Matematisk Analyse og Sandsynlighedsteori i Datalogi

Lecture 1
06.09.2022

François Lauze - with a lot of material originally from Pawel Winter and other people

The lecture starts at 8.15

Who are we?

Lecturers:

- **François Lauze, francois@di.ku.dk**
- **Stefan Sommer (course responsible), Kasra Arnavaz**

MASD and MAD: rationale and focus

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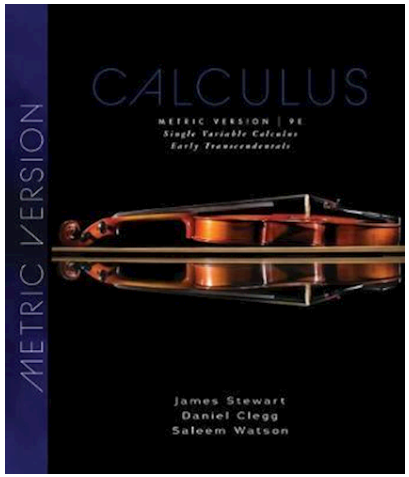
- Both courses rely on calculus, linear algebra and general math skills – these are your most fundamental tools, and we will use them!
- After MASD+MAD, you will be ready to use basic statistics, machine learning and signal/image processing in practical applications.

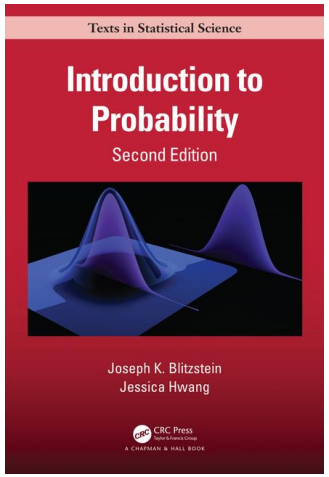
Teaching

- Tuesdays 8.15-10.00
- Thursdays 10.15-12.00
- Lecture plan on Absalon.

Teaching material

- **Textbooks**





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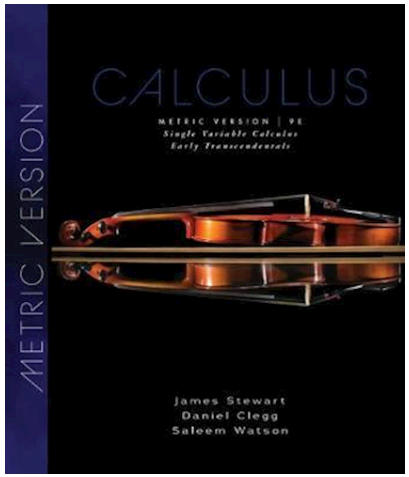

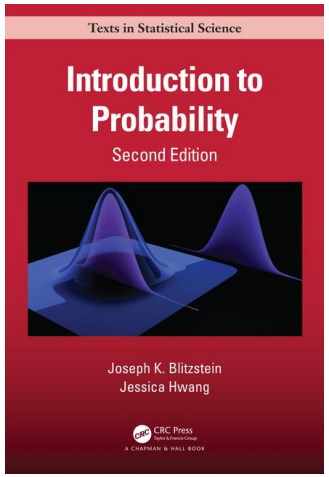

- **Software:**
  - Programming: Python 3.x in all materials handed out to you
  - E.g.: Anaconda distribution (`https://www.continuum.io/downloads`); it contains both Python and the most useful libraries for data science (we will need numpy, scipy, matplotlib, skimage, and likely more)
  - Jupyter notebook – you will need this. Get it via Anaconda or from `http://jupyter.org/`

Assignments and Exam form

- Assignments: Weekly, released on Tuesdays at noon, due on Thursdays at 10.00.

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- Use the Friday studie-cafes. **Take Poll 1 now**

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- Limits and Continuity, Sections 2.1-2.6

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  - Clarity for a human reader.
- You will be checking your fellow students' proofs against these 4 requirements in peer-review of Assignment 1, Exercise 1.

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- We obtained a contradiction. **Therefore P must be infinite**.

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- If
  - Basis: $P(b)$ is TRUE,
  - Inductive step: $P(k)$ is TRUE ==> $P(k+1)$ is TRUE, for every k, $k \geq b$,

  then $P(n)$ is TRUE for **all** natural numbers $n \geq b$.

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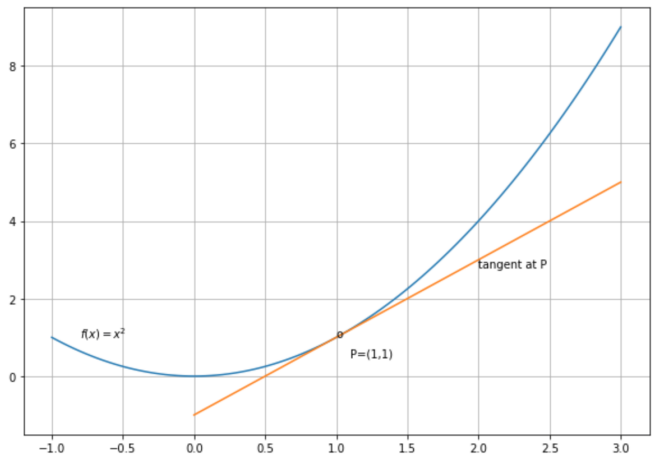
$$(k+1)(k/2+1) = (k+1)(k+2)/2$$

- By the principle of mathematical induction, **we proved** that $P(n)$ is true for all natural numbers.

Tangents



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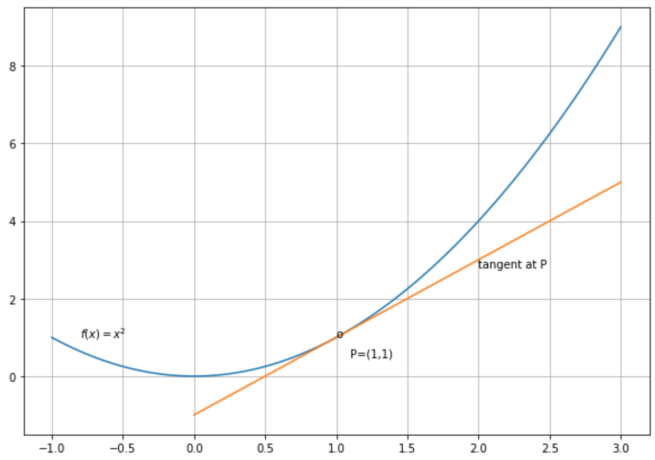

