Topological sorting, O, and Θ

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Outline

1. Topological sorting
2. O as a relation
3. Θ as a relation

Recap and intro

A partial order on set A is a relation R which is reflexive, antisymmetric, and transitive.

- A poset $(A, \leqslant)$
- Posets can contain incomparable elements.
  Recall: Elements 5 and 3 are incomparable in poset $(\mathbb{Z}^+, |)$.
- If any two elements in a poset $(A, \leqslant)$ are comparable, then we say that $(A, \leqslant)$ is totally ordered.
- If $(A, \leqslant)$ is finite and totally ordered, then its elements can be sorted as

$$a_1, \ldots, a_n$$

  where $|A| = n$ and $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$

Q: How can we sort the elements of a partial order?

Sorting elements of a poset $(A, \leqslant)$

Goal: Given an n-element poset $(A, \leqslant)$, produce an ordering

$$a_1, \ldots, a_n \tag{1}$$

which is consistent with the "$\leqslant$" partial order:

$$a_i \leqslant a_j \Rightarrow i \leqslant j \tag{2}$$

Def. We refer to the ordering in (1) as topological sorting of poset $(A, \leqslant)$.

- Contrapositive of (2): $i > j \Rightarrow a_i \nleqslant a_j$
- What does $a_i \nleqslant a_j$ mean?

Topological sorting of a poset $(A, \leqslant)$

...is an ordering of its elements

$$a_1, a_2, \ldots, a_n$$

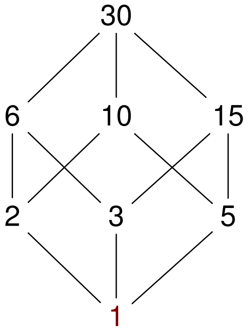
where $i > j \Rightarrow a_i \nleqslant a_j$.

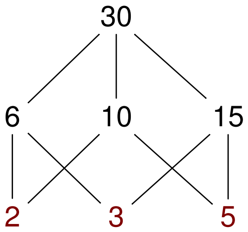
Equivalently:

$$\begin{aligned} i>1 &\Rightarrow a_i \nleqslant a_1 \\ i>2 &\Rightarrow a_i \nleqslant a_2 \\ i>3 &\Rightarrow a_i \nleqslant a_3 \\ &\vdots \\ i>(n-1) &\Rightarrow a_i \nleqslant a_{n-1} \end{aligned}$$

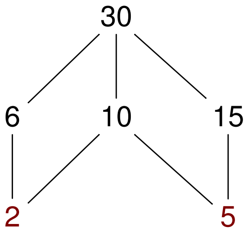
How to sort

Consider poset $\left(\{1, 2, 3, 5, 6, 10, 15, 30\}, \mid\right)$

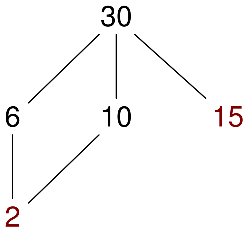




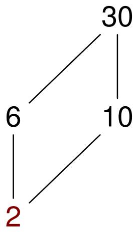
1,



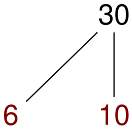
1,3,



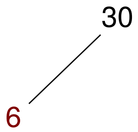
1,3,5,



1,3,5,15,



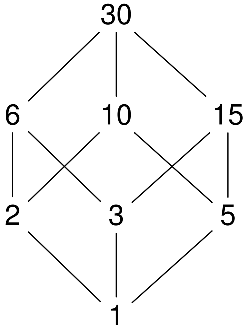
1,3,5,15,2,



1,3,5,15,2,10

30

1,3,5,15,2,10,6

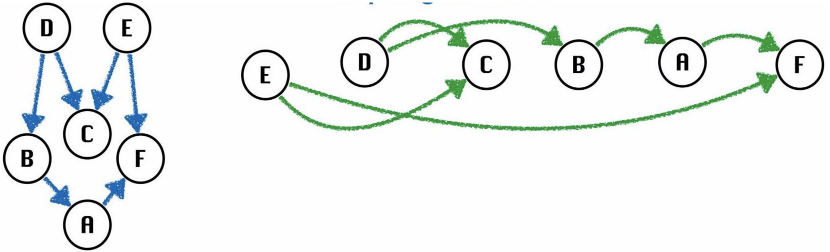


1,3,5,15,2,10,6,30

Note: The ordering produced by topological sorting is not unique.

