MASD 2020, Written Exam

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This 4 hours written exam consists of seven equally weighted problems. Use separate sheets of paper for each problem. Suggestion 1: Start with problems that you think would be easiest. Suggestion 2: Handwriting is usually faster than typing. Make it as readable as possible.
Open book. You are permitted to use all kinds of aid materials including electronic devices, provided that you do not access internet and do not communicate with others.
IMPORTANT: Justify your statements. In particular, you may refer to the textbooks used in the course, to slides (if available on absalon) and to the lecture notes and assignments given during the course. It is for example permitted to justify a statement by writing that it derives trivially from a result in the textbook (if this is the case). Always provide precise locations. References to other books or any other sources will not be accepted.
Notation. $]2,4[$ is used to represent an open interval between 2 and 4 in $\mathbb{R}$.

Problem 1 (Function Limits, Continuity). Consider the function $f(x) = \frac{3x^2-5x-2}{5x^2-20}$.

a) Is f defined for all $x \in \mathbb{R}$? If not, specify for which $x \in \mathbb{R}$ the function f is not defined.

b) Determine $\lim_{x\to a} f(x)$ for any $a \in \mathbb{R}, |a| \neq 2$.

c) Determine $\lim_{x\to 2} f(x)$ or decide that it does not exist. Hint: Start by factoring both polynomials.

d) Determine $\lim_{x\to -2} f(x)$ or decide that it does not exist.

e) Is f continuous for all $x \in \mathbb{R}$?

f) We now restrict the domain of f to the open interval $]-2,2[$. Is restricted f differentiable everywhere in $]-2,2[$?

Solution:

a) No. It is defined neither for $x = 2$ nor for $x = -2$ as then the denominator is 0.

b) We use the limit law 5 (p. 95) stating that the limit of a rational function is equal to the ratio of the limits of numerator and denominator polynomials provided that the latter is not 0. So the limit is $\frac{3a^2-5a-2}{5a^2-20}$. This can be simplified to $\frac{3a+1}{5a+10}$.

c) We observe that $x = 2$ is the root of both $3x^2 - 5x - 2 = 0$ and $5x^2 - 20 = 0$. Therefore $3x^2 - 5x - 2 = 3(x-2)(x+\frac{1}{3})$ and $5x^2 - 20 = 5(x-2)(x+2)$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{3(x+\frac{1}{3})}{5(x+2)} \frac{x-2}{x-2} = \lim_{x\to 2} \frac{3(x+\frac{1}{3})}{5(x+2)} \lim_{x\to 2} \frac{x-2}{x-2} = \frac{3(2+\frac{1}{3})}{5(2+2)} \times 1 = \frac{7}{20}$$

using that a limit of a product is the product of the limits (p. 95), and direct substitution for rational functions (p. 97).

d) Then the numerator goes to 20 and the denominator goes to 0. The limit does not exist. Notice that $\lim_{x\to -2^-} f(x) = -\infty$ and $\lim_{x\to -2^+} f(x) = \infty$

e) Polynomials are continuous everywhere (p. 120). Rational functions are continuous everywhere apart when the denominator is 0 (p. 120). Neither $f(2)$ nor $f(-2)$ are defined and therefore f is continuous neither at 2 nor at -2 (p. 115).

f) Both numerator and denominator are polynomials and therefore differentiable everywhere. The denominator is $\neq 0$ in $]-2,2[$. Hence, f' exists for all $x \in]-2,2[$.

Problem 2 (Functions of 2 and More Variables). Consider the function $f(x,y,z) = (x+y)^2 + (y+z)^2 + (z+x)^2$ defined for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$.

a) What is the gradient vector of f at the point $(2,-1,2)$?

b) What is the directional derivative of f in the gradient vector direction at the point $(2,-1,2)$?

c) What are the critical points of f?

d) Classify each critical point as local minimum, local maximum or neither. Hint: Start by deciding what is the range of f.

