DMA: Trees

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Directed trees

Reading: KBR 7.1

Digraphs

Recall: We used digraphs to visually represent relations.

Def. A digraph is an ordered pair $G = (V, E)$ where

- V is a set of vertices (or nodes)
- E is a set[1] of ordered pairs (elements of $V \times V$) called (directed) edges (or arrows)

Note: Digraphs and relations are different ways to look at the same mathematical object

$$G = (V, E) \Leftrightarrow \text{Relation } E \text{ on the set } V$$

[1] or a multiset if we want to allow multiple arrows

Directed trees

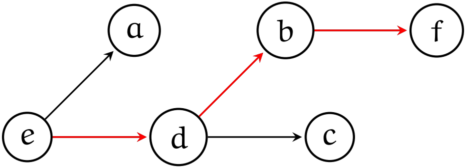
Def. A directed tree is a digraph T which has a vertex, v_0, called a root, such that there is a unique path from v_0 to any other vertex of T and no path from v_0 to itself.

Notes

- We write: (T, v_0).
- **Recall:** A (directed) path[2] of length $n \in \mathbb{Z}^+$ is a sequence

$$v_0, v_1, \ldots, v_n$$

where $(v_{i-1}, v_i) \in E$ for all $i \in \{1, \ldots, n\}$.



[2] KBR allows vertices to repeat. Other texts do not.

Properties of directed trees

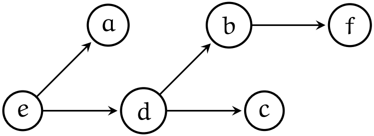
Def. A directed tree is a digraph T which has a vertex, v_0, called a root, such that there is a unique path from v_0 to any other vertex of T and no path from v_0 to itself.

Thm. (**Properties**) Let (T, v_0) be a directed tree. Then

- There are no cycles in T.
- v_0 is the only root of T.
- The in-degree of v_0 is 0 and the in-degree of all the other vertices is 1.

Number of vertices and edges in a tree

Thm. If $T = (V, E)$ is a directed tree on $n = |V|$ vertices, then $|E| = n - 1$.



$|V| = 6, |E| = 5$ ✓

Proof.

- The number of arrows is the sum of the in-degrees of all the vertices:

$$|E| = \sum_{v \in V} \text{indeg}(v)$$

- Since the in-degree of the root is 0 and all the other vertices have in-degree 1, we get

$$\sum_{v \in V} \text{indeg}(v) = |V| - 1 = n - 1.$$

Directed trees as relations

Thm. Consider a directed tree, T, with a vertex set V and let R_T be the corresponding relation on V. Then

1. R_T is irreflexive.
2. R_T is asymmetric.
3. If $(a, b) \in R_T$ and $(b, c) \in R_T$, then $(a, c) \notin R_T$ for all a, b, and c in V.

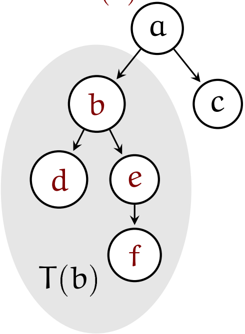
Tree terminology

Let (T, v_0) be a tree and v be a vertex of T.

- Leaves: vertices with no out-going arrows
- Children (or offspring) of v: the out-neighbors of v
- Parent of v: the in-neighbour of v
- Siblings of v: the other children of the parent p of v.
- Descendants of v: vertices that can be reached by a path starting from v
- Level of a vertex v: the length of the path from v_0 to v.
  - Define root to be at level 0.
- Height of a tree: maximum length of a path from the root to a leaf.

Subtrees

- Let $T = (V, E)$ be a tree and $b \in V$ be a vertex of T.
- Let $D(b)$ be the set containing vertex b and all its descendants.
- Remove vertices not in $D(b)$, along with their in- and out-going arrows to get $T(b)$.
- **Thm.** $T(b)$ is a tree with root b



$$D(b) = \{b, d, e, f\}$$