Introduction to Nuclear and Particle Physics

Quantum mechanics background

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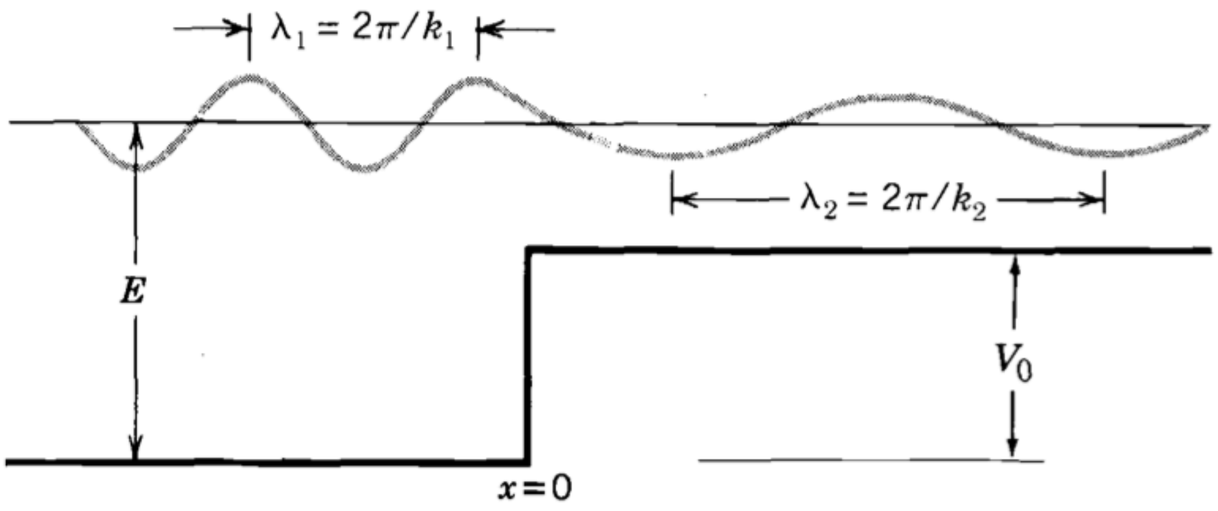

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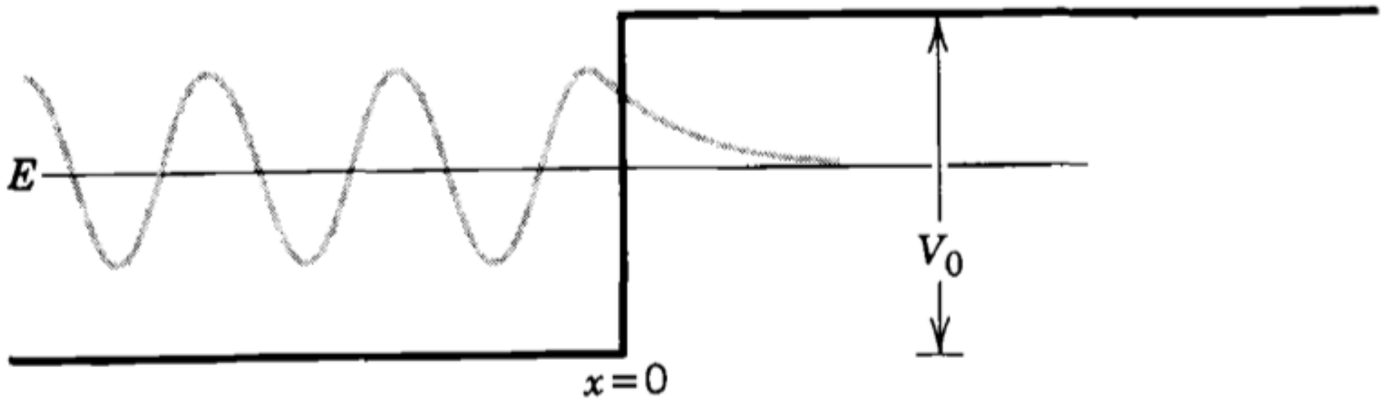

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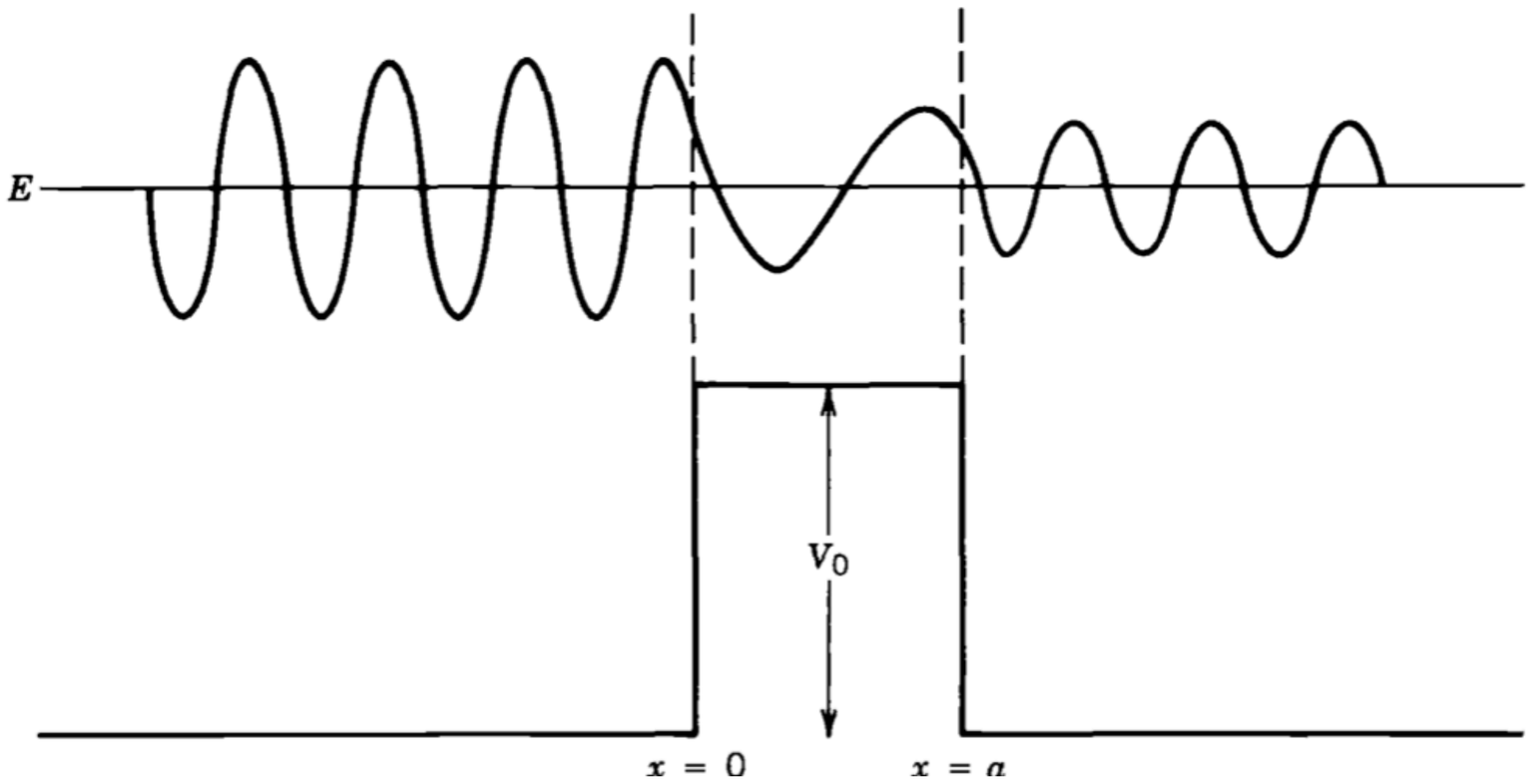

