MASD

Lecture 2
08.09.2022

François Lauze

Lecture will start at 10.15

Objectives

After today's lecture, you should

- Know the definition of the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ and have an intuitive understanding of what it means (Sections 2.7-2.8).

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- Know some differentiation rules (Sections 3.1 and 3.2). More rules and their use will follow next week.

Limits of Functions - From Lecture 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$.

- What does the statement $\lim_{x \to a} f(x) = L$ mean?

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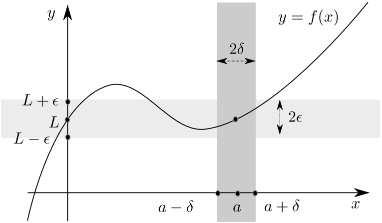
- Pick any $\varepsilon > 0$.
- $\lim_{x \to a} f(x) = L$ if there is $\delta > 0$ such that for all x

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$



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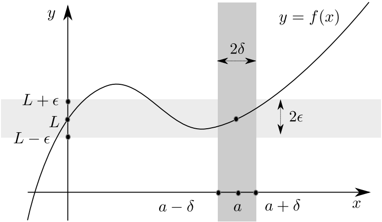

- $f(x)$ can be made arbitrarily close to L if x is close enough to a but not equal to a.

Continuous Functions - From Lecture 1

- A function f defined on an interval I (could be entire $\mathbb{R}$) is *continuous* at $a \in I \iff lim_{x \to a} f(x) = f(a)$. This implicitly requires that
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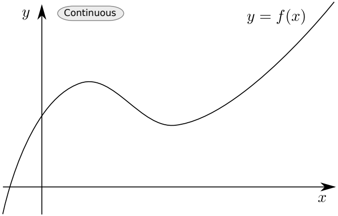
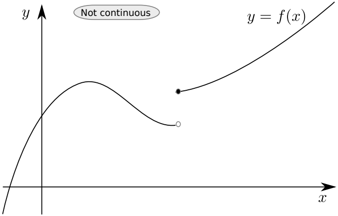
- Remember that x can approach a from both the left and the right side and limits from left and from right both **must be equal** to $f(a)$ (unless I is a closed interval with a being one of its endpoints - then only right limit at a or only left limit at b is needed).





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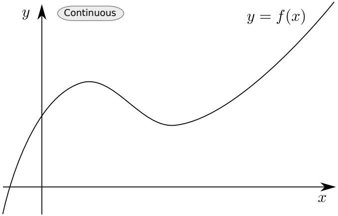
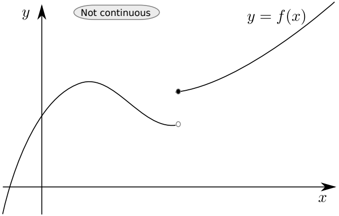
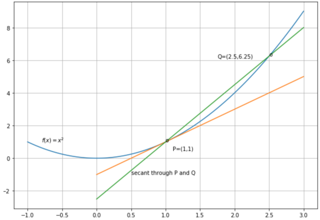




- The function f is *continuous* if it is continuous at every point a of its domain.

Tangents Revisited



- When x approaches a then $Q = (x, f(x))$ approaches $P = (a, f(a))$ on C defined by $y = f(x)$. The slope m_{PQ} of the secant line approaches the slope of the tangent line.

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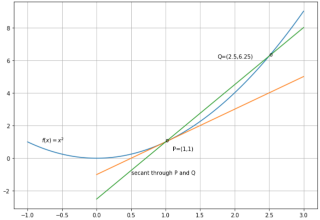

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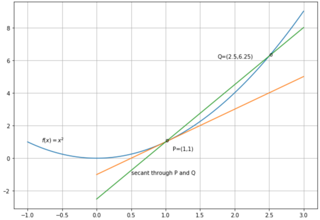

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- The slope of the tangent line of C at P is given by

$$m_P = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Tangents Revisited

- Consider the function $f(x) = x^2$. What is the tangent slope at point $P = (2, 4)$?

$$m_P = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} x+2 = 4$$

Tangents Revisited

- Let $h = x - a$. Then m_{PQ} and m_P can be written as

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- Consider the function $f(x) = \frac{3}{x}$. What is the tangent slope at point $P = (3, 1)$?

$$m_P = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\frac{3-(3+h)}{3+h}}{h} =$$

$$\lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

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$$\text{av. velocity} = \frac{\text{distance travelled}}{\text{time elapsed}} = \frac{s(t) - s(t_0)}{t - t_0} = 3\frac{t^2 - t_0^2}{t - t_0}$$

