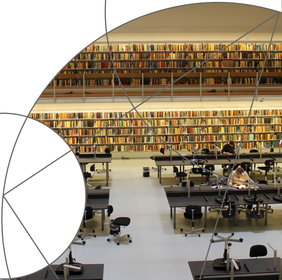
DMA: Lattices and Boolean Algebras

Jurij Volčič
Institut for Matematiske Fag



Outline

- Lattices
  - Definition and examples
  - Hasse diagrams of lattices
  - Isomorphic lattices
- Boolean algebras

Reading: KBR 6.3 and 6.4

Lattices

Def. A poset $(L, \leqslant)$ is a lattice if all of its two-element subsets, $\{a, b\}$, have a least upper bound (LUB) and a greatest lower bound (GLB).

- Take 1-2 minutes to understand the definition. What pre-defined terms does it include? Recall their definitions. How would I check that $(L, \leqslant)$ is a lattice?

Lattices: the join and the meet

Def. A poset $(L, \leqslant)$ is a lattice if all of its two-element subsets, $\{a, b\}$, have a least upper bound (LUB) and a greatest lower bound (GLB).

We denote

- LUB$(\{a, b\})$ by $a \vee b$ and call it the join of a and b.
- GLB$(\{a, b\})$ by $a \wedge b$ and call it the meet of a and b.

Recall: (Uniqueness Thm) If $(A, \leqslant)$ is a poset and $\{a, b\} \in A$ admits a LUB (GLB), then it is unique.

Examples of lattices

Example: Consider the power set $P(\{1,2,\ldots,n\})$ of $\{1,2,\ldots,n\}$. $(P(\{1,2,\ldots,n\}),\subseteq)$ is a poset. It is also a lattice with

$$A \wedge B = A \cap B \qquad A \vee B = A \cup B$$

- Why is $A \cup B$ the LUB of $\{A, B\}$?
- Why is $A \cap B$ the GLB of $\{A, B\}$?

Example. Let $D_m = \{n \in \mathbb{Z}^+ |\ n \mid m\}$.
The poset $(D_m, |)$ is a lattice and

$$a \wedge b = \text{GCD}(a, b) \qquad a \vee b = \text{LCM}(a, b)$$

Recall: We can specify a poset by its Hasse diagram

To get the Hasse diagram of $(A, \leqslant)$:

1. Start with the digraph.
2. Remove loops.
3. Remove the arrows implied by transitivity.
4. Make all the arrows point upwards and replace them with lines.

Exercise: $A = \{1, 2, 3, 4, 6, 24\}$.

- Draw the Hasse diagram of $(A, |)$.
- Is $(A, |)$ a lattice?
- List all pairs $\{a, b\}$ of incomparable elements of A.

Setting the stage for **Boolean algebras**

- Product lattices
- Isomorphic lattices

Product lattices

Thm. Let $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ be lattices and consider the product partial order[1], $\leqslant$, on $L_1 \times L_2$. Then $(L_1 \times L_2, \leqslant)$ is a lattice.

Example

Consider lattice $(\{0, 1\}, \leqslant)$.

(Here, $\leqslant$ is comparison of numbers.)

$$\{0,1\}^n = \overbrace{\{0,1\} \times \cdots \times \{0,1\}}^{n}$$

$(\{0,1\}^n, \leqslant)$ is a lattice with

$$x_1 \dots x_n \wedge y_1 \dots y_n = \min(x_1, y_1) \dots \min(x_n, y_n)$$

$$x_1 \dots x_n \vee y_1 \dots y_n = \max(x_1, y_1) \dots \max(x_n, y_n)$$

[1] Recall: $(a_1, a_2) \leqslant (b_1, b_2) \Leftrightarrow (a_1 \leqslant_1 b_1 \text{ and } a_2 \leqslant_2 b_2)$

The (binary string) lattice $\left(\{0,1\}^n, \leqslant\right)$

Consider lattice $\left(\{0,1\}, \leqslant\right)$.

(Here, $\leqslant$ is comparison of numbers.)

$$\{0,1\}^n = \overbrace{\{0,1\} \times \cdots \times \{0,1\}}^{n}$$

$\left(\{0,1\}^n, \leqslant\right)$ is a lattice with

$$x_1 \ldots x_n \wedge y_1 \ldots y_n = \min(x_1, y_1) \ldots \min(x_n, y_n)$$

$$x_1 \ldots x_n \vee y_1 \ldots y_n = \max(x_1, y_1) \ldots \max(x_n, y_n)$$

Questions

- Find $010 \wedge 110$ and $010 \vee 110$
- How many elements does this lattice have?

Isomorphic lattices

Posets $(A_1, \leqslant_1)$ and $(A_2, \leqslant_2)$ are isomorphic if there is a bijection $f : A_1 \to A_2$ such that for all $a, b \in A_1$ we have

$$a \leqslant_1 b \Leftrightarrow f(a) \leqslant_2 f(b)$$

Def. We say that lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ are isomorphic if they are isomorphic as posets.

An example of isomorphic lattices

Thm. For any $n \in \mathbb{N}$, the lattices $\left(\{0,1\}^n, \leqslant\right)$ and $(P(\{1,\ldots,n\}), \subseteq)$ are isomorphic.

Boolean algebras

Def. A lattice $(L, \leqslant)$ is called a Boolean algebra if it is isomorphic to $\left(\{0,1\}^n, \leqslant\right)$ for some n.

Examples of Boolean algebras

- $\left(\{0,1\}^n, \leqslant\right)$ for any $n \in \mathbb{N}$.
- $(P(\{1,\dots,n\}), \subseteq)$ for any $n \in \mathbb{N}$.
- **Thm.** If $m = p_1 p_2 \dots p_n$, where all the p_i's are distinct primes, then $(D_m, |)$ is a Boolean algebra.

Properties[2] of Boolean algebras

Let $(L, \leqslant)$ be a Boolean algebra. Then we have

- $|L| = 2^n$ for some n.
- L has a greatest element, denoted I, and a least element, denoted 0.
- Every element $x \in L$ has a unique complement (i.e., there is a unique x', s.t. $x \vee x' = I$ and $x \wedge x' = 0$)

Tips for exercises: To show that a poset $(A, \leqslant')$

- is a Boolean algebra, argue that it is isomorphic to $\left(\{0,1\}^n, \leqslant\right)$ for some $n \in \mathbb{N}$ by exhibiting a suitable bijection $f : A \to \{0,1\}^n$.
- is not a Boolean algebra, find a property that it violates.

[2] See p. 247 in [KBR]