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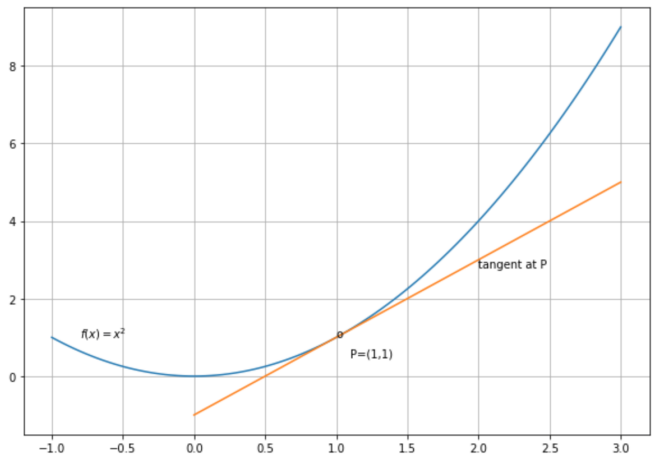

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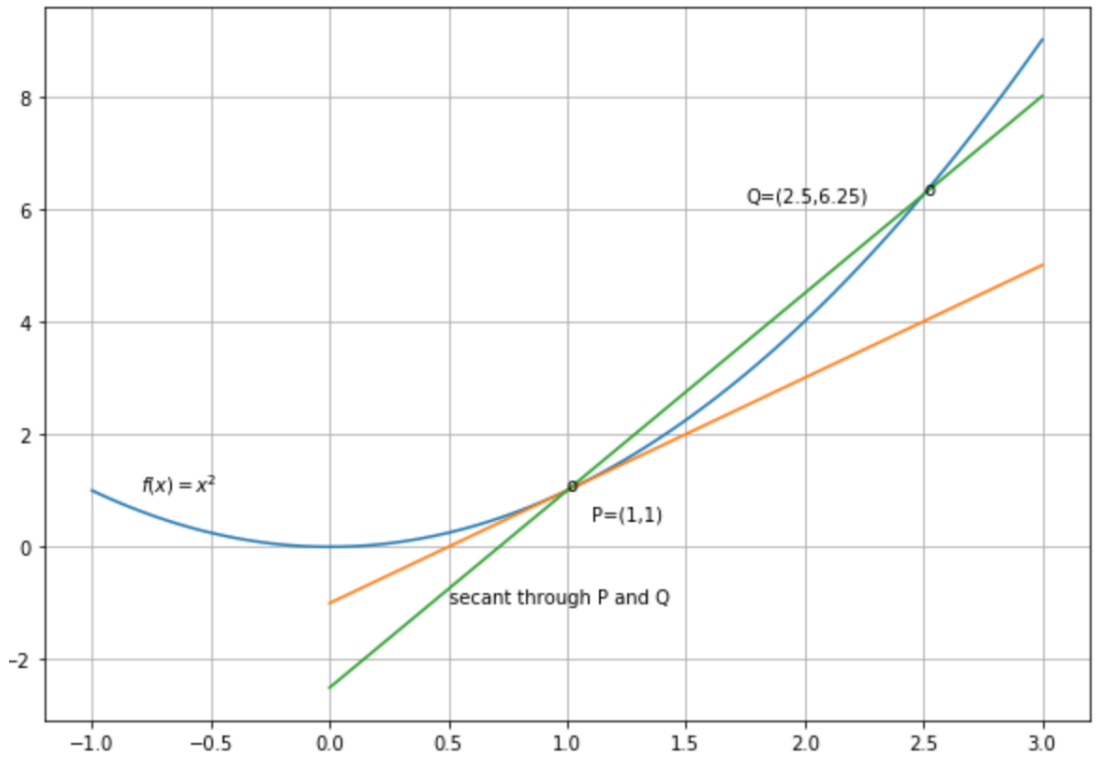
- Two points are needed to determine m. But we have only one.

Secants



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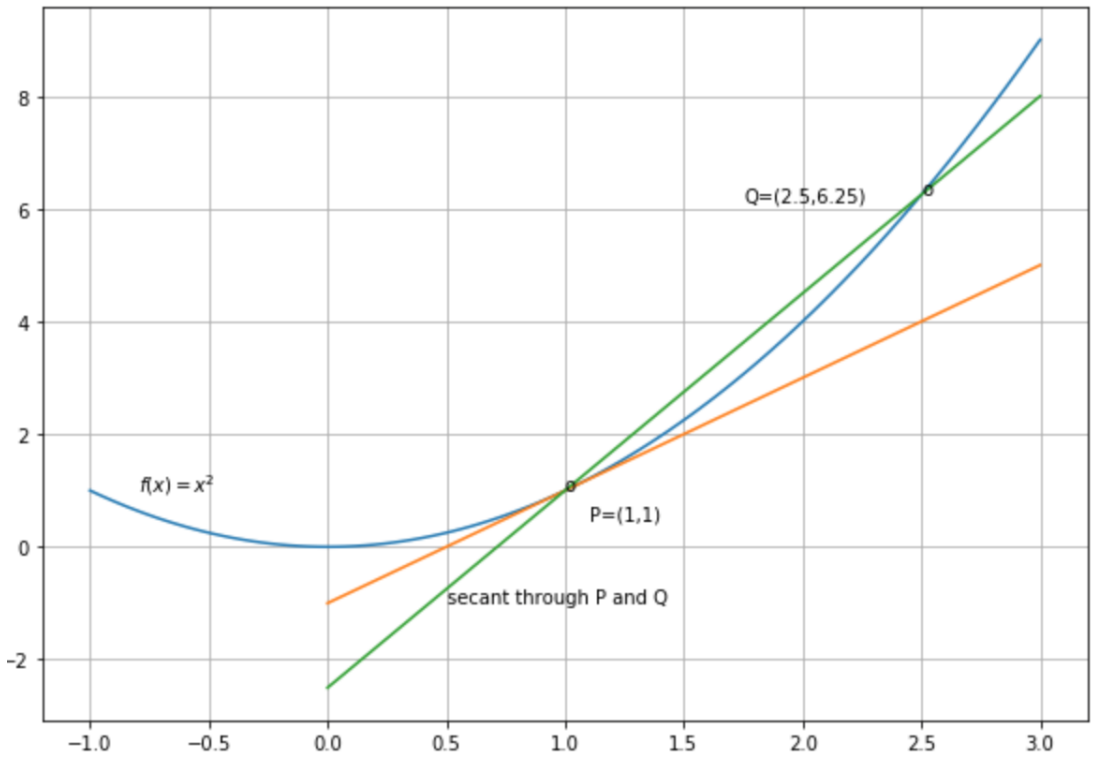

- What happens with the slope of the secant (da. sekant) when point Q gets closer and closer to point P?
- Slopes of secant lines get closer and closer to the slope of the tangent line through P.

