DMA: Proof Techniques

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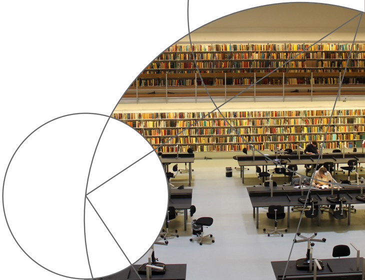

KBR: “Construction of proofs is an art and must be learned in part from observation and experience.”

Outline

- Proving an implication

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- Proving an implication
  - by a direct proof
  - by proving the contrapositive

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  - by proving the contrapositive
- Proving a biconditional statement

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- Proving an implication
  - by a direct proof
  - by proving the contrapositive
- Proving a biconditional statement
- Proof by contradiction

Proving an implication

$$p \Rightarrow q$$

Proving an implication: Direct proof

Task: Prove that $p \Rightarrow q$.

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Proof template

- Assume p holds.

Proving an implication: Direct proof

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- Assume p holds.
- Use relevant definitions and previously proven statements to argue that q must hold.

Proving an implication: Direct proof (example)

Task: Prove that if $x, y \in \mathbb{Z}$ are odd, then $x + y$ is even.

Proving an implication: Direct proof (example)

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By definition:

$x \in \mathbb{Z}$ is even if we can write it as $x = 2n$ for some $n \in \mathbb{Z}$.

$x \in \mathbb{Z}$ is odd if we can write it as $x = 2n + 1$ for some $n \in \mathbb{Z}$.

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Proof. Assume that x and y are odd integers.

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Proof. Assume that x and y are odd integers.

So we can write them as $x = 2n + 1$ and $y = 2m + 1$ for some $n, m \in \mathbb{Z}$.

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Proof. Assume that x and y are odd integers.

So we can write them as $x = 2n + 1$ and $y = 2m + 1$ for some $n, m \in \mathbb{Z}$.

Then $x + y = (2n + 1) + (2m + 1) = 2(n + m + 1)$.

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Then $x + y = (2n + 1) + (2m + 1) = 2(n + m + 1)$.

Hence, $x + y$ is even. $\square$

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- Prove the contrapositive, $(\sim q) \Rightarrow (\sim p)$, by a direct proof:

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- Prove the contrapositive, $(\sim q) \Rightarrow (\sim p)$, by a direct proof:
    - Assume $\sim q$ holds.
    - Use relevant definitions and previously proven statements to argue that $\sim p$ must hold.

Proving an implication: via contrapositive (example)

Task: Let $a, b, n \in \mathbb{Z}$. Prove that if $n \nmid (ab)$, then $n \nmid a$ and $n \nmid b$.

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Implication: $n \nmid (ab) \;\Rightarrow\; (n \nmid a \text{ and } n \nmid b)$

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Contrapositive: $(n \mid a \text{ or } n \mid b) \ \Rightarrow\ n \mid (ab)$

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Proof. We prove the contrapositive: If $n \mid a$ or $n \mid b$, then $n \mid (ab)$.

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Assume that $n \mid a$ or $n \mid b$.

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Assume that $n \mid a$ or $n \mid b$. Let us analyze the cases when $n \mid a$ and when $n \mid b$ separately.

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(finish on the board)

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- Write: "We prove p implies q and vice versa".

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- Write: "We prove p implies q and vice versa".
- Write: "First we show $p \Rightarrow q$": prove the implication.

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Note: A different proof technique can be used for each implication.

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Task: Let $a \in \mathbb{Z}$. Prove that a is even if and only if a^2 is even.

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Assume that a is even.

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We first show that if a is even then a^2 is even.

Assume that a is even. Then $a = 2k$ for some $k \in \mathbb{Z}$.

Hence, $a^2 = (2k)^2 = 4k^2 = 2(2k^2)$

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(other direction on the board)

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- Write: "We use proof by contradiction."
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Task: Prove that q holds.

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- Write: "We use proof by contradiction."
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- Deduce something known to be false (a contradiction).

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- Write: "We have reached a contradiction. Hence, q holds."

Proof by contradiction (example)

Task: Prove that $\sqrt{2}$ is an irrational number.

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Assume that $\sqrt{2}$ is rational. So by definition, this means that $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}^+$.

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