DMA: Probability

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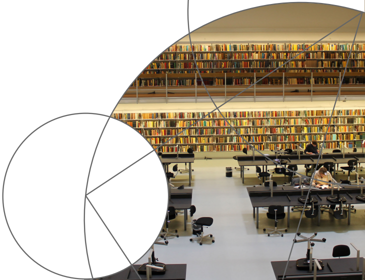

Plan for today

1. Poker hands and their probabilities
2. Expected value of an experiment

Recap

In how many ways can we choose r elements from an n-element set?

	with repetitions	without repetitions
ordered	n^r	${}_nP_r$
unordered	${}_{n+r-1}C_r$	${}_nC_r$

where

$$ {}_nP_r = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!} $$

and

$$ {}_nC_r = \binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 1} = \frac{n!}{r!(n-r)!} $$

Poker hands

A **poker hand** consists of 5 unordered cards drawn **uniformly at random** from a standard deck of 52. There are

- **4 suits:**

$$\spadesuit \text{ (spades)} \quad \heartsuit \text{ (hearts)} \quad \clubsuit \text{ (clubs)} \quad \diamondsuit \text{ (diamonds)}$$

- **13 ranks:**

$$2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$$

Question: How many different poker hands are there?
Answer: ${}_{52}C_5 = \frac{52!}{5!47!} = 2,598,960.$

Special poker hands

Experiment: Draw a random poker hand

Sample space: $A = \{\text{all possible hands } h\}$

Probability function: $p(h) = \frac{1}{|A|} = \frac{1}{{}_5C_{52}}$ (equally likely outcomes)

Events E:

1. Straight flush
2. **Four of a kind**
3. **Full house**
4. Flush
5. Straight
6. Three of a kind
7. **Two pairs**
8. A pair

Recall: $p(E) = \frac{|E|}{|A|}$

Four of a kind

Four of a kind: 4 cards of the same rank and one other card.

Examples: $\{A\spadesuit, A\heartsuit, A\diamondsuit, A\clubsuit, 5\clubsuit\}$, $\{7\heartsuit, 7\diamondsuit, 7\spadesuit, 7\clubsuit, K\heartsuit\}$

Task: Find the probability of drawing four of a kind.

Solution

- Event $E = \{\text{hands, } h\text{, that are four of a kind}\}$
- $p(E) = \frac{|E|}{|A|}$
- To choose a hand that is four of a kind:
  - Choose the rank for the 4 cards (task 1)
  - Choose the last card (task 2)
- $|E| = {}_{13}C_1 \cdot {}_{48}C_1 = 13 \cdot 48 = 624$
- $p(E) = \frac{|E|}{|A|} = \frac{624}{{}_{52}C_5} \approx 0.00024$

Two pairs

Two pairs: 2 pairs of different ranks and a card of a third rank (AABBC).

Example: $\{3\clubsuit, 3\heartsuit, 5\heartsuit, 5\spadesuit, K\clubsuit\}$

Task: Find the probability of drawing two pairs (event E)

Solution

- To choose two pairs:
  1. Choose the ranks for the two pairs ($_{13}C_2$ ways)
  2. Choose the suits for the two pairs ($_4C_2 \cdot {}_4C_2$ ways)
  3. Choose the last card of another rank ($_{44}C_1$ ways)
- $|E| = {}_{13}C_2 \cdot {}_4C_2 \cdot {}_4C_2 \cdot {}_1C_{44} = 123,552$
- $p(E) = \frac{|E|}{|A|} = \frac{123{,}552}{_{52}C_5} \approx 0.0475$

Two pairs

Two pairs: 2 pairs of different ranks and a card of a third rank (AABBC).

Example: $\{3\clubsuit, 3\heartsuit, 5\heartsuit, 5\spadesuit, K\clubsuit\}$

Task: Find the probability of drawing two pairs (event E)

Solution

- To choose two pairs:
  1. Choose the ranks for the two pairs ($_{13}C_2$ ways)
  2. Choose the suits for the two pairs ($_4C_2 \cdot {}_4C_2$ ways)
  3. Choose the last card of another rank ($_{44}C_1$ ways)
- **Q:** Could we perform (1) and (2) above as
  - (a) choose the 1st rank and its suites ($13 \cdot {}_4C_2$ ways)
  - (b) choose the 2nd rank and its suites ($12 \cdot {}_4C_2$ ways)

Two pairs

Two pairs: 2 pairs of different ranks and a card of a third rank (AABBC).

