DMA: Matrices

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Announcements

- No lectures or exercise classes on Friday.
  (See Absalon!)

Plan for today

- Matrices: definition and examples
- Matrix operations (addition, multiplication)
- Boolean matrices and Boolean matrix operations (join, meet, Boolean product)
- Additive and multiplicative inverses
- Tuesday: Applications of matrices

Reading: KBR 1.5

Matrices: Definition

Def. An $m \times n$ matrix is a rectangular array of elements (numbers or symbolic expressions):

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

with m rows and n columns.

Examples:

$$A = \begin{bmatrix} 0 & 1/2 & 7 \\ -\sqrt{2} & 4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2x+3 & 1 \\ -5y & x \end{bmatrix}$$

A is a 2×3 matrix. B is a 2×2 matrix.

Matrices: Notes

Consider an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

- $m, n \in \mathbb{Z}^+$
- a_{ij} is the (i, j)-element/entry of A
- The first index refers to rows and the second to columns.
- $m \times n$ is the size or dimension of A.
- Notation $A = [a_{ij}]$: matrix with elements a_{ij}.
- If $m = n$, then $a_{11}, a_{22}, \ldots, a_{nn}$ form the main diagonal of A (a_{ij} with $i \neq j$ are called the off-diagonal elements).

Special classes of matrices

- **Diagonal matrices**: $n \times n$ matrices with 0 off-diagonal entries.
  - **Identity matrix**, I, has all diagonal entries equal to 1.
    (I_n is the $n \times n$ identity matrix)
  - **Zero matrix**, 0, has all entries equal to zero.
    (0_n is the $n \times n$ zero matrix)

- **vectors**: $m \times n$ matrices where $m = 1$ or $n = 1$.
  - **Row vectors** have $m = 1$.
  - **Column vectors** have $n = 1$.

Examples: $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & & 3+x \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Addition of matrices

Def. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then

$$A + B = [a_{ij} + b_{ij}].$$

Warning: Only matrices of the same dimension can be added!

Example:

$$\begin{bmatrix} 0 & 1 & 7 \\ -2 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 1+2 & 7+3 \\ -2-1 & 4-2 & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 10 \\ -3 & 2 & -2 \end{bmatrix}$$

Properties of matrix addition

Thm. Let A, B, C be matrices of the same size. Then

- $A + B = B + A$ (commutativity of $+$)
- $(A + B) + C = A + (B + C)$ (associativity of $+$)
- $A + 0 = A$ (0 is the additive identity)

Note: These are similar to the properties for addition of numbers.
(Things will change for multiplication)

Matrix multiplication

Def. If $A = [a_{ij}]$ is $m \times p$ and $B = [b_{ij}]$ is $p \times n$ then $AB = C$ where $C = [c_{ij}]$ is a $m \times n$ matrix and

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

Example:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 + 1\cdot 4 & 0\cdot 2 + 1\cdot 5 & 0\cdot 3 + 1\cdot 6 \\ 2\cdot 1 + 3\cdot 4 & 2\cdot 2 + 3\cdot 5 & 2\cdot 3 + 3\cdot 6 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 5 & 6 \\ 14 & 19 & 24 \end{bmatrix}$$

Exercise

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

1. Calculate AB, BA and AC

The order matters!

- For matrix multiplication the order matters!
- Unlike matrix addition or multiplication of numbers, matrix multiplication is not commutative.

Properties of matrix multiplication

Let A, B, C be matrices of compatible dimensions. Then

1. $(AB)C = A(BC)$ (associativity)
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $AI_n = I_mA = A$ for any $m \times n$ matrix A

Question: What does "compatible dimensions" mean for item 2?

Boolean matrices and their operations

Boolean matrices

Def. We say that $A = [a_{ij}]$ is a Boolean matrix if all the $a_{ij} \in \{0, 1\}$.

