DMA 2021

– Notes week 5 –

Preliminaries

Let us try to prove that for every $n \in \mathbb{N}$, we have that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

Proving this gives a summation formula for the series of the first n square numbers.

The naive approach is to start from the beginning of the series. Explicit calculation gives us that

$$\begin{aligned} 1^2 = 1 = \frac{6}{6} &= \frac{1(1+1)(2+1)}{6}, \\ 1^2 + 2^2 = 5 = \frac{30}{6} &= \frac{2(2+1)(4+1)}{6}, \\ 1^2 + 2^2 + 3^2 = 14 = \frac{84}{6} &= \frac{3(3+1)(6+1)}{6}, \end{aligned}$$

It is tempting to continue in this manner, calculate a couple of statement more, and then just write

$$\vdots$$
$$\textit{etc.}$$

and say that the statment has been proven. This is, however, not a mathematical proof and is actually bordeline false as illustrated by the following example.

Example 1. *Consider the following two sequences explicitly defined by*

$$a_n = \left\lfloor \frac{2n}{\ln 2} \right\rfloor$$

and

$$b_n = \left\lceil \frac{2}{2^{1/n} - 1} \right\rceil .$$

Both of them start as

$$2, 5, 8, 11, 14, 17, 20, 23, 25, 28, 31, 34, 37, 40, 43, 46, 49, 51, 54, 57,$$
$$60, 63, 66, 69, 72, 75, 77, 80, 83, 86, 89, 92, 95, 98, 100, 103, 106,$$
$$109, 112, 115, 118, 121, 124, 126, 129, 132, 135, 138, 141, 144$$

It is actually easy to check on a computer that the two sequences have identical elements for the first couple of million values of n. It is therefore tempting to conclude that $a_n = b_n$ for all n but it turns out that

$$a_{777451915729368} = 2243252046704766 \neq 2243252046704767 = b_{777451915729368}$$

The above example illustrates that even if we used a computer to check that the formula (1) was true for the all n between 1 and 100000, we would still not be able to answer the following two questions:

(i) Is the formula valid for *all* $n \in \mathbb{N}$?

(ii) Why is it true?

These two questions are highly connected and it is difficult to answer (i) without also answering (ii).

We will therefore try another approach. Let us define a sequence (a_n) by

$$a_n = \frac{n(n+1)(2n+1)}{6}$$

(note this is the right-hand side of the formula (1)) and try to prove the statement

$$a_n + (n+1)^2 = a_{n+1} \tag{2}$$

instead. This turns out to be relatively easy since by lifting the parenthesis, we get

$$\begin{aligned} a_n + (n+1)^2 &= \frac{1}{6}\left[n(n+1)(2n+1) + 6(n+1)^2\right] \\ &= \frac{1}{6}\left[2n^3 + 9n^2 + 13n + 6\right] \end{aligned}$$

and

$$\begin{aligned} a_{n+1} &= \frac{1}{6}\left((n+1)(n+1+1)(2(n+1)+1)\right) \\ &= \frac{1}{6}\left[2n^3 + 9n^2 + 13n + 6\right] \end{aligned}$$

The formula (2) implies that **if** we know that (1) applies to some particular value of n then we can conclude that it also applies to $n+1$ since

$$\begin{aligned}
1^2 + \cdots + n^2 + (n+1)^2 &= a_n + (n+1)^2 && \textit{[Formula (1) is valid for } n\textit{]} \\
&= a_{n+1} && \textit{[Formula (2)]} \\
&= \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6} && \textit{[Def. of } a_n\textit{]}
\end{aligned}$$

We have already shown that the formula is valid for $n = 3$. The above argument then shows that it is also valid for $n = 4$. This means that it is also valid for $n = 5$, which means that it is valid for $n = 6$ ... *etc.*

We are now getting close to answering questions (i) and (ii). But our argument still contains a sequence of dots and the statement "et cetera (etc.)". Mathematical induction, which we will describe below, is the standard method to deal with "... *etc.*".

Note how the statement (2) turned out to be the *key* for finding a proof. It is worth noticing how one can get the idea to construct such a statement by *analyzing* the problem. Because **if** (1) was true then

$$a_{n+1} = 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = a_n + (n+1)^2.$$

In more advanced proofs by induction, a significant part of the task is to find statements, which can act as the key to a proof in the same way as (2).

Mathematical induction

Let us now formalize the above ideas as *the principle of mathematical induction.*

> If a collection of statements $P(n)$, where n takes the values from $\{n_0, n_0+1, \dots\}$, satisfy that
>
> $$P(n_0) \text{ is true} \tag{S}$$
>
> and that for all $n \geq n_0$, we have that
>
> $$\text{If } P(n) \text{ is true, then } P(n+1) \text{ is true} \tag{T}$$
>
> then $P(n)$ is true for all $n \geq n_0$.