Example: $\{3\clubsuit, 3\heartsuit, 5\heartsuit, 5\spadesuit, K\clubsuit\}$

Task: Find the probability of drawing two pairs (event E)

Solution

- To choose two pairs:
  1. Choose the ranks for the two pairs ($_{13}C_2$ ways)
  2. Choose the suits for the two pairs ($_4C_2 \cdot {}_4C_2$ ways)
  3. Choose the last card of another rank ($_{44}C_1$ ways)
- **Q:** Could we perform (3) above as
  - (a) choose the 3rd rank ($_{11}C_1$ ways)
  - (b) choose the suite for the 3rd rank ($_4C_1$ ways)

Full house

Full house: three cards of one rank, and two cards of another rank (AAABB).

Example: $\{3\clubsuit, 3\heartsuit, K\heartsuit, K\spadesuit, K\clubsuit\}$

Task: Find the probability of drawing a full house.

Solution

- To choose full house: did you
  1. Choose the rank for the pair and choose their suites ($13 \cdot {}_4C_2 = 13 \cdot 6$ ways)
  2. Choose the rank for the triple and choose their suites ($12 \cdot {}_4C_3 = 12 \cdot 4$ ways)

  or

  1. Choose the two ranks (${}_{13}C_2 = 13 \cdot 6$ ways)
  2. Choose the suits for the triple and for the pair (${}_4C_3 \cdot {}_4C_2 = 6 \cdot 4$ ways)

Full house

Full house: three cards of one rank, and two cards of another rank (AAABB).

Example: $\{3\clubsuit, 3\heartsuit, K\heartsuit, K\spadesuit, K\clubsuit\}$

Task: Find the probability of drawing a full house.

Solution

- To choose full house:
  1. Choose the rank for the pair and choose their suites ($13 \cdot {}_4C_2$ ways)
  2. Choose the rank for the triple and choose their suites ($12 \cdot {}_4C_3$ ways)
- $|E| = 13 \cdot {}_4C_2 \cdot 12 \cdot {}_4C_3 = 13 \cdot 6 \cdot 12 \cdot 4 = 3,744$
- $p(E) = \frac{|E|}{|A|} = \frac{3{,}744}{{}_{52}C_5} \approx 0.00144$

Probabilities of poker hands

Kind of hand	Count	Probability
Straight flush		
Four of a kind	624	0.00024
Full house	3,744	0.00144
Flush		
Straight		
Three of a kind		
Two pairs	123,552	0.0475
A pair		

Expected value of an experiment

Definition: Let $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}$ be the sample space for an experiment X. Let p_k be the probability of getting outcome a_k, for all $1 \leqslant k \leqslant n$. Then, the **expected value** of the experiment is

$$E[X] = a_1 p_1 + a_2 p_2 + \ldots + a_k p_k.$$

Observation: We have assumed that a_k is a real number.

Expected value represents the average of a large number of independent realizations of the experiment.

Expected value of an experiment

Example: Suppose we toss a coin and record a 0 in case of heads and 1 in case of tails. Our sample space is $A = \{0, 1\}$. For a fair coin $p_0 = p_1 = \frac{1}{2}$ and the expected value is

$$E[X] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Observation: The expected value itself is not necessarily a valid outcome.

Expected value of an experiment

Example: Suppose we toss a coin and record a -1 in case of heads and 1 in case of tails. Our sample space is $A = \{-1, 1\}$. For a fair coin

$$E[X] = (-1) \cdot \frac{1}{2} + (1) \cdot \frac{1}{2} = 0$$

In contrast, if, for example, $p(-1) = \frac{1}{10}$ and $p(1) = \frac{9}{10}$, then

$$E[X] = (-1) \cdot \frac{1}{10} + (1) \cdot \frac{9}{10} = \frac{8}{10} = 0.8$$

Expected value of an experiment

Exercise 3.4.37. An array of length n is searched for a randomly placed keyword. On the average, how many steps will it take to find the key?

Solution: The number of steps that it can take to find a key is $1, 2, \ldots, n$. Hence, sample space, $A = \{1, 2, \ldots, n\}$. The key is equally likely to be found at any position. Hence, $p_k = \frac{1}{n}$. Therefore, the expected value of steps is:

$$\begin{aligned} 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + \ldots + n \cdot \frac{1}{n} &= \frac{(1 + 2 + \ldots + n)}{n} \\ &= \frac{1}{n} \frac{n(n+1)}{2} \\ &= \frac{n+1}{2} = \Theta(n) \end{aligned}$$