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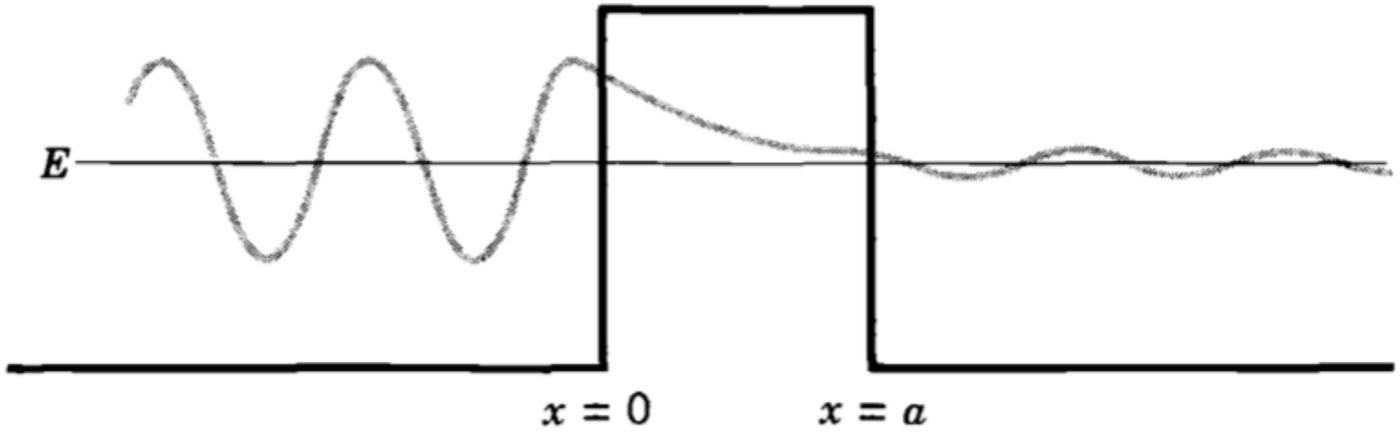
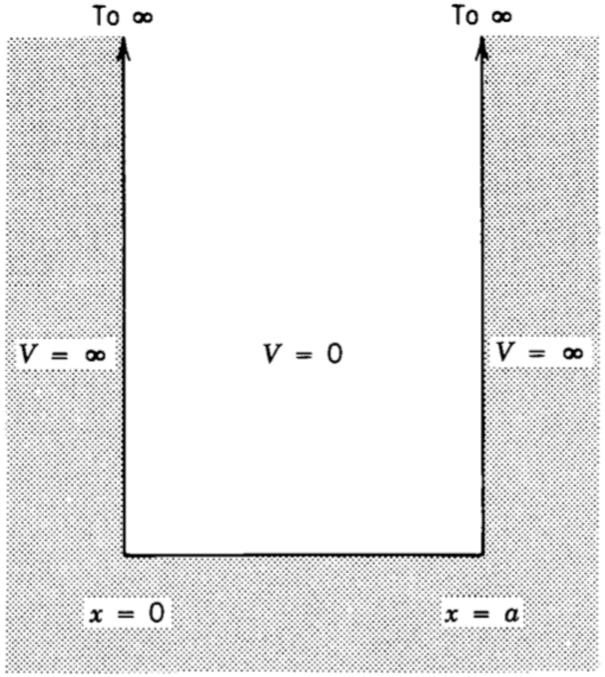


Infinite potential well



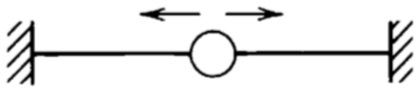


Figure 2.6 A particle moves freely in the one-dimensional region $0 \leq x \leq a$ but is excluded completely from $x < 0$ and $x > a$. A bead sliding without friction on a wire and bouncing elastically from the walls is a simple physical example.

