DMA 2021

– Week 7 –

Work instructions

The first topic of this week is proof techniques. You have already seen mathematical induction which is one example of a specific proof technique. This week we will also see proof from contradiction, proof by case analysis and other proof methods. It is good to note, however, that many (or perhaps even most) proofs don't follow any specific proof technique or might combine different proof techniques to prove intermediate statements.

In the second half of this week we return to the asymptotic big-O notation that we have already seen in Week 3. This time, we are better equipped to understand the formal definitions involving quantifiers and formally prove or disprove the statements of the form "f is $O(g)$". You should expect this to be challenging at first. As always, remember that it is ok not to succeed on the first try and rather aim to improve over successive tries and exercises. You will see the asymptotic notation later on in your studies and in this course our goal is to just give a first introduction to the topic.

Assigned reading

- KBR 2.3. We will use this material on Monday and Tuesday.
- Notes for Week 7 (available on Absalon). Some of the material is repeated from Week 3 notes. Carefully examine the formal definitions of big-O, little-o, big-Ω, big-Θ. Memorize the definition of big-O. **Read/do this before the class on Tuesday.**
- [CLRS] Chapter 3 up to o-notation (pages 43–51). Read this material **carefully before the class on Tuesday**. If needed, read it twice.

Lecture plan

Monday Oct. 25th 09:15–10:00

Proof methods with examples.

Tuesday Oct. 26th, 13:15–15:00

- Additional proof methods with examples
- Asymptotic notation: formal definitions

Friday Oct. 29th, 09:15–10:00

Proving statements involving asymptotic notation.

Exercise plan

Monday Oct. 25th, 10:15–12:00

1. Let $a, b \in \mathbb{Z}^+$. Prove that if $a \mid b$ and $a \mid c$ then for any $m, n \in \mathbb{Z}$ we have that $a \mid (mb + nc)$.

2. Let k be a positive integer. Prove the following statement by proving its contrapositive: If k^3 is odd then k is odd. Make sure to start by explicitly stating the contrapositive.

3. Let k be a positive integer. Show that k^3 is odd if and only if k is odd. *You can make use of your solution to the previous problem.*

4. Solve KBR exercises 2.3.27, 2.3.31, 2.3.34.

5. Let $a, b, n \in \mathbb{R}^+$. Consider the following statement.

$$\text{If } ab = n \text{ then } a \leq \sqrt{n} \text{ or } b \leq \sqrt{n}. \tag{1}$$

   (a) State the negation of (1) as simply as possible (the simplified statement should not involve negation).

   (b) Use proof by contradiction to prove (1). Make sure to start your proof with the negation of (1) from part (a).

   Time permitting, the instructor provides a sample solution on the board.

6. Use proof by contradiction to prove that $\log_3(4)$ is irrational. *If you are stuck, ask your instructor for a hint.*

7. Let A, B be sets. Show that $A \subseteq B$ if and only if $A \cup B = B$.

Tuesday Oct. 26th, 15:15–17:00

1. Solve KBR exercises 2.3.23 and 2.3.24
2. Let $p > 3$ be a prime. Prove that
$$p^2 \equiv 1 \mod 3$$
by separately analyzing the following three cases: (i) $p \equiv 0 \mod 3$, (ii) $p \equiv 1 \mod 3$, and (iii) $p \equiv 2 \mod 3$.
3. You will prove the same statement in two different ways.
   (a) Use the definition of big-O to show that $x^3 + 12x - 3$ is $O(x^3)$.
   (b) Use the rules from the weekly notes to show that $x^3 + 12x - 3$ is $O(x^3)$.
4. You will prove the same statement in two different ways.
   (a) Use the definition of big-O to show that $x^3 + 12x - 3$ is **not** $O(x^2)$. *Make sure to start by writing down what it means that $x^3 + 12x - 3$ is not $O(x^2)$.*
   (b) Use the rules and theorems from the weekly notes to show that $x^3 + 12x - 3$ is **not** $O(x^2)$.
5. Use the definition of big-O to prove or disprove the following
   (a) 2^{x+1} is $O(2^x)$.
6. Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. Consider the following two statements:
   (a) $g(x)$ is $O(f(x))$ and $f(x)$ is $O(g(x))$.
   (b) There exist constants $c_1, c_2 > 0$ and x_0 such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for all $x \ge x_0$.

