DMA 2021

– Notes for Week 7 –

1 Asymptotic growth of functions

1.1 Collection of definitions

In this section we collect the definitions regarding asymptotic growth of functions. Further explanations are available in the [CLRS] book and during lectures.

Definition 1. We say that a function $f : \mathbb{R}^+ \to \mathbb{R}$ is *asymptotically positive* if there exists $x_0 \in \mathbb{R}^+$ such that $0 < f(x)$ for all $x \geq x_0$.

We will also apply the above definition for functions that are defined on some subset of the positive reals. A common choice of such a subset will be positive integers $\mathbb{Z}^+$ or natural numbers $\mathbb{N}$.

Definition 2 (Asymptotic notation)**.** Let f and g be asymptotically positive functions.

- We say that $f(x)$ *is* $O(g(x))$ if there exists a constant $c > 0$ and x_0 such that
$$f(x) \leq cg(x)$$
for all $x \geq x_0$.
Think of this as **"g grows at least as fast as f asymptotically"**.

- We say that $f(x)$ *is* $\Omega(g(x))$ if $g(x)$ is $O(f(x))$.
Think of this as **"g grows no faster than f asymptotically"**.

- We say that $f(x)$ *is* $\Theta(g(x))$ if f is $O(g(x))$ and $g(x)$ is $O(f(x))$.
Think of this as **"g and f grow at the same rate asymptotically"**.

- We say that $f(x)$ *is* $o(g(x))$ if for any constant $c > 0$ we can find x_0 such that
$$f(x) < cg(x)$$
for all $x \geq x_0$.
Think of this as **"g grows (strictly) faster than f asymptotically"**.

One can use the definition of big-O to show that big-Θ can equivalently be defined in the following manner.

Definition 3 (Second definition of big-Θ)**.** Let f and g be asymptotically positive functions. We say that $f(x)$ *is* $\Theta(g(x))$ if there exist constants $c_1, c_2 > 0$ and x_0 such that

$$c_1 g(x) \leq f(x) \leq c_2 g(x)$$

for all $x \geq x_0$.

It can be useful to *informally*(!) think of the above defined asymptotic notions as being analogous to comparison of numbers. Specifically,

$$\begin{array}{lll} f(x) \text{ is } O(g(x)) & \text{is like} & \text{“}f \leq g\text{”} \\ f(x) \text{ is } \Omega(g(x)) & \text{is like} & \text{“}g \leq f\text{”} \\ f(x) \text{ is } o(g(x)) & \text{is like} & \text{“}f < g\text{”} \\ f(x) \text{ is } \Theta(g(x)) & \text{is like} & \text{“}f = g\text{”} \end{array}$$

One must be careful and only use the above analogy to build intuition as some properties that hold for comparison of numbers do not carry over for functions. For instance, if a and b are numbers, then we have that $a \leq b$ or $b \leq a$. For functions, however, we can have a situation where f is not $O(g)$ and g is not $O(f)$. Can you think of such an example?

Big-Ω provides a lower bound, while both little-o and big-O give upper bounds. Intuitively, little-o gives a strict upper bound while big-O gives an upper bound that is potentially not strict. More formally, we have the following.

Theorem 4. *Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions such that $f(x)$ is $o(g(x))$. Then we have that*

1. *$f(x)$ is $O(g(x))$ and*
2. *$g(x)$ is* not *$O(f(x))$.*

1.2 Collection of rules

Theorem 5. *Let $f, g, h, p : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions.*

(R1) **"Overall constant factors can be ignored"**
If $c > 0$ is a constant then $cf(x)$ is $\Theta(f(x))$.

(R2) **"For polynomials only the highest-order term matters"**
If $p(x)$ is a polynomial of degree d, then $p(x)$ is $\Theta(x^d)$.

(R3) **"The fastest growing term determines the growth rate"**
If $f(x)$ is $o(g(x))$ then $c_1 g(x) + c_2 f(x)$ is $\Theta(g(x))$, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants.

(R4) **"Logarithms grow faster than constants"**
If $c > 0$ is a constant then c is $o(\log_a(x))$ for all $a > 1$.