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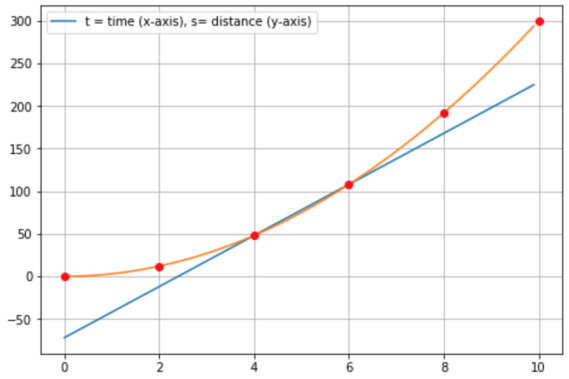
- What is the velocity of the object at $t = 4$? **Take Poll 2 here**

Average and Instantaneous Velocity



- What is the **average** velocity between $t_s = 4$ and $t_e = 6$?

$$av(4,6) = \frac{\text{distance travelled}}{\text{time elapsed}} = \frac{s(6) - s(4)}{6 - 4} = \frac{108 - 48}{2} = 30$$

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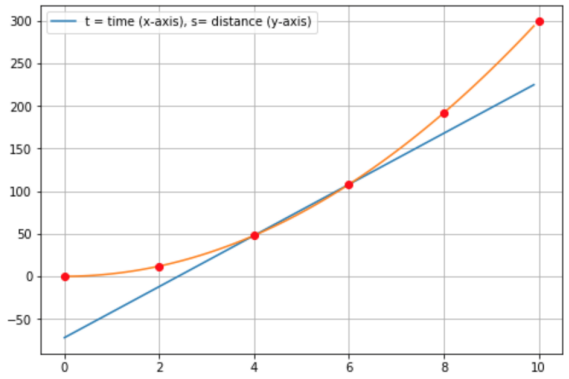

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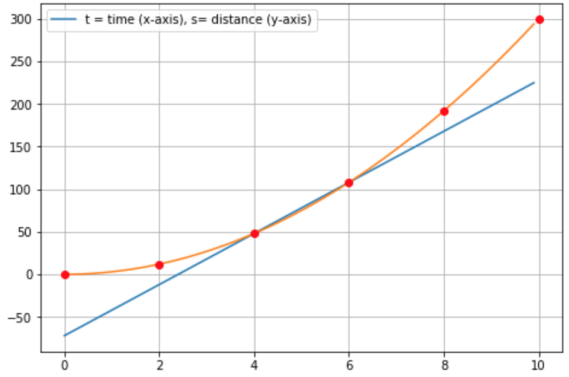

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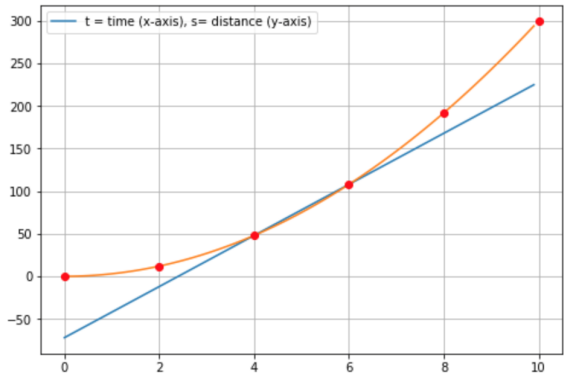

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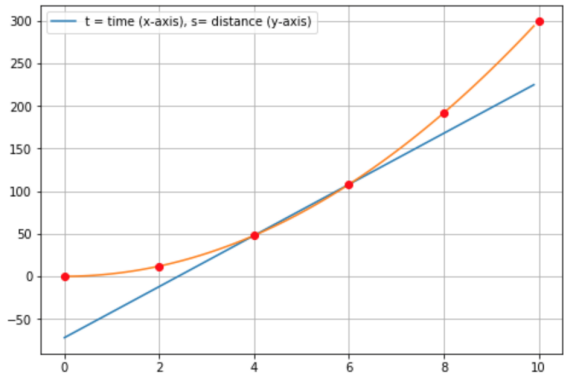

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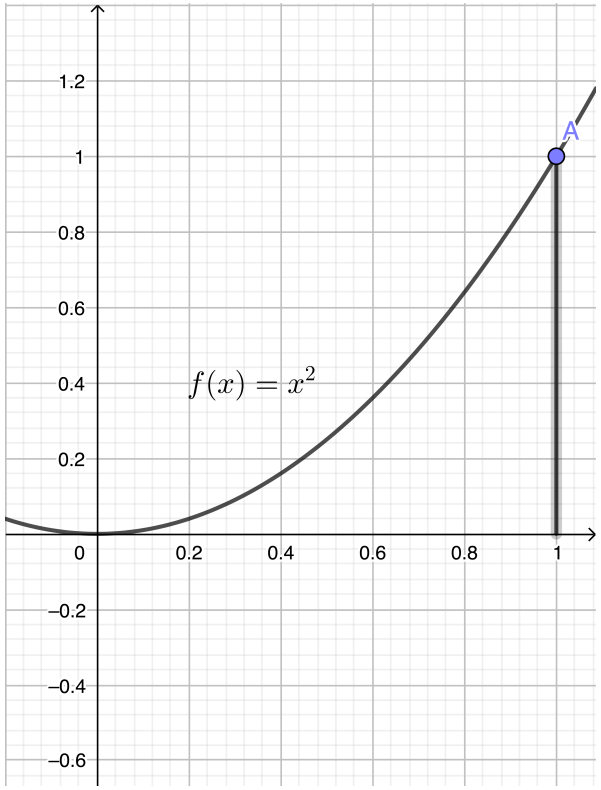
- Limits play central role in differentiation - main concept in calculus needed for example to determine the **instantaneous** velocity at time $t = 4$.

Area



- What is the area of the region under the curve $y = x^2$ and above the x-axis for x between 0 and 1?

