MINIMUM SPANNING TREES

$\boxed{G = (V, E)}$ undirected, connected graph
$\boxed{w : E \to \mathbb{R}}$ weight function

} Throughout today's lecture

Ex 1) V, cities, E road network

2) V pins of circuit that should be wired together to be electronically equivalent

E pairwise cost for pins

Want to find

1) smallest total length of roads to keep clear in case of snowstorm, say

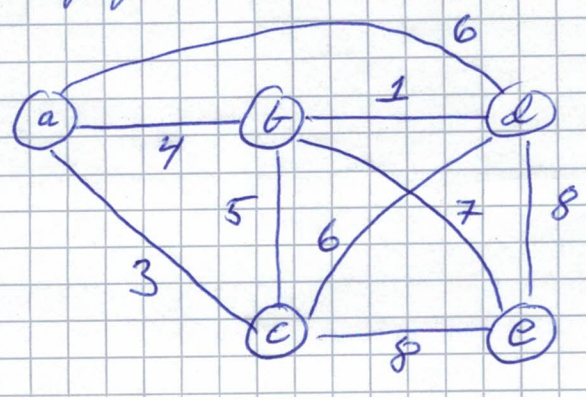
2) cheapest wiring

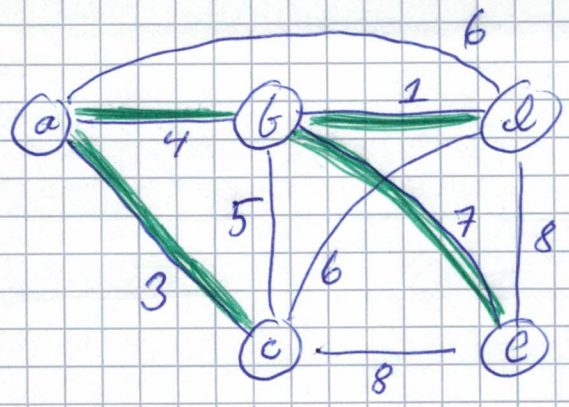
MINIMUM SPANNING TREE (MST)

$$\boxed{\begin{array}{l} \text{Spanning tree } T \text{ for } G \text{ such that} \\ w(T) = \sum_{(u,v) \in E(T)} w(u,v) \quad \text{minimal} \end{array}}$$

Actually minimum-<u>weight</u> spanning tree, but drop the word "weight"

Ex What is a minimum spanning tree for this graph?





Minimum spanning trees need NOT be unique.

Will build MSTs using $\boxed{\text{GREEDY APPROACH:}}$

Do what seems best locally at the moment

Greedy algorithms do not give optimal results in general, but here we can show that sequence of locally optimal choices give globally optimal result

General idea:

- Maintain subset of edges $A \subseteq E$ that is part of some MST
- At each step, add edge (u,v) that maintains this invariant, i.e., $A \cup \{(u,v)\}$ also part of some MST

Such edge (u,v) is a $\boxed{\text{SAFE EDGE}}$ for A

```
GENERIC-MST ( G, w)
    A := ∅
    while ( A not spanning tree)
        find edge (u,v) safe for A
        A := A ∪ {(u,v)}
    return A
```

Suppose $|V| = n$, $|E| = m$

How many iterations in while loop?

$n-1$, because any $\boxed{\text{tree on } n \text{ vertices}}$ contains $\boxed{n-1 \text{ edges}}$.

At all times, maintain $A \subseteq$ some MST

When $|A| = n-1$, the edges in A form MST.

Also note that always possible to add safe edge by definition, since always $A \subseteq$ some MST but A not yet tree

But how to find safe edges?

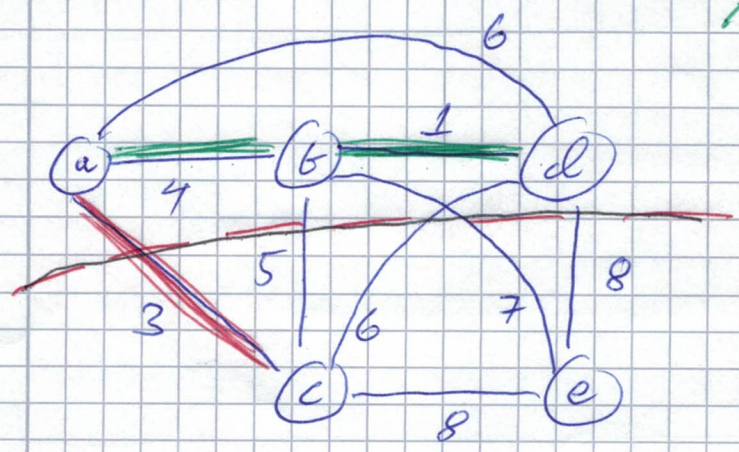
DEFINITIONS

$\boxed{\text{CUT}}$ $(S, V \setminus S)$: partition of vertices V

Edge (u,v) $\boxed{\text{CROSSES}}$ cut if $u \in S$, $v \in V \setminus S$

Cut $(S, V \setminus S)$ $\boxed{\text{RESPECTS}}$ $A \subseteq E$ if no edges in A cross the cut

