DMA: Logic

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Warm-up: Which of the following are true propositions?

1. Every $n \in \mathbb{Z}^+$ can be expressed as a product of primes and $n \log_2(n) \in O(n^2)$.
2. If $1 + 4 = 7$ then 4 is a prime.
3. Please, take a seat.
4. If someone in this class has a birthday today, then $\text{LCM}(12, 8) = 24$.

Plan for this class

- Properties of logical operations
- Predicates
- Logical statements with quantifiers
- Negation of statements with quantifiers

Properties of logical operations

Recall: $p \equiv q$ means that p, q always have the same truth value.
Example: $(r \Rightarrow s) \equiv \big((\sim s) \Rightarrow (\sim r)\big)$

Properties of AND, OR, NOT [Thm 2.2.1]

(1) $p \vee q \equiv q \vee p$

(2) $p \wedge q \equiv q \wedge p$

(3) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(4) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(7) $p \vee p \equiv p$

(8) $p \wedge p \equiv p$

(9) $\sim(\sim p) \equiv p$

(10) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

(11) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

We can prove the above by comparing the truth tables of LHS and RHS.

Properties invloving $\Rightarrow$ and $\Leftrightarrow$ [Thm. 2.2.2]

(a) $(p \Rightarrow q) \equiv (\sim p \vee q)$

(b) $(p \Rightarrow q) \equiv ((\sim q) \Rightarrow (\sim p))$

(c) $(p \Leftrightarrow q) \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

(d) $\sim(p \Rightarrow q) \equiv (p \wedge (\sim q))$

(e) $\sim(p \Leftrightarrow q) \equiv (p \wedge (\sim q)) \vee (q \wedge (\sim p))$

We proved (a) and (b) on Monday.
Items (d) and (e) can be proven the previous properties.

Thm 2.2.4 gives more properties...

Predicates

Predicates

Consider the sentence: "$x^2 + y^2 = 4$"
It has a truth value only after we fix x and y.

Def. A predicate is a declarative sentence whose truth value depends on one or more (free) variables.

Notation: $P(x), Q(x), P(x, y)$ etc.

Examples of predicates

- $Q(x, y)$: $x^2 + y^2 = 4$
- $P(n)$: $3^n + 6n - 1$ is divisible by 4

Defining sets using predicates

$S' = \{x \in S \mid \mathbf{P}(x)\}$ denotes the collection of all $x \in S$ for which the statement $\mathrm{P}(x)$ is true.

- Let $S = \mathbb{Z}$ and $\mathrm{P}(x): \; x > 0$. We have

$$\{x \in \mathbb{Z} \mid x > 0\} = \{1, 2, 3, 4, \ldots\} = \mathbb{Z}^+$$

- We can define the positive reals, $\mathbb{R}^+$, as $\{x \in \mathbb{R} \mid x > 0\}$

- Let $\mathrm{Q}(x, y): \;\; x^2 + y^2 = 4$. What is the set

$$\{(x, y) \in \mathbb{R}^2 \mid \mathrm{Q}(x, y)\}?$$

Quantifiers

Quantifiers

Given a predicate $P(x)$ where $x \in S$, we can form two propositions:

1. Proposition
$$\forall x \in S \;\; P(x)$$
is true if $P(x)$ is true for all $x \in S$.
Pronounced: "for all elements x of S, $P(x)$ is true"

2. Proposition
$$\exists x \in S \;\; P(x)$$
is true if $P(x)$ is true for some $x \in S$.
Pronounced: "there exists $x \in S$ such that $P(x)$ is true"

Quantifiers: examples

1. Let $P(n): \; n^2 + 1 \geqslant 2$. We can write the statement "*For all positive integers n, we have $n^2 + 1 \geqslant 2$*" as

$$\forall n \in \mathbb{Z}^+ \;\; P(n)$$

or as

$$\forall n \in \mathbb{Z}^+ \;\; n^2 + 1 \geqslant 2$$

2. We can also specify sets using inequalities and write

$$\forall x \in \mathbb{R}^+ \;\; x^2 + 1 \geqslant 2$$

as

$$\forall \text{real } x > 0 \;\; x^2 + 1 \geqslant 2$$

Note: The statements in (1) are true, but the ones in (2) are false.

Order of quantifiers

Example: Let

$$S = \{\text{Living and deceased people}\}$$

Consider the following statements:

1. $\forall p \in S\ \exists m \in S\ \ m$ is the mother of p
2. $\exists m \in S\ \forall p \in S\ \ m$ is the mother of p

Are these different? Yes!