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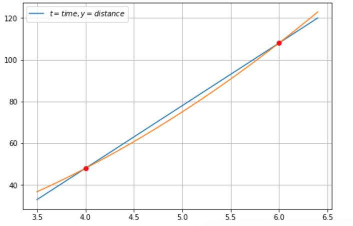
- What is the instantaneous velocity at time t_0?

$$\text{inst. velocity} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Velocity Revisited



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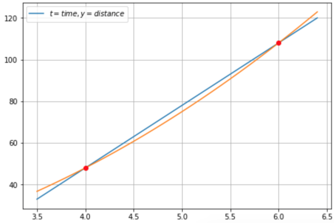

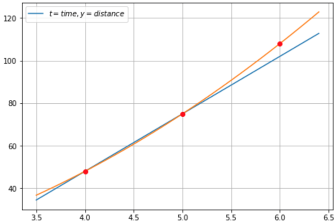

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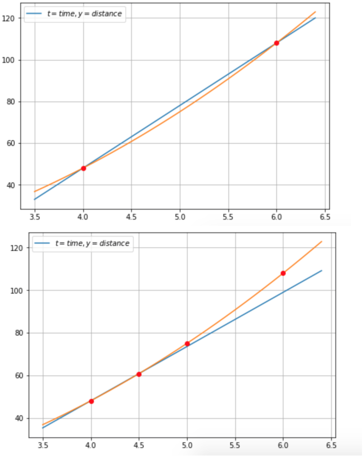

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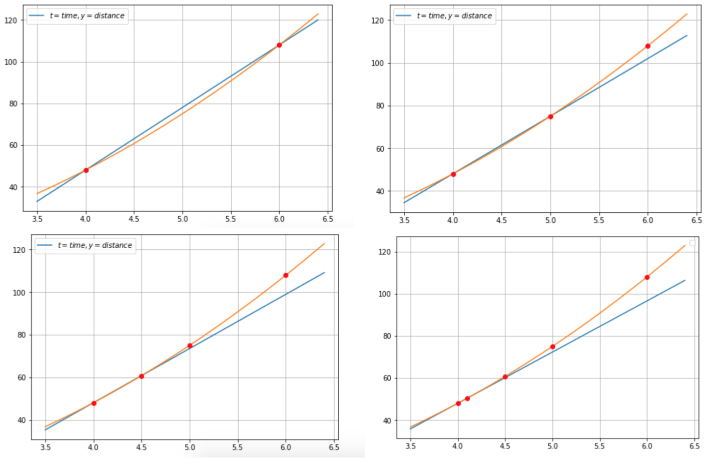

- Limits play central role in differentiation - main concept in calculus.

Derivatives

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- Many problems in sciences and engineering require calculations of rates of change.
- Given some function f, its *derivative at some a* (da. afledte i a) in the domain of f is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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- Derivative $f'(a)$ at a is a number.

Derivatives as Functions

- Given a function $f(x)$, its *derivative function* (da. afledt funktion) f' is defined by

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- Leibnitz notation: $\frac{dy}{dx}$

Differentiability

- The *derivative* of a function $f : \mathbb{R} \to \mathbb{R}$ at some x_0 in the domain of f is the limit

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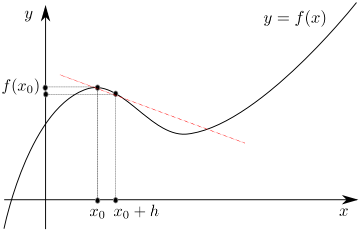

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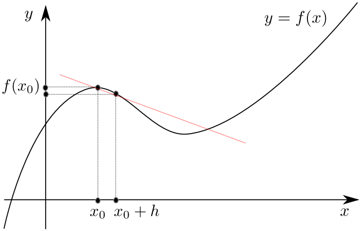

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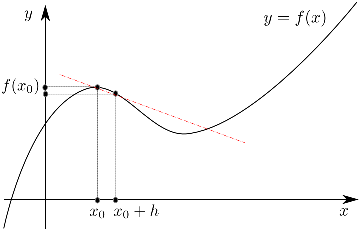
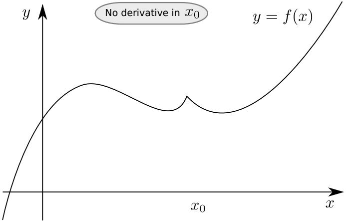
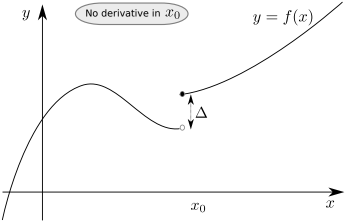


- A function f is *differentiable at some* x_0 in the domain of f if $f'(a)$ exists.
- When the derivative f' of f exists in every point x_0 in an open interval $]a, b[\in \mathbb{R}$, then f is *differentiable* in $]a, b[$ (a could be $-\infty$ and/or b could be ∞).

