# MASD

Lecture 4
15.09.2022

François Lauze (Pawel again...)

Lecture starts at 10.15

# Objectives

- Functions of several variables.
- Limits and continuity for functions of several variables.
- Partial derivatives.
- Tangent planes
- Chain rules

We cover material in Sections 14.1-14.5

# Functions of Two Variables

- A function $f$ assigns to each (ordered) pair of numbers $(x, y)$ in a domain $D$ a unique value $f(x, y)$. The range of $f$ is $R_f = \{f(x, y) | (x, y) \in D\}$

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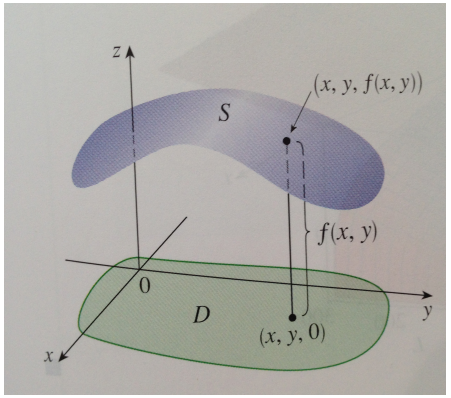
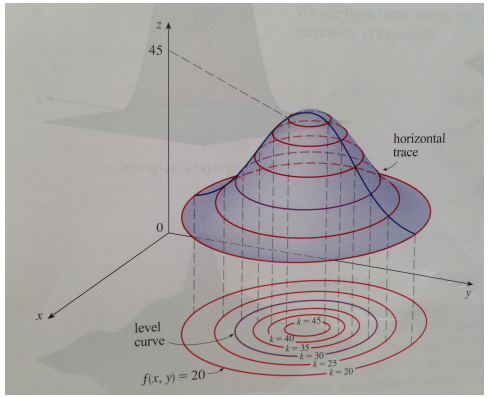
- A function with two variables can be visualized by
  - a 2-dimensional table where (selected) $x$ and $y$ values correspond to rows and columns while the function value $f(x, y)$ is in the appropriate entry of the table.
  - a graph which is the set of all points $(x, y, z)$ where $(x, y) \in D$ and $z = f(x, y)$.
  - level curves which are 2-dimensional curves with equations $f(x, y) = k$ for appropriately chosen values of $k$ (in the range of $f$).

**Table 1** Wind-chill index as a function of air temperature and wind speed

Wind speed (km/h)

| Actual temperature (°C) $T$ \ $v$ | 5 | 10 | 15 | 20 | 25 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 4 | 3 | 2 | 1 | 1 | 0 | −1 | −1 | −2 | −2 | −3 |
| 0 | −2 | −3 | −4 | −5 | −6 | −6 | −7 | −8 | −9 | −9 | −10 |
| −5 | −7 | −9 | −11 | −12 | −12 | −13 | −14 | −15 | −16 | −16 | −17 |
| −10 | −13 | −15 | −17 | −18 | −19 | −20 | −21 | −22 | −23 | −23 | −24 |
| −15 | −19 | −21 | −23 | −24 | −25 | −26 | −27 | −29 | −30 | −30 | −31 |
| −20 | −24 | −27 | −29 | −30 | −32 | −33 | −34 | −35 | −36 | −37 | −38 |
| −25 | −30 | −33 | −35 | −37 | −38 | −39 | −41 | −42 | −43 | −44 | −45 |
| −30 | −36 | −39 | −41 | −43 | −44 | −46 | −48 | −49 | −50 | −51 | −52 |
| −35 | −41 | −45 | −48 | −49 | −51 | −52 | −54 | −56 | −57 | −58 | −60 |
| −40 | −47 | −51 | −54 | −56 | −57 | −59 | −61 | −63 | −64 | −65 | −67 |



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- Formally: Let $f$ be a function with two variables defined in an arbitrarily small open disk $D$ around $(a, b)$ (but not necessarily in $(a, b)$). Then

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ for every $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$

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- Limit lows for functions with one variable extend in natural way to functions with two or more variables.

# Continuity of a Function of Two Variables

- A function $f$ of two variables is continuous at $(a, b)$ if

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- Intuition: Graphs of continuous functions of 2 variables have no holes or breaks.