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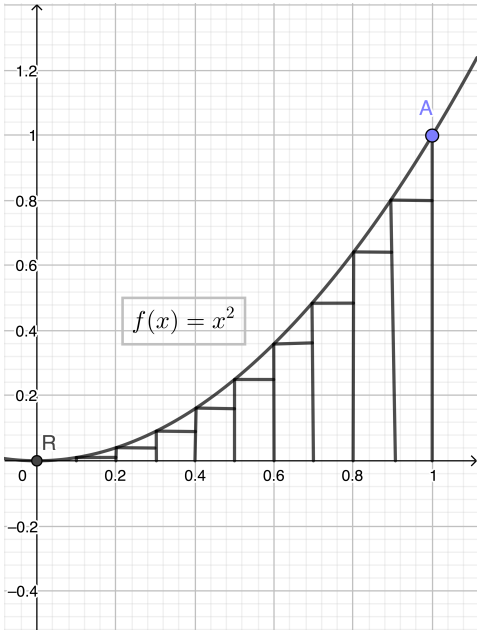
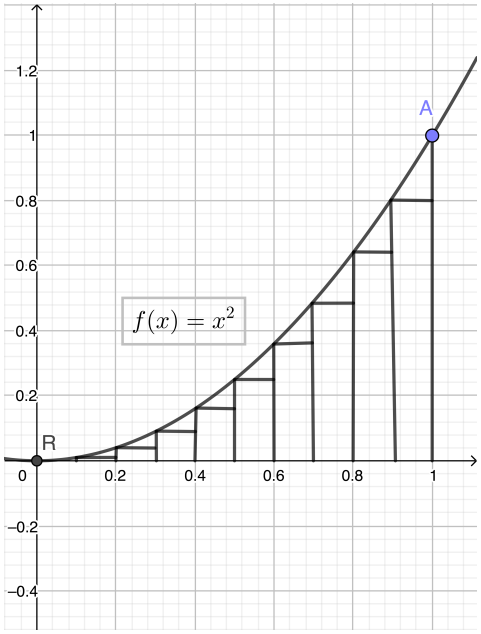


$$0.1 * (0.01 + 0.04 + 0.09 + 0.16 + 0.25 + 0.36 + 0.49 + 0.64 + 0.81) = 0.285$$

$$\frac{1}{n} * (\frac{1^2}{n^2} + \frac{2^2}{n^2} + ... + \frac{(n-1)^2}{n^2}) = \frac{1^2 + 2^2 + ... + (n-1)^2)}{n^3} =$$

$$\frac{n * (n-1) * (2n-1)}{6n^3} = \frac{2n^3 - 3n^2 + n}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

Area



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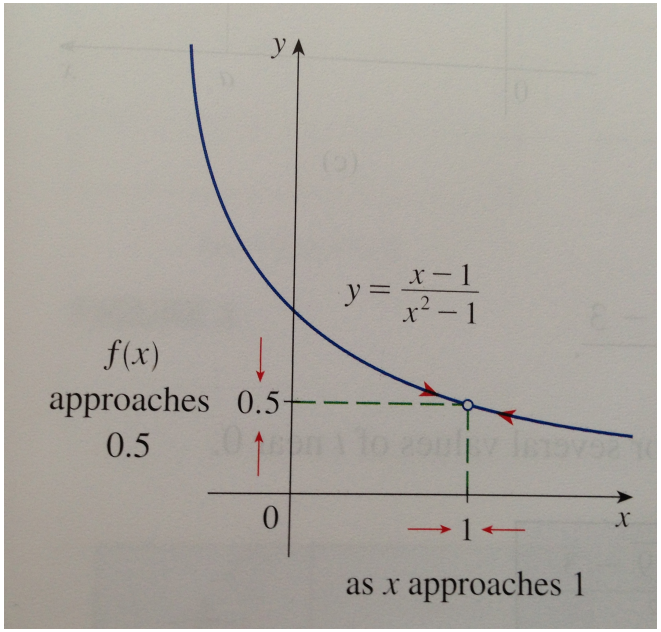
- Limits play central role in integration - another main concept in calculus needed for example to determine area and volume of various objects.