Does a derivative always exist?





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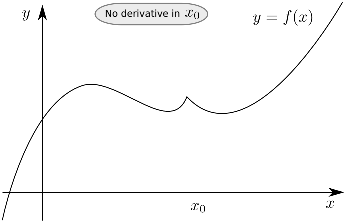

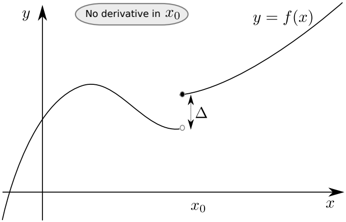

- If f is differentiable at x_0 then it is continuous in x_0.
- There are continuous functions that are not differentiable (e.g. $f(x) = |x|$ is not differentiable at 0).

Proof: f is differentiable in a $\implies$ f is continuous in a

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$$\lim_{x \to a} f(a) + f'(a) * 0 = f(a)$$

Higher Order Derivatives

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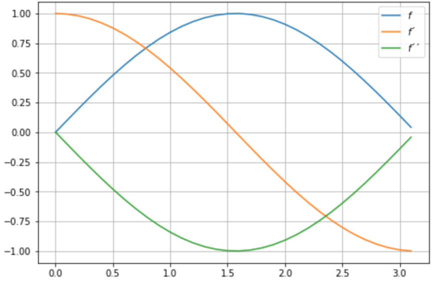
- f'' is the *second order* derivative of f and the derivative of f'. It can be interpreted as the rate of change of the slope of the curve defined by f.
- If f is the distance covered by a moving object as the function of time, f' is its instantaneous velocity as the function of time (or the rate of position change), and f'' is its instantaneous accelaration as the function of time (or the rate of velocity change).



Derivatives of Constant Functions

- **Constant Rule**: Let $f(x) = c$ where c is an arbitrary constant. Then $f'(x) = 0$. Let $a \in \mathbb{R}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{c - c}{h} = 0$$

Derivatives of Power Functions

- **Power Rule**: Let $f(x) = x^n$ where n is a positive integer. Then $f'(x) = nx^{n-1}$. Let $a \in \mathbb{R}$

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$$a^{n-1} + a^{n-2}a + ... + aa^{n-2} + a^{n-1} = na^{n-1}$$

Derivatives of Power Functions

- **Power Rule**: Let $f(x) = x^n$ where n is a positive integer. Then $f'(x) = nx^{n-1}$. Let $a \in \mathbb{R}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} =$$

$$\lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + ... + xa^{n-2} + a^{n-1})}{x - a} =$$

$$\lim_{x \to a} (x^{n-1} + x^{n-2}a + ... + xa^{n-2} + a^{n-1}) =$$

$$a^{n-1} + a^{n-2}a + ... + aa^{n-2} + a^{n-1} = na^{n-1}$$

- Generalizes to n being any real number (proof in Section 3.6).

New Derivatives from Old Derivatives

- **Constant Multiple Rule**: Let $g(x) = cf(x)$ where c is a constant and f is a differentiable function.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} =$$

$$\lim_{h \to 0} c \frac{f(x+h) - f(x)}{h} = c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = cf'(x)$$

where the second last equality follows from one of the limit laws.

Sum Rule

- Let $F(x) = f(x) + g(x)$ where both f and g are differentiable.

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(h)}{h} =$$

$$\lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} =$$

$$\lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] =$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} =$$

$$f'(x) + g'(x)$$

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$$f'(x) + g'(x)$$

  where the second last equality follows from one of the limit laws.

- **Difference Rule**: similar.

Sum Rule

- Let $F(x) = f(x) + g(x)$ where both f and g are differentiable.

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$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} =$$

$$f'(x) + g'(x)$$

  where the second last equality follows from one of the limit laws.
- **Difference Rule**: similar.
- These rules can be used to differentiate any polynomial.

Differentiating Polynomials

- Let $f(x) = x^7 - 4x^5 + 13x^4 - x + 19$

  $f'(x) = 7x^6 - 4*5x^4 + 13*4x^3 - 1x^0 + 0 = 7x^6 - 20x^4 + 52x^3 - 1$

Exponential Functions

- Let $f(x) = b^x$..

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} =$$

$$\lim_{h \to 0} \frac{b^x b^h - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h \to 0} \frac{b^h - 1}{h} = b^x f'(0)$$

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- This shows that if $f(x) = b^x$ is differentiable at 0 then it is differentiable everywhere.

