MASD

Lecture 3
13.09.2022

François Lauze (material from Pawel Winter and others)

Lecture will start at 8.15

Objectives

We cover sections 3.4-3.6 and 3.10

- Differentiation of composite functions: Chain rule
- Implicit differentiation
- Differentiation of inverse functions
- Differentiation of logarithmic functions
- Logarithmic differentiation
- Linear approximations of functions
- Differentials

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- F is differentiable at x and $F'(x) = f'(g(x))g'(x)$.
- Let $u = g(x)$ and $y = f(u)$. Then using Leibnitz notation

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

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  $3(4x^5 - 7x^3 + 14x^2 - 5)^2(20x^4 - 21x^2 + 28x)$

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- $\frac{du}{dx} = \frac{d}{dx}(3x^2 - 7x + 12) = 6x - 7$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(6x - 7) = \frac{6x - 7}{2\sqrt{3x^2 - 7x + 12}}$$

Chain Rule - Example III

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- Let $y = \sin(u)$ and $u = 4x$
- $\frac{dy}{du} = \frac{d}{du}(sin(u)) = \cos(u)$
- $\frac{du}{dx} = \frac{d}{dx}(4x) = 4$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos(u) * 4 = 4\cos(4x)$$

Chain Rule - Example IV

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- $\frac{du}{dx} = \frac{d}{dx}(\cos(x)) = -\sin(x)$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(-\sin(x)) = -3\cos^2(x)\sin(x)$

Chain Rule - Example V

- Let $F(x) = e^{-x}$. This is a composite function.

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- $\frac{du}{dx} = \frac{d}{dx}(-x) = -1$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(-1) = -e^{-x}$

Chain Rule - Example VI

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Chain Rule - Example VI

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- Let $y = e^u$ and $u = 4x^3 - 6x + 1$
- $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$
- $\frac{du}{dx} = \frac{d}{dx}(4x^3 - 6x + 1) = 12x^2 - 6$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(12x^2 - 6) = (12x^2 - 6)e^{4x^3-6x+1}$

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- Applying chain rule yields

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{(\ln b)x}) = e^{(\ln b)x}\frac{d}{dx}[(\ln b)x] = e^{(\ln b)x}\ln b = b^x \ln b$$

where the 2. equality is due to the chain rule while the 3. equality holds because $\ln b$ is a constant.

Chain Rule - Proof Attempt

- Definition of the derivative of $F = f \circ g$ at some a where g if differentiable at a and f is differentiable at $g(a)$:

$$F'(a) = (f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

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- If $g(x) \neq g(a)$ for any x near a, then

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \to a} \left[\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} \right] =$$

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- Unfortunately, there are functions such that $g(x) - g(a) = 0$ when x is arbitrarily close to a. For example $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ when x is close to 0.

Chain Rule - Proof

- Let

$$Q(g(x)) = \begin{cases} \frac{f(g(x)) - f(g(a)}{g(x) - g(a)} & \text{if } g(x) \neq g(a) \\ f'(g(a)) & \text{if } g(x) = g(a) \end{cases}$$

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$$\lim_{x \to a}\left[Q(g(x))\frac{g(x)-g(a)}{x-a}\right] = \lim_{x \to a} Q(g(x)) \lim_{x \to a} \frac{g(x)-g(a)}{x-a} =$$

$$\left[\lim_{x \to a} Q(g(x))\right] g'(a) \text{ (since } g \text{ is differentiable at } a)$$

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- Therefore $Q \circ g$ is continuous at a.

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- g is continuous at a because it is differentiable at a .
- Therefore $Q \circ g$ is continuous at a.
- So $\lim_{x \to a} Q(g(x)) = Q(g(a)) = f'(g(a))$.

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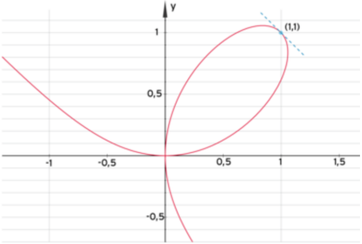
- $x^2 + y^2 = 25$ implicitly defines 2 functions

$$g_1(x) = \sqrt{25 - x^2} \text{ and } g_2(x) = -\sqrt{25 - x^2}$$

on the closed interval $[-5, 5]$. g_1 defines the function whose graph is the upper half-circle while g_2 defines the function whose graph is the lower half-circle.

Implicit Differentiation

- For the equation $x^3 + y^3 = 2xy$ it is however not that easy to identify implicitly defined functions.



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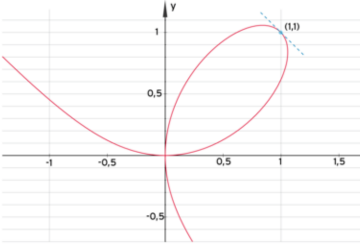

- But such equations define curves. One may still need to determined derivatives (slopes of tangents) of differentable functions without bothering about the functions itself.

