MASD 2022, Assignment 1

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Guidelines for the assignment.

- This is a group assignment. Hand-in in groups of 2 or 3 latest 14.09.2022 at 21.59. One submission per group and remember to include the name of all group members.
- The assignment report must be uploaded in PDF format, we strongly recommand the use of LaTeX to create the PDF.
- Please pay careful attention to the plagiarism rules, see `https://absalon.ku.dk/courses/61325/pages/course-information`.

Exercise 1 – (Writing proofs). In this exercise you will practice writing proofs. Remember that that the proofs in this exercise (and any other proof) must satisfy:

- Clearly stated assumptions (if there are any),
- Clearly stated claims – what you want to prove,
- Clearly stated logical arguments leading from assumptions to claims.

1. Show by induction on $n \in \mathbb{N}$ that
$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}.$$
2. A function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a limit a when x approaches $x_0 \in D$ if for **each** number $\varepsilon > 0$, there is a number $\delta > 0$ such that
$$0 < |x - x_0| < \delta \implies |f(x) - a| < \varepsilon.$$
Show that if it is true for a given $\varepsilon_0 > 0$, then it is also true for each $\varepsilon > \varepsilon_0$.
3. Let f and g two functions from $D \subset \mathbb{R} \to \mathbb{R}$ and let $x_0 \in D$. Assume that $\lim_{x\to x_0} f(x) = a$ and that $\lim_{x\to x_0} g(x) = b$, a and b being real numbers (i.e., not infinite). Using ε and δ, show that $\lim_{x\to x_0} f(x)g(x) = ab$.
4. Show by induction that $f(x) = x^n$ is continuous for any $n \geq 0$.
5. Using the *intermediate values theorem* (book, section 2.5, page 122), show that a continuous function $f : [a, b] \to \mathbb{R}$ which takes only finitely many values is actually constant.

Deliverable: The proofs.

Exercise 2 (Limits and Continuity). Consider the function

$$f(x) = \begin{cases} 10 & \text{when } x < -4 \\ -\frac{x^3}{32} & \text{when } x \in [-4, 0] \\ \frac{x^2}{3} & \text{when } x > 0. \end{cases}$$

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- In the supplied Jupyter notebook template `A1template.ipynb`, plot the function $f(x)$ on the interval $[-5, 5]$. Base on this plot, decide if there a point where f is not continuous, and which point, in case your answer is positive. Include the plot and your claim of discontinuity in your report.
- Prove that your observations from 1) are corrects:
  - Prove that f is continuous at all $a \in [-5, 5]$ where you claim it is,

- Prove that f is discontinuous at those points $a \in [-5, 5]$ where you claim it is.

Your proofs can use any results of Section 2.5 from the textbook.

Deliverables. Submit the filled-out Jupyter template and include the plot and the non continuous points, if any, in your report. The proofs, following the same guidelines as the previous exercise.

Exercise 3 (Limits and area of a disk). Some preliminaries first.

Definition 0.1 (Converging sequence.) *A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ converges to a real x^*, if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$n \geq n_0 \quad \implies \quad |x_n - x^*| < \varepsilon.$$

One writes

$$\lim_{n\to\infty} x_n = x^* \text{ or } x_n \to x^* \text{ when } n \to \infty.$$

Let $n \in \mathbb{N}$, $n \geq 3$ be the number of sides of a regular polygon P_n inscribed in a circle C of radius r and center O. As $n \to \infty$, the area S_n of P_n approximates the area of C. We know it is πr^2. Prove that

$$S = \lim_{n\to\infty} S_n = \pi r^2.$$

Hint: you may want to use that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, or in terms of a sequence $(x_n)_n$, if $x_n \to 0$ when $n \to \infty$, then $\lim_{n\to\infty} \frac{\sin x_n}{x_n} = 1$.

Deliverable: The proof.