DMA: Combinatorics

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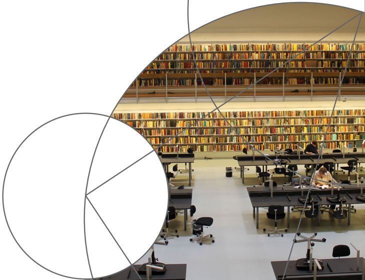

What is combinatorics?

How many (different) ways are there to

1. choose an 8-digit phone number?
2. pick first, second, and third place from 10 contestants?
3. roll three (indistinguishable) dice?
4. choose a hand of 5-cards in a card game?

Plan

1. Multiplication principle
   - Two tasks
   - More tasks
   - Application: Counting subsets
2. Ordered/unordered, with/without repetition
3. Order matters: sequences
   - Sequences (with repetitions)
   - Sequences without repetition
   - Application: Permutations
4. Order doesn't matter
   - Combinations without repeats (subsets)
   - Combinations with repeats (multisets)

Reading: KBR 3.1 and 3.2

Multiplication principle (Thm. 3.3.1)

Example: How many different outfits can we make from a wardrobe with 2 pants (blue, black) and 3 shirts (blue, green, red)?

(blue, blue)	(blue,green)	(blue, red)
(black, blue)	(black,green)	(black, red)

There are $2 \cdot 3 = 6$ choices.

Multiplication Principle: Suppose there are n_1 ways of doing Task T_1 and n_2 ways of doing task T_2. Then there are

$$n_1 n_2$$

ways of performing both tasks together.

We can handle more tasks similarly

Example. How many possibilities are there for a Danish license plate?

$$\boxed{\text{AB 12345}}\ \boxed{\text{AA 10000}},\ \boxed{\text{AA 10001}},\ \cdots,\ \boxed{\text{ZZ 99999}}$$

- 1. letter (24 choices – not I,Q,Æ, Ø, Å)
- 2. letter (23 choices– not I,O,Q,Æ, Ø, Å)
- 1. number (9 choices)
- 2. number (10 choices)
- 3. number (10 choices)
- 4. number (10 choices)
- 5. number (10 choices)

There are $24 \cdot 23 \cdot 9 \cdot 10^4 = 49,680,000$ possibilities

(General) Multiplication Principle

Multiplication Principle: Suppose we have k tasks and there are n_i different ways of doing Task T_i, where $i \in \{1, \ldots, k\}$. Then there are

$$n_1 n_2 \ldots n_k = \prod_{i=1}^{k} n_i$$

ways of performing all k tasks together.

Application: Counting subsets

Question: How many subsets does $A = \{1, 2, 3\}$ have?

Answer: A has 8 subsets:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$$

Thm. Any set with n elements has 2^n different subsets.

Proof. To pick a subset of an n-element set A, for each $x \in A$ we need to choose whether to include it or not. Call this task T_x. There are 2 ways to do task T_x. By Multiplication Principle there are $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ ways of performing all the tasks together. □

Outline

Sequences, sets, multisets

Sequence (string): an ordered collection of elements.
Ex: $(1,2),(2,2,2),(2,1),(1,1,2,3,5,8,13)$
Note: $(1,2) \neq (2,1)$

Set: an unordered collection of elements without repetitions.
Ex: $\{1,2\},\{2,1,3\},\{2,1\},\{1,2,3,5,8,13\}$
Note: $\{1,2\} = \{2,1\}$

Multiset: an unordered collection of possibly repeating elements.
Ex: $[1,2],[2,1,2],[2,1],[1,1,2,3,5,8,13]$
Note: $[1,2] = [2,1] \neq [2,1,2]$

Today, each of the elements will be chosen from some set A.

Ordered/unordered, with/without repetition

How many (different) ways are there to

1. choose an 8-digit phone number?
2. pick first, second, and third place from 10 contestants?
3. roll three (indistinguishable) dice?
4. choose a hand of 5-cards in a card game?

In all the above, we are choosing r elements out of n.

- Does the order matter?
- Are repetitions allowed?

	with repetitions	without repetitions
ordered	1 (sequences)	2 (sequences w/o repetitions)
unordered	3 (multisets)	4 (sets)

Outline

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Order matters (sequences)

Question: How many **sequences of length** r can we form using elements from an n-element set?

Distinguish between cases when elements in the sequence

1. **can repeat**
   Ex. 1: In a phone number digits can repeat.
2. **cannot repeat**
   Ex. 2: The same contestant cannot get the first and the second place.