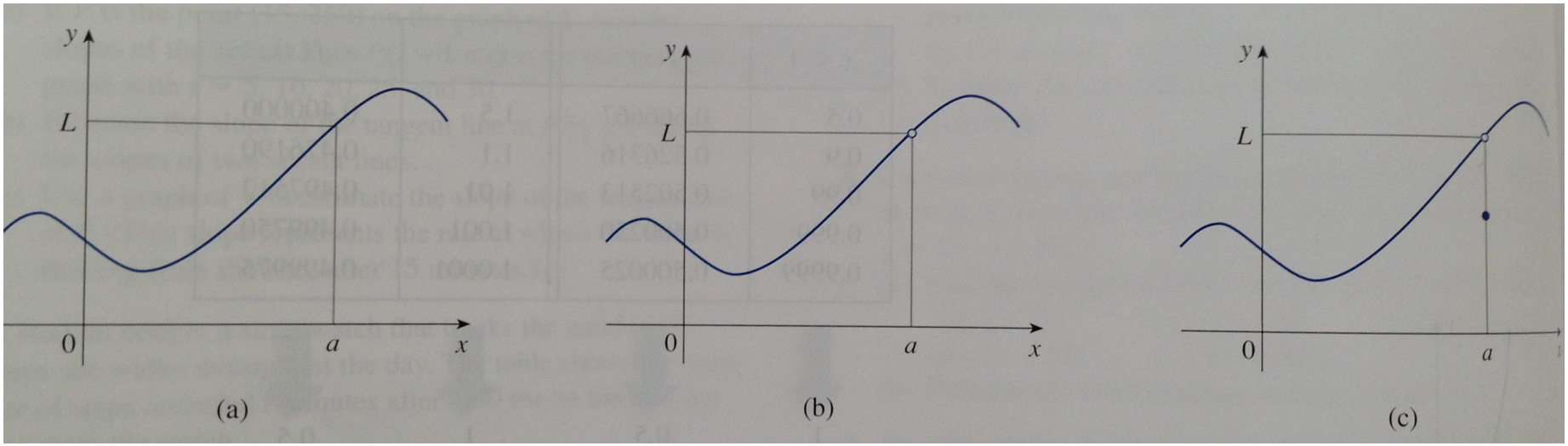
Limit of a Function



$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975
1	0.5	1	0.5

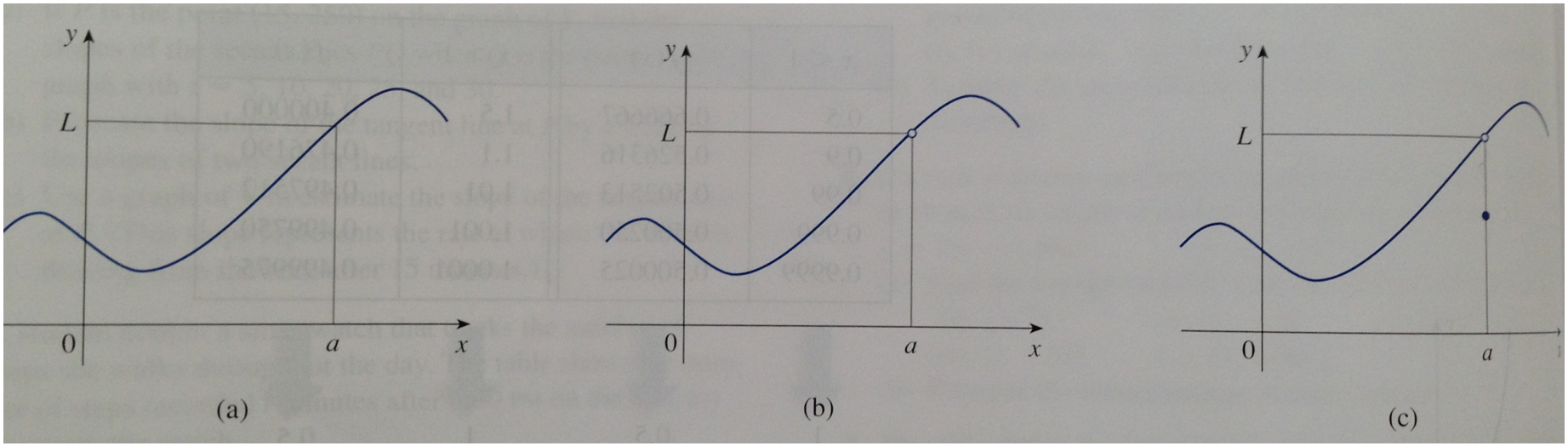
$$f(x) = \frac{x-1}{\quad} \qquad \forall x \in \mathbb{R} \setminus \{-1, 1\} \qquad \lim \frac{x-1}{\quad} = 0.5$$

Limit of a Function



What is $\lim_{x \to a} f(x)$ in the above three cases?

Limit of a Function



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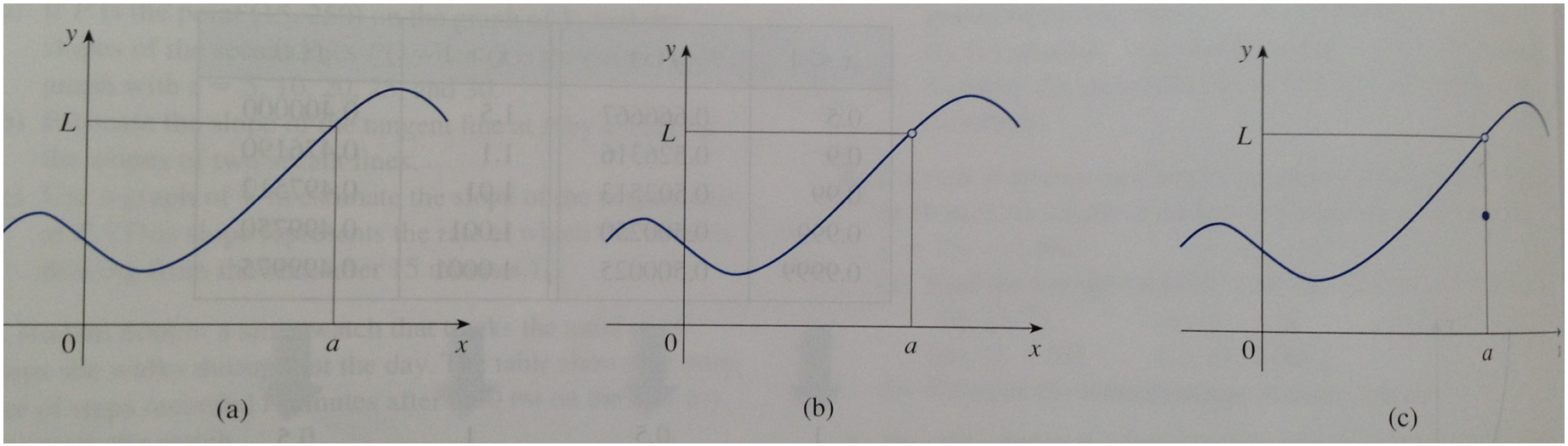

What is $\lim_{x \to a} f(x)$ in the above three cases?

- Figure (a) The limit of $f(x)$ is L as x approaches a.
- Figure (b) The limit of $f(x)$ is L as x approaches a eventhough f is not defined for a. **Take a poll here**

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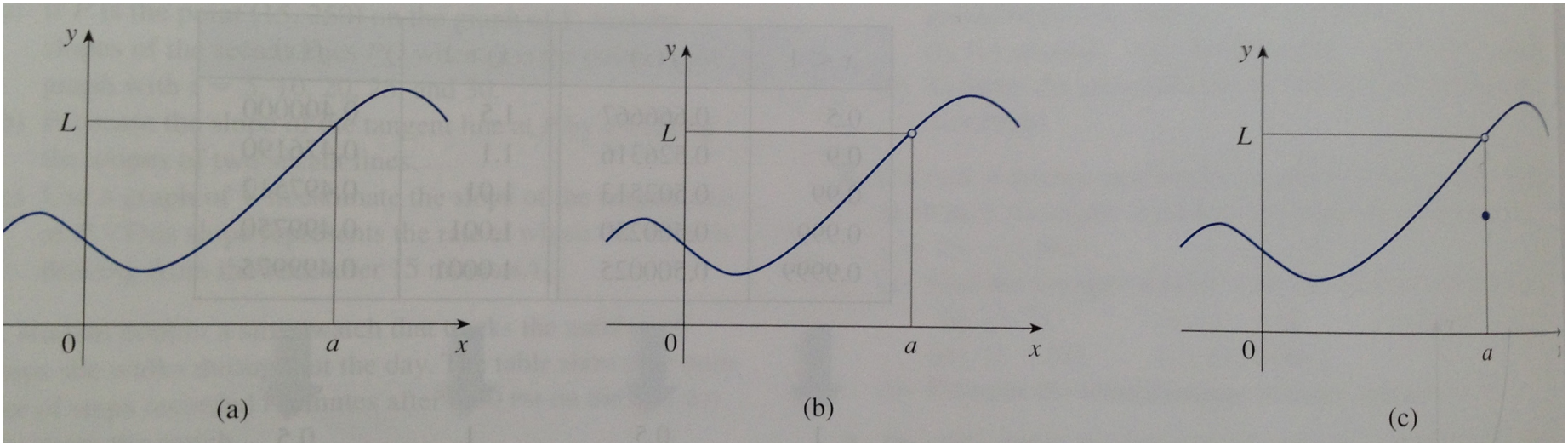

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- Figure (c) The limit of $f(x)$ is L as x approaches a eventhough f is defined for a but $f(a) \neq L$.