Infinite potential well

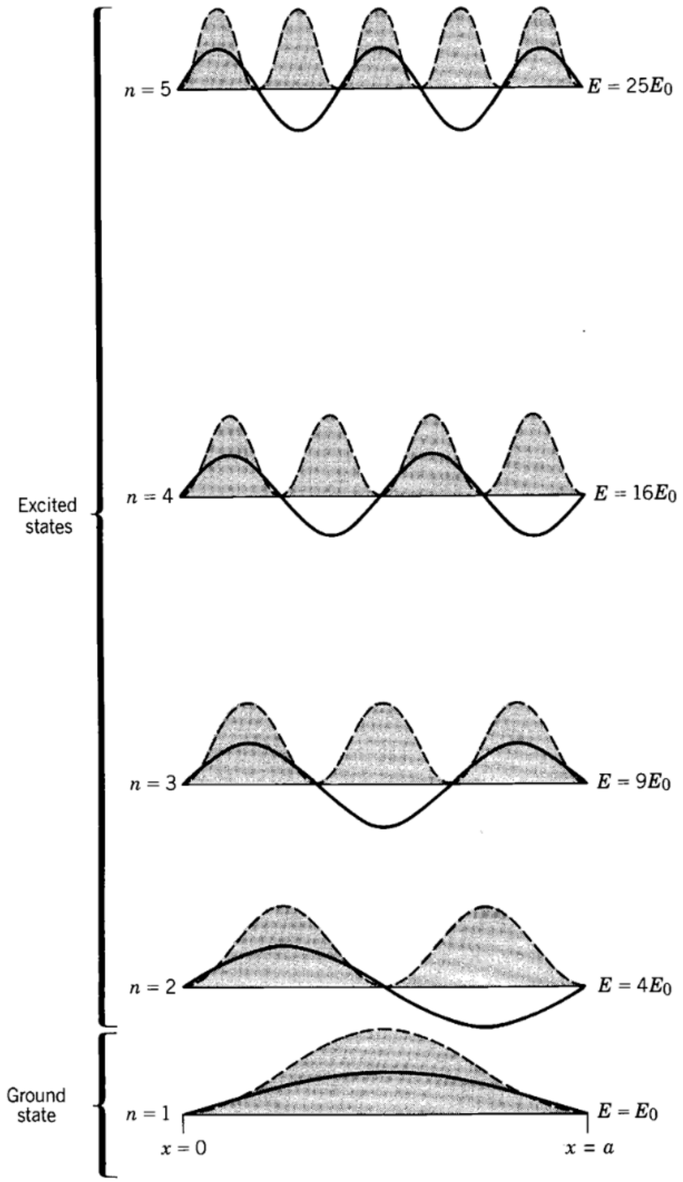


Figure 2.7 The permitted energy levels of the one-dimensional infinite square well. The wave function for each level is shown by the solid curve, and the shaded region gives the probability density for each level. The energy E_0 is $\hbar^2\pi^2 / 2ma^2$.

Finite potential well

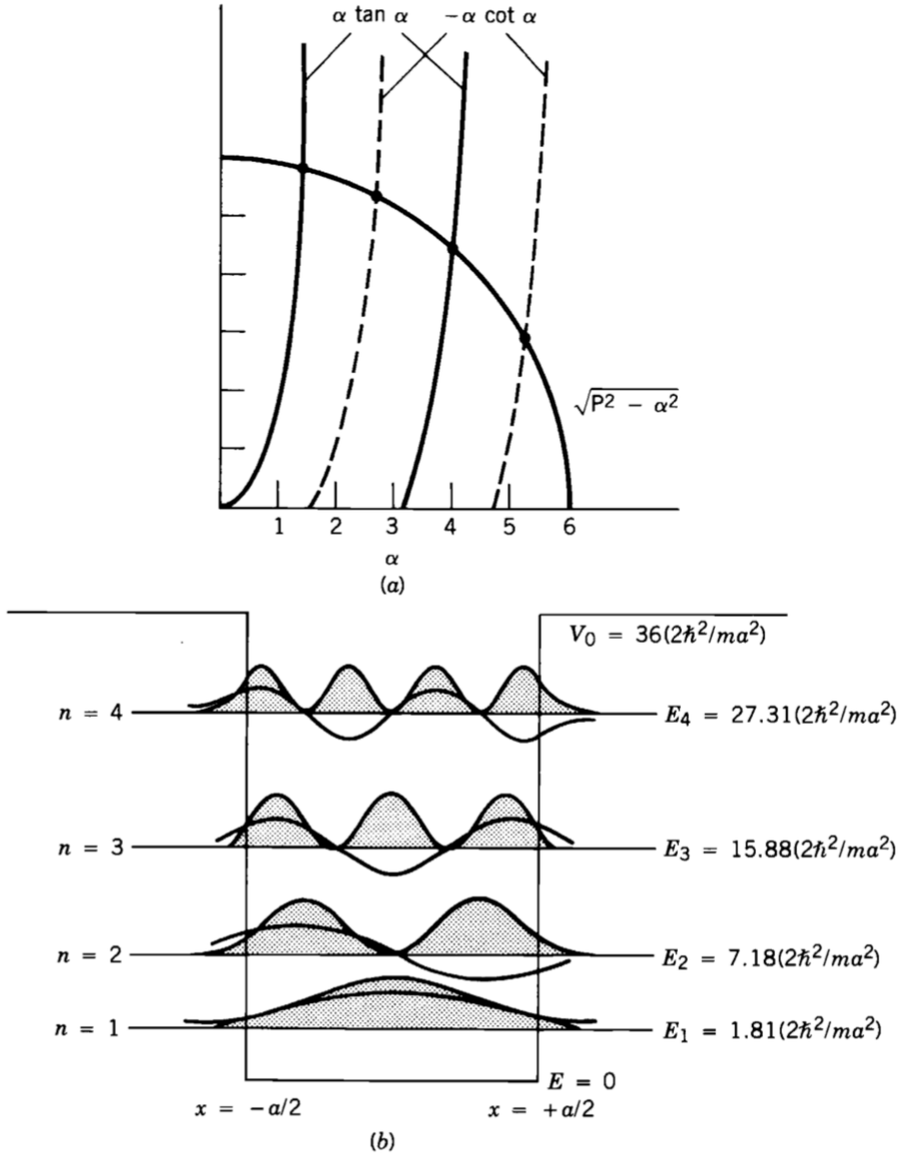


Figure 2.8 (*a*) The graphical solution of Equations 2.49*a* and 2.49*b*. For the case of $P = 6$ (chosen arbitrarily) there are four solutions at $\alpha = 1.345$, 2.679, 3.985, and 5.226. (*b*) The wave functions and probability densities (shaded) for the four states. (Compare with the infinite well shown in Figure 2.7.)