A $\boxed{\text{LIGHT EDGE}}$ crossing a cut is a smallest weight edge crossing a cut (need not be unique)



$A = \{(a,b), (b,d)\}$

CUT $(\{a,b,d\}, \{c,e\})$ respects A

(a,c) is LIGHT EDGE for this cut

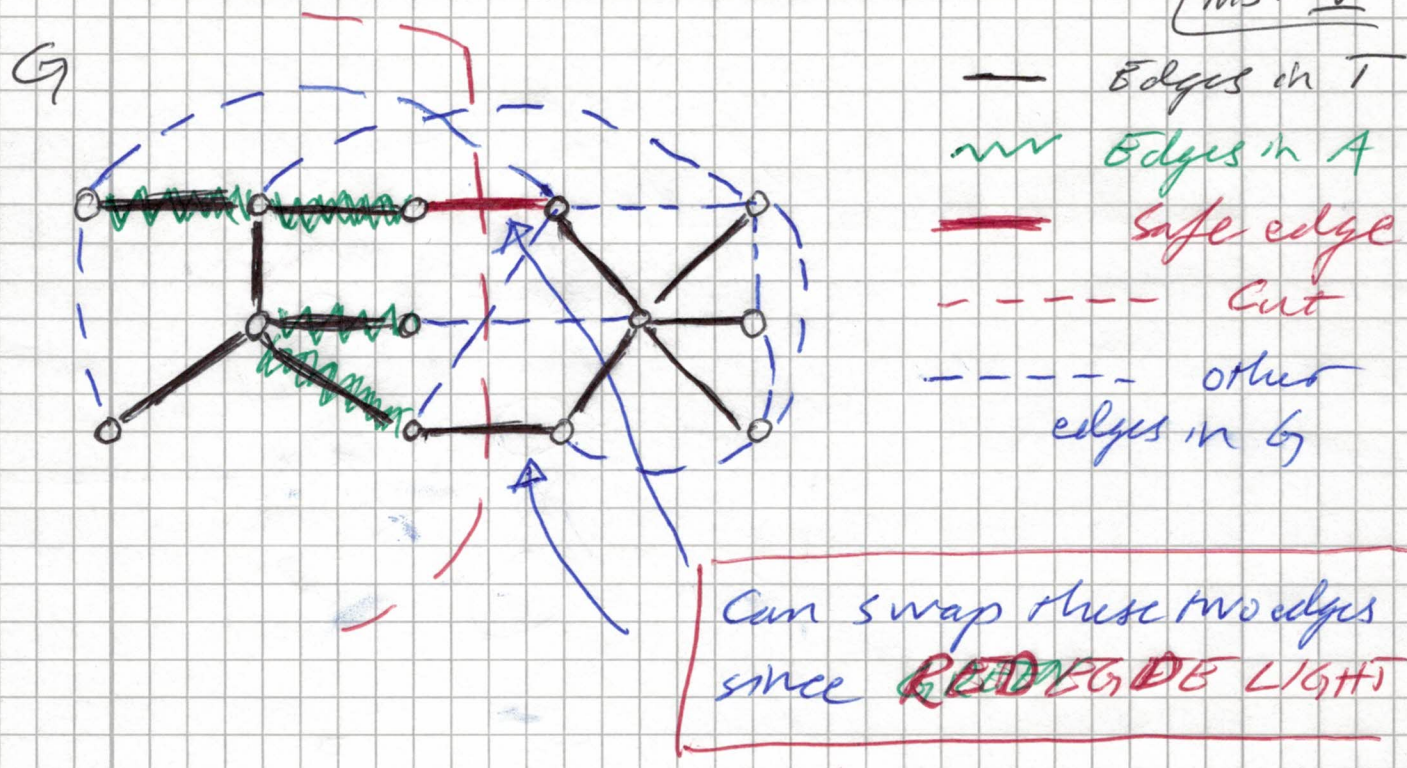
THEOREM 1

Let $G=(V,E)$ connected, undirected graph
$w: E \to \mathbb{R}$ weight function
Let $A \subseteq E$ be such that A is included in some MST
Let $(S, V \setminus S)$ cut that respects A
Let (u,v) light edge crossing $(S, V \setminus S)$
THEN (u,v) is safe for A