Limit of a Function by Guessing

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	$0.162277\ldots$
± 0.5	$0.165525\ldots$
± 0.1	$0.166620\ldots$
± 0.05	$0.166655\ldots$
± 0.01	$0.166666\ldots$

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$$

(this is a qualified **guess**)

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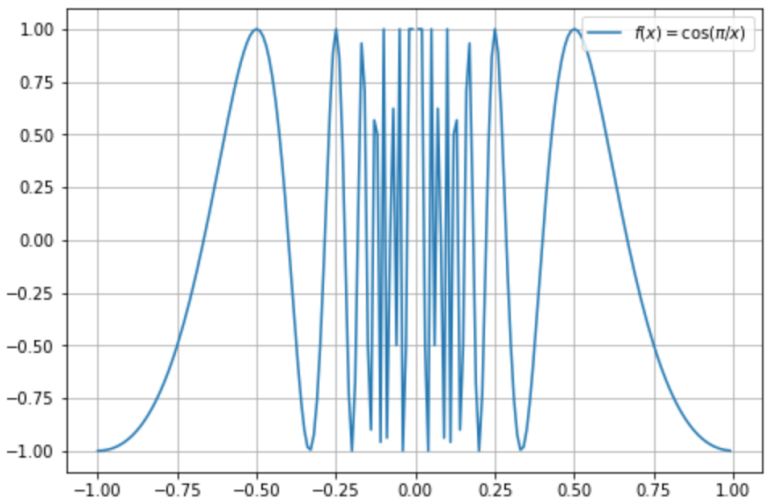
We cannot do such calculations each time we want to compute a limit of some function for some value a.

Limit of a Function - Danger of Guessing



Sometimes a guess can be completely wrong. Consider for example the function $f(t) = \cos(\frac{\pi}{t})$ defined for all $t \in \mathbb{R} \setminus \{0\}$.

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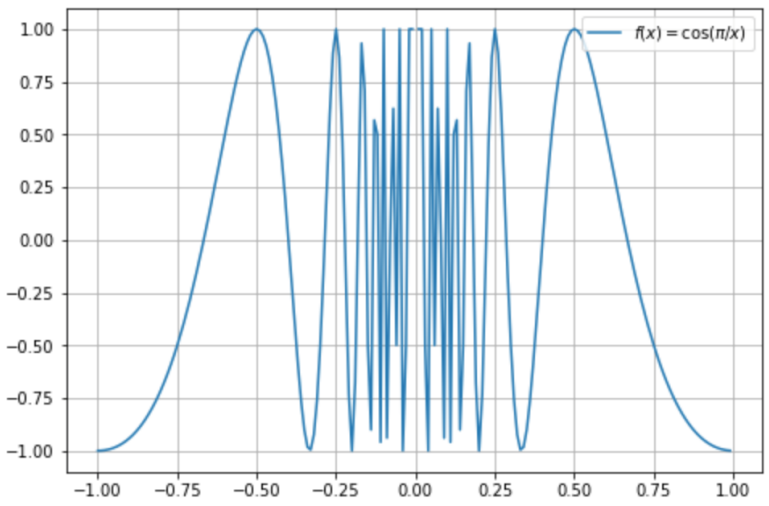

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$$\cos(\pi/0.1) = \cos(\pi/0.01) = \cos(\pi/0.001) = 1$$

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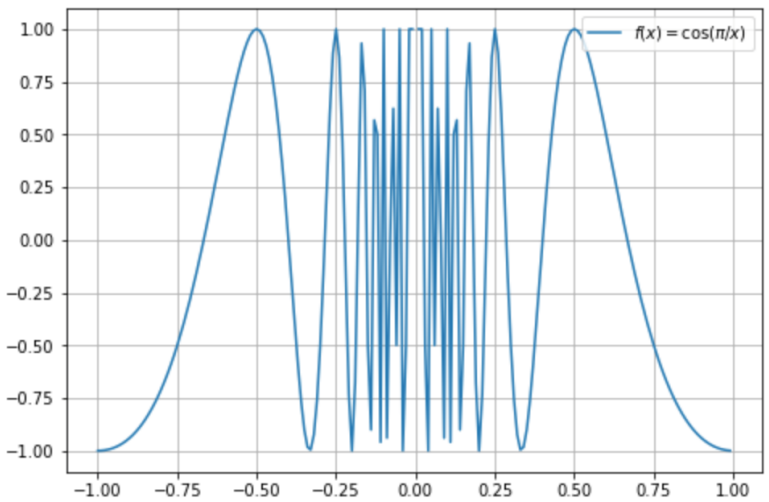

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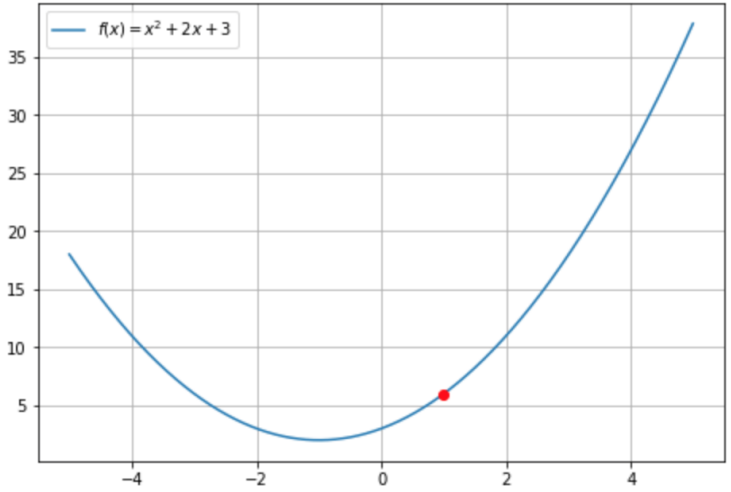
but 1 is not the limit of this function at $t = 0$ (actually the limit at $t = 0$ does not exist).