Solution:

a) $\nabla(f(x,y,z)) = \langle f_x, f_y, f_z\rangle$ at $(2,-1,2)$. So we need to determine partial derivatives f_x, f_y, f_z:

- $f_x(x,y,z) = 2(x+y) + 2(z+x)$
- $f_y(x,y,z) = 2(x+y) + 2(y+z)$
- $f_z(x,y,z) = 2(y+z) + 2(z+x)$

Plugging $(2,-1,2)$ gives the gradient vector at that point: $\nabla(f(2,-1,2)) = \langle 10,4,10\rangle$.

b) The unit vector in the direction of the gradient is $\mathbf{u} = \langle \frac{10}{6\sqrt{6}}, \frac{4}{6\sqrt{6}}, \frac{10}{6\sqrt{6}}\rangle^T = \langle \frac{5}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}, \frac{5}{3\sqrt{6}}\rangle^T$. $D_u f(2,-1,2) = \nabla f(2,-1,2)\cdot \mathbf{u} = \frac{108}{3\sqrt{6}} = \frac{36}{\sqrt{6}} = 6\sqrt{6}$.

c) $f_x(x,y,z) = 0, f_y(x,y,z) = 0, f_z(x,y,z) = 0$ has only one solution: $x = y = z = 0$. f is defined everywhere so there is only one critical point: $(0,0,0)$.

d) Since f has only non-negative values and $f(0,0,0) = 0$, this critical point is a (global) minimum.

Problem 3 (Taylor and Maclaurin Series). Consider the function

$$f(x) = (1+x)^s$$

where s is an arbitrary real number other than 0.

a) What is the Maclaurin series for $f(x)$?

b) What is the radius of convergence of this Maclaurin series? Hint: Use the ratio test (p. 774).

c) When s is a positive integer then it can be shown that f is equal to the sum of its Maclaurin series for $x \in]-1,1[$. Suppose that you are asked to approximate $f(0.07)$ for any positive integer s. Explain (without doing any calculations) how would you do it.

Solution:

a)

$$f(0) = 1$$
$$f'(x) = s(1+x)^{s-1} \text{ and } f'(0) = s$$
$$f''(x) = s(s-1)(1+x)^{s-2} \text{ and } f''(0) = s(s-1)$$
$$f'''(x) = s(s-1)(s-2)(1+x)^{s-3} \text{ and } f'''(0) = s(s-1)(s-2)$$

In general

$$f^{(n)}(x) = s(s-1)(s-2)\dots(s-n+1)(1+x)^{s-n} \text{ and } f^{(n)}(0) = s(s-1)(s-2)\dots(s-n+1)$$

Maclaurin series is therefore

$$1 + \frac{s}{1!}x + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!}x^3 + \dots$$

b) We have

$$a_{n+1} = \frac{s(s-1)...(s-n)}{(n+1)!}x^{n+1}$$
$$a_n = \frac{s(s-1)...(s-n+1)}{n!}x^n$$

and

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{s(s-1)...(s-n)}{(n+1)!} \cdot \frac{n!}{s(s-1)...(s-n+1)} \cdot \frac{x^{n+1}}{x^n}\right| = \left|\frac{s-n}{n+1}x\right| = \left|\frac{\frac{s}{n}-1}{1+\frac{1}{n}}x\right|$$

Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$$

By the ratio test, this series converges for all $x \in]-1,1[$ and the radius of convergence is 1.

c) Take the n-th-degree Taylor polynomial $T_n(x)$ of f at 0 for some low n (for example $n = 2$ or $n = 3$) and then determine $T_n(0.07)$. Higher n, $n \leq s$ has to be taken to obtain better approximations.

Problem 4 (Integration). Remember to justify all non-trivial details. In particular, clearly state if you use the substitution rule or integration by parts.

a) Determine $\int x(x-1)(x-2)dx$ and $\int_0^1 x(x-1)(x-2)dx$.

b) Determine $\int \frac{x-3}{x^2-6x+5}dx$.

c) Determine $\int x^{10}\ln x dx$.

Solution:

a)

$$\int x(x-1)(x-2)dx = \int(x^3-3x^2+2x)dx = \int x^3dx - 3\int x^2dx + 2\int xdx =$$
$$\frac{x^4}{4} - 3\frac{x^3}{3} + 2\frac{x^2}{2} + C = \frac{x^4}{4} - x^3 + x^2 + C$$

and by FTC2:

$$\int_0^1 x(x-1)(x-2) = \frac{1}{4} - 1 + 1 - \frac{0}{4} - 0 + 0 = \frac{1}{4}$$

b) The roots of $x^2 - 6x + 5 = 0$ are $x_1 = 1$ and $x_2 = 5$. Suppose that $x \neq 1$ and $x \neq 5$. We notice that the derivative of the denominator is $f'(x) = 2x - 6$ which differs by factor 2 from the numerator. We will use the substitution rule where the outer function f is the identity function. Let $u = g(x) = x^2 - 6x + 5$. Then

$$\int \frac{x-3}{x^2-6x+5}dx = \int \frac{1}{x^2-6x+5} \cdot \frac{1}{2}(2x-6)dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-6x+5| + C$$

using the Table of Indefinite Integrals (p. 410).