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- The derivative function of $g_c$ is $g_c'(x) = 3x^2 + 2c^3x$
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- $g_c'(a)$ is the *partial derivative* of $f$ with respect to $x$ at $(a, c)$. It is denoted by $f_x(a, c)$.

$$f_x(a, c) = g_c'(a) = \lim_{h \to 0} \frac{g_c(a + h) - g_c(a)}{h} = \lim_{h \to 0} \frac{f(a + h, c) - f(a, c)}{h}$$

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- Let $x = c$. $f_y(c, b)$ is the partial derivative of $f$ w.r.t. $y$ at $(c, b)$ and the function $f_y$ is the partial derivative function of $f$ w.r.t. $y$.

# Partial Derivatives and Notation

- Let $z = f(x, y)$ be a function of two variables. Partial derivatives of $f$ are:

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- There are many different ways to denote partial derivatives:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

# Partial Derivatives - Example I

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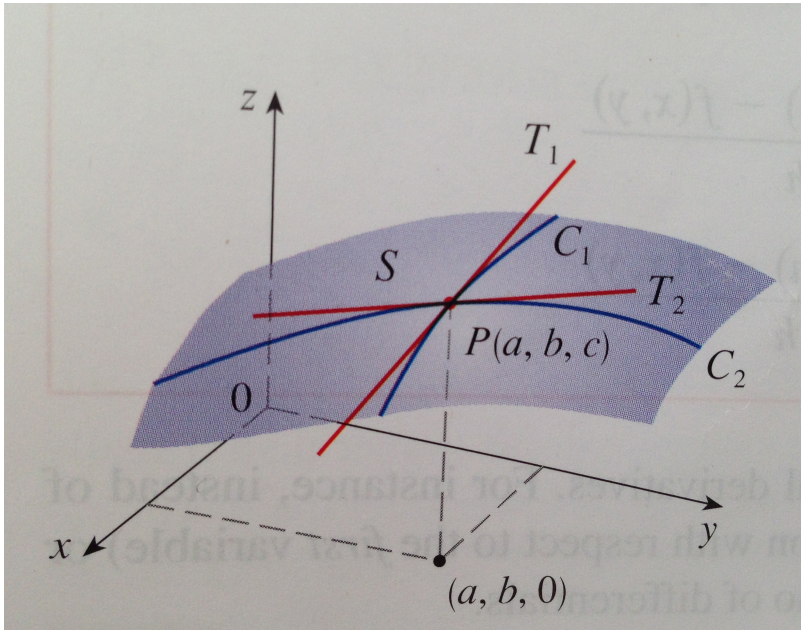
# Partial Derivatives - Geometric Interpretation

- Partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are the slopes of tangent lines at $P(a, b, c)$ for curves $f(x, b)$ and $f(a, y)$ on the surface graph.



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$$\frac{\partial u}{\partial z} = 10(3x^2y + z^4)^9 * 4z^3 = 40z^3(3x^2y + z^4)^9$$

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$$\frac{\partial z}{\partial y} = x^y \ln x \text{ for } x > 0$$

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To understand the last partial derivative, let $b = \ln x$. Then we have to show that

$$\frac{\partial}{\partial y}(b^{\sin y}) = b^{\sin y} (\ln b) \cos y$$

Using chain rule with $u = \sin y$, we get $\frac{d}{du}(b^u) = b^u \ln b$ and $\frac{d}{dy}(\sin y) = \cos y$. Backsubstituting $u$ and $b$ proofs the claim.

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$f_{xx} = 6x + 2y^3$

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# Higher Partial Derivatives - Example

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$f_x = 3x^2 + 2xy^3$

$f_y = 3x^2y^2 - 4y$

$f_{xx} = 6x + 2y^3$

$f_{xy} = 6xy^2$

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$f_{yy} = 6x^2y - 4$

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$f_{xy} = 6xy^2$

$f_{yx} = 6xy^2$

$f_{yy} = 6x^2y - 4$

- Note that $f_{xy} = f_{yx}$. This holds in general (Clairaut's Theorem).

# Partial Differential Equations

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- Let $u(x, y) = e^x \sin y$. Then

$$u_x = e^x \sin y, u_y = e^x \cos y, u_{xx} = e^x \sin y, u_{yy} = -e^x \sin y$$

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$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