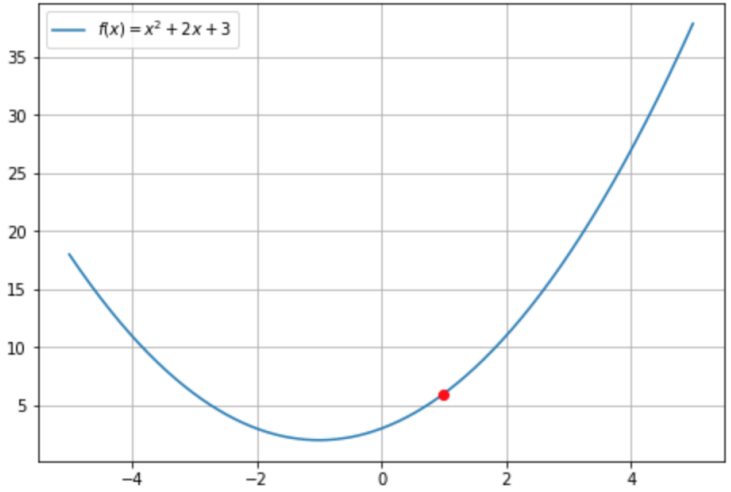
Limit of "Nice" Functions



If a function f is "nice" around a and defined in a , then

$$\lim_{x \to a} f(x) = f(a)$$

Limit of "Nice" Functions



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The limit can then be determined by plugging a into f. For instance,

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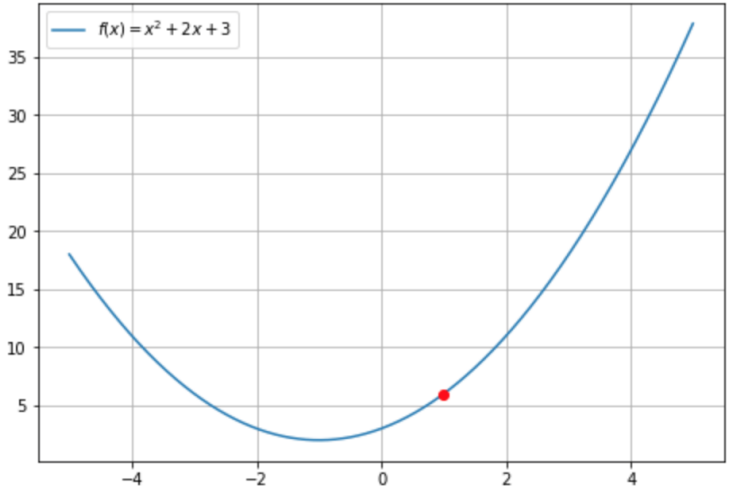
Limit of "Nice" Functions



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As we will see shortly, many functions are "nice" but there are also many that are not "nice".

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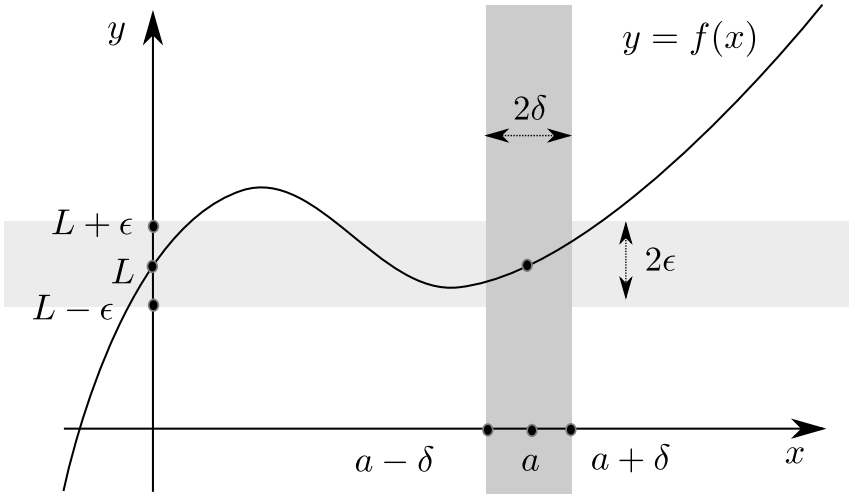

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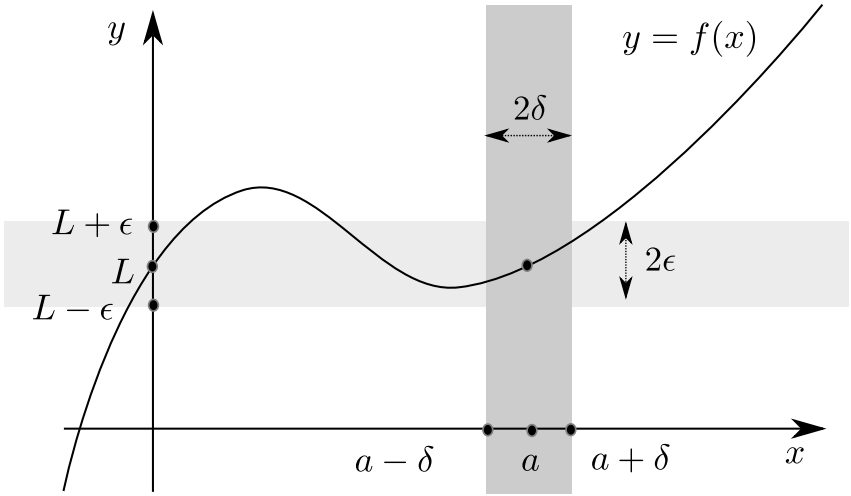

