DMA: Relations

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Plan for today

1. Recap: Relations and their representations
2. Paths in relations
   - Resulting relations
3. Functions
   - Special classes of functions
4. Properties of relations
   - Reflexive relations
   - Symmetric relations
   - Transitive relations

Reading for today: KBR 4.3-4.5 and 5.1

Recap: Relations

Recall:

- A relation R from A to B is a set of ordered pairs (a,b), where $a \in A$ and $b \in B$. So $R \subseteq A \times B$.
- If $R \subseteq A \times A$, we say that R is a relation on A

Example. If $A = \{1,2,3,4,5\}$, then

$$R = \{(2,1),(3,1),(4,1),(5,1),(4,2),(5,2)\} \subseteq A \times A$$

is a relation on A.

Q: What is the relationship expressed by R?
A: $(a,a') \in R \Leftrightarrow a \geqslant 2a'$

Recap: Matrix and digraph of a relation

Example. $A = \{1, 2, 3, 4, 5\}$

$$R = \{(2,1), (3,1), (4,1), (5,1), (4,2), (5,2)\}$$

- The matrix of R is $\mathbf{M}_R = [m_{ij}]$ where
$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$
- The digraph, D_R, of R consists of
  - **vertices** corresponding to elements of A
  - **directed edges** corresponding to pairs $(a, a') \in R$

Outline

Paths

Def. Let R be a relation on A and let $a, b \in A$.
A path of length n in R from a to b is a sequence

$$a, x_1, x_2, \ldots, x_{n-1}, b$$

where $x_1, \ldots, x_{n-1} \in A$ and

$$aRx_1 \;,\; x_1Rx_2 \;,\; \ldots \;,\; x_{n-1}Rb$$

$$a \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow b$$

Note: The length of a path corresponds to the number of arrows.

Paths give rise to new relations

Let R be a relation on A.

Def. For $n \in \mathbb{Z}^+$, the relation R^n on A is given by

$aR^nb \Leftrightarrow$ there exists a path of length n in R from a to b.

Example. If $R = \{(2,1),(3,1),(4,1),(5,1),(4,2),(5,2)\}$

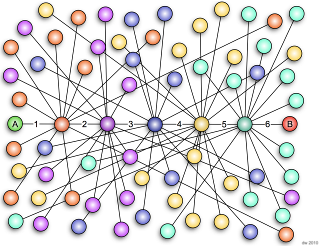
then $R^2 =$

Shrinking world: Six degrees of separation

$$A = \{\text{people in the world}\}$$

Relation on A: $aRb \Leftrightarrow a$ knows b.

Claim (informal). On average, we have aR^6b.



Feb 2016:

On average, the shortest path-length for two Facebook users is 4.57.

(1.6 billion users; $> \frac{1}{5}$ of world population).

Given R, how do we find R^n?

Let R be a relation on $A = \{a_1, \ldots, a_m\}$.

$aR^nb \Leftrightarrow$ there exists a path of length n in R from a to b

Idea 1: Draw digraph D_R and find all paths of length n.
Idea 2: Use the matrix $\mathbf{M}_R$ to find $\mathbf{M}_{R^2}$

Thm. $\mathbf{M}_{R^2} = \mathbf{M}_R \odot \mathbf{M}_R$.

Recall: $A \odot B = C$ where

$$c_{ij} = \begin{cases} 1 & \text{if } \exists k \text{ such that } a_{ik} = 1 \text{ and } b_{kj} = 1 \\ 0 & \text{otherwise} \end{cases}$$

How would you prove the above theorem?

Given R, how do we find R^n?

Let R be a relation on $A = \{a_1, \ldots, a_m\}$.

$aR^nb \Leftrightarrow$ there exists a path of length n in R from a to b

Thm. For any $n \geqslant 1$ we have

$$\mathbf{M}_{R^n} = \underbrace{\mathbf{M}_R \odot \cdots \odot \mathbf{M}_R}_{n}.$$

Proof method: Induction on n

Connectivity relation

Let R be a relation on $A = \{a_1, \ldots, a_m\}$.