1. Every p has a mother
2. There is an m that is the mother for all p

The order matters! In general $\forall x \exists y\ \ P(x, y)$
is different from $\exists y \forall x\ \ P(x, y)$

Negation of statements with quantifiers

How can we rewrite $\sim(\forall x \in S \;\; P(x))$ mean?

Example.

q : **All** students in this class have read KBR Chapter 2.

What is $\sim q$?

$\sim q$: **There is** a student in this class who has not read KBR Chapter 2.

In general, we have

$\sim(\forall x \in S \;\; P(x)) \equiv \big(\exists x \in S \;\; \sim P(x)\big)$ Thm 2.2.3 (a)

$\sim(\exists x \in S \;\; P(x)) \equiv \big(\forall x \in S \;\; \sim P(x)\big)$ Thm 2.2.3 (b)

Negation of Big-Θ using quantifiers

Big-Θ: Let $f, g\colon \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive. Then f is $\Theta(g)$ if there exists $c_1, c_2, x_0 > 0$ such that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ for all $x \geqslant x_0$.

With quantifiers:

$$\exists c_1 > 0\ \exists c_2 > 0\ \exists x_0 > 0\ \forall x \geqslant x_0\ c_1 g(x) \leqslant f(x) \leqslant c_2 g(x).$$

The negation:

$$
\begin{aligned}
&\sim\big[\exists c_1 > 0\ \big(\exists c_2 > 0\ \exists x_0 > 0\ \forall x \geqslant x_0\ c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)\big)\big] \\
\equiv\ &\forall c_1 > 0\ \sim\big[\exists c_2 > 0\ \big(\exists x_0 > 0\ \forall x \geqslant x_0\ c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)\big)\big] \\
\equiv\ &\forall c_1 > 0\ \forall c_2 > 0\ \sim\big[\exists x_0 > 0\ \big(\forall x \geqslant x_0\ c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)\big)\big] \\
\equiv\ &\forall c_1 > 0\ \forall c_2 > 0\ \forall x_0 > 0\ \sim\big[\forall x \geqslant x_0\ \big(c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)\big)\big] \\
\equiv\ &\forall c_1 > 0\ \forall c_2 > 0\ \forall x_0 > 0\ \exists x \geqslant x_0\ \sim\big[c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)\big] \\
\equiv\ &\forall c_1 > 0\ \forall c_2 > 0\ \forall x_0 > 0\ \exists x \geqslant x_0 \\
&\qquad\qquad \big[c_1 g(x) > f(x) \vee f(x) > c_2 g(x)\big]
\end{aligned}
$$

What is going on here?

> Three mathematicians walk into a bar.
> The bartender asks: “Do all three of you want a beer?”
> The first mathematician says: “I don’t know.”
> The second one says: “I don’t know.”
> The third one says: “Yes!”

Let S be the set of three mathematicians.
Consider the predicate $P(x)$: x wants to have a beer.
The bartender asks if the following is true

$$\forall x \in S \;\; \mathsf{P}(x) \tag{1}$$

Why don’t the first two mathematicians know the truth value of (1) but the last one does?

Implications involving predicates

Let $\mathsf{P}(x)$ and $\mathsf{Q}(x)$ be predicates concerning $x \in S$.

It is a convention that with $\mathsf{P}(x) \Rightarrow \mathsf{Q}(x)$ we mean the statement

"*For all* $x \in S$ *we have that* $\mathsf{P}(x) \Rightarrow \mathsf{Q}(x)$*.*"

Example. Statement

$$x > 1 \Rightarrow x^2 > 1$$

means

"*For all* x *we have that if* $x > 1$ *then* $x^2 > 1$*.*"

You should be able to:

- Rewrite sentences using logical connectives and quantifiers.
- Translate logical statements into everyday language.
- Evaluate whether a given proposition is true or false.
- Rewrite and simplify logical statements using properties of logical operations and quantifiers.

Puzzle: The blue-eyed islanders

The islanders: 60 with blue eyes and 70 with brown eyes.

They know: everyone on the island is an excellent logician.

They don't know: their own eye color, the total number of blue-eyed people. Also they cannot talk to each other.

The pirate captain: visits every night and will free any *blue-eyed* person who can tell him their own eye color. (He will "dispose of" anyone who guesses incorrectly.)

The guru: proclaims on Day 1 that someone on the island has blue eyes.

The question: Who gets off the island and when?