Topological sorting and digraphs

Informally speaking, a topological sorting of an acyclic digraph is an ordering of the vertices, where all the arrows "point to the right".

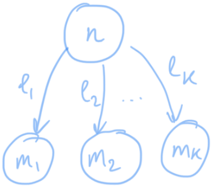


$$A = \{A, B, C, D, E, F\}$$
$$R = \{(A, F), (B, A), (D, B), (D, C), (E, C), (E, F)\}$$

Finding the next minimal element efficiently

Kahn's algorithm with $O(|V| + |E|)$ runtime

```
L ← Empty list that will contain the sorted elements
S ← Set of all nodes with no incoming edge
while S is non-empty do
    remove a node n from S
    add n to tail of L
    for each node m with an edge e from n to m do
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S
```



O as a relation

O as a relation

Recall: For asymptotically positive sequences (a_n), (b_n), we say that (a_n) is $O((b_n))$ if $\exists C > 0$ and $\exists k \geqslant 1$ such that $\forall n \geqslant k$

$$a_n \leqslant C b_n$$

How to think of O as a relation on some set A:

- $A = \{$all asymptotically positive sequences (c_n) of numbers$\}$
- $(a_n) O (b_n) \Leftrightarrow \exists C > 0 \exists k \geqslant 1 \left(n \geqslant k \Rightarrow a_n \leqslant C b_n\right)$

Question: Is O a partial order?

Is O a partial order? ✗

Reflexive, $(a_n)O(a_n)$? ✓

For $C = 1, k = n_0$ we have $(n \geqslant 1 \Rightarrow a_n \leqslant 1 \cdot a_n)$

Transitive, $(a_n)O(b_n) \wedge (b_n)O(c_n) \Rightarrow (a_n)O(c_n)$? ✓

- $\exists C > 0\ \exists k \geqslant 1\ (n \geqslant k \Rightarrow a_n \leqslant Cb_n)$
- $\exists D > 0\ \exists \ell \geqslant 1\ (n \geqslant \ell \Rightarrow b_n \leqslant Dc_n)$
- Let $E = CD$, $m = \max\{k, \ell\}$. Assume $n \geqslant m$. Then

$$a_n \leqslant Cb_n \leqslant CDc_n = Ec_n$$

So $n \geqslant m \Rightarrow a_n \leqslant Ec_n$

Antisymmetric, $(a_n)O(b_n) \wedge (b_n)O(a_n) \Rightarrow (a_n) = (b_n)$? ✗

$(n)O(2n)$ and $(2n)O(n)$, but $(n) \neq (2n)$

Θ as a relation

Θ as a relation

Recall: We say that (a_n) is $\Theta((b_n))$ if (a_n) is $O((b_n))$ and (b_n) is $O((a_n))$

- $A = \{$all asymptotically positive sequences (c_n) of numbers$\}$
- $(a_n)\Theta(b_n) \Leftrightarrow (a_n)O(b_n)$ and $(b_n)O(a_n)$
- So $\Theta = O \cap O^{-1}$

Question: Is Θ an equivalence relation[1]?

[1] reflexive, symmetric, and transitive

Is $\Theta = \mathrm{O} \cap \mathrm{O}^{-1}$ an equivalence relation?

- O is reflexive and transitive.
- **Thm.** If R is reflexive/transitive then R^{-1} is also reflexive/transitive.

  So O^{-1} is reflexive and transitive.
- Recall from last week: **Thm.** If R and S are reflexive/transitive then $\mathrm{R} \cap S$ is reflexive/transitive.

  So $\Theta = \mathrm{O} \cap \mathrm{O}^{-1}$ is reflexive and transitive.
- **Final observation:** $\Theta = \mathrm{O} \cap \mathrm{O}^{-1}$ is symmetric.
- **Answer:** Θ is an equivalence relation.