Def. The connectivity relation R^∞ on A is given by

$$aR^\infty b \;\Leftrightarrow\; \exists n \in \mathbb{Z}^+ \text{ such that } aR^n b$$

Equivalently: $R^\infty = R \cup R^2 \cup R^3 \cup \ldots = \bigcup_{i=1}^{\infty} R^i$

Q: How large powers of R do we need to consider?
Is $R^\infty = R \cup R^2 \cup R^3$? Or can we stop at some power $f(m)$?

Connectivity relation

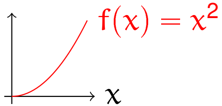
Let R be a relation on $A = \{a_1, \ldots, a_m\}$.

Thm. If R is a relation on $A = \{a_1, \ldots, a_m\}$ then

$$R^\infty = R \cup \cdots \cup R^m.$$

What is a function?

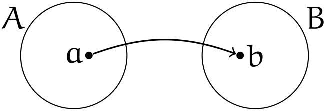
$$f(x) = x^2 \quad ,$$



We can specify f by listing all pairs $(x, f(x))$

Functions as relations

Def. Let A and B be sets. A relation $f \subseteq A \times B$ is called a function from A to B if for every $a \in A$ there is at most one $b \in B$ such that $(a, b) \in f$.



$b = f(a)$

Notes

- $f\colon A \to B$ denotes that f is a function from A to B.
- Warning: [KBR] doesn't require that $\mathrm{Dom}(f) = A$. (Often this is included in the definition of a function.)

Exercise

Determine which of the following relations are functions:

- Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ and let

$$f = \{(a, 1), (b, 1)\} \subseteq A \times B$$

The relation f is a function. $\text{Dom}(f) = \{a, b\}$.

- Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ and let

$$g = \{(a, 1), (b, 2), (c, 2), (c, 3)\} \subseteq A \times B$$

The relation g is not a function. $\text{Dom}(g) = A$

- Let $A = B = \mathbb{R}$ and let

$$h = \{(x, x^2) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$$

The relation h is a function. $\text{Dom}(h) = \mathbb{R}$.

Special classes of functions

Def. Let $f : A \to B$ be a function. We say that f is

- surjective (or "onto") if $\text{Ran}(f) = B$
- injective (or "one to one") if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$
- everywhere defined (KBR) if $\text{Dom}(f) = A$
- bijective if it is surjective, injective, and everywhere defined.

Example. Let $A = \{a, b, c\}$, $B = \{1, 2\}$ and consider

$$f = \{(a, 1), (b, 1)(c, 2)\} \subseteq A \times B$$

Exercise. Determine if f is (a) injective, (b) surjective, (c) bijective.

Reflexive relations

Def. A relation R on a set A is reflexive if $(a, a) \in R$ for all $a \in A$. We say that R is irreflexive, if $(a, a) \notin R$ for all $a \in A$.

Examples.

- The equality relation Δ on set A given by $a\Delta a' \Leftrightarrow a = a'$ is reflexive.
- The relation "$\leqslant$" on $\mathbb{R}$ is reflexive.
- The relation "$<$" on $\mathbb{R}$ is irreflexive.

Exercise. How can we tell whether a relation is reflexive by looking at its matrix $\mathbf{M}_R$?

Symmetric relations

Def. A relation R on a set A is

- symmetric if $(a, b) \in R$ implies that $(b, a) \in R$
- asymmetric if $(a, b) \in R$ implies that $(b, a) \notin R$
- antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Examples.

- The equality relation Δ is symmetric.
- **Exercise:** Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1)\}$. Is R symmetric, asymmetric, or antisymmetric?
- **Exercise:** Can a relation be both symmetric and asymmetric? How about symmetric and antisymmetric?

Transitive relations

Def. A relation R on a set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, we also have that $(a, c) \in R$.

Example. Let $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ be given by

$$aRb \Leftrightarrow a \mid b \quad (a \text{ divides } b)$$

If $a \mid b$ and $b \mid c$ then $a \mid c$. Thus, R is transitive.

Thm. R is transitive if and only if $R^2 \subseteq R$.

Summary

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