Pigeonhole statements

(1) Every $(N+1)$ - subset of $\{1, \ldots, 2N\}$ contains a pair of numbers where one divides the other.

<u>Proof</u>. Every natural number can be written as $n = m \cdot 2^k$, where m is odd.

If $n \leq 2N$, then $m \in \{1, 3, \ldots, 2N-1\}$.

Thus if $n_1, \ldots, n_{N+1} \in \{1, \ldots, 2N\}$ and $n_i = m_i \cdot 2^{k_i}$, the $N+1$ odd numbers $m_1, \ldots, m_{N+1}$ belong to a set of size N.

Pigeonhole principle: there are $n_i < n_j$ such that $m_i = m_j$. Then n_i divides n_j.

(2) Given 5 points on a sphere, there is a hemisphere containing 4 of them.

<u>Proof</u>. Pick 2 points. There is a great circle ("equator") through them, diving the sphere into two hemispheres.

By the PHP, one of them contains at least 2 of the remaining 3 points.

So this hemisphere contains at least 4 points.

(3) Every sequence of n^2+1 distinct real numbers contains a monotone subsequence of length $n+1$.

<u>Proof</u>. Let $r_1, \dots, r_{n^2+1}$ be distinct.

To each r_i assign a pair (a_i, b_i),

where a_i = the length of the longest increasing subseq ending with r_i,

and b_i = —||— decreasing

Observe: all these pairs are distinct.

Indeed, let $i<j$. If $r_i < r_j$ then $a_i < a_j$;

if $r_i > r_j$ then $b_i < b_j$.

There are at most $n \cdot n$ pairs (a_i, b_i) with $1 \le a_i, b_i \le n$. Since we have n^2+1 numbers, the PHP implies that some pair (a_k, b_k) is not like this.

So $a_k \geq n+1$, in which case there is an increasing $(n+1)$-subseq;

or $b_k \geq n+1$, —||— decreasing.

Monty Hall

- there are 3 doors: 2 goats & 1 car.
- the host knows where the car is.
- the contestant points to a door, say no 1.
- the host opens one of the other doors, revealing a goat.
- Should the contestant stick with her choice?

There are 3 scenarios:

Door No 1	C	G	G
Door No 2	G	C	G
Door No 3	G	G	C

← points

After the contestant points to Door No 1, the host opens 2 or 3 in 1st scenario, 3 in 2nd one, and 2 in 3rd one.

If she changes her choice, she wins in the second two scenarios, so in 2 out of 3 cases.