Taylor series

$$V(x) = V(x_0) + \left(\frac{dV}{dx}\right)_{x=x_0}(x - x_0) + \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x=x_0}(x - x_0)^2 + \cdots$$

Wave functions for harmonic oscillator

Table 2.1 Sample Wave Functions of the One-Dimensional Simple Harmonic Oscillator

n	E_n	$\psi_n(x)$
0	$\frac{1}{2}\hbar\omega_0$	$\pi^{-1/4}\, e^{-\alpha^2 x^2/2}$
1	$\frac{3}{2}\hbar\omega_0$	$2^{-1/2}\pi^{-1/4}(2\alpha x)\, e^{-\alpha^2 x^2/2}$
2	$\frac{5}{2}\hbar\omega_0$	$2^{-3/2}\pi^{-1/4}(4\alpha^2 x^2 - 2)\, e^{-\alpha^2 x^2/2}$
3	$\frac{7}{2}\hbar\omega_0$	$(1/4\sqrt{3}\,\pi^{1/4})(8\alpha^3 x^3 - 12\alpha x)\, e^{-\alpha^2 x^2/2}$
4	$\frac{9}{2}\hbar\omega_0$	$(1/8\sqrt{6}\,\pi^{1/4})(16\alpha^4 x^4 - 48\alpha^2 x^2 + 12)\, e^{-\alpha^2 x^2/2}$

$$E_n = \hbar\omega_0(n + \tfrac{1}{2})$$

$$\psi_n(x) = (2^n n!\sqrt{\pi})^{-1/2} H_n(\alpha x)\, e^{-\alpha^2 x^2/2}$$

where $H_n(\alpha x)$ is a Hermite polynomial

Simple harmonic oscillator

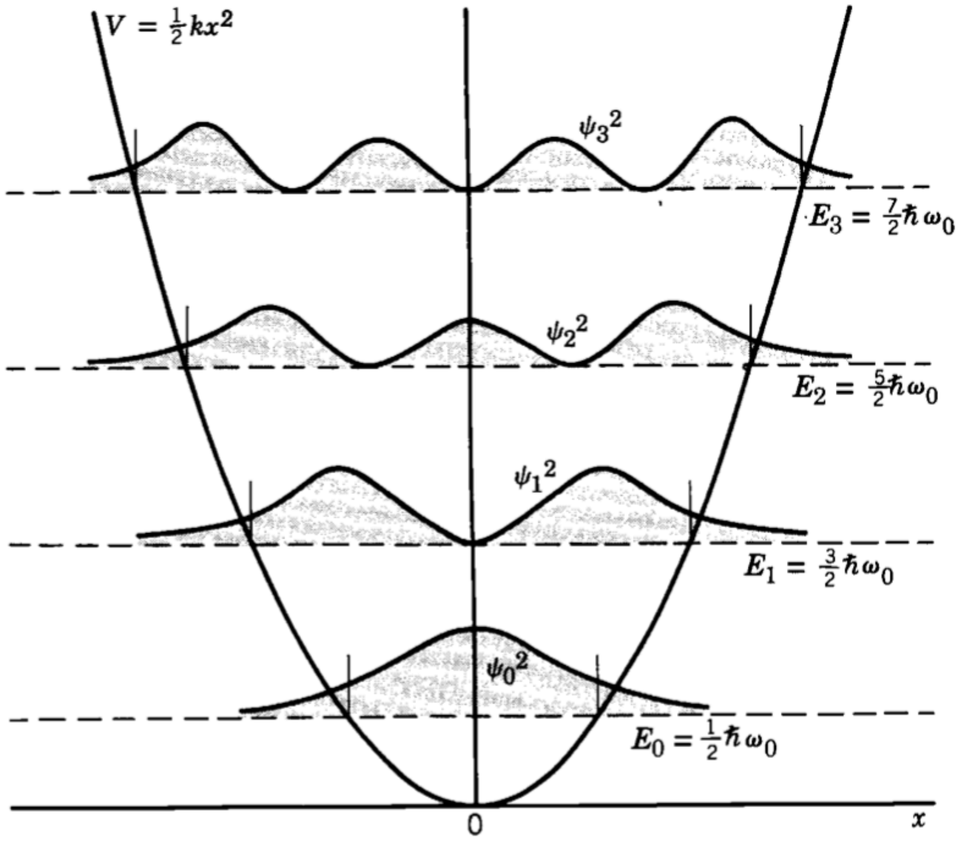


Figure 2.9 The lowest few energy levels and corresponding probability densities of the harmonic oscillator.

Summary

By studying these one-dimensional problems, we learn several important details about the wave properties of particles.

1. Quantum waves can undergo **reflection and transmission** when they encounter a potential barrier; this behavior is very similar to that of classical waves.

2. A **wave packet can penetrate into the classically forbidden region and appear beyond a potential** barrier that it does not have enough energy to overcome.

3. Wave functions oscillate whenever $E > V(x)$ and decay exponentially whenever $E < V(x)$.

4. When a potential confines a particle to a region of space, **bound-state wave functions can result**. The particle is permitted only a set of discrete energy values; the number of allowed energy values is determined by the depth of the potential well.