Proof Fix MST T including edges in A.
If (u,v) edge in T, then done, so suppose not.
Plan: Find edge (x,y) in T. Swap with (u,v). Argue still MST

u & v connected in T by (unique) path $P: v \rightsquigarrow u$
Add (u,v) to get cycle $C: v \rightsquigarrow u \to v$
Edge (u,v) crosses cut, so at least one more edge in C crosses cut
Fix any such edge (x,y)

Since (x,y) crosses cut, $(x,y) \notin A$ (respectfulness)
Remove edge (x,y) and add edge (u,v) to get T'. T' has $|V|-1$ edges and is connected (since T was connected)
[Prove this!]
Hence T' is tree.
Since (u,v) & (x,y) both cross cut and (u,v) light
$w(u,v) \leq w(x,y)$ Hence
$w(T') = w(T) - w(x,y) + w(u,v) \leq w(T)$
so T' also MST $\boxtimes$



A is always acyclic
Collection of trees — FOREST
A safe edge connects two subtrees in forest to form new subtree

COROLLARY 2

Let $A \subseteq E$ included in some MST
Let $C = (V_c, E_c)$ be connected component (i.e., subtree) in forest $G_A = (V, A)$
If (u,v) is light edge connecting C to some other component, then (u,v) safe for A.

<u>Proof</u> The cut $(V_c, V \setminus V_c)$ respects A
(u,v) is a light edge for this cut
Apply Theorem 1 $\boxtimes$

PRIM'S ALGORITHM

Start in arbitrary root vertex r

Grow tree rooted in r.

In every step, add light edge to currently disconnected vertex

Use priority queue for vertices v

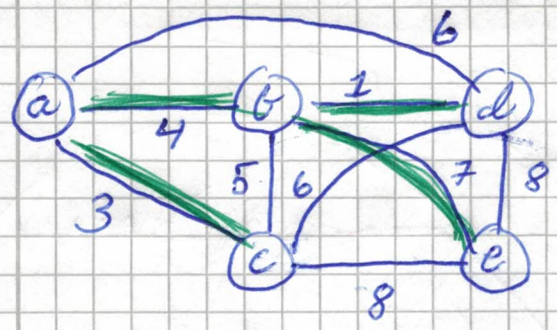
$\boxed{v.key}$ = min weight of edge connecting v to current tree (or ∞)

$v.\pi$ = parent of v in tree

Set A will be $A = \{ (v, v.\pi) \mid v \in V \setminus (Q \cup \{r\}) \}$

```
MST-PRIM (G, w, r)
    for v ∈ V(G)                                  n times
        v.key := ∞
        v.π   := NIL                              O(1) time
        v.marked := FALSE                                          O(n)
Q := BUILD-MIN-HEAP(V)                  — O(n)
    Q.DECREASE-KEY(r, 0)                O(log n)
    while (NOT (Q.EMPTY))               n times
        u := Q.EXTRACT-MIN              — O(log n)
        u.marked := TRUE                — O(1)
        for v ∈ N(u)                    — total of O(|E|) times   O(m)
            if (NOT(v.marked) AND w(u,v) < v.key)
O(1)            v.π := u
O(log n)        Q.DECREASE-KEY(v, w(u,v))
```

At termination MST edges are $\{ (v, v.\pi) \mid v \in V \setminus \{r\} \}$



min queue element taken care of next

Key values in queue

a	b	c	d	e	
0	∞	∞	∞	∞	Initialize starting in a
0	4	3	6	∞	Dequeue a $b.\pi := a$; key 4 $c.\pi := a$; key 3 $d.\pi := a$; key 6
0	4	3	6	8	Dequeue c EDGE (a,c) $e.\pi := c$; key 8
0	4	3	1	7	Dequeue b EDGE (a,b) $d.\pi := b$; key 1 $e.\pi := b$; key 7
0	4	3	1	7	Dequeue d EDGE (b,d)
					Dequeue e EDGE (b,e)

Correctness

Follows from Theorem 1 + generic algorithm
We grow a tree from starting vertex
Always add light edge connecting this tree to new vertex

