DMA: Pigeonhole Principle
Probability

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Pigeonhole Principle

(Danish: skuffeprincippet)

KBR 3.3

Pigeonhole Principle (Thm. 3.3.1)

Thm. If n pigeons are put into m pigeonholes, where $n > m$, then at least one pigeonhole contains two or more pigeons.

Proof. We use proof by contradiction. Assume that all m pigeonholes contain at most one pigeon. Then the total number of pigeons is at most $m \cdot 1 = m$. In other words, $n \leqslant m$. This is a contradiction, since $n > m$. Hence, there exists a pigeonhole containing two or more pigeons.

Pigeonhole Principle in action

To use Pigeonhole Principle, we must identify

- the pigeons (objects)
- the pigeonholes (categories)

Are there two people in this class with the same number of hair? In Copenhagen? In the world?

Answer. Research says that a person has at most 200,000 hairs.
The population of Copenhagen is 602,481.
By Pigeonhole Principle, there are two people in Copenhagen that have the same number of hairs.
(pigeons = people, pigeonholes= “the number of hairs”)

Example 3.3.3

Claim. No matter which 11 numbers are chosen from $\{1, 2, \ldots, 20\}$, one of them will always be a multiple of the other.

Example: Choose $7, 9, 11, 12, 13, 15, 16, 17, 18, 19, 20$

Proof outline

- Every integer $n \in \mathbb{Z}^+$ can be expressed as

$$n = 2^k m$$

  where $k \in \mathbb{N}$ and $m \in \mathbb{Z}^+$ is odd (we call m the odd part of n).
- Pigeons = "11 chosen numbers" pigeonholes = "the odd parts"
- There are 10 odd numbers (pigeonholes) between 1 and 20.
- By Pigeonhole Principle, (at least) two of the chosen numbers must have the same odd part.
- $2^{k_1} m$ divides $2^{k_2} m$, where $k_1 < k_2$.

The Extended Pigeonhole Principle

Thm. If n pigeons are put into m pigeonholes, then there exists a pigeonhole containing at least

$$\lfloor (n-1)/m \rfloor + 1$$

pigeons.

Proof (by contradiction). If every pigeonhole has at most $\lfloor (n-1)/m \rfloor$ pigeons, then the total number of pigeons is

$$m \cdot \lfloor (n-1)/m \rfloor \leqslant m \cdot (n-1)/m = n-1$$

Contradiction. □

Fact: $\lfloor (n-1)/m \rfloor + 1 = \lceil n/m \rceil$ (check at home).

Example 3.3.6

Claim. If 30 books have 61327 pages in total, then one of the books has at least 2045 pages.

Solution: Books are pigeonholes (m) and pages are pigeons (n). By the Extended Pigeonhole Principle, there is a book with at least

$$\lfloor (61327 - 1)/30 \rfloor + 1 = \lfloor 2044.33\ldots \rfloor + 1 = 2045$$

pages.

More advanced samples of PHP

Claim. Given any 5 points on a sphere, there is a hemisphere that contains 4 of them.

points can be on the boundary of the hemisphere

Claim. Every sequence of $n^2 + 1$ distinct real numbers contains a monotone subsequence of length $n + 1$.

monotone = increasing / dicreasing; subsequence - not necessarily consecutive

E.g. $(n = 2; n^2 + 1 = 5, n + 1 = 3)$ $4, 7, -1, 3, 1$

Probability

KBR 3.4

Sample space of an experiment

Experiment 1: Roll a 6-sided die and record the outcome.
Experiment 2: Randomly draw 3 cards out of a deck of 52.

- A sample space is a set A consisting of outcomes some experiment.
- Probability function $p : A \to \mathbb{R}$, where
  - $p(a) \geqslant 0$ for all $a \in A$
  - $\sum_{a \in A} p(a) = 1$
- We refer to (A, p) as probability space.

Example: $A_1 = \{1, 2, 3, 4, 5, 6\}$
- For a fair die: $p(a) = \frac{1}{6}$ for any $a \in A_1$.
- Biased die, e.g.:
  $p(a) = 1/9$ if $a \in \{1, 3, 5\}$, $p(a) = 2/9$ if $a \in \{2, 4, 6\}$.

Events

Definition. An event is a subset of the sample space. We refer to outcomes as elementary events.

Examples:

- "Rolling an even number" is an event: $E_1 = \{2, 4, 6\}$
- "Rolling a 3" is an elementary event: $E_2 = \{3\}$

Given probability space (A, p), the probability of an event E is

$$p(E) = \sum_{a \in E} p(a)$$

Examples: $p(E_1) = p(2) + p(4) + p(6) \qquad p(E_2) = p(3)$

When (A, p) is the probability space for rolling a fair die:

$$p(E_1) = 3 \cdot \frac{1}{6} = \frac{1}{2} \qquad p(E_2) = \frac{1}{6}$$

Equally likely outcomes

Let (A, p) be a probability space with $p(a) = \frac{1}{|A|}\ \forall a \in A$. Then,

$$p(E) = \sum_{a \in E} p(a) = \sum_{a \in E} \frac{1}{|A|} = \frac{|E|}{|A|}$$

In other words,

$$p(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes}}$$

Experiment: Randomly draw a 3-card hand from a deck of 52.

- $A =$ all possible three-card hands. $|A| = {}_{52}C_3$
- $E =$ the hand consists of only Kings
  $= \{\{\heartsuit K, \spadesuit K, \diamondsuit K\}, \{\heartsuit K, \spadesuit K, \clubsuit K\},$
  $\{\heartsuit K, \clubsuit K, \diamondsuit K\}, \{\clubsuit K, \spadesuit K, \diamondsuit K\}\}$
- $p(E) = \frac{{}_4C_3}{{}_{52}C_3} = \frac{4}{22100} = \frac{1}{5525}$

Set operations with events

Let (A, p) be a probability space and $E, F \subseteq A$ be events.

- Union: $E \cup F = \{a \in A \mid a \in E \text{ or } a \in F\}$
- Intersection: $E \cap F = \{a \in A \mid a \in E \text{ and } a \in F\}$
- Complementation: $\overline{E} = \{a \in A \mid a \notin E\}$

Probabilities:

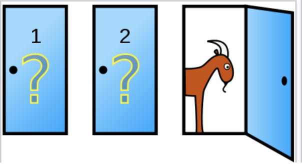
$$p(\overline{E}) = 1 - p(E) \qquad p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

Example: (A, p): the probability space for a fair die

- Event of rolling an odd number: $E = \{1, 3, 5\}$ $p(E) = \frac{1}{2}$
- Event of rolling a prime: $F = \{2, 3, 5\}$ $p(F) = \frac{1}{2}$

$E \cup F = \{1, 2, 3, 5\}$ $E \cap F = \{3, 5\}$ $\overline{E} = \{2, 4, 6\}$ $\overline{F} = \{1, 4, 6\}$

Monty Hall problem



- Based on a game show: three doors, one has a car the others have goats
- Player picks a door (say, it is door 1)
- Next, host opens a different door, revealing a goat
  (He knows where the car is and opens a non-car door on purpose)

The player is asked if she would like to pick a different door. Should she

(a) stick with her initial choice (door 1); or

(b) switch to door 2; or

(c) it doesn't matter