Permutations (again)

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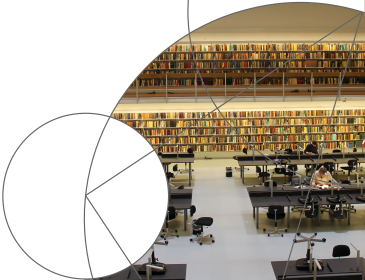


Outline

- Recap of functions and related notions
- Permutation functions
  - Definition and examples
  - Composing permutation
  - Inverse permutation
  - Cycles

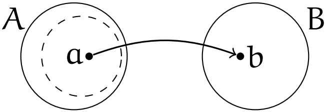
Reading: KBR 5.1 (repetition) and KBR 5.4

Recap: Functions

- Composition of functions
- Inverse of a function
- Bijections

Functions

A **function** $f\colon A \to B$, assigns a unique element $f(a) \in B$ to each $a \in \mathrm{Dom}(f) \subseteq A$



$b = f(a)$

Notes

- Today we focus on everywhere-defined functions (i.e. $\mathrm{Dom}(f) = A$).
- We can view a function $f\colon A \to B$ as a relation

$$\{(a, f(a)) \mid a \in A\} \subseteq A \times B$$

Specifying a function $f\colon A \to B$

Using a formula or a rule:

- $f\colon \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2$
- $g\colon \{1,2,3,4\} \to \{0,1\}$ and $g(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$

As a subset of $A \times B$

- $f = \{(x, x^2) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$
- $g = \{(1,1),(2,0),(3,1),(4,0)\} \subseteq \{1,2,3,4\} \times \{0,1\}$

Composition of functions

Take (everywhere-defined) functions $f\colon A \to B$ and $g\colon B \to C$.

Their composition, $g \circ f$, is a function $t\colon A \to C$, where

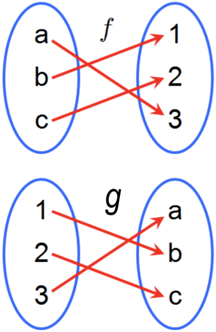
$$t(a) = g(f(a))$$

We write $t = g \circ f$ and $t(x) = (g \circ f)(x)$.

Example. Let $f, g\colon \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$, $g(y) = y + 1$

$$(f \circ g)(z) = (z+1)^2 = z^2 + 2z + 1$$
$$(g \circ f)(z) = z^1 + 1$$

The inverse function



The inverse function (call it g) “reverses” the function f:

If $a \overset{f}{\mapsto} b$ then $b \overset{g}{\mapsto} a$.

Def. Let f be a function with $\mathrm{Dom}(f) = A$ and $\mathrm{Ran}(f) = B$. A function $g : B \to A$ is the inverse of f if for all $a \in A$

$$g(f(a)) = a$$

i.e. $g \circ f$ is the identity function on A.

The inverse function: Example

Def. Let f be a function with $\mathrm{Dom}(f) = A$ and $\mathrm{Ran}(f) = B$. A function $g : B \to A$ is the inverse of f if for all $a \in A$

$$g(f(a)) = a$$

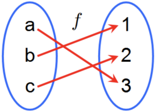
Example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x + 5$. Its inverse function is $g(y) = (y - 5)/2$ since

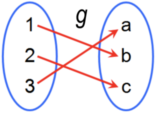
$$g(f(x)) = \frac{f(x) - 5}{2} = \frac{2x}{2} = x$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. $\mathrm{Ran}(f) = \mathbb{R}^+ \cup \{0\}$.
Q: What is the inverse function of f? Is it $g(y) = \sqrt{y}$?
No because, for example,

$$g(f(-3)) = \sqrt{f(x)} = \sqrt{9} = 3 \neq -3$$

The inverse function (notes)





- The inverse function[1] of f doesn't always exist!
- If it exists, the inverse function is unique and we say that f is invertible.
- KBR: f is invertible, if the inverse relation of f is a function.
- **Thm.** Function f is invertible $\Leftrightarrow$ f is onto & one-to-one[2].

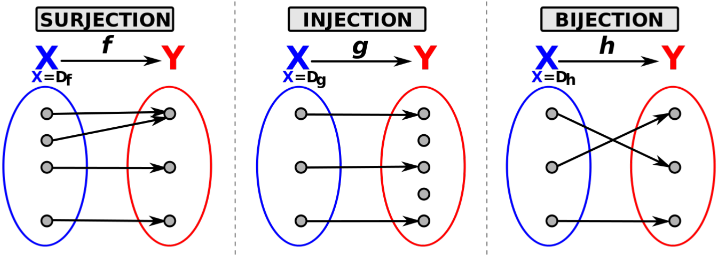
[1] In contrast, the inverse relation of f always exists but might not be a function.