(R5) **"Powers (and polynomials) grow faster than logarithms"**
$\log_a(x)$ is $o(x^b)$ for all $a > 1$ and $b > 0$.

(R6) **"Exponentials grow faster than powers (and polynomials)"**
x^a is $o(b^x)$ for all a and all $b > 1$.

(R7) **"Larger powers grow faster"**
x^a is $o(x^b)$ if $a < b$.

(R8) **"Exponentials with a bigger base grow faster"**
a^x is $o(b^x)$ if $0 < a < b$.

Informally, we can summarize rules (R4)–(R6) from above as

Constants < Logarithms < Polynomials < Exponentials

Example 6. Let us use the above rules to show that $2^x + x$ grows asymptotically faster than $3x^2 + 5x$ (*i.e.* $3x^2 + 5x$ is $o(2^x + x)$).

(1) First, use *(R2)* to conclude that $3x^2 + 5x$ is $\Theta(x^2)$.
(In other words, $3x^2 + 5x$ and x^2 grow at the same rate asymptotically.)

(2) Use *(R6)* observe that x^2 is $o(2^x)$.
(In other words, 2^x grows faster x^2 asymptotically.)

(3) Combine the statements from (1) and (2), to conclude that $3x^2 + 5x$ is $o(2^x)$.
(In other words, 2^x grows faster than $3x^2 + 5x$ asymptotically.)

(4) Use *(R6)* again to observe that x is $o(2^x))$. By *(R3)*, we can now say that $2^x + x$ is $\Theta(2^x)$.
(In other words, $2^x + x$ and 2^x grow at the same rate asymptotically.)

(5) Finally, combine the statements from (3) and (4), to get that $3x^2 + 5x$ is $o(2^x + x)$.
(In other words, $2^x + x$ grows faster than $3x^2 + 5x$ asymptotically,)

In Step (3) of the above example we, implicitly and without any formal justification, used the fact that if f_1 grows faster than f_2 which grows at the same rate as f_3 then f_1 grows faster than f_3. The formal statement is that if f_2 is $o(f_1)$ and f_2 is $\Theta(f_3)$ then f_3 is $o(f_1)$. Similar formal statements hold for other combinations of the asymptotic notation. We summarize these statements in the following theorem:

Theorem 7. *Let $f, g, h : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions.*

(R9) **Transitivity**
If $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$ then $f(x)$ is $O(h(x))$.
If $f(x)$ is $o(g(x))$ and $g(x)$ is $o(h(x))$ then $f(x)$ is $o(h(x))$.
If $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$ then $f(x)$ is $\Theta(h(x))$.

(R10) **Chaining**
If $f(x)$ is $O(g(x))$ and $g(x)$ is $\Theta(h(x))$ then $f(x)$ is $O(h(x))$.
If $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $O(h(x))$ then $f(x)$ is $O(h(x))$.
If $f(x)$ is $o(g(x))$ and $g(x)$ is $O(h(x))$ then $f(x)$ is $o(h(x))$.
If $f(x)$ is $O(g(x))$ and $g(x)$ is $o(h(x))$ then $f(x)$ is $o(h(x))$.
If $f(x)$ is $o(g(x))$ and $g(x)$ is $\Theta(h(x))$ then $f(x)$ is $o(h(x))$.
If $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $o(h(x))$ then $f(x)$ is $o(h(x))$.

Example 8. All logarithms have the same asymptotic order. This is seen from the following general property of logarithms

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)},$$

which shows that two logarithms are equal up to multiplication with a constant. It then follows from (*R1*) that all logarithms have the same asymptotic order.

Example 9 tells us that we do not need to worry about the base of logarithms when dealing with asymptotic growth. We will therefore simply write $\log x$ when the base is irrelevant.

The final rule that we introduce applies in situations where we the functions whose asymptotic growth we are interested to compare are multiples of some other function h.

Theorem 9. *Let $f, g, h : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions.*

(R11) **Multiplication**
If $f(x)$ is $O(g(x))$, then $h(x)f(x)$ is $O(h(x)g(x))$.
If $f(x)$ is $\Omega(g(x))$, then $h(x)f(x)$ is $\Omega(h(x)g(x))$.
If $f(x)$ is $o(g(x))$, then $h(x)f(x)$ is $o(h(x)g(x))$.
If $f(x)$ is $\Theta(g(x))$, then $h(x)f(x)$ is $\Theta(h(x)g(x))$.

