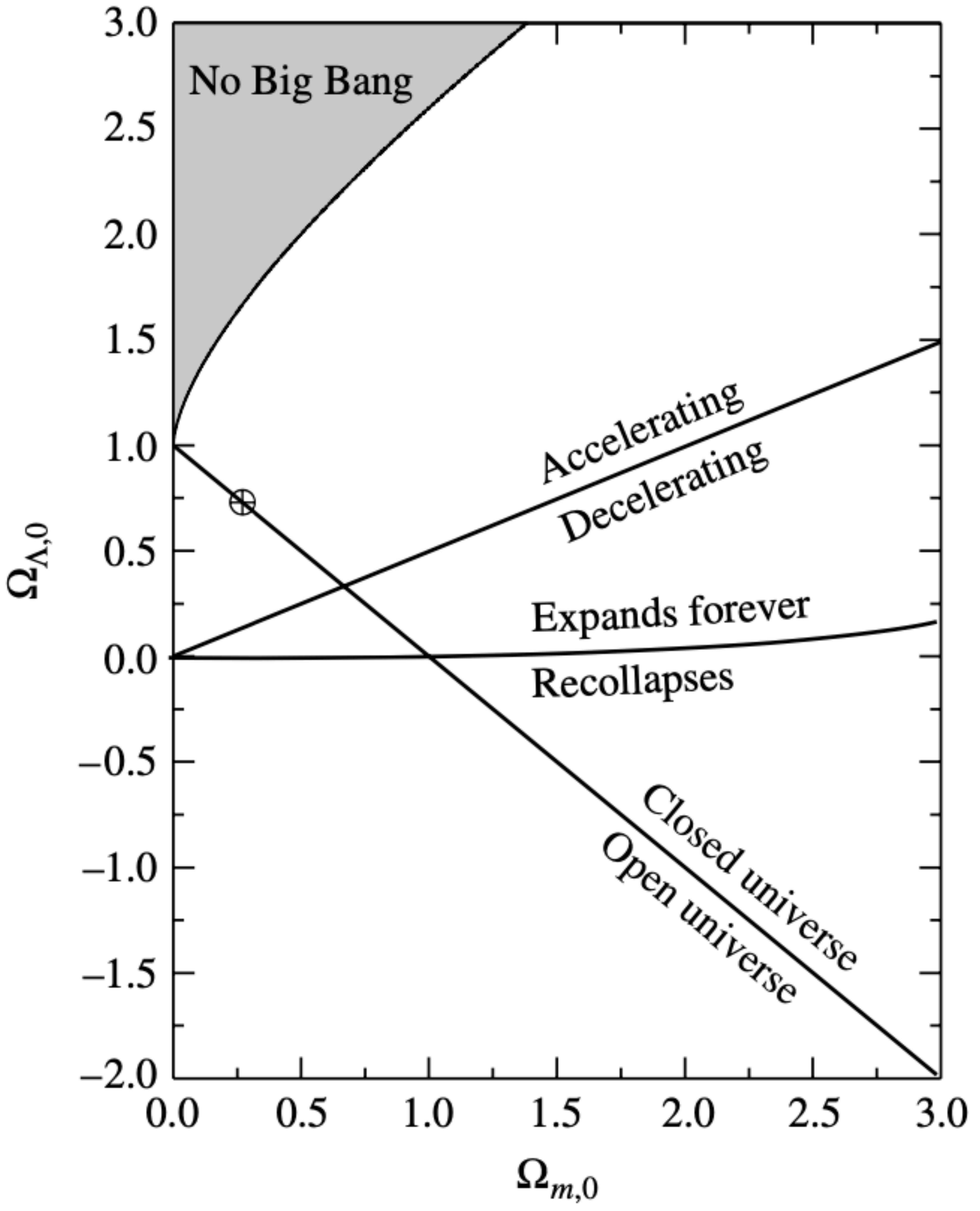


FIGURE 21 Model universes on the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane. Every point on this plane represents a possible universe. The point ($\Omega_{m,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$) is indicated by the circle.

Model universes

Every model of a three-component universe (matter, relativistic particles, and dark energy) is specified by the values of the three density parameters $\Omega_{m,0}$, $\Omega_{rel,0}$, and $\Omega_{\Lambda,0}$. At the present epoch $\Omega_{rel,0}$ is negligible, so we can **consider a two-dimensional plot of $\Omega_{m,0}$ vs. $\Omega_{\Lambda,0}$,** as shown in Fig. 21.

