# DMA: Order Relations

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# Outline



**Reading:** KBR 6.1 and 6.2

# Recap

A relation $R$ on set $A$ is a collection of ordered pairs $(a, a') \in A \times A$.

**Example.** $A = \{2, 3, 4, 5, 6, 15\}$

Divisibility relation: $aRb \Leftrightarrow a \mid b$

$$R = \{(2,2), (2,4), (2,6), (3,3), (3,6), (3,15), (4,4), (5,5), (6,6)(5,15), (15,15)\}$$

# Partially ordered **sets (posets)**

**Motivating example:** The "$\leqslant$" relation on $\mathbb{Z}$ is

- reflexive: $a \leqslant a$
- transitive: if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$
- antisymmetric: if $a \leqslant b$ and $b \leqslant a$ then $a = b$

**Def.** A relation $R$ on a set $A$ is a partial order, if it is reflexive, antisymmetric, and transitive. The pair $(A, R)$ is called a poset.

## Notes

- We often use $\leqslant$ to denote a partial order and write $(A, \leqslant)$ instead of $(A, R)$.
- Write $a < b$ to denote that $a \leqslant b$ and $a \neq b$.

# Caution!

Use context to distinguish if "$\leqslant$", in expresion like $a \leqslant b$, is used to denote

- comparison of numbers;
- a partial order, $(A, \leqslant)$, which is not necessarily comparison of numbers.

# Examples: partial orders and posets

Given a set $S$, consider its power set $\mathrm{P}(S) = \{\mathrm{T} : \mathrm{T} \subseteq S\}$.

- The set inclusion, $\subseteq$, is a partial order on $\mathrm{P}(S)$.
- The pair $\left(\mathrm{P}(S), \subseteq\right)$ is a poset.

**Exercise:** Is "$<$" a partial order on $\mathbb{R}$?

# Comparable elements

**Def.** Let $(A, \leqslant)$ be a poset. We say that elements $a, b \in A$ are comparable if $a \leqslant b$ or $b \leqslant a$.

**Example.** Consider poset $(\mathbb{Z}^+, |)$.

- 3 and 12 are comparable (3 divides 12).
- 64 and 16 are comparable (16 divides 64).
- Are any two elements of $\mathbb{Z}^+$ comparable in the poset $(\mathbb{Z}^+, |)$?

**Example:** Consider poset $(\mathbb{Z}^+, \leqslant)$. For $a, b \in \mathbb{Z}^+$, we have $a \leqslant b$ or $b \leqslant a$. Any two elements are comparable!

# Total (linear) orders

**Def.** Let $(A, \leqslant)$ be a poset. We say that relation $\leqslant$ is a total (or linear) order, if any two elements of $A$ are comparable.

- Posets $(\mathbb{Z}, \leqslant)$ and $(\mathbb{R}, \leqslant)$ are totally ordered.
- Poset $(\mathbb{Z}^+, |)$ is not totally ordered ($5 \nmid 3$ and $3 \nmid 5$).
- **Exercise:** Is the poset $(\mathrm{P}(S), \subseteq)$ totally ordered?

# Product partial orders

Let $(A, \leqslant_1)$ and $(B, \leqslant_2)$ be posets. Define a new relation $\leqslant_3$ on $A \times B$ by

$$(a, b) \leqslant_3 (a', b') \Leftrightarrow a \leqslant_1 a' \text{ and } b \leqslant_2 b'$$

**Thm.** $(A \times B, \leqslant_3)$ is a poset.

**Example:** Consider $(\mathbb{Z} \times \mathbb{Z}, \leqslant_3)$ where $\leqslant_1, \leqslant_2$ be the usual (less-or-equal-than) comparison of numbers.

Note: Even if $(A, \leqslant_1)$ and $(B, \leqslant_2)$ are totally ordered, $(A \times B, \leqslant_3)$ is not (except for the case when $|A|, |B| \leqslant 1$).

# Hasse diagrams

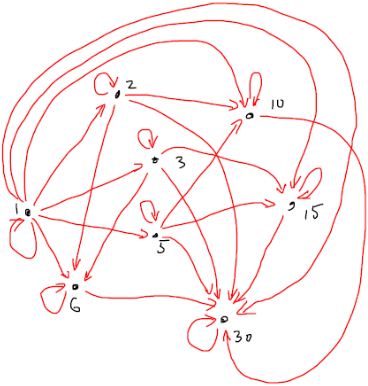
A Hasse diagram is a compact visual representation of a poset.