[2] $\mathrm{Ran}(f) = B$ and $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$.

Special classes of functions

Def. A function $f\colon X \to Y$ is

- surjection (or "onto") if $\mathrm{Ran}(f) = Y$
- injection (or "one-to-one") if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- bijection if it is surjective, injective, and everywhere defined.



Test yourself

You should be able to

- Recognize whether a relation $R \subseteq A \times B$ is a function.
- Compose functions.
- Determine whether a function is invertible and if so find its inverse.
- Recognize injective, surjective, and bijective functions

Permutations

Permutations

Def. A function $p\colon A \to A$ is a permutation of A[3] if it is a bijection.

We can specify permutation p of $A = \{a_1, \dots, a_n\}$ using a

- **two-line** notation as $\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ p(a_1) & p(a_2) & \dots & p(a_n) \end{pmatrix}$
- **one-line** notation as $p(a_1)\, p(a_2) \dots p(a_n)$

Examples of permutations of $A = \{1,2,3,4,5\}$

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

Recall: There are $n!$ permutations of an n-element set.

[3] We assume A is finite. Usually $A = \{1, \dots, n\}$

Composition (product) of permutations

Example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} =$$

Exercise: Verify that p_2 is the inverse function of p_1.

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

Cycles

Def. A permutation p of A is a cycle of length $r \geqslant 2$ if there are r distinct elements $a_1, \ldots, a_r \in A$ such that

$$p(a_1) = a_2,\ p(a_2) = a_3,\ \ldots,\ p(a_r) = a_1$$

and for all $a \in A - \{a_1, \ldots, a_r\}$ we have $p(a) = a$.
Notation: $p = (a_1, \ldots, a_r)$

Ex. The following permutation is a cycle of length 3:

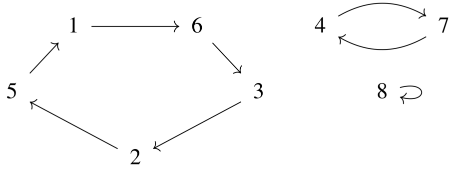
$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

Task: Express p in the "cycle format" as (a_1, a_2, a_3)

Disjoint cycles

Consider $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 2 & 7 & 1 & 3 & 4 & 8 \end{pmatrix}$

- Draw numbers 1 to n and arrows $i \to p(i)$



- Every number has exactly one incoming and one outgoing arrow. So the drawing splits into (disjoint) cycles.
- We can write $p = (1, 6, 3, 2, 5) \circ (4, 7)$
- Let $c_{n,k}$ be the number of permutations of $\{1, \ldots, n\}$ whose drawing has exactly k cycles (including the loops).

A recursive formula for $c_{n,k}$

What permutations can be formed by inserting $n = 6$ into $(1,4,2)(3,5)$ (a permutation of size $n-1$)?

- **Case 1**: Insert 6 into an existing cycle in one of $n-1 = 5$ ways:
$$(1,6,4,2)(3,5)$$
$$(1,4,6,2)(3,5)$$
$$(1,4,2,6)(3,5) = (6,1,4,2)(3,5)$$
$$(1,4,2)(3,6,5)$$
$$(1,4,2)(3,5,6) = (1,4,2)(6,3,5)$$
- **Case 2**: Insert 6 as a loop: $(1,4,2)(3,5)(6)$

In general: To obtain k cycles, insert n into a permutation of $n-1$ with k cycles (if added to an existing cycle) or $k-1$ cycles (if added as a new loop).

A recursive formula for $c_{n,k}$

Pick a permutation of size n with k cycles by inserting n into a permutation of size $n-1$.

- **Case 1**: n is not alone in a cycle
  Pick permutation of size $n-1$ with k cycles ($c_{n-1,k}$ ways)
  Insert n into an existing cycle ($n-1$ ways)
  Subtotal: $(n-1)c_{n-1,k}$
- **Case 2**: n is a "loop"
  Pick permutation of size $n-1$ with $k-1$ cycles ($c_{n-1,k-1}$ ways)
  and add a new loop (n) (one way)
  Subtotal: $c_{n-1,k-1}$

Overall

$$c_{n,k} = (n-1)c_{n-1,k} + c_{n-1,k-1}$$