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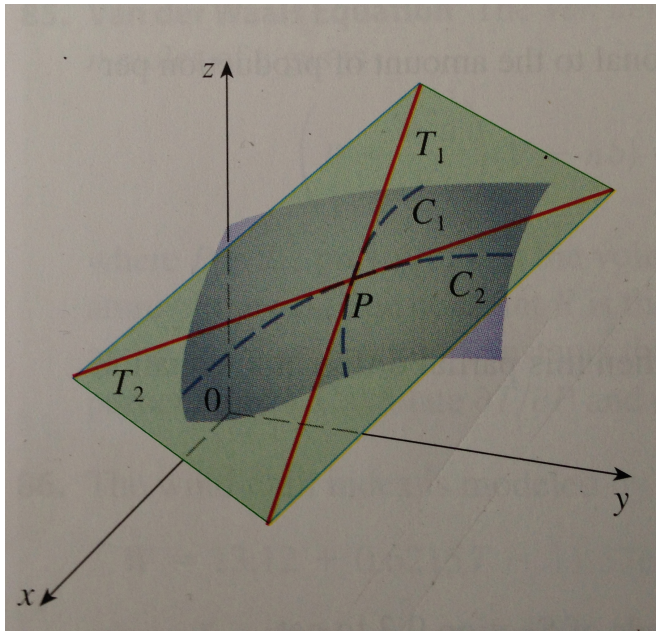
- Wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is another important partial differential equation.

# Tangent Plane

- $S$: surface defined by equation $z = f(x, y)$. $P(x_0, y_0, z_0)$: point on $S$. Assume that $f$ has continuous first partial derivatives $f_x$ and $f_y$.



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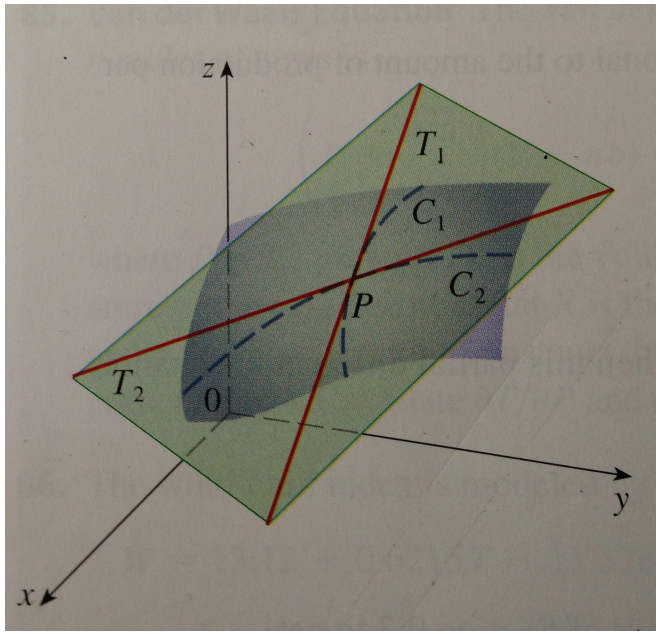


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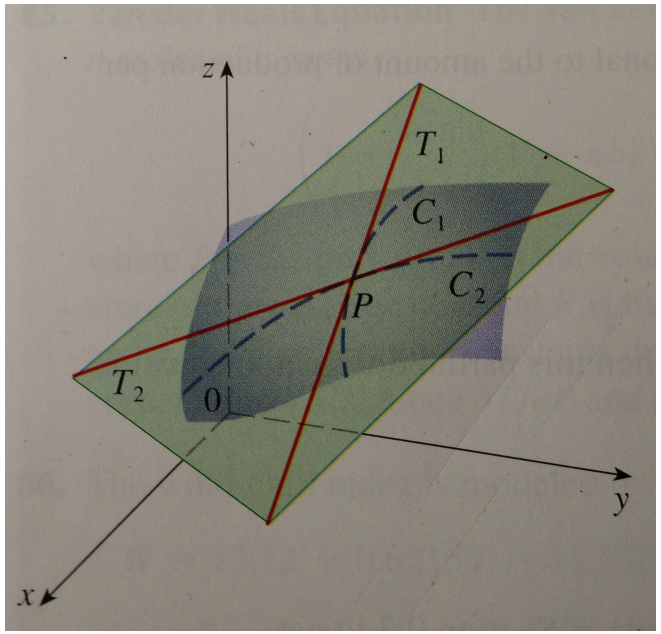


