Prop. $M_{R^2} = M_R \odot M_R$ (R relation on A)

Proof. Let $A = \{a_1, \dots, a_m\}$.

$(M_{R^2})_{ij} = 1 \iff a_i R^2 a_j$

$\iff$ there is a path of length 2 from a_i to a_j

$\iff$ there is a_k such that $a_i R a_k$ and $a_k R a_j$

$\iff$ there is k such that $(M_R)_{ik} = 1$ and $(M_R)_{kj} = 1$

$\iff (M_R \odot M_R)_{ij} = 1$.

Since i,j are arbitrary and M_{R^2}, $M_R \odot M_R$ are Boolean matrices, it follows $M_{R^2} = M_R \odot M_R$.

$\blacksquare$

Thm. $M_{R^n} = \underbrace{M_R \odot \cdots \odot M_R}_{n}$ for $n \geq 1$.

Proof. It suffices to show

(★) $M_{R^n} = M_R \odot M_{R^{n-1}}$ for $n \geq 2$;

the statement of the thm then follows by induction.

We prove (★) as before:

$$
\begin{aligned}
(M_{R^n})_{ij} = 1 &\iff a_i R^n a_j \\
&\iff \exists a_k : a_i R a_k \ \& \ a_k R^{n-1} a_j \\
&\iff \exists k : (M_R)_{ik} = 1 \ \& \ (M_{R^{n-1}})_{kj} = 1 \\
&\iff (M_R \odot M_{R^{n-1}})_{ij} = 1 .
\end{aligned}
$$

∎

Thm. Let $A = \{a_1, \ldots, a_m\}$ and let R be a relation on A. Then

$$R^\infty = R \cup \cdots \cup R^m.$$

Proof.

Clearly $R^\infty \supseteq R \cup \cdots \cup R^m$. To show the converse inclusion, let $(a,b) \in R^\infty$ be arbitrary. Then there is a path from a to b, and we can consider a shortest one:

$$a \to x_1 \to \cdots \to x_{n-1} \to x_n = b \quad \text{of length } n.$$

If $n > m$, then two terms of the sequence $(x_1, \ldots, x_n)$ have to be the same, since $|A| = m < n$. Therefore $x_j = x_k$ for some $j < k$. But then

$$a \to x_1 \to \cdots \to x_j = x_k \to x_{k+1} \to \cdots \to x_n = b \quad \text{is}$$

also a path from a to b, of length $n + (k-j)$. This contradicts our choice of a shortest path.

Therefore $n \le m$, and so $(a,b) \in R \cup \cdots \cup R^m$.

Since (a,b) in R^∞ was arbitrary, we have

$$R^\infty = R \cup \cdots \cup R^m.$$

Two comments on previous thm:

(1) if A is not finite it can happen that

$R^{\infty} \neq R \cup \dots \cup R^{n}$ for every $n \in \mathbb{N}$.

For example: $A = \mathbb{Z}$, $R = \{(x, x+1) : x \in \mathbb{Z}\}$

(2) Sometimes a lower power (than m) of R suffices; e.g. if R is transitive then $R^{\infty} = R$.

Let R be a relation on A.

(i) R is symmetric & asymmetric $\Leftrightarrow R = \emptyset$

(ii) R is symmetric & antisymmetric

$\Leftrightarrow \quad R \subseteq \{(a,a) : a \in A\}$