UNIVERSITY OF COPENHAGEN

DMA: Relations

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Plan for today

1. Sets and their products
   - Subsets and the power set
2. Relations
   - Domain and range
3. How do we represent a relation?
   - Matrix representation
   - Digraph representation
4. R-relative sets

Reading for today: KBR 4.1–4.2

Sets vs tuples

Recall: A set is a well-defined collection of elements.

Examples:
$\{a, b, c\}$, $\mathbb{Z}^+$, $\mathbf{M}_n(\mathbb{R})$, {all students currently in Aud 1}

Def. An n-tuple is an ordered collection of n elements. We refer to 2-tuples as ordered pairs.

Examples:
(a, b, c), $(1, 1, 2, 3, 5, 8, 13)$, (Mette, Jonas, 5, a)

Warning: Order matters for tuples: $(a, b, c) \neq (b, c, a)$.
Also, $\{a, b, c\} \neq (a, b, c)$ (types don't match)

Product sets

Def. The (Cartesian) product of sets A and B consists of ordered pairs of elements from A and B:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Example

For $A = \{a, b, c\}$ and $B = \{1, 2\}$ we get

$$A \times B = \Big\{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\Big\}$$

Note: Order matters, so

- $(1, a) \notin A \times B$
- $A \times B \neq B \times A$

Products of many sets

Def. The (Cartesian) product of sets $A_1, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i\}$$

Thm. Given n finite sets $A_1, \ldots, A_n$, we have

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1|\,|A_2| \cdots |A_n|$$

Example: Vectors

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$
  vectors in a plane (2 dimensions)
- $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
  vectors in 3 dimensions.

Subsets and the power set

Recall: A set B' is a subset of B , denoted as $B' \subseteq B$, if all the elements of B' are contained in B.

Examples:
$\{1,2,3\} \subseteq \{1,2,3,4\}$, $\mathbb{Z} \subseteq \mathbb{R}$, $\{\pi, 3\} \nsubseteq \mathbb{Q}$

Def. Let B be a set. The power set of B, denoted as $P(B)$ or 2^B, is the set of all subsets of B.

Example. The power set of $B = \{1,2,3\}$ is

$$\Big\{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

Q: $|P(A)| = ?$

Outline

Examples of relations

Goal: Provide a rigorous way to describe relations between elements of a set A and a set B.

Examples

- $A = \{\text{all women}\}$, $B = \{\text{all people}\}$
  Relation: a is b's mother.
- $A = B = \mathbb{Z}^+$
  Relation: $a \leqslant b$.

Relations: definition

Def. Let A and B be sets. A relation R from A to B is a subset of $A \times B$ (i.e. $R \subseteq A \times B$).

Notes

- R consists of (some) pairs (a, b), where $a \in A, b \in B$.
- If $(a, b) \in R$, we say that a is related to b by R and write
$$aRb$$
- If $(a, b) \notin R$, we write $a \not R b$.
- If $R \subseteq A \times A$, we say that R is a relation on A.

Examples

- $A = \{\text{all women}\}$, $B = \{\text{all people}\}$
  Relation: $aR_1b \Leftrightarrow a$ is b's mother.
  $R_1 = \{\ldots, (\text{Mette}, \text{Frederik}), \ldots\}$

- $A = B = \mathbb{Z}^+$
  Relation: $aR_2b \Leftrightarrow a \leqslant b$.
  $R_2 = \{(1,1), (1,2), (2,2), (1,3), (2,3), \ldots\}$

- $A = \{2,3\}$, $B = \{1,2,3,4,5,6\}$
  Relation: $aR_3b \Leftrightarrow a$ divides b
  Exercise: $R_3 = ?$

Databases as relations n-ary relations

Table: Customers

CustomerID	ContactName	City	Country
1	Maria Anders	Berlin	Germany
2	Ana Trujillo	México D.F.	Mexico
3	Antonio Moreno	México D.F.	Mexico
4	Thomas Hardy	London	UK
5	Christina Berglund	Luleå	Sweden
6	Hanna Moos	Mannheim	Germany
7	Frédérique Citeaux	Strasbourg	France
8	Martín Sommer	Madrid	Spain
9	Laurence Lebihans	Marseille	France
10	Elizabeth Lincoln	Tsawassen	Canada

SQL query:

```
SELECT ContactName, City FROM Customers WHERE Country="France";
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Domain and range

Def. Let R be a relation from A to B, i.e., $R \subseteq A \times B$.
The domain of R is $\mathrm{Dom}(R) = \{a \in A \mid \exists b \in B : aRb\}$
The range of R is $\mathrm{Ran}(R) = \{b \in B \mid \exists a \in A : aRb\}$

Example. Recall $A = \{2, 3\}$, $B = \{1, 2, 3, 4, 5, 6\}$, and

$$R_3 = \{(2,2),(2,4),(2,6),(3,3),(3,6)\}.$$

Exercise: Find $\mathrm{Dom}(R_3)$ and $\mathrm{Ran}(R_3)$.

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Matrix of a relation

Def. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_n\}$, and let R be a relation from A to B. The matrix of R is an $m \times n$ Boolean matrix $\mathbf{M}_R = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example

Let $A = \{1, 2, 3\}$, $B = \{r, s\}$ and $R = \{(1, r), (2, s), (3, r)\}$.

Then $\mathbf{M}_R$ is a 3×2 matrix:

$$\begin{array}{cc} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} ?1 & ?0 \\ ?0 & ?1 \\ ?1 & ?0 \end{bmatrix} \end{array}$$

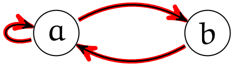
Directed graphs: visualizing relations

Let $R \subseteq A \times A$ be a relation on a finite set A.

Def. The digraph of R consists of

- vertices, each representing a unique element of A
- directed edges, each representing a unique element of R.

Example. Consider $A = \{a, b\}$ and $R = \{(a,a),(a,b),(b,a)\}$.
The digraph of R:



Note: Digraph of R contains all the information about R.

The digraph and matrix of R

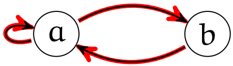
Example

Let $A = \{a, b\}$ and $R = \{(a, a), (a, b), (b, a)\}$.

The matrix of R:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The digraph of R:



Outline

R-relative sets

Def. Let R be a relation from A to B. Also, let $a \in A$ and $A_1 \subseteq A$.

- The R-relative set of a is $R(a) = \{b \in B \mid aRb\}$.
- The R-relative set of A_1 is

$$R(A_1) = \{b \in B \mid \exists a \in A_1 : aRb\} = \bigcup_{a \in A_1} R(a)$$

Examples. Let $A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5, 6\}$ and consider $R \subseteq A \times B$ given by $aRb \Leftrightarrow$ "a divides b".

$$R(2) = \{2, 4, 6\}$$
$$R(3) = ?$$
$$R(\{2, 3\}) = ?$$

Properties of R-relative sets

Thm. Let R be a relation from A to B and $A_1, A_2 \subseteq A$.

- If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$
- $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$
- $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Exercise: Find an example of sets A, A_1, A_2, B, and relation R where

$$R(A_1 \cap A_2) \neq R(A_1) \cap R(A_2)$$

What we've seen today

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