What does Exp(n) return?

```
Exp(n)
    If (n = 0) then
        return 1
    Elseif (n % 2) = 0 then        (n is even)
        x ← Exp(n/2)
        return x*x
    Else                           (n is odd)
        x ← Exp((n-1)/2)
        return 3*x*x
```

DMA: Strong Induction

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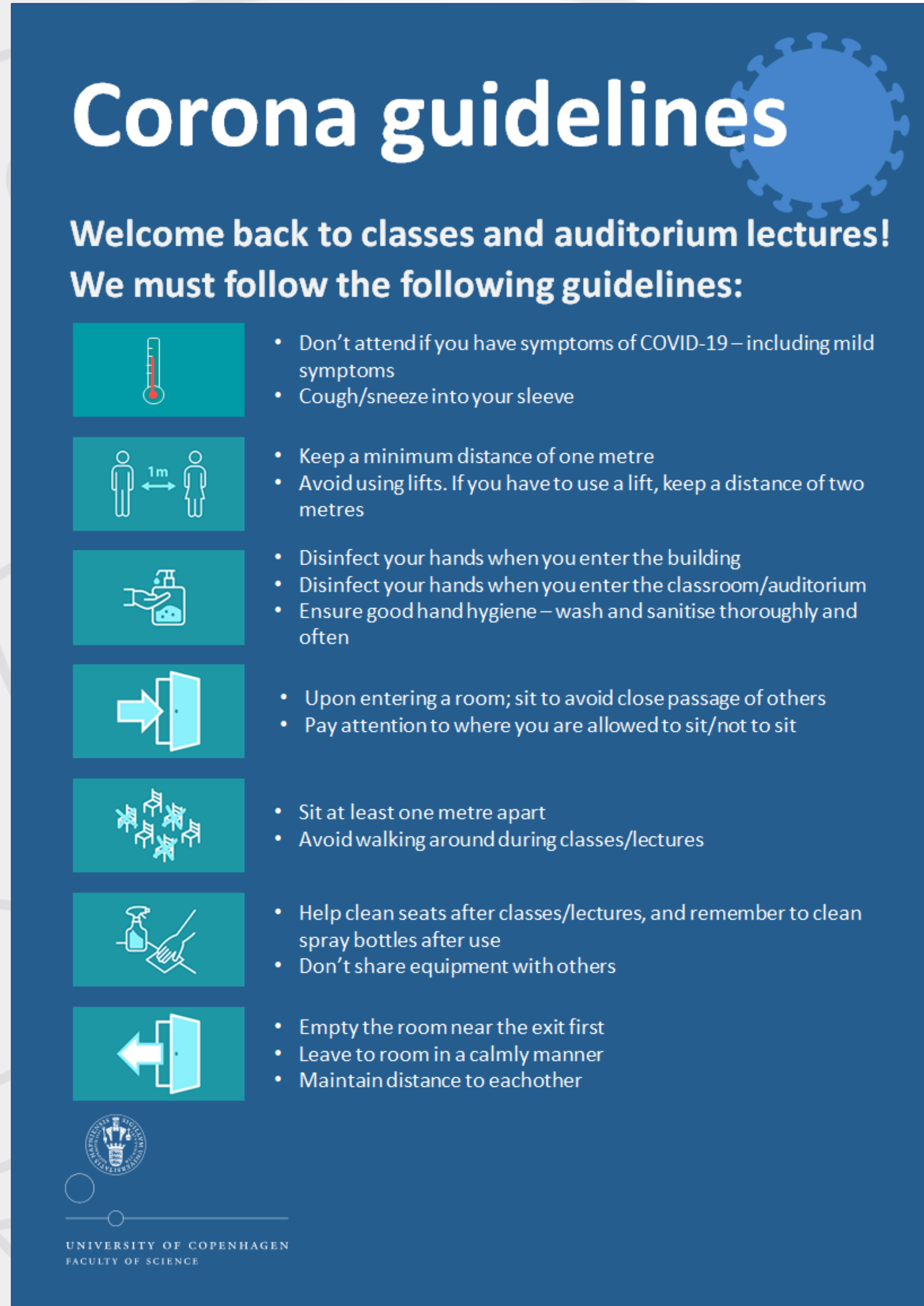