Thm. 3.1.3: Sequences (with repetitions)

Theorem. Let A be an n-element set. The number of length-r sequences that can be formed from elements of A, allowing repetitions, is n^r.

Ex. 1: How many 8-digit phone numbers are there?

$$00000000, 00000001, 00000002, \ldots, 99999999$$

Order matters, digits can repeat. So we count sequences.

- $A = \{0, \ldots, 9\}$, $n = 10$, and $r = 8$.
- By Thm 3.1.3, there are 10^8 **phone numbers**.

Proof. We need to choose each of the r elements in the sequence (r tasks). There are n ways to do each task. By Multiplication Principle, there are n^r ways of performing all tasks together (choosing a sequence of length r).

Thm. 3.1.4: Sequences without repeats

Theorem. Let A be an n-element set and let $1 \leqslant r \leqslant n$. The number of length-$r$ sequences that can be formed from elements of A, without repetition, is

$$_nP_r = n \cdot (n-1) \cdots (n-r+1).$$

Ex. 2: Choose 1st, 2nd, 3rd place from 10 contestants

$$\{\text{Alice}, \text{Bob}, \dots, \text{John}\}$$

Any choice corresponds to a length-3 sequence. E.g.

(Carol, John, Alice)

No repeats, order matters.

By Thm 3.1.4, the total number of choices is

$$_{10}P_3 = 10 \cdot 9 \cdot 8 = 720.$$

Thm. 3.1.4: Sequences without repeats

Theorem. Let A be an n-element set and let $1 \leqslant r \leqslant n$. The number of length-$r$ sequences that can be formed from elements of A, without repetition, is

$$ {}_nP_r = n \cdot (n-1) \cdots (n-r+1) $$

- ${}_nP_r$ is called the number of permutations of n objects taken r at a time. Note that ${}_nP_r = \frac{n!}{(n-r)!}$

Proof (sketch). We have n choices for the first element in the sequence (Task 1). For the second element, we have $n-1$ choices left (Task 2), and so on. By Multiplication Principle, we have $n \cdot (n-1) \cdots (n-r+1)$ choices in total. $\square$

Summary so far

In how many ways can we choose r elements from an n-element set?

	with repetitions	without repetitions
ordered	n^r (sequences)	${}_nP_r$ (sequences w/o repetitions)
unordered	? (multisets)	? (sets)

where

$${}_nP_r = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Application: Permutations

Question: How many sequences contain each element of A exactly once?

Such a sequence is called a permutation of A.
For example, $(2,3,1)$ and $(3,1,2)$ are permutations of $\{1,2,3\}$.

Rewording of the question: In how many ways can we rearrange (permute) the elements of A?

Answer: By Thm. 3.1.4, we have

$$ {}_nP_n = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n! $$

different ways to permute the elements of A.

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Thm 3.2.1: Combinations without repeats

Theorem. Any set with n elements has

$$_nC_r = \frac{n!}{r!(n-r)!}$$

subsets of size r, where $0 \leqslant r \leqslant n$.

Ex 4: How many different 5-card hands can be dealt?
No repeats, order doesn't matter:

$$\{\heartsuit 8, \clubsuit 5, \spadesuit Q, \diamondsuit 3, \heartsuit 9\} = \{\spadesuit Q, \diamondsuit 3, \heartsuit 9, \heartsuit 8, \clubsuit 5\}$$

So we count subsets of size $r = 5$.
Deck has 52 cards. By Thm 3.2.1, the answer is

$$_{52}C_5 = \frac{52!}{5!47!} = 2,598,960.$$

Thm 3.2.1: Combinations without repeats

Theorem. Any set with n elements has

$$_nC_r = \frac{n!}{r!(n-r)!}$$

subsets of size r, where $0 \leqslant r \leqslant n$.

Notes

- $_nC_r$ is called the number of combinations of n objects taken r at a time
- Many texts use symbol $\binom{n}{r}$ instead of $_nC_r$.

Thm 3.2.1: Combinations without repeats

Theorem. Any set with n elements has

$$_nC_r = \frac{n!}{r!(n-r)!}$$

subsets of size r, where $0 \leqslant r \leqslant n$.

Proof idea: Count in two different ways the number of **length-r sequences made up from non-repeating** elements of $S = \{s_1, \ldots, s_n\}$:

1. By Thm 3.1.4, there are $_nP_r$ such sequences.
2. Task A: Choose an r-element subset of S.
   Task B: Pick a permutation of the chosen r elements.