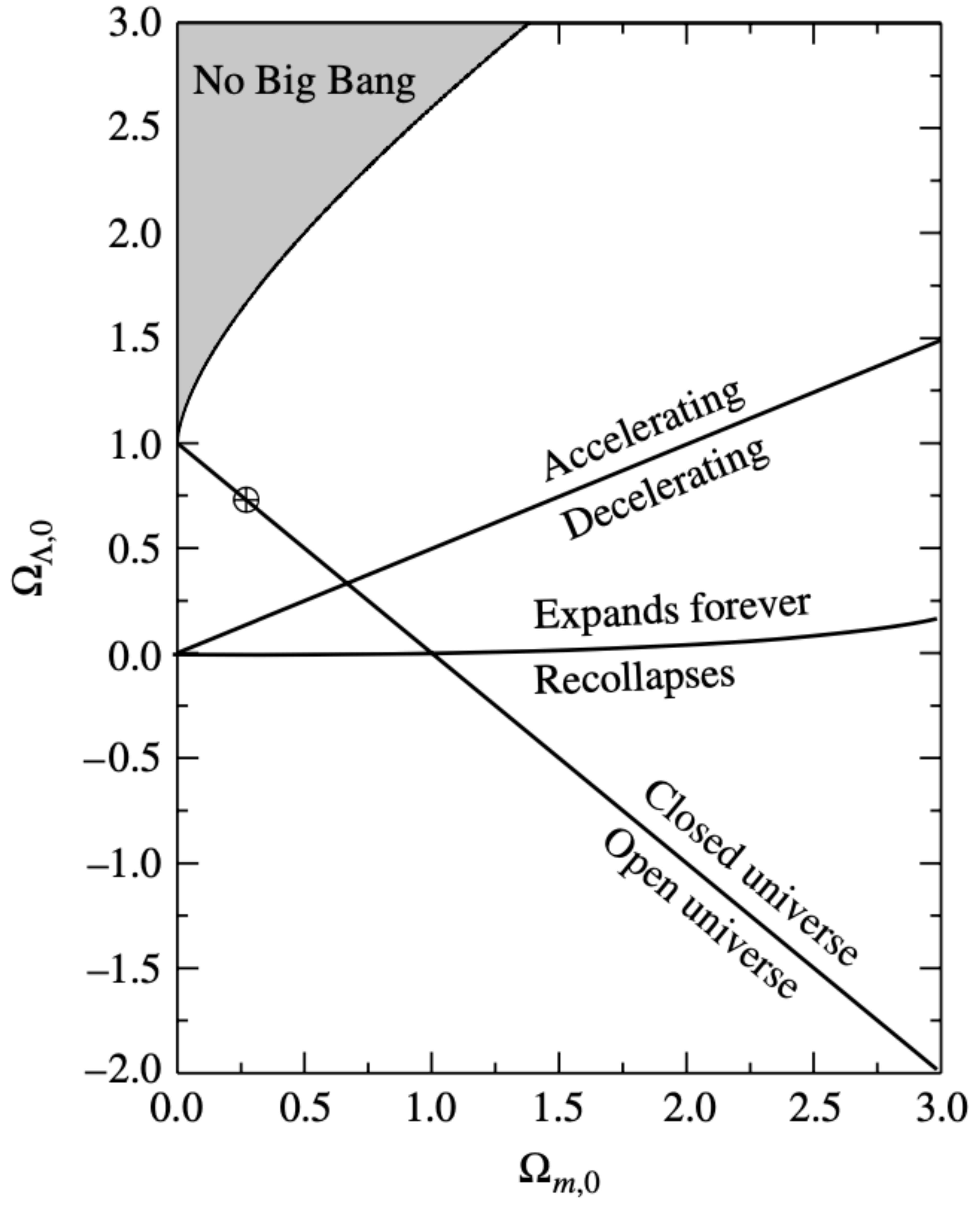


FIGURE 21 Model universes on the $\Omega_{m,0}$–$\Omega_{\Lambda,0}$ plane. Every point on this plane represents a possible universe. The point ($\Omega_{m,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$) is indicated by the circle.