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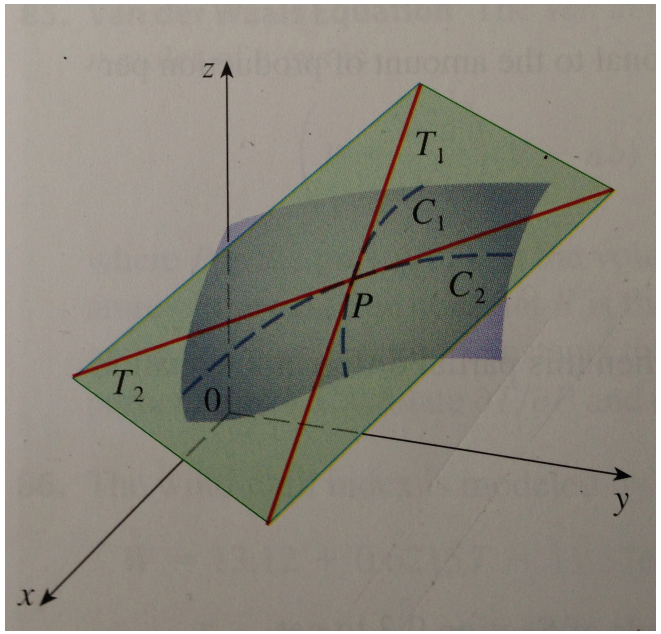


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- $T_1$ and $T_2$: tangent lines of $C_1$ and $C_2$ at $P$.
- Tangent plane of the surface $S$ at $P$ is defined by $T_1$ and $T_2$.

# Tangent Plane Equation

- A plane with the normal vector $[A, B, C]^T$ and through the point $P = (x_0, y_0, z_0)$ has the equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

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$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

- Dividing the equation by $C$ and setting $a = -A/C$ and $b = -B/C$ yields

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

# Tangent Plane Equation

- A plane with the normal vector $[A, B, C]^T$ and through the point $P = (x_0, y_0, z_0)$ has the equation

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- Dividing the equation by $C$ and setting $a = -A/C$ and $b = -B/C$ yields

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- Intersection of this tangent plane with the line $T_1$ given by $y = y_0$ yields a tangent line

$$z - z_0 = a(x - x_0)$$

with $a$ being its slope. Therefore $a = f_x(x_0, y_0)$. Similarly, $b = f_y(x_0, y_0)$. Therefore

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

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- Tangent planes which are linear functions of two variables can be used to approximate such surfaces of differentiable functions of two variables.
- The equation of the tangent plane of a function $f$ at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

and the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called a **linearization** of $f$ at $(a, b)$. It is the tangent plane approximation of $f$ at $(a, b)$.

# Linear Approximations and Differentiable Functions

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$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

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- If the partial derivatives $f_x$ and $f_y$ exist near $(a, b)$ and are continuous at $(a, b)$, then $f$ is differentiable at $(a, b)$.

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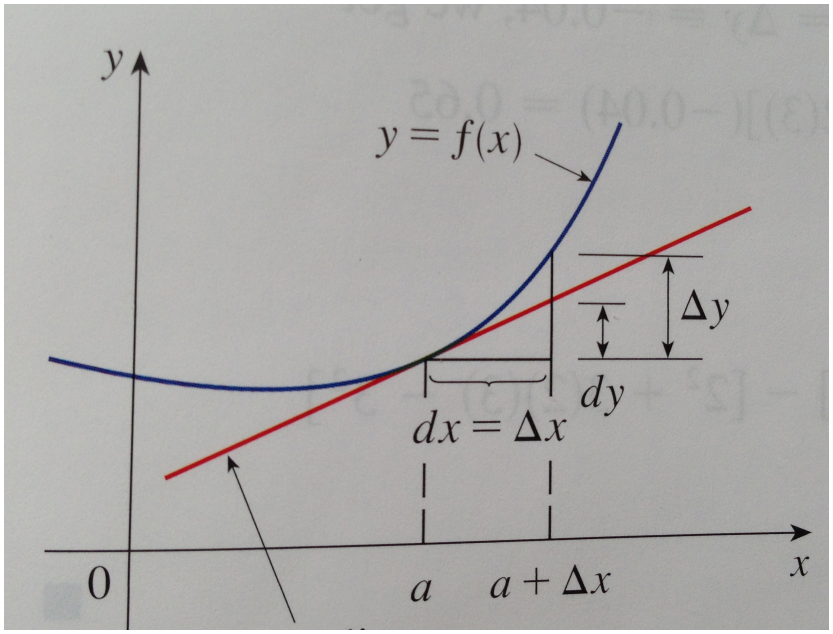
- $\Delta y$ represents the change in height of the curve $y = f(x)$ while $dy$ represent the change in height of the tangent line when $x$ changes by $dx = \Delta x$.