3D cubical box

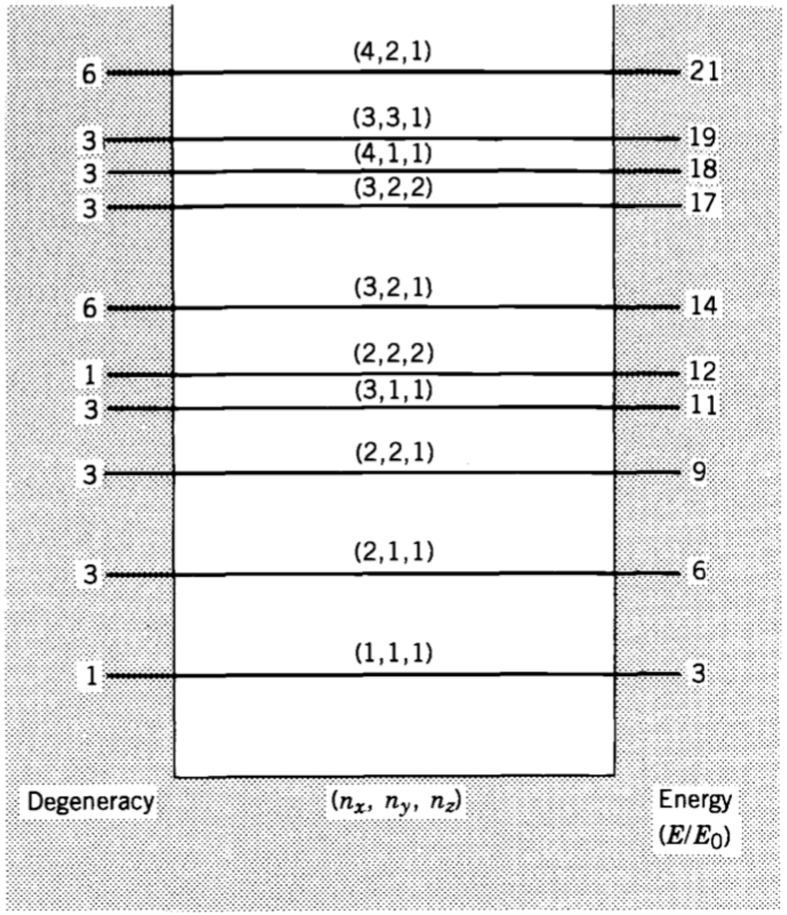


Figure 2.10 Energy levels of a particle confined to a three-dimensional cubical box. The energy is given in units of $E_0 = \hbar^2\pi^2 / 2ma^2$.

3D cubical box

Table 2.2 Spherical Harmonics for Some Low ℓ Values

ℓ	m_ℓ	$Y_{\ell m_\ell}(\theta, \phi) = \Theta_{\ell m_\ell}(\theta)\, \Phi_{m_\ell}(\phi)$
0	0	$(1/4\pi)^{1/2}$
1	0	$(3/4\pi)^{1/2} \cos\theta$
1	± 1	$\mp(3/8\pi)^{1/2} \sin\theta\, e^{\pm i\phi}$
2	0	$(5/16\pi)^{1/2}(3\cos^2\theta - 1)$
2	± 1	$\mp(15/8\pi)^{1/2} \sin\theta \cos\theta\, e^{\pm i\phi}$
2	± 2	$(15/32\pi)^{1/2} \sin^2\theta\, e^{\pm 2i\phi}$

$$\Phi_{m_\ell}(\phi) = \frac{1}{\sqrt{2\pi}}\, e^{i m_\ell \phi}$$

$$\Theta_{\ell m_\ell}(\theta) = \left[\frac{2\ell + 1}{2}\, \frac{(\ell - m_\ell)!}{(\ell + m_\ell)!}\right]^{1/2} P_\ell^{m_\ell}(\theta)$$

where $P_\ell^{m_\ell}(\theta)$ is the associated Legendre polynomial

Table 2.3 Spherical Bessel Functions — Sample Expressions and Limits

$$j_0(kr) = \frac{\sin kr}{kr}$$

$$j_1(kr) = \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}$$

$$j_2(kr) = \frac{3 \sin kr}{(kr)^3} - \frac{3 \cos kr}{(kr)^2} - \frac{\sin kr}{kr}$$

$$j_\ell(kr) \cong \frac{(kr)^\ell}{1 \cdot 3 \cdot 5 \cdots (2\ell + 1)} \qquad kr \to 0$$

$$j_\ell(kr) \cong \frac{\sin(kr - \ell\pi/2)}{kr} \qquad kr \to \infty$$

$$j_\ell(kr) = \left(-\frac{r}{k}\right)^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell j_0(kr)$$

3D spherical box

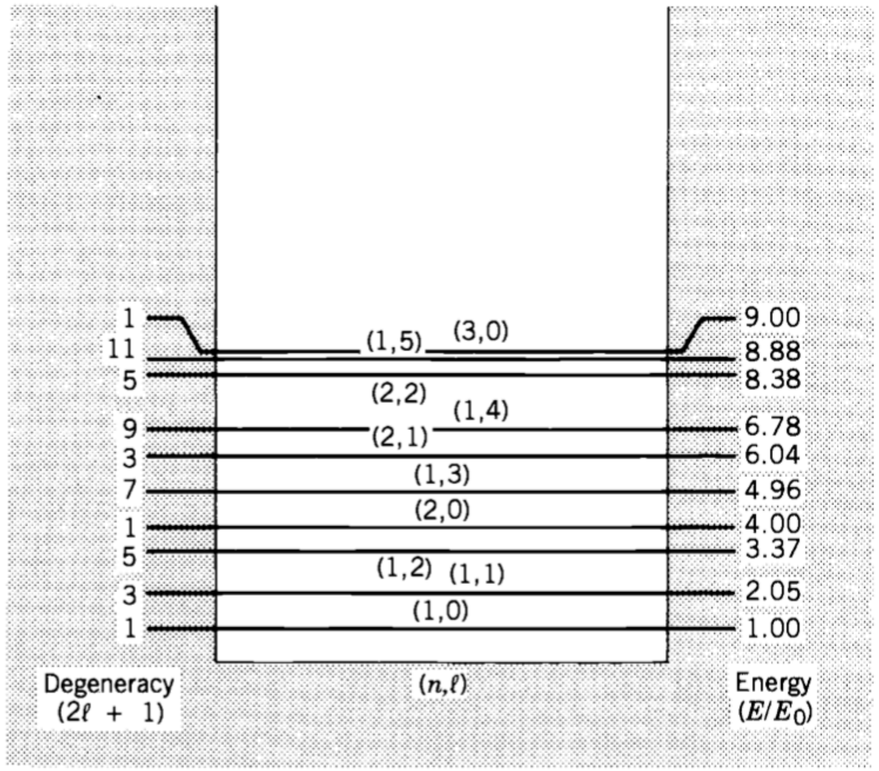


Figure 2.11 Energy levels of a particle confined to a three-dimensional spherical container. The energy is given in units of $E_0 = \hbar^2\pi^2 / 2ma^2$. Compare with the spacings and degeneracies of Figure 2.10. The quantum number n does not arise directly in the solution in this case; it serves to number the states of a given ℓ.