How to draw a Hasse diagram of a poset $(A, \leqslant)$?

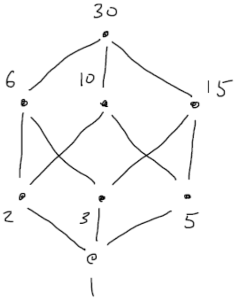
1. Start with the corresponding digraph.
2. Remove all the loops.
3. Remove all the edges (arrows) implied by transitivity.
4. Make all the arrows point upwards and then replace them with lines.

$(\{1, 2, 3, 5, 6, 10, 15, 30\}, \quad |)$

Digraph :




Hasse diagram :



# Outline

1. Order Relations
   - Definition, examples, and properties
   - Hasse diagrams
2. Maximal and minimal elements
   - Greatest and least elements
   - Upper and lower bounds

# Maximal and minimal elements

**Def.** Let $(A, \leqslant)$ be a poset. We say that $z \in A$ is

- a maximal element of $A$ if we cannot find $a \in A$ such that $z < a$.
- a minimal element of $A$ if we cannot find $a \in A$ such that $a < z$.

**Thm.** Any finite, nonempty poset $A$ has a maximal and a minimal element.

- $(\mathbb{Z}, \leqslant)$ does not have a maximal or a minimal element.
- **Question:** What (if any) are the minimal and maximal elements of $(\mathbb{Z}^+, \leqslant)$?
- **Exercise (HW):** What (if any) are the minimal and maximal elements of $\big(\mathrm{P}(\mathbb{Z}), \subseteq\big)$?

# Greatest and least elements

**Def.** Let $(A, \leqslant)$ be a poset. We say that $z \in A$ is

- the greatest element of $A$ if $a \leqslant z$ for all $a \in A$.
- the least element of $A$ if $z \leqslant a$ for all $a \in A$.

Note: The greatest element of $A$ is always maximal, but the converse doesn't hold.

Example: $(\{1, 2, 3, 4\}, |)$ has maximal elements 3,4 but **no** greatest element; 1 is its least (and minimal) element.

**Thm. (Uniqueness)** A poset $(A, \leqslant)$ has at most one greatest element and at most one least element.

# Upper and lower bounds

**Def.** Let $(A, \leqslant)$ be a poset and $B \subseteq A$. We say that $a \in A$ is

- an upper bound for $B$ if $b \leqslant a$ for all $b \in B$.
- a lower bound for $B$ if $a \leqslant b$ for all $b \in B$.

**Def.** Let $(A, \leqslant)$ be a poset and $B \subseteq A$. We say that $a \in A$ is

- the least upper bound (LUB) for $B$ if $a$ is an upper bound on $B$ and $a \leqslant a'$ for any upper bound $a'$ of $B$.
- the greatest lower bound (GLB) for $B$ if $a$ is an lower bound on $B$ and $a' \leqslant a$ for any lower bound $a'$ of $B$.

Example: consider the interval $B = [0, 1)$ in $(\mathbb{R}, \leqslant)$. Then 5 is an upper bound, $-1$ is a lower bound, 1 is the least upper bound, and 0 is the least lower bound. **Thm.** (**Uniqueness**) Let $(A, \leqslant)$ be a poset. A subset $B \subseteq$ has at most one LUB and at most one GLB.

# Summary of Posets

- Order relation $\mathcal{R}$: reflexive, antisymmetric, and transitive.
- Poset is a pair $(A, \leqslant)$, where "$\leqslant$" is an order relation.
- Posets can have incomparable elements.
- We can visualize posets using Hasse diagrams.
- Given a poset we can look for extremal elements :
  - Maximal/minimal
  - Greatest/least
  - Upper/Lower bounds

# Exercises: LUB and GLB

Consider the poset $(\mathbb{Z}^+, |)$

- What is the LUB of $\{6, 9\}$?
- What is the LUB of $\{a, b\} \subseteq \mathbb{Z}^+$?
- What is the GLB of $\{6, 9\}$?
- What is the GLB of $\{a, b\} \subseteq \mathbb{Z}^+$?