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- This shows that if $f(x) = b^x$ is differentiable at 0 then it is differentiable everywhere.
- If $f(x) = 2^x$ then $f'(x) = f'(0)2^x$ and $f'(0)$=0.693...

Exponential Functions

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- This shows that if $f(x) = b^x$ is differentiable at 0 then it is differentiable everywhere.
- If $f(x) = 2^x$ then $f'(x) = f'(0)2^x$ and $f'(0)$=0.693...
- If $f(x) = 3^x$ then $f'(x) = f'(0)3^x$ and $f'(0)$=1.099...

Exponential Functions

- Let $f(x) = b^x$..

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$$\lim_{h\to 0} \frac{b^x b^h - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h\to 0} \frac{b^h - 1}{h} = b^x f'(0)$$

- This shows that if $f(x) = b^x$ is differentiable at 0 then it is differentiable everywhere.
- If $f(x) = 2^x$ then $f'(x) = f'(0)2^x$ and $f'(0)$=0.693...
- If $f(x) = 3^x$ then $f'(x) = f'(0)3^x$ and $f'(0)$=1.099...
- Determine constant e such that the exponential function $f(x) = e^x$ satisfies $f'(0) = 1$. This happens for e=2.71828... Then $f'(x) = f(x)$.

Product Rule (da. Produktreglen)

- Let $F(x) = f(x)g(x)$ where both f and g are differentiable. Then

$$F'(x) = f(x)g'(x) + f'(x)g(x)$$

Product Rule (da. Produktreglen)

- Let $F(x) = f(x)g(x)$ where both f and g are differentiable. Then

$$F'(x) = f(x)g'(x) + f'(x)g(x)$$

- Let $F(x) = x^3 \cos x$. Hence, $f(x) = x^3$ and $g(x) = \cos x$

$$F'(x) = x^3(-\sin x) + 3x^2 \cos x = -x^3 \sin x + 3x^2 \cos x$$

Product Rule - Proof

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} =$$

$$\lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0} \frac{g(x)[f(x+h) - f(x)]}{h} =$$

$$\lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h}\right] + \lim_{h\to 0} \left[g(x)\frac{f(x+h) - f(x)}{h}\right] =$$

$$\lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

$$f(x)g'(x) + g(x)f'(x)$$

Quotient Rule (da. Brøkreglen)

- Let $F(x) = \frac{f(x)}{g(x)}$ where both f and g are differentiable.

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Quotient Rule (da. Brøkreglen)

- Let $F(x) = \frac{f(x)}{g(x)}$ where both f and g are differentiable.

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

- Let $F(x) = \frac{2-x^2}{2x^3+x+3}$. Hence $f(x) = 2 - x^2$ and $g(x) = 2x^3 + x + 3$.

$$F'(x) = \frac{-2x(2x^3 + x + 3) - (2 - x^2)(6x^2 + 1)}{(2x^3 + x + 3)^2} =$$

$$\frac{2x^4 - 13x^2 - 6x - 2}{(2x^3 + x + 3)^2}$$

Quotient Rule - Proof

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

Quotient Rule - Proof

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

$$\lim_{h \to 0} \left\{ \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right\} =$$

Quotient Rule - Proof

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

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Quotient Rule - Proof

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

$$\lim_{h\to 0} \left\{ \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right\} =$$

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$$\lim_{h\to 0} \left\{ \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \right\} =$$

Quotient Rule - Proof

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

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$$\lim_{h \to 0} \left\{ \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right\} =$$

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$$\lim_{h \to 0} \left\{ \frac{1}{g(x+h)g(x)} \left[g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h} \right] \right\} =$$

Quotient Rule - Proof

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

$$\lim_{h\to 0} \left\{ \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right\} =$$

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$$\lim_{h\to 0} \left\{ \frac{1}{g(x+h)g(x)} \left[g(x) \frac{f(x+h) - f(x)}{h} - f(x) \frac{g(x+h) - g(x)}{h} \right] \right\} =$$

$$\lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left[\lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \right] =$$

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$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

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$$\lim_{h\to 0} \left\{ \frac{1}{g(x+h)g(x)} \left[g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h} \right] \right\} =$$

$$\lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left[\lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \right] =$$

$$\frac{g(x)f'(x) - f(x)g'(x)}{\lim_{h\to 0} g(x+h) \lim_{h\to 0} g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Summary

By now, you should be familar with

- the definition of function limits,
- the most familiar rules for function limits, and be able to use them for deriving limits of new functions,
- the definition of continuity, and be able to use it to prove continuity of other functions,
- the definition of the derivative of a function $f : \mathbb{R} \to \mathbb{R}$,
- basic differentiation rules.