Let $f(x) = x$ and $g(x) = x^{1+\sin(\pi x)}$.

Observe: if x is even then $\sin(\pi x) = 1$ and so $g(x) = x^2$;

if x is odd then $\sin(\pi x) = -1$ and so $g(x) = 1$.

(1) Suppose $f = O(g)$. So there are $c, x_0 > 0$ such that $f(x) \leq c \cdot g(x)$ for all $x > x_0$.

However, if x is an odd number larger than c and x_0, then $f(x) = x > c = c \cdot 1 = c \cdot g(x)$, a contradiction! So f is not $O(g)$.

(2) Suppose $g = O(f)$. Then there are $c, x_0 > 0$ such that $g(x) \leq c \cdot f(x)$ for all $x > x_0$.

So for every even x greater than x_0,

$$x^2 = g(x) \leq c \cdot x$$

On the other hand, $x = o(x^2)$.

This in particular means that for the constant $\frac{1}{c}$ there exists $\tilde{x}_0 > 0$ such that

$$x < \frac{1}{c} x^2 \quad \text{for all } x > \tilde{x}_0.$$

But this contradicts the previous statement for any even x larger than x_0 and $\tilde{x}_0$!

So g is not $O(f)$.

$f = \Theta(g)$:

def (1) : $f = O(g)$ & $g = O(f)$

def (2) : there are $c_1, c_2, x_0 > 0$ such that

$c_1 g(x) \le f(x) \le c_2 g(x)$ for all $x > x_0$.

(1) $\Rightarrow$ (2) :

$f = O(g)$: $\exists\ c', x_0'$: $f(x) \le c' g(x)$ $\forall x > x_0'$

$g = O(f)$: $\exists\ c'', x_0''$: $g(x) \le c'' f(x)$ $\forall x > x_0''$.

Take $c_1 = \frac{1}{c''}$, $c_2 = c'$ and $x_0 = \max(x_0', x_0'')$.

Then (2) holds.

(2) $\Rightarrow$ (1) : if (2) holds, then

$f(x) \le c_2\ g(x)$ $\forall x > x_0$, so $f = O(g)$,

and

$g(x) \le \frac{1}{c_1} f(x)$ $\forall x > x_0$, so $g = O(f)$.

Multiplicativity: if f, g, h are asymptotically positive, then $f = O(g)$ implies $fh = O(gh)$.

Suppose $f = O(g)$. So there exist $c, x_0' > 0$ such that $f(x) \le cg(x)$ for all $x > x_0'$.

Since h is asympt positive, there is $x_0'' > 0$ such that $h(x) > 0$ for all $x > x_0''$.

Let $x_0 = \max(x_0', x_0'')$. Then for all $x > x_0$: $f(x)h(x) \le cg(x)h(x)$.

Therefore $fh = O(gh)$.

Let $f(x) = 3^x \cdot x$ and $g(x) = 2^x(x^2 + 42)$.

Let us show that $g = o(f)$.

First, R2 says $x^2 + 42 = \sim(x^2)$,
so R11 implies $2^x(x^2+42) = \sim(2^x x^2)$.

R6: $x = o\left(\left(\frac{3}{2}\right)^x\right)$ (since $\frac{3}{2} > 1$)

R11: $2^x x^2 = 2^x x \cdot x = o\left(2^x x \left(\frac{3}{2}\right)^x\right)$

$= o\left(3^x \cdot x\right)$

R10: (putting together): $2^x(x^2+42) = o(3^x \cdot x)$.

$\sum_{k=1}^{n} \frac{1}{k}$ is $\Theta(\log n)$.

Proof: it suffices to consider $\log_2$.

Let $n \geq 1$. Then there is $m \in \mathbb{N}$ such that $2^m \leq n < 2^{m+1}$.

Since $\log_2$ is monotone: $m \leq \log_2 n < m+1$.

Denote $f(n) = \sum_{k=1}^{n} \frac{1}{k}$. f is also clearly monotone, so $f(2^m) \leq f(n) < f(2^{m+1})$.

$$\frac{1}{2} \leq \overbrace{\qquad}^{} \leq 1$$

$$f(2^m) = \sum_{k=1}^{2^m} \frac{1}{k} = 1 + \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4} \leq \quad + \left(\frac{1}{3} + \frac{1}{4}\right) \leq \frac{1}{2} + \frac{1}{2} = 1$$

$$\frac{1}{2} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \leq \quad + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

$$+ \cdots$$

$$\frac{1}{2} \leq \quad + \left(\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}\right) \leq 1$$

$$\Rightarrow$$

$$1 + \underbrace{\left(\frac{1}{2} + \cdots + \frac{1}{2}\right)}_{m} \leq f(2^m) \leq 1 + \overbrace{1 + \cdots + 1}^{m}.$$

$$\Rightarrow 1 + \frac{m}{2} \leq f(2^m) \leq f(n) < f(2^{m+1}) \leq 1 + m + 1$$

$$\Rightarrow \frac{1}{2}\log_2 n + \frac{1}{2} = 1 + \frac{1}{2}(\log_2 n - 1) \leq 1 + \frac{m}{2} \leq f(n)$$

and $f(n) \leq 2 + m \leq \log_2 n + 2$

$$\Rightarrow f(n) = \Theta(\log n)$$