   Show that (a) holds if and only if (b) holds thus establishing the equivalence of the two definition of big-Θ from this week's notes.

   Remember that proving an "if and only if" statement involves showing two directions.
7. [*] Using the definition of little-o, use proof by contradiction to show that x is **not** $o(5x)$.

Time permitting, the instructor provides sample solutions for exercises 4, (parts of) 6, and 3b on the board.

Friday Oct. 29th 10:15–12:00

1. Let $f, g, h : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. Use the definition of big-O to show that if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$ then $f(x)$ is $O(h(x))$.

2. Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. Prove the following:

   - **(Θ-relation is symmetric)** $f(x)$ is $\Theta(g(x))$ if and only if $g(x)$ is $\Theta(f(x))$.
   - If $f(x)$ is $o(g(x))$ then $g(x)$ is *not* $\Theta(f(x))$. *Hint: You can make use of any of the theorems from the notes.*

3. Algorithm A spends $T(n) = n^2$ microseconds for a problem of size n, while algorithms A_1, A_2, and A_3 spend

   $$T_1(n) = n^2 + 2^n, \quad T_2(n) = (n+2)^2, \quad \text{and} \quad T_3(n) = n^3 - n$$

   microseconds, respectively, for a problem of size n. Which of the three algorithms, A_1, A_2, A_3 have the same running time asymptotically as the quadratic-time algorithm A? Justify your answer using the appropriate asymptotic notation to formally compare the relevant running times. You can use any rules and theorems from this week's notes. Remember that you also need to justify if you claim that some of the algorithms A_i do **not** have the same asymptotic running time as A. *Hint: To argue that a function is* **not** $\Theta(n^2)$*, it is helpful to recall from the previous exercises that if* $f(n)$ *is* $o(g(n))$*, then* $g(n)$ *is not* $\Theta(f(n))$*. Time permitting, instructor provides sample solution for parts of this exercise on the board.*

4. Let $f(n) = 2^{\log_3(n)}$ and $g(n) = n$. Find a such that $f(n) = n^a$. Then prove or disprove the following statements

   (a) $f(n)$ is $O(g(n))$

   (b) $g(n)$ is $O(f(n))$.

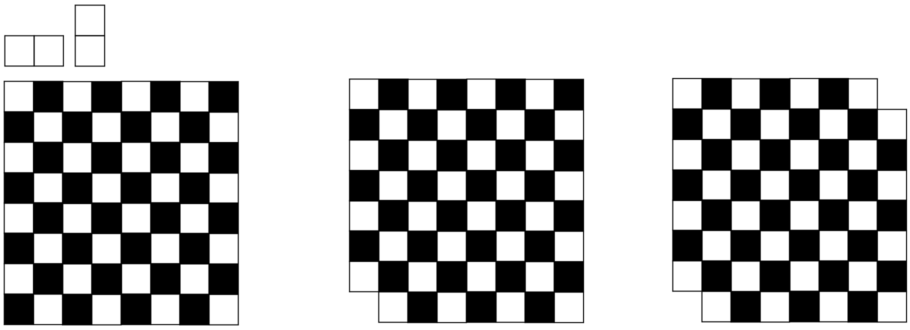
5. Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. Prove rule (*R3*) from this week's notes. In your proof you can use the rest of the rules and theorems from the notes as well as the following rule:

   (R3') If $f(x)$ is $o(g(x))$ then $g(x) \pm f(x)$ is $\Theta(g(x))$.

Finish any left-over exercises from the previous days.

1 Extra exercises

(1) For each of the three checkerboards pictured in Fig. 1, decide if it can be tiled with 1×2 dominoes. Justify your answer in each case.



(a) (b) (c)

Figure 1: Three boards.

(2) [**] Find the mistake in the proof given in KBR Example 2.4.7 (page 73).