# Differentials for Functions with Two Variables

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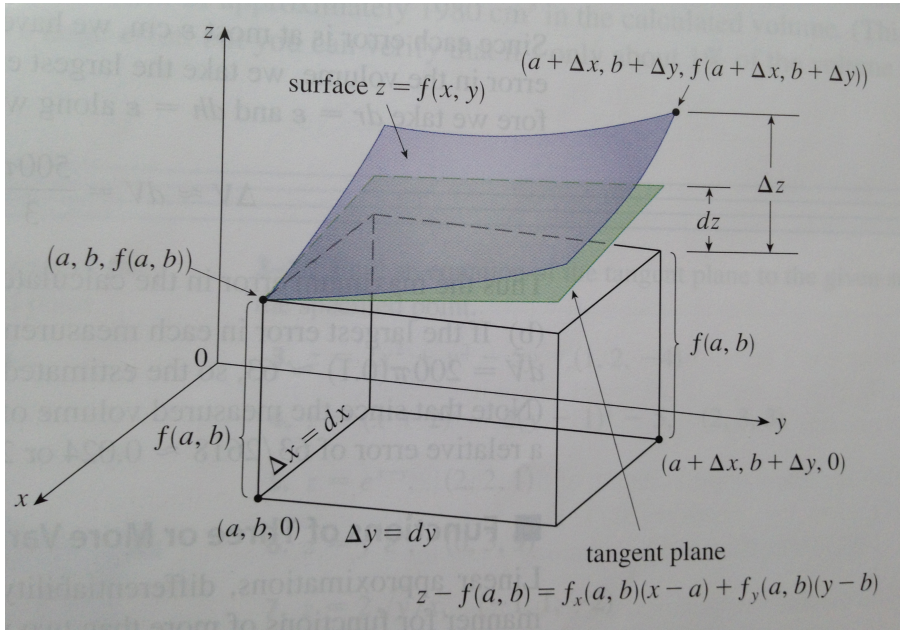
- $dz$ represent the change in height of the tangent plane while $\Delta z$ represents the change in height of the surface $z = f(x, y)$ when $(x, y)$ changes from $(a, b)$ to $(a + \Delta x, b + \Delta y)$.



# Chain Rules for Functions of Two Variables

- Chain rule for functions of one variable applies to composite functions $z = F(t) = f(g(t))$ where $y = f(x)$ and $x = g(t)$ are both differentiable functions. The rule states that $F$ is a differentable function of $t$ and $F'(t) = f'(g(t))g'(t)$ or $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

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  - $z = f(g(s,t), h(s,t))$ - inner functions are functions of two variables.
  - $z = f(g_1(t_1, t_2, ..., t_m), g_2(t_1, t_2, ..., t_m), ..., g_n(t_1, t_2, ..., t_m))$ - general case where $f$ is a differentiable function of $n$ variables $x_1, x_2, ..., x_n$ and each $x_i = g_i$ is a differentiable function of $m$ variables $t_1, t_2, ..., t_m$

# Chain Rule - Case 1

- $z = f(x, y)$ is differentiable function of $x$ and $y$, where $x = g(t)$ and $y = h(t)$ are differentiable functions of $t$. Then $z$ is indirectly a differentiable function of $t$ and (proof omitted)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- Example: $z = x^2y + 3xy^4, x = \sin 2t, y = \cos t$

$$\frac{dz}{dt} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

- For $t = 0$, we observe that $x = \sin 2t = 1$ and $\cos t = 1$. Hence,

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2 * 1) + (0 + 0)(0) = 6$$

# Chain Rule - Case 2

- $z = f(x, y)$ is differentiable function of $x$ and $y$, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of $s$ and $t$. Then $z$ is indirectly a function of $s$ and $t$ and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

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$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

- This follows almost immediately from Case 1: when computing $\frac{\partial z}{\partial t}$, we hold $s$ fixed ($g$ and $h$ become functions of one variable). Similarly when computing $\frac{\partial z}{\partial s}$.

# Chain Rule - General Case

- $z = f(g_1(t_1, t_2, ..., t_m), g_2(t_1, t_2, ..., t_m), ..., g_n(t_1, t_2, ..., t_m))$ - general case where $f$ is a differentiable function of $n$ variables $x_1, x_2, ..., x_n$ and each $x_i = g_i$ is a differentiable function of $m$ variables $t_1, t_2, ..., t_m$. Then

$$\frac{\partial z}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, ..., m$.

# Summary

- We covered various topics related to function of two variables such as:
  - limits and continuity,
  - partial derivatives,
  - tangent planes and linear approximations,
  - chain rules.