c) We assume that $x > 0$. We use integration by parts with $u = \ln x$, $dv = x^{10}dx$. Hence, $du = \frac{1}{x}dx$, $v = \int x^{10}dx = \frac{1}{11}x^{11}$. We get

$$\int x^{10}\ln x dx = \frac{1}{11}x^{11}\ln x - \int \frac{1}{11}x^{11} \cdot \frac{1}{x}dx = \frac{1}{11}x^{11}\ln x - \frac{1}{11}\int x^{10}dx =$$

$$\frac{1}{11}x^{11}\ln x - \frac{1}{11}\frac{1}{11}x^{11} + C = \frac{1}{11}x^{11}(\ln x - \frac{1}{11}) + C$$

Problem 5 (Combinatorics). We sample n balls enumerated with the numbers $1, 2, \ldots, n$ one by one from a hat without replacement, i.e. we blindly pick a ball in each of n rounds.

a) Provide an appropriate model for the probability space underlying this experiment.

b) What is the probability of at least one ball having the number on it as the round it was picked in?

c) What is the probability p_n that the balls with numbers $1, 2$ are picked consecutively at some point during our experiment, i.e. without any other balls being picked in between them? What does p_n converge to as $n \to \infty$?

Solution:

a) $\Omega = S_n$ the permutations of the numbers $\{1, \ldots, n\}$ with the uniform distribution is an appropriate model.

b) We define $A_i = \{\sigma \in S_n : \sigma_i = i\}$ and use the inclusion-exclusion principle to see

$$\begin{aligned}
\mathbb{P}(\cup_{i=1}^n A_i) &= \sum_{\emptyset \neq I \subset \{1,\ldots,n\}} (-1)^{|I|+1}\mathbb{P}(\cap_{i\in I} A_i) \\
&= \sum_{\emptyset \neq I \subset \{1,\ldots,n\}} (-1)^{|I|+1}\frac{(n-|I|)!}{n!} \\
&= \sum_{k=1}^{n} \binom{n}{k}(-1)^{k+1}\frac{(n-k)!}{n!} \\
&= 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!} .
\end{aligned}$$

c) Let A_i be the event that $1, 2$ are the picks in rounds $i, i+1$. Then

$$p_n = \mathbb{P}(\text{We see } 1,2 \text{ consecutively}) = \mathbb{P}(\cup_{i=1}^{n-1} A_i) = \sum_{i=1}^{n-1}\mathbb{P}(A_i) = \sum_{i=1}^{n-1}\frac{(n-2)!}{n!} = \frac{(n-1)!}{n!} = \frac{1}{n} .$$

In particular, $p_n \to 0$ as $n \to \infty$.

Problem 6 (Random Variables and Distributions).

a) Let (X, Y) be a 2-dimensional Gaussian random vector with

$$\mathbb{E}X = 0\,, \qquad \mathbb{E}Y = 0\,, \qquad \operatorname{Var} X = 2\,, \qquad \operatorname{Var} Y = 4\,, \qquad \operatorname{Cov}(X, Y) = 1\,.$$

What is the **joint** distribution of $X + Y$ and $X - Y$?

Hint: Recall that linear transformations of Gaussian vectors are again Gaussian vectors.

b) Let X_1, X_2 be independent random variables with distribution Exp(1). Determine the probability density function (pdf) of $X_1 + X_2$ and $\min\{X_1, X_2\}$.

Solution:

a) Since $(X+Y, X-Y)$ is a linear transformation of a centred Gaussian vector, it is also a centred Gaussian vector. Thus, it suffices to determine the covariance matrix.

$$\begin{aligned}
\operatorname{Var}(X+Y) &= \operatorname{Var} X + \operatorname{Var} Y + 2\operatorname{Cov}(X, Y) = 2 + 4 + 2 = 8\,, \\
\operatorname{Var}(X-Y) &= \operatorname{Var} X + \operatorname{Var} Y - 2\operatorname{Cov}(X, Y) = 2 + 4 - 2 = 4\,, \\
\operatorname{Cov}(X+Y, X-Y) &= \operatorname{Var} X - \operatorname{Var} Y = 2 - 4 = -2
\end{aligned}$$