We refer to the statement (S) as *the basis step* and the statement (T) as *the induction step*.

Let us complete the argument for (1) formally using this framework. We let $P(n)$ be the statement

$$P(n): \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

and we wish to show that $P(n)$ is true for all $n \geq n_0 = 1$.
Basis step: We need to show that $P(1)$ is true, i.e., that

$$\sum_{k=1}^{1} k^2 = \frac{1(1+1)(2 \cdot 1+1)}{6}.$$

This follow trivially since both sides of the equality evaluate to 1.
Induction step: We need to show that if $P(n)$ is true then $P(n+1)$ is also true. We can therefore assume that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We see that

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{1}{6}\left[n(n+1)(2n+1) + 6(n+1)^2\right] \\
&= \frac{1}{6}\left[2n^3 + 9n^2 + 13n + 6\right] \\
&= \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6},
\end{aligned}$$

which means that $P(n+1)$ is true.
Conclusion: Using the principle of mathematical induction we see that $P(n)$ is true for all $n \geq n_0$.

In some situations, it is necessary to use the *strong* principle of induction where the induction step does not only assume that $P(n)$ is true but that all $P(k)$, with $k \leq n$, are true. Formally:

If a collection of statements $P(n)$ indexed from the set $\{n_0, n_0+1, \dots\}$ satisfy that

$$P(n_0) \text{ is true} \tag{S}$$

and that for all $n \geq n_0$, we have that

$$\text{If } P(n_0), \dots, P(n) \text{ are all true, then } P(n+1) \text{ is true} \tag{T'}$$

then $P(n)$ is true for all $n \geq n_0$.

Depending on how the induction step is carried out in some cases as part of the basis step it may be necessary to check extra statements in addition to $P(n_0)$. One case where this happens is if we need to make use of more than one of the previous statements $P(n_0), \dots, P(n)$ in order to complete the induction step.

We will conclude with an example of how to use the principle of strong induction. We define a sequence (b_n) recursively by setting $b_1 = 0$ and

$$b_n = b_{\lfloor n/2 \rfloor} + 1$$

for all $n > 1$. We want to show that

$$P(n): \qquad b_n = \lfloor \log_2(n) \rfloor.$$

Basis: We need to show that $P(1)$ is true, i.e,

$$b_1 = \lfloor \log_2(1) \rfloor,$$

which follows because both sides of the equality are 0.

Induction step: We use the principle of strong induction and thus, we have to show that if all statements $P(1), P(2), \dots P(n)$ are true, then $P(n+1)$ is also true. Let us set $m = \lfloor (n+1)/2 \rfloor$. Then we have $b_{n+1} = b_m + 1$. We will use that $P(m)$ is true, which follows from our assumption since $m = \lfloor (n+1)/2 \rfloor \leq n$.

If n is odd then $n+1$ is even and we have that $m = (n+1)/2$ and

$$\begin{aligned}
b_{n+1} &= b_m + 1 \\
&= \lfloor \log_2(m) \rfloor + 1 && \textit{[Here we use that } P(m) \textit{ is true!]} \\
&= \lfloor \log_2(m) + 1 \rfloor \\
&= \lfloor \log_2(m) + \log_2(2) \rfloor \\
&= \lfloor \log_2(2m) \rfloor \\
&= \lfloor \log_2(n+1) \rfloor.
\end{aligned}$$

It follows that $P(n+1)$ is true.

If n is even then $n+1$ is odd and we have that $m = n/2$ and

$$\begin{aligned} b_{n+1} &= b_m + 1 \\ &= \lfloor \log_2(m) \rfloor + 1 && \textit{[Here we use that } P(m) \textit{ is true!]} \\ &= \lfloor \log_2(m) + 1 \rfloor \\ &= \lfloor \log_2(m) + \log_2(2) \rfloor \\ &= \lfloor \log_2(2m) \rfloor \\ &= \lfloor \log_2(n) \rfloor. \end{aligned}$$

This is not yet the desired statement $P(n+1)$. However, note that for all n between the even number 2^k and the odd number $2^{k+1}-1$, we have that $\lfloor \log_2(n) \rfloor = k$. Hence, for all even $n > 0$, we have $\lfloor \log_2(n) \rfloor = \lfloor \log_2(n+1) \rfloor$. So we conclude that $b_{n+1} = \lfloor \log_2(n+1) \rfloor$ and thus $P(n+1)$ is true in this case too.

Conclusion: Using the principle of strong induction we see that $P(n)$ is true for all $n \geq 1$.