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- Differentiate both sides with respect to x (applying the chain rule to y^2) and solve w.r.t. $\frac{dy}{dx}$.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) \iff \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = 0 \iff$$

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 0 \iff 2x+2y\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = -\frac{x}{y}$$

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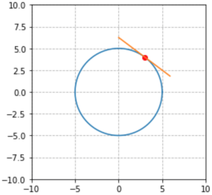
- At point $(3,4)$, $x = 3$ and $y = 4$ and $\frac{dy}{dx} = -\frac{3}{4}$



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$$2x - 2y - 2(x-y)\frac{dy}{dx} = 1 + \frac{dy}{dx} \iff$$

$$2x - 2y - 1 = (2x - 2y + 1)\frac{dy}{dx} \iff$$

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$$2x-2y-2(x-y)\frac{dy}{dx} = 1+\frac{dy}{dx} \iff$$

$$2x-2y-1 = (2x-2y+1)\frac{dy}{dx} \iff$$

$$\frac{dx}{dy} = \frac{2x-2y-1}{2x-2y+1}$$

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  $$3x^2 + 3y^2 y' = 2xy' + 2y$$

  where $y' = \frac{dy}{dx}$.

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  where $y' = \frac{dy}{dx}$.
- Solve for y'

  $$3y^2 y' - 2xy' = 2y - 3x^2 \iff (3y^2 - 2x)y' = 2y - 3x^2 \iff$$

  $$y' = \frac{2y - 3x^2}{3y^2 - 2x} \text{ if } 3y^2 - 2x \neq 0$$

Implicit Differentiation - Example III: $x^3 + y^3 = 2xy$

- Changed notation.
- Regard y as a function of x. Differentiate both sides with respect to x (applying chain rule to y^3 on the left side and product rule to the right side).

  $$3x^2 + 3y^2 y' = 2xy' + 2y$$

  where $y' = \frac{dy}{dx}$.
- Solve for y'

  $$3y^2 y' - 2xy' = 2y - 3x^2 \iff (3y^2 - 2x)y' = 2y - 3x^2 \iff$$

  $$y' = \frac{2y - 3x^2}{3y^2 - 2x} \text{ if } 3y^2 - 2x \neq 0$$

- Point $(1, 1)$ satisfies the equation and therefore the derivative y' (slope of the tangent) in that point is

  $$y' = \frac{2 * 1 - 3 * 1^2}{3 * 1^2 - 2 * 1} = -1$$

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- $f(x) = e^x$. We know that $f'(x) = e^x$ and $g(y) = \ln y$ is the inverse of f. Hence, $g'(y) = \frac{1}{f'(\ln y)} = \frac{1}{e^{\ln y}} = \frac{1}{y}$

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- If $b = e$, we get

$$\frac{dy}{dx}(\ln x) = \frac{1}{x}$$

Derivatives of Logarithmic Functions - Example I

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- It follows that $f'(x) = \frac{1}{x}$ for all $x \neq 0$.

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- Determine $f'(x) = \ln(\sin x)$. Let $u = \sin x$ and $y = \ln u$. Then

$$\frac{d}{dx}\ln(sinx) = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x = \cot x$$

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- Solving for y' and back substituting

$$y' = \sqrt{\frac{x-1}{x^4+1}} \left(\frac{1}{2x-2} - \frac{2x^3}{x^4+1} \right)$$

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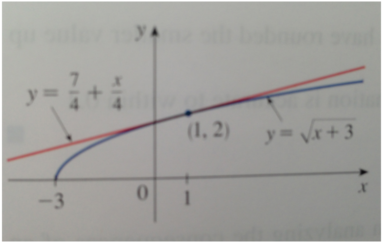
- Equation of the tangent line of a differentable function at a is $y = f(a) + f'(a)(x - a)$
- The linear function given by $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at a.



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- If the permitted error is < 0.1, then $x \in]-1.1, 3.9[$

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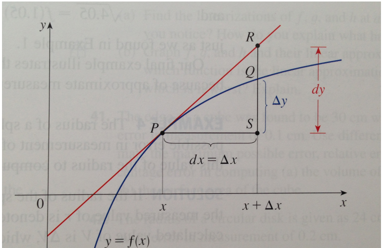
- The differential dy is then the dependent variable given by $dy = f'(x)dx$.
- Let $P = (x, f(x))$ and $Q = (x + \Delta x, f(x + \Delta x))$ and let $dx = \Delta x$. Δy represents how much the curve falls or rises when x changes by $\Delta x = dx$ while dy how much the tangent line falls or rises by the same change of x.



Summary

You should after this lecture be familar with:

- Chain rule
- Implicit differentiation
- Logarithmic differentiation
- Linear approximation