(finish on the board)

Correspondence between subsets and n-bit strings

Let $A = \{a_1, a_2, \ldots, a_n\}$. To a subset $B \subseteq A$ associate the n-bit string

$$b_1 b_2 \ldots b_n \quad \text{where } b_i = \begin{cases} 1 & \text{if } a_i \in B \\ 0 & \text{otherwise} \end{cases}$$

Example. $A = \{1, 2, 3, 4, 5\}$

Subset	$\{2,3\}$	$\varnothing$	$\{4\}$	$\{1,2,3,4,5\}$
String	01100	00000	00010	11111

Corollary.

$$\left(\begin{array}{c}\text{The number of } n\text{-bit} \\ \text{strings with exactly } k \text{ ones}\end{array}\right) = \left(\begin{array}{c}\text{The number of} \\ \text{size-}k \text{ subsets of } A\end{array}\right) = {}_nC_k$$

Thm 3.2.2: Combinations (with repeats)

Theorem. Let A be an n-element set. There are ${}_{n+r-1}C_r$ multisets of size r made up from elements of A.

Recall: multiset is a collection of unordered, potentially repeating elements.

Ex 3: How many ways can we roll three (indistinguishable) dice?

Two dice can come up the same. Hence repeats are ok. Order doesn't matter. So we count multisets.

$A = \{1, \ldots, 6\}$ and $r = 3$. The total number of ways is:

$$ {}_{6+3-1}C_3 = \frac{8!}{3!5!} = 56 $$

Counting multisets - proof of Thm 3.2.2

Correspondence between multisets of $A = \{a_1, a_2, \ldots, a_n\}$ of size r and $(r+n-1)$-bit strings with exactly r ones.

To a multiset $B = [\overbrace{a_1, \ldots, a_1}^{k_1}, \overbrace{a_2, \ldots, a_1}^{k_2}, \cdots, \overbrace{a_n, \ldots, a_n}^{k_n}]$
assign the string $\overbrace{1\cdots1}^{k_1}\,0\,\overbrace{1\cdots1}^{k_2}\,0\cdots0\,\overbrace{1\cdots1}^{k_n}$

Example. $A = \{1, 2, 3, 4\}$

Multiset	$[1, 1, 2, 4]$	$\varnothing$	$[3, 3, 3]$	$[1, 2, 3, 4]$
String	1101001	000	001110	1010101

$$\left(\begin{array}{c}\text{\# size-}r\\ \text{multisets of } A\end{array}\right) = \left(\begin{array}{c}\text{\# } (n+r-1)\text{-bit strings}\\ \text{with exactly } r \text{ ones}\end{array}\right) = {}_{n+r-1}C_r$$

Summary

In how many ways can we choose r elements from an n-element set

	with repetitions	without repetitions
ordered	n^r	${}_nP_r$
unordered	${}_{n+r-1}C_r$	${}_nC_r$

where

$$ {}_nP_r = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!} $$

and

$$ {}_nC_r = \binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 1} = \frac{n!}{r!(n-r)!} $$

Permutations with indistinguishable objects

Q1: How many “words” can be made from the letters of “BOK”?

Ans1: $3! = 6$

Q2: How about “BOOO”? Is it 24?

Ans2: $\frac{4!}{3!} = 4$ BOOO,BOOO,BOOO, BOOO, BOOO, BOOO

Thm. The number of distinguishable permutations that can be formed from n objects where the 1st object appears k_1 times, the 2nd object k_2 times and so on, is

$$\frac{n!}{k_1!\,k_2!\ldots k_t!} \quad \text{where } k_1 + \ldots + k_t = n$$

- **Q3:** How many “words” can be formed from the letters of “BOOKKEEPER”? **Ans3:** $\frac{10!}{2!\,2!\,3!}$

Exercise

An urn contains **8 red** and **7 black** different balls. How many ways are there to choose 5 balls so that at least 2 are red.

Solution 1

- X_i = # of ways to choose 5 balls so that exactly i are red.
- By Multiplication Principle, $X_i = {}_8C_i \times {}_7C_{5-i}$
- The answer to the exercise is
  $${}_{15}C_5 - X_1 - X_0 = {}_{15}C_5 - {}_8C_1 \times {}_7C_4 - {}_8C_0 \times {}_7C_5 = 2702$$

Solution 2

Task 1: Choose 2 red balls (${}_8C_2 = 28$ ways)

Task 2: Choose 3 balls from the remaining 13 balls (${}_{13}C_3 = 286$ ways)

- By Multiplication Principle, we have $28 \cdot 286 = 8008$ ways