DMA: Proofs

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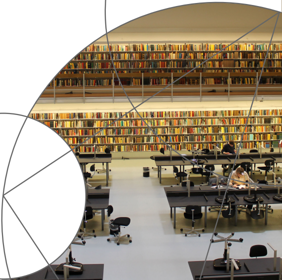

Outline

- Proof method: case analysis

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- To prove or to disprove?

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- The axiomatic method
- “The blue-eyed islanders”

Proof by case analysis

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Proof template

- Write: "We use case analysis."
- Identify a list of conditions, at least one of which must hold.

  (If it is not obvious that the list is exhaustive, you must prove it.)
- For each condition:
  - State the condition.
  - Prove q assuming that the condition holds.

Proof by case analysis (example)

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Q: Why is the list exhaustive?

(finish on the board)

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To disprove a "for all"-type statement, we only need a counterexample.

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Proof is a a sequence of logical deductions (valid arguments).

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- **Q:** What additional assumption do we need, to conclude that q is true?
- Indeed, $(p \wedge (p \Rightarrow q)) \Rightarrow q$ is a tautology (check!).

Logical deductions

Def. Given logical statements $p_1, \ldots, p_n$ and q, we say that q logically follows from $p_1, \ldots, p_n$ if

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Notation:

$$\begin{array}{rll} & p_1 & \\ & p_2 & \text{Hypotheses} \\ & \vdots & \\ & p_n & \\ \hline \therefore & q & \text{Conclusion} \end{array}$$

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Example (modus ponens)

$$\begin{array}{l} p \\ p \Rightarrow q \\ \hline q \end{array}$$

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- Assume (for contradiction) that $\sqrt{2}$ is rational, *i.e.* $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}^+$, where $\gcd(a, b) = 1$.
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So we had established that

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Then we concluded that a must be even. This was modus ponens.

A logical deduction?

Example.

If today is Wednesday, then Mette has POP today.

Today is not Wednesday.

Mette does not have POP today.

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Not a valid argument, since

$$((p \Rightarrow q) \wedge (\sim p)) \Rightarrow (\sim q)$$

is *not* a tautology. (When does it fail to be true?)

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Example.

If I cycle to university, then I arrive tired.

I did not arrive tired.

I did not cycle to university.

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Valid argument, since

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is a tautology (check!).

Let's "prove" that $1/8 > 1/4$

Bogus proof

$$\begin{aligned} 3 &> 2 && \Leftrightarrow \\ 3\log_{10}(1/2) &> 2\log_{10}(1/2) && \Leftrightarrow \\ \log_{10}(1/2)^3 &> \log_{10}(1/2)^2 && \Leftrightarrow \\ (1/2)^3 &> (1/2)^2 \end{aligned}$$

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What's wrong here?

A common mistake

Let $a, b \in \mathbb{R}^+$. It is a fact that the Arithmetic Mean is at least as large as the Geometric Mean, namely,

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Take-away: NEVER start with what you want to prove.

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