3D spherical box

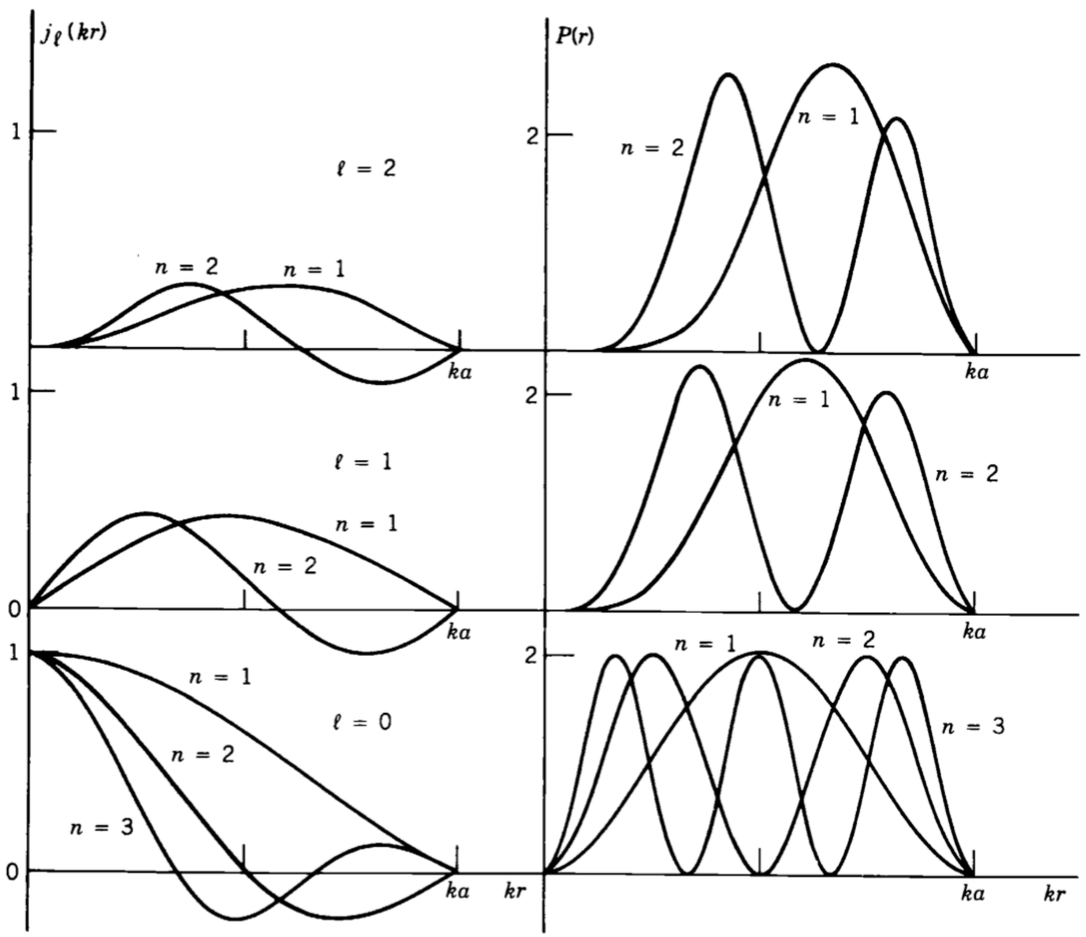


Figure 2.12 The left side shows, for some of the lower energy levels, the unnormalized $j_\ell(kr)$, adjusted so that $j_\ell(ka) = 0$. The right side shows the corresponding normalized radial probability density, r^2R^2. Note that all j_ℓ vanish at the origin except j_0, and that all probability densities vanish at $r = 0$. Also, note how the "centrifugal repulsion" pushes corresponding maxima in $P(r)$ away from the origin as ℓ increases.