Example:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Boolean operations: OR

For $a, b \in \{0, 1\}$ let

$$a \vee b = \begin{cases} 1 & \text{if } a = 1 \text{ OR } b = 1 \\ 0 & \text{otherwise} \end{cases}$$

Def. If A, B are Boolean matrices of the same size, then

$$A \vee B = [a_{ij} \vee b_{ij}]$$

Example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Boolean operations: AND

For $a, b \in \{0, 1\}$ let

$$a \wedge b = \begin{cases} 1 & \text{if } a = 1 \text{ AND } b = 1 \\ 0 & \text{otherwise} \end{cases}$$

Def. If A, B are Boolean matrices of the same size, then

$$A \wedge B = [a_{ij} \wedge b_{ij}]$$

Example

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Boolean operations: Boolean product $\odot$

Def. If $A = [a_{ij}]$ is $m \times p$ and $B = [b_{ij}]$ is $p \times n$ then $A \odot B = C$ where $C = [c_{ij}]$ is a $m \times n$ matrix and

$$c_{ij} = \begin{cases} 1 & \text{if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for some } k \in \{1, \dots, p\} \\ 0 & \text{otherwise} \end{cases}$$

The (usual) matrix product versus Boolean product

$$AB = [a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}]$$
$$A \odot B = [(a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ip} \wedge b_{pj})]$$

Multiplication by a scalar and inverses

Multiplication by a Scalar

Def. For $c \in \mathbb{R}$ and a matrix $A = [a_{ij}]$, we define

$$cA = Ac = [ca_{ij}].$$

Example

$$2\begin{bmatrix} 2 & 1 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & -10 \end{bmatrix}$$

Additive inverses

- For $x \in \mathbb{R}$ its additive inverse is $-x$. This is because

$$x + (-x) = 0$$

and 0 is the additive identity.

- For any $m \times n$ matrix A, its additive inverse is the $m \times n$ matrix $A' = [-a_{ij}]$, since

$$A + A' = [a_{ij} - a_{ij}] = 0$$

Example

$$A = \begin{bmatrix} 2x & 1 & 0 \\ -1 & -5 & 0 \end{bmatrix} \quad A' = \begin{bmatrix} -2x & -1 & 0 \\ 1 & 5 & 0 \end{bmatrix}$$

Multiplicative inverses

- For $x \in \mathbb{R}$ its multiplicative inverse is $x' \in \mathbb{R}$ such that

$$xx' = 1$$

- **Fact:** $x \in \mathbb{R}$ has a multiplicative inverse $\Leftrightarrow x \neq 0$.
- For an $n \times n$ matrix A, its multiplicative inverse, denoted by A^{-1}, is the $n \times n$ matrix such that

$$AA^{-1} = A^{-1}A = I_n$$

Warning: Not every nonzero $n \times n$ matrix has an inverse!

Transpose of a matrix

Def. The transpose of an $m \times n$ matrix $A = [a_{ij}]$, is the $n \times m$ matrix, denoted A^T, whose (ij)-entry is given by a_{ji}.

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Def. We say that a matrix A is symmetric if $A = A^T$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is not symmetric but $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ is.

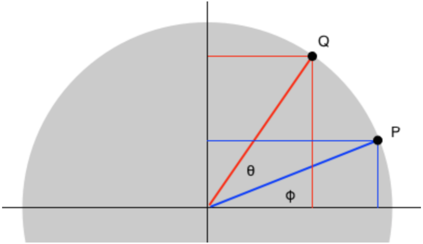
Summary

- An $m \times n$ matrix is an array $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$
- We can add and multiply matrices.
- Matrix multiplication is not commutative!
- Some matrices have (multiplicative) inverses: $AA^{-1} = I = A^{-1}A$.

Example applications

Application 1: computer graphics

- We can use matrix multiplication to represent geometric transformations.
- Simple example: Rotate a point $p = (x, y)$ by angle θ counter-clockwise around the origin.
- Task: determine $q = (x', y')$ from $p = (x, y)$ and θ.