The rule *(R11)* is useful when we want to compare the asymptotic growth of, for example, $n \log n$ and n^2. From rule *(R5)* we know that $\log n$ is $o(n)$. Applying *(R11)* with $h(n) = n$, we can conclude that $n \log n$ is $o(n^2)$. This, for example, tells us that Merge sort is a better sorting algorithm than Insertion sort when it comes to asymptotic worst-case complexity.

2 Asymptotic growth of sequences

We can treat the asymptotic behaviour of sequences completely analogously to functions. Simply replace the functions $f(x)$, $g(x)$ with sequences (a_n), (b_n) in section 1 in all definitions and rules.

We are going to be particularly interested in situations where a sequence (a_n) describes the worst-case runtime of an algorithm on an input of size n. It is very common that such sequences are of asymptotic order

$$\Theta(1), \Theta(\log n), \Theta(n), \Theta(n \log n), \Theta(n^2), \Theta(n^2 \log n), \Theta(n^3), \Theta(2^n)$$

During the course, we will repeatedly see that if one can replace an algorithm with another of lower asymptotic order, a significant improvement can be achieved.

An essential challenge when analyzing the efficiency of algorithms is to find the asymptotic order of recursively defined sequences and series, in particular. For now, let us only note that our sum rules imply that $\sum_{k=1}^{n} k$ is of the same asymptotic order as n^2 and that $\sum_{k=1}^{n} k^2$ is of same asymptotic order as n^3. We will also make use of the following relation:

Theorem 10. $\log(n!)$ *is* $\Theta(n \log n)$

Proof. As mentioned previously, it is irrelevant what base we choose for the logarithm. We will therefore choose to use $\log_2$. We want to show that

$\log_2(n!)$ is $O(n \log_2 n)$ and that $n \log_2 n$ is $O(\log_2(n!))$. We will present two different proofs.

The first proof is easy since we have that

$$n! = 1 \cdot 2 \cdots n \leq n \cdot n \cdots n \leq n^n$$

and thus,

$$\log_2(n!) \leq \log_2(n^n) = n \log_2 n,$$

where the inequality follows from the fact that the logarithm is an increasing function and the equality is a fundamental property of logarithms. This shows that $\log_2(n!)$ is $O(n \log_2 n)$ (with $c = 1$ in the definition).

For the second proof, we need to be a bit more resourceful. When $1 \leq i \leq n$, we have that $i - 1 \geq 0$ and $n - i \geq 0$. Thus,

$$(i-1)(n-i) \geq 0.$$

Expanding, we get

$$i \cdot n - i^2 - n + i \geq 0$$

or

$$i \cdot n - i^2 + i \geq n,$$

which can be written as

$$i(n - i + 1) \geq n.$$

This means that all products of the form $1 \cdot n$, $2 \cdot (n-1), 3 \cdot (n-2), \ldots, n \cdot 1$ are greater than or equal to n. This means that

$$\begin{aligned}
n \log_2 n &= \log_2 n + \log_2 n + \cdots + \log_2 n \\
&\leq \log_2(1 \cdot n) + \log_2(2 \cdot (n-1)) + \cdots + \log_2(n \cdot 1) \\
&= [\log_2 1 + \log_2 n] + [\log_2 2 + \log_2(n-1)] + \cdots + [\log_2 n + \log_2 1] \\
&= 2 \log_2 1 + 2 \log_2 2 + \cdots + 2 \log_2 n \\
&= 2 \sum_{i=1}^{n} \log_2 i \\
&= 2 \log_2(1 \cdot 2 \cdots n) \\
&= 2 \log_2(n!)
\end{aligned}$$

where we have rearranged the terms at the third equality sign but otherwise just used the standard rules for logarithms. This shows that $n \log_2 n$ is $O(\log_2(n!))$ (with $c = 2$ in the definition). $\square$

We end the notes with the following result that we will state without a proof:

Theorem 11. *The series*

$$\sum_{k=1}^{n} \frac{1}{k}$$

is $\Theta(\log n)$.