3D spherical box

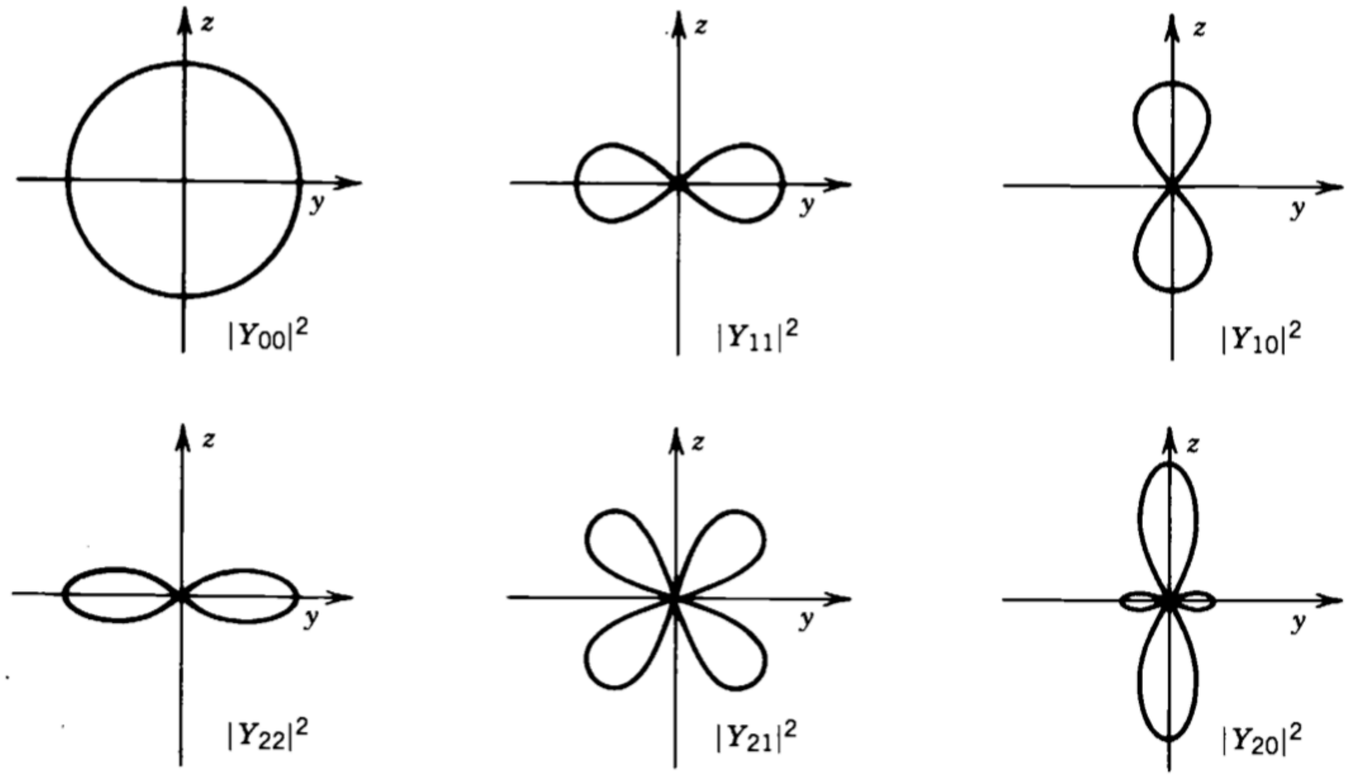


Figure 2.13 Spatial probability distributions resulting from the $Y_{\ell m_\ell}$. The three-dimensional representations can be obtained by rotating each figure about the z axis.

3D spherical box

Table 2.5 Coulomb (Hydrogenic) Radial Wave Functions

n	ℓ	$R(r)$
1	0	$2(Z/a_0)^{3/2}\, e^{-Zr/a_0}$
2	0	$(Z/2a_0)^{3/2}(2 - Zr/a_0)\, e^{-Zr/2a_0}$
2	1	$3^{-1/2}(Z/2a_0)^{3/2}(Zr/a_0)\, e^{-Zr/2a_0}$
3	0	$\frac{2}{3}(Z/3a_0)^{3/2}[3 - 2Zr/a_0 + 2(Zr/3a_0)^2]\, e^{-Zr/3a_0}$
3	1	$(4\sqrt{2}/9)(Z/3a_0)^{3/2}(Zr/a_0)(1 - Zr/6a_0)\, e^{-Zr/3a_0}$
3	2	$(2\sqrt{2}/27\sqrt{5})(Z/3a_0)^{3/2}(Zr/a_0)^2\, e^{-Zr/3a_0}$

The radial wave functions have the mathematical form of associated Laguerre polynomials multiplied by exponentials. The Bohr radius a_0 is $4\pi\epsilon_0\hbar^2/me^2$. For a discussion of these solutions, see L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (New York: McGraw-Hill, 1935), Chapter 5.

3D spherical box

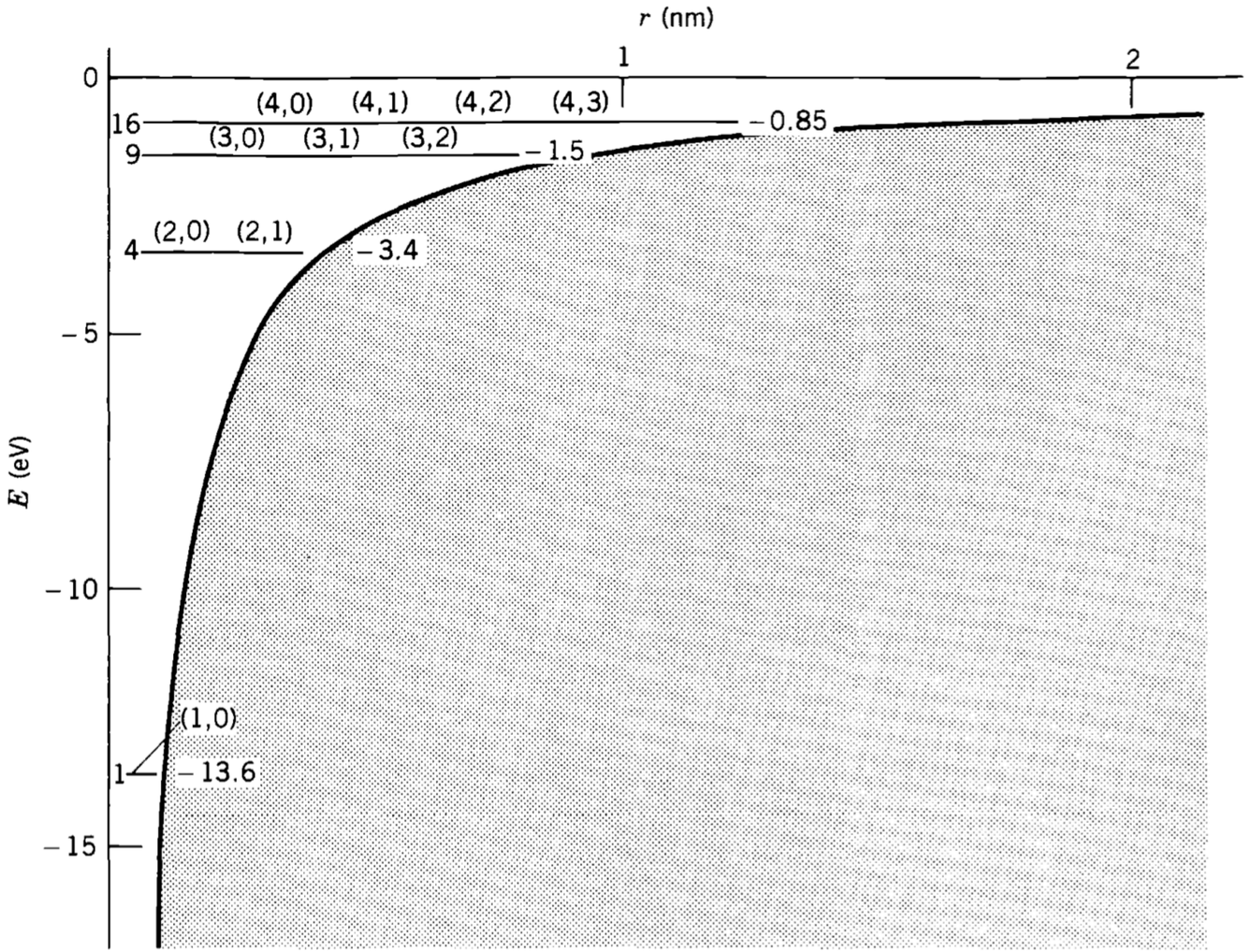


Figure 2.16 The lower energy levels in a Coulomb potential, shown for $Z = 1$ (hydrogen atom). The states are labeled with (n, ℓ); the degeneracies are indicated on the left and the energy values on the right.