Time cplx (recall $|V|=n$, $|E|=m$)

Initialization: $O(n)$

while loop: n times

$O(\log n)$ time except loop over adjacency list

Time in adjacency list for loop

total of $2m$ iterations

each iteration $O(\log n)$ time

Summing up: $O(n) + O(n \log n) + O(m \log n)$

$= \boxed{O(m \log n)}$

[$m \geq n-1 = \Omega(n)$ since graph is connected]

Faster priority queue would give faster algorithm. In coming algorithm courses, you will see how to use FIBONACCI HEAPS to get time $\boxed{O(m + n \log n)}$

— clearly better for dense graphs ($m \gg n$)

KRUSKAL'S ALGORITHM

Maintain sets of subtrees = FOREST

Add least-weight edge that connects two distinct trees in forest (T_1 & T_2)

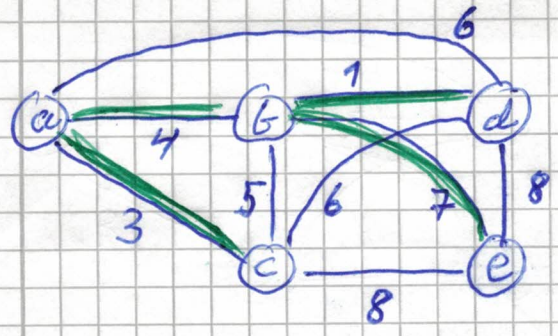
Correctness: If (u,v) is lightest edge connecting any two trees, say T_1 & T_2, then

- $(V(T_1), V \setminus V(T_1))$ is a cut
- This cut respects all edges so far
- (u,v) is a light edge crossing this cut.

Formalizing this as an inductive proof, it follows that algorithm below must be correct.

```
MST-KRUSKAL (G, w)
    A := ∅
    for v ∈ V(G)                          n times
        BUILD-SET ({v})
    L := all edges in E                   O(m)
    SORT (L)        [in nondecreasing order by edge weight]   O(m log m)
    for (u,v) in L in order               m times
        if (NOT SAME-SET (u, v)))
            A := A ∪ {(u,v)}              O(1)
            JOIN-SETS (u,v)
    return A
```

MST IX ½



Sort edges

(b, d)	1
(a, c)	3
(a, b)	4
(b, c)	5
(a, d)	6
(c, d)	6
(b, e)	7
(a, d)	8
(d, e)	8

Edge	Forest	
	{a} {b} {c} {d} {e}	
(b, d)	{a} {b, d} {c} {e}	Add (b, d)
(a, c)	{a, c} {b, d} {e}	Add (a, c)
(a, b)	{a, b, c, d} {e}	Add (a, b)
(b, c)	b & c in same set	—
(a, d)	a & d in same set	—
(c, d)	c & d in same set	—
(b, e)	{a, b, c, d, e}	Add (b, e)
(a, d)	a & d in same set	—
(d, e)	d & e in same set	—

To implement Kruskal efficiently, need data structure for disjoint sets

Discussed in CLRS Chapter 21

Recap of basic idea:

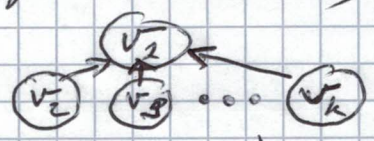
For each set, maintain tree with all edges oriented so that walk along edges ends in root

Root is unique $\boxed{\text{representative}}$ of set

Operations:

$\boxed{\text{SAME-SET}(u,v)}$ check if u & v is in same set (i.e. if same representative)

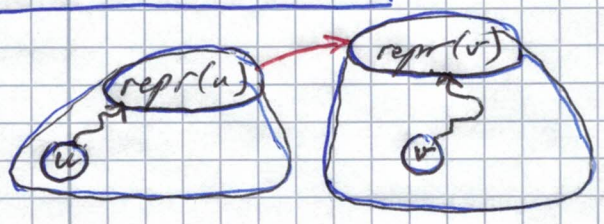
$\boxed{\text{BUILD-SET}(\{v_1, \ldots, v_k\})}$ Build tree



(representative is v_1)

$\boxed{\text{JOIN-SETS}(u,v)}$ Find representatives repr(u) and repr(v)

Add edge from repr(u) to repr(v)



(Can optimize further by moving pointers of all encountered elements to representative)

Running time of Kruskal dominated by sorting $O(m \log m)$. But $m \le n^2$ and $\log n^2 = 2\log n$

So running time $\boxed{O(m \log n)}$