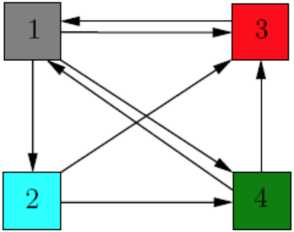
Solution:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Application 2: PageRank algorithm (Page-Brin)

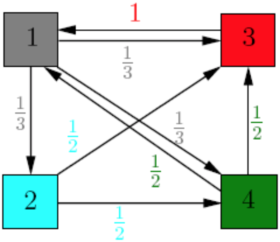
- A good search engine orders the results by relevance.
- Idea: The importance of a site is depends on the sites which link to it.
  - The more links point to a site, the more important it is
  - For a given link, its importance depends on the importance of its source site

Example: 4 sites. If site i links to j, we draw an arrow from i to j.



Weighted links

A site transfers its importance uniformly to all the sites it links to:



- n_i = # of outgoing arrows (links) from i
- Describe the weighted diagram with $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} \frac{1}{n_j} & \text{if } j \text{ links to } i \\ 0 & \text{otherwise} \end{cases}$$

The first algorithm

Goal: Rank the sites according to their importance.

Use $v = \begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix}$ where i_ℓ represents the importance of site ℓ.

1. Set $v_0 = \frac{1}{\text{\# of sites}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$
2. Updated guess: $v_1 = Av_0$
3. Given i-th guess v_i, let $v_{i+1} = Av_i$

After 8 updates

$$\mathbf{v} = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}, \quad \mathbf{Av} = \begin{pmatrix} 0.37 \\ 0.08 \\ 0.33 \\ 0.20 \end{pmatrix}, \quad \mathbf{A}^2\mathbf{v} = \mathbf{A}\,(\mathbf{Av}) = \mathbf{A}\begin{pmatrix} 0.37 \\ 0.08 \\ 0.33 \\ 0.20 \end{pmatrix} = \begin{pmatrix} 0.43 \\ 0.12 \\ 0.27 \\ 0.16 \end{pmatrix}$$

$$\mathbf{A}^3\mathbf{v} = \begin{pmatrix} 0.35 \\ 0.14 \\ 0.29 \\ 0.20 \end{pmatrix}, \quad \mathbf{A}^4\mathbf{v} = \begin{pmatrix} 0.39 \\ 0.11 \\ 0.29 \\ 0.19 \end{pmatrix}, \quad \mathbf{A}^5\mathbf{v} = \begin{pmatrix} 0.39 \\ 0.13 \\ 0.28 \\ 0.19 \end{pmatrix}$$

$$\mathbf{A}^6\mathbf{v} = \begin{pmatrix} 0.38 \\ 0.13 \\ 0.29 \\ 0.19 \end{pmatrix}, \quad \mathbf{A}^7\mathbf{v} = \begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}, \quad \mathbf{A}^8\mathbf{v} = \begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}$$

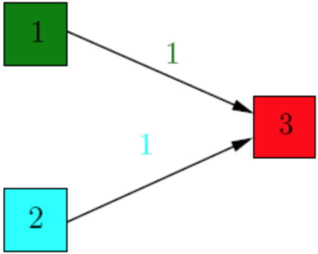
Idea (Wish): The process should stabilize as the vectors get closer and closer to v such that

$$Au = u$$

and the entries of u are all positive and add up to 1.

Exercise

Consider 3 sites described by



1. Which site do you think is the most important?
2. Find the corresponding A-matrix.
3. Compute v_0, v_1, v_2, v_3.

PageRank algorithm

Def. Given a damping factor $p \in (0, 1)$, the PageRank or (Google) matrix is given by

$$M = (1 - p)A + pB,$$

where $B = [b_{ij}]$, $b_{ij} = \frac{1}{n}$ and n is the number of sites.

1. Set $v_0 = \frac{1}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$
2. Updated guess: $v_1 = Mv_0$
3. Given i-th guess v_i, let $v_{i+1} = Mv_i$