Thus $(X+Y, X-Y) \sim N(0, A)$ with

$$A = \begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix}.$$

b) Here $\alpha = 1$. We compute the cdf of $X_{\min} = \min_{i=1}^{2} X_i$ as

$$\begin{aligned}
\mathbb{P}(X_{\min} \le x) &= 1 - \mathbb{P}(X_{\min} > x) \\
&= 1 - \mathbb{P}(X_1 > 0 \text{ and } X_2 > 0) \\
&= 1 - \mathbb{P}(X_1 > 0)^2 \\
&= 1 - (1 - F_{X_1}(x))^2 \\
&= 1 - \mathrm{e}^{-2\alpha x}.
\end{aligned}$$

Thus $X_{\min} \sim \operatorname{Exp}(2\alpha)$. Now we use the convolution formula to determine the pdf for $X_1 + X_2$ and get

$$\rho_{X_1+X_2}(x) = \alpha^2 \int_0^x \mathrm{e}^{-\alpha y}\mathrm{e}^{-\alpha(x-y)}\mathrm{d}y\,\mathbb{1}_{(0,\infty)}(x) = \alpha^2 x\,\mathrm{e}^{-\alpha x}\,\mathbb{1}_{(0,\infty)}(x)\,.$$

Problem 7 (Expectation and Variance). Compute the following (each item is a separate problem):

a) Expectation $\mathbb{E}X$ and variance $\operatorname{Var} X$ of a random variable X with $X = 2Y + 1$, where Y is uniformly distributed on $[0, 1]$.

b) Variance $\operatorname{Var} X$ of the random variable $X = \mathrm{e}^{-tY}$ for $t > 0$, where $Y \sim \operatorname{Exp}(1)$.

c) Expectation $\mathbb{E}X$ of a random variable X that has cumulative distribution function (cdf)

$$F_X(x) = 1 - \mathrm{e}^{-\alpha\lfloor x \rfloor}$$

for any $x \ge 0$, where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \le x\}$ is rounding down and $\alpha > 0$ is a constant. Hint: First decide whether the random variable X is discrete or continuous.

Solution:

a) We have

$$\mathbb{E}Y = \int_0^1 y\,\mathrm{d}y = \frac{1}{2}, \quad \mathbb{E}Y^2 = \int_0^1 y^2\,\mathrm{d}y = \frac{1}{3}, \quad \operatorname{Var} Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \frac{1}{12}\,.$$

Thus, $\mathbb{E}X = 2\mathbb{E}Y + 1 = 2$ and $\operatorname{Var} Y = \operatorname{Var}(2X) = 4\operatorname{Var} X = \frac{1}{3}$.

b) Here $\alpha = 1$. We compute the expectation using the pdf of an exponentially distributed random variable

$$\mathbb{E}X = \mathbb{E}\mathrm{e}^{-tY} = \alpha \int_0^\infty \mathrm{e}^{-ty}\mathrm{e}^{-\alpha y}\mathrm{d}y = \frac{\alpha}{t+\alpha} = \frac{1}{1+t}\,.$$

Performing the same calculation with t replaced by $2t$ yields $\mathbb{E}X^2 = \frac{\alpha}{2t+\alpha}$. Thus

$$\operatorname{Var} X = \frac{\alpha}{2t+\alpha} - \frac{\alpha^2}{(t+\alpha)^2} = \frac{t^2}{(1+2t)(1+t)^2}\,.$$

c) From the cdf we see that

$$\mathbb{P}(X = x) = \mathrm{e}^{-\alpha(x-1)} - \mathrm{e}^{-\alpha x} = \mathrm{e}^{-\alpha x}(\mathrm{e}^{\alpha} - 1)$$

for every $x \in \mathbb{N}$. Thus, we compute

$$\begin{aligned}
\mathbb{E}X &= \sum_{x\in\mathbb{N}} x\mathbb{P}(X = x) \\
&= (\mathrm{e}^{\alpha} - 1)\sum_{x\in\mathbb{N}} x\mathrm{e}^{-\alpha x} \\
&= -(\mathrm{e}^{\alpha} - 1)\frac{\mathrm{d}}{\mathrm{d}\alpha}\sum_{x\in\mathbb{N}} \mathrm{e}^{-\alpha x} \\
&= -(\mathrm{e}^{\alpha} - 1)\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{1-\mathrm{e}^{-\alpha}} \\
&= \frac{(\mathrm{e}^{\alpha} - 1)\mathrm{e}^{-\alpha}}{(1-\mathrm{e}^{-\alpha})^2} \\
&= \frac{1}{1-\mathrm{e}^{-\alpha}}\,.
\end{aligned}$$