Summary

The mathematical techniques of finding and using solutions to the Schrodinger equation in three dimensions are similar to the techniques we illustrated previously for one-dimensional problems,

There are two important new features in the three-dimensional calculations that do not arise in the one-dimensional calculations:

(1)The energy levels are degenerate-several different wave functions can have the same energy. The degeneracies will have the same effect in the nuclear shell model that the $\boldsymbol{l}$ and $\boldsymbol{m_l}$ degeneracies of the energy levels of the Coulomb potential have in the atomic shell model - they tell us how many particles can occupy each energy level.

(2) When the potential depends only on $\boldsymbol{r}$, the wave functions can be assigned a definite angular momentum quantum number $\boldsymbol{t'}$.

Angular momentum quantum number

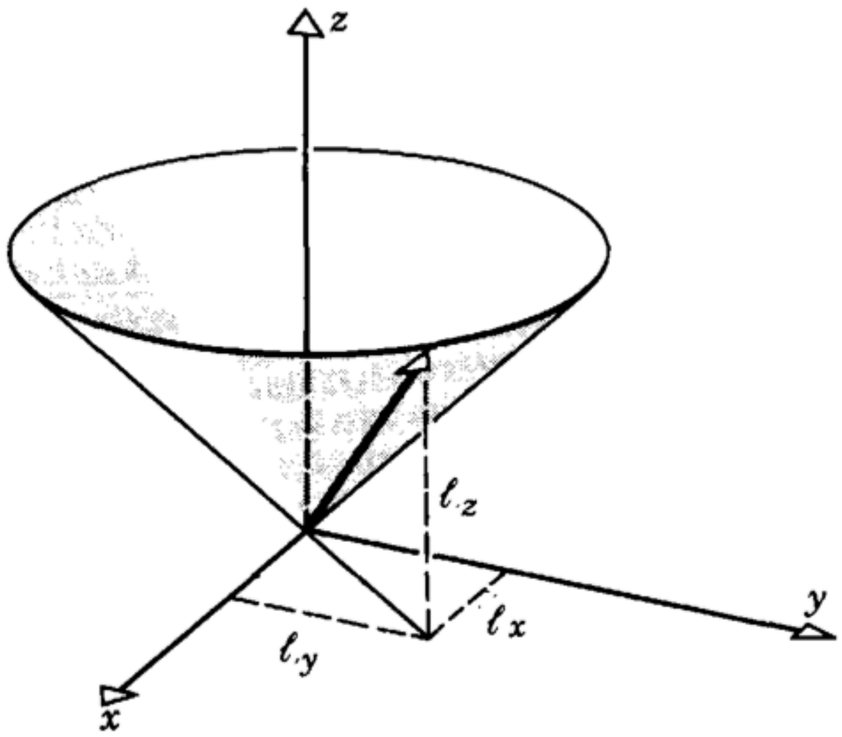


Figure 2.18 The vector ℓ precesses rapidly about the z axis, so that ℓ_z stays constant, but ℓ_x and ℓ_y are variable.

Angular momentum quantum number

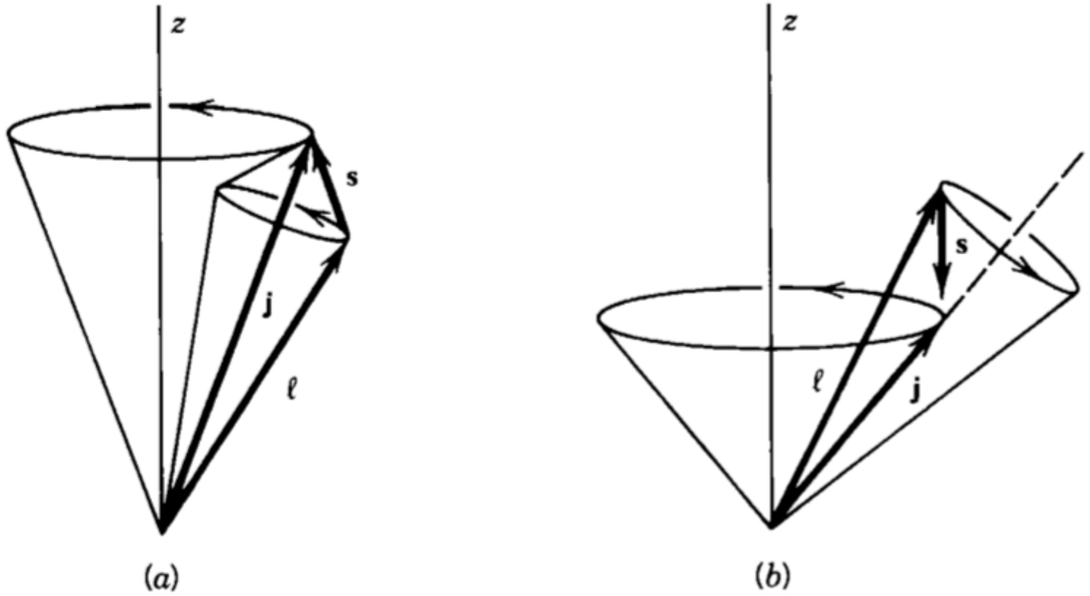


Figure 2.19 The coupling of orbital angular momentum $\boldsymbol{\ell}$ to spin angular momentum $\boldsymbol{s}$ giving total angular momentum $\boldsymbol{j}$. (*a*) Coupling giving $j = \ell + \frac{1}{2}$. The vectors $\boldsymbol{\ell}$ and $\boldsymbol{s}$ have definite lengths, as does $\boldsymbol{j}$. The combined $\boldsymbol{\ell}$ and $\boldsymbol{s}$ vectors rotate or precess about the direction of $\boldsymbol{j}$; in this coupling the z components ℓ_z and s_z thus do not have definite values. The vector $\boldsymbol{j}$ precesses about the z direction so that j_z has a definite value. (*b*) The similar case of $j = \ell - \frac{1}{2}$. In interpreting both figures, keep in mind that all such representations of vectors governed by the rules of quantum mechanics are at best symbolic and at worst misleading.