Plan for today

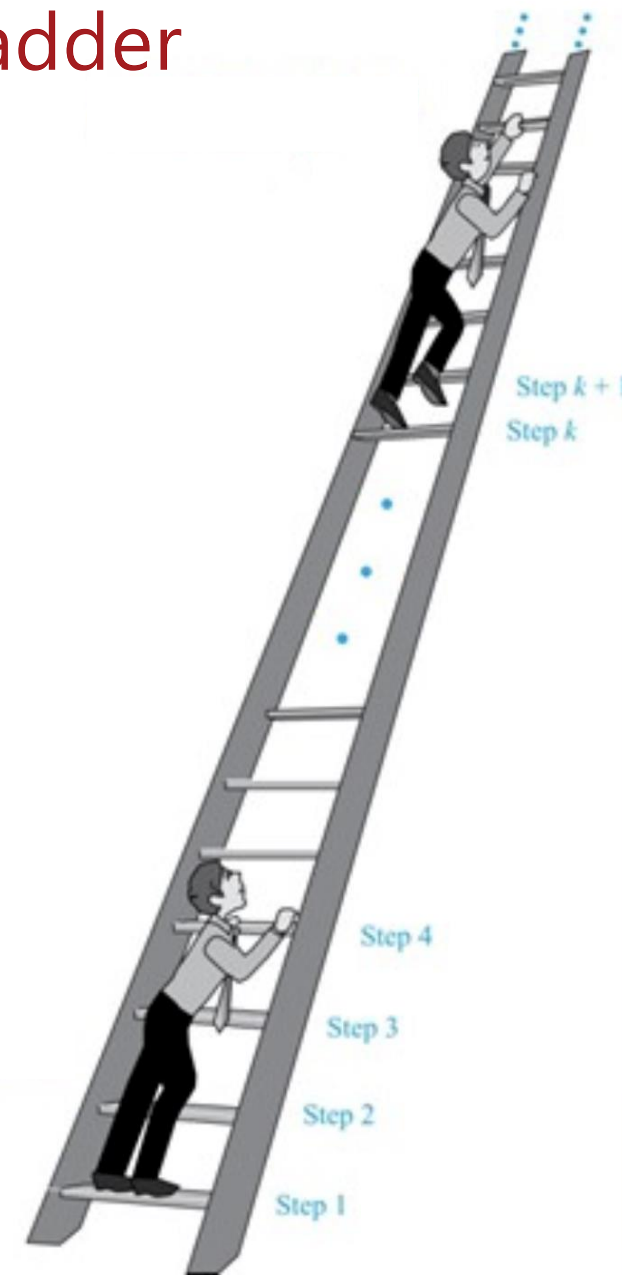
- Recap of the principle of induction
- Strong induction
- Examples

Reading: Notes for Week 5 (on Absalon)

Analogy: climbing an infinite ladder

- Suppose you can reach the first rung

- If you are on a particular rung k you can get on the next rung $k+1$



The principle of mathematical induction

Let $P(n)$ be a predicate (statement) defined for integers $\{n_0, n_0 + 1, \dots\}$. If

a) [Base case] $P(n_0)$ is true and

b) [Induction step] For any $n \geq n_0$, we have that $P(n)$ being true implies that $P(n+1)$ is true

Then $P(n)$ is true for all integers $n \geq n_0$.

We used induction to show that

- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{Z}^+$
- $5|(6^n - 5n + 4)$ for all $n \in \mathbb{Z}^+$
- A diagonally moving robot cannot reach position (1,0) when starting from (0,0).

Strong induction

The principle of strong induction

Let $P(n)$ be a predicate (statement) defined for integers $\{n_0, n_0 + 1, \dots\}$. If

1. **[Base case]** $P(n_0)$ is true and
2. **[Induction step]** for any $n \geq n_0$, we have that all of $P(n_0), \dots, P(n)$ being true implies that $P(n+1)$ is true

then

- $P(n)$ is true for all integers $n \geq n_0$.

The Strong of Inductia: making change

- Inductia's currency is Strongs (Sg)
- They only have 3- and 5- Strong coins

- Which of these amounts can Inductians make:

  4, 5, 6, 7, 8, 9, 10, 11?

Claim. Inductians can make any integer amount $n \geq 8$.

Proof by strong induction.

Let $P(n)$ be the statement that Inductians can make n using their coins. Take $n_0 = 8$.

The Strong of Inductia: proof continued

- Base case: $P(8)$ holds since 8=3+5
- Induction step.
  - Take $n = 8$. If $P(8)$ holds then $P(9)$ holds.
  - Take $n = 9$. If $P(8)$ and $P(9)$ hold then $P(10)$ holds.
  - Assume that all of $P(8), P(9), \ldots, P(n)$ hold for some $n \geq 10$. (**Strong Induction Hypothesis**)
    Need to show that $P(n+1)$ holds.

What does Exp(n) return?

```
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```

Computing 3^n with repeated squaring

```
Exp(n)
    If (n = 0) then
        return 1
    Elseif (n % 2) = 0 then          (n is even)
        x ← Exp(n/2)
        return x*x
    Else                             (n is odd)
        x ← Exp((n-1)/2)
        return 3*x*x
```

Claim. `Exp(n)` returns 3^n for any $n \in \mathbb{Z}^+$.

Let $P(n)$ be the statement that `Exp(n)` returns 3^n.

Detour: All horses are the same color
(“induction example”)

All horses have the same color

$P(n)$: in any set of n horses, all horses have the same color.

Base case: $P(1)$ is true since in a set with only one horse, all (one) horses are of the same color.

Inductive step: Suppose $P(n)$ is true for some $n \geq 1$ **(IH)**
So in any set of n horses all horses have the same color.

Let us argue that $P(n+1)$ is true. Let

$$\{h_1, h_2, \dots, h_{n+1}\}$$

be a set of $n+1$ horses.

Have we shown that all horses have the same color?

Corona – guidelines

Rengøring mellem undervisning

Cleaning between lectures

- Rengør bordflader, stolerygge og andre kontaktpunkter
- Spray på og tør af med papir efter ca. 30 sek
- Husk at rengøre sprayflasken efter brug
- Sprit eller vask dine hænder

- *Clean tables, the backs of chairs and other contact surfaces in the room*
- *Spray and wipe with paper after approx. 30 secs*
- *Remember to clean the spray bottle after use*
- *Disinfect or wash your hands*

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Tak for hjælpen

Thanks for helping out