DMA 2021

– Week 16 –

Work instructions

This week we consider two independent subjects: **permutations** and **recurrence relations**.

We will see that recurrences commonly arise from counting problems and when analyzing the runtime of recursive algorithms. Given a counting problem, we will practice writing down the corresponding recurrence and then consider several methods for solving recurrences.

One general method for solving recurrence relations with **constant** coefficients is described in KBR 3.5. We will apply this method to find explicit expressions of second degree recurrence relations. It will, for example, allow us to explicitly express the nth Fibonacci number as

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

For analyzing runtimes of recursive, and especially divide-and-conquer type algorithms (e.g. Merge Sort), we will use the recursion-tree method and the Master theorem described in CLRS 4.4. and 4.5, respectively. The material in these sections can be harder to follow than our usual readings and therefore we recommend that you read these sections at least twice: once before and once after Tuesday's class.

We have already discussed permutations. For example, we have seen that there are $n!$ ways to permute (re-arrange) n elements. On Friday we will see how to view permutations as **bijective functions**, how to compose them, and how to describe them using the cycle notation. All of this is covered in KBR 5.4.

Lecture plan

Monday Jan. 10th 09:00–09:45

Recurrence relations. Characteristic polynomials. (KBR 3.5)

Tuesday Jan. 11th, 13:00–14:45

Recursion trees and Master theorem. Examples. (CLRS 4.4 and 4.5)

Friday Jan. 14th, 09:00–09:45

Permutations: Composition, inverse, and cycle notation. (KBR 5.4)

Exercise plan

Monday Jan. 10th, 10:15–12:00

- The instructor presents KBR example 3.5.7.
- Solve KBR exercises 3.5.4, 3.5.5, 3.5.9, 3.5.11, 3.5.18, 3.5.19., 3.5.20, 3.5.21, 3.5.22
- The instructor outlines how KBR theorem 3.5.1 can be generalized to linear recurrence relations of higher order when all roots of the characteristic polynomial are distinct. (KBR page 118).
- Solve KBR exercise 3.5.34, 3.5.35

Tuesday Jan. 11th, 15:15–17:00

- The goal of this exercise is to use the recursion-tree method to find a good asymptotic upper bound for the recurrence

  $$\begin{cases} T(1) = 1 \\ T(n) = 3\,T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \quad \text{for } n \geq 2 \end{cases} \tag{1}$$

  and then verify the answer using the Master theorem.

  1. Draw the recursion tree.
  2. Find the height, d, of the tree and express the per-level cost at level $k \in \{0, 1, \ldots, d\}$ as a function of k.
  3. Find the total cost by adding all the level costs. Express the total cost as $O(n^\beta)$ for appropriate choice of β.
  4. Use the Master theorem to find an asymptotically tight bound for $T(n)$. Compare this with your answer in the previous part.

- Follow the same steps as in the previous part to find a good asymptotic upper bound for the recurrence

$$\begin{cases} T(1) = 1 \\ T(n) = 2\,T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \quad \text{for } n \geq 2 \end{cases}$$

  and then check your answer using the Master theorem.

- Solve CLRS 4.5-1. When using a specific part of the Master theorem, don't forget to check the assumptions and justify why that part applies.

Friday Jan. 14th, 10:15–12:00

- Solve KBR exercises 5.1.19, 5.1.21, 5.4.5, 5.4.6, 5.4.9, 5.4.12, 5.4.13

Extra exercises

(1) [*] KBR exercises 3.5.7, 3.5.32, 3.5.26