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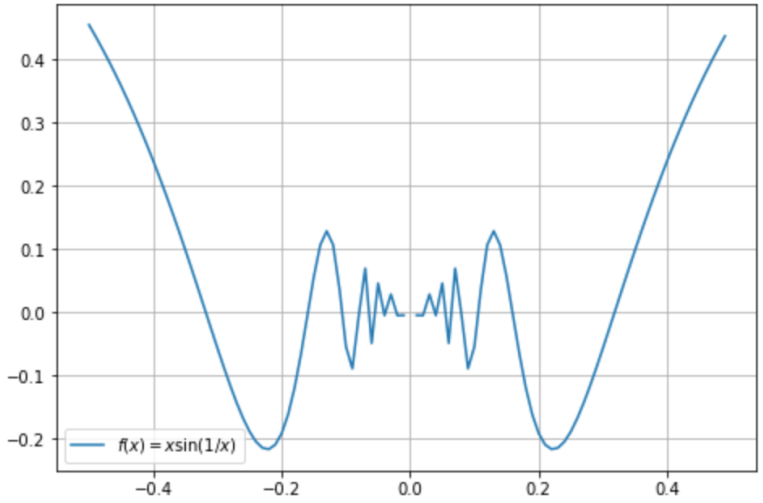
Limit of a Function - ϵ/δ Definition - Example I

- Show that $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$



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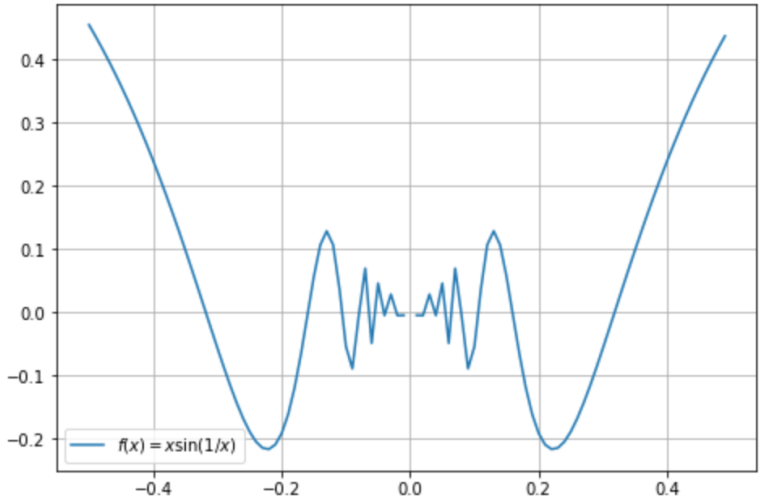

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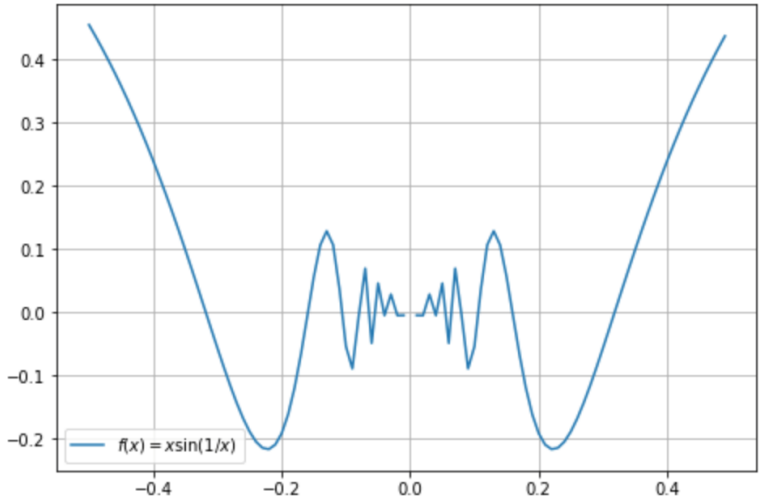

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- Let $\delta = \epsilon$. For any x satisfying $0 < |x| < \delta$, we get

$$\left| x\sin\left(\frac{1}{x}\right)\right| = |x|\left|\sin\left(\frac{1}{x}\right)\right| < \delta * 1 = \delta = \epsilon$$

Limit of a Function - ϵ/δ Definition - Example II

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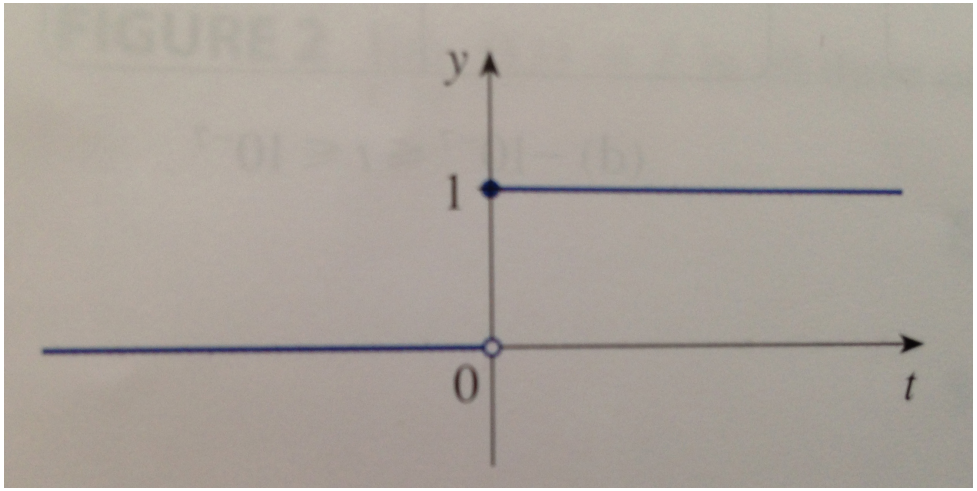
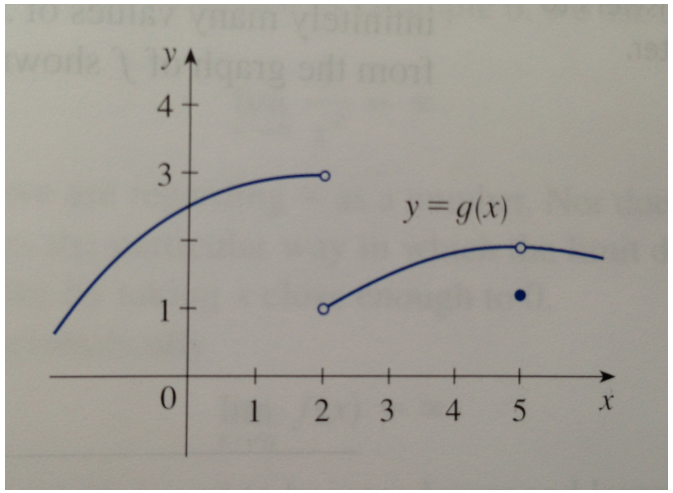
- We have $|x-2| < \delta \leq \epsilon/C$ and $|x+2| < 5 = C$. Hence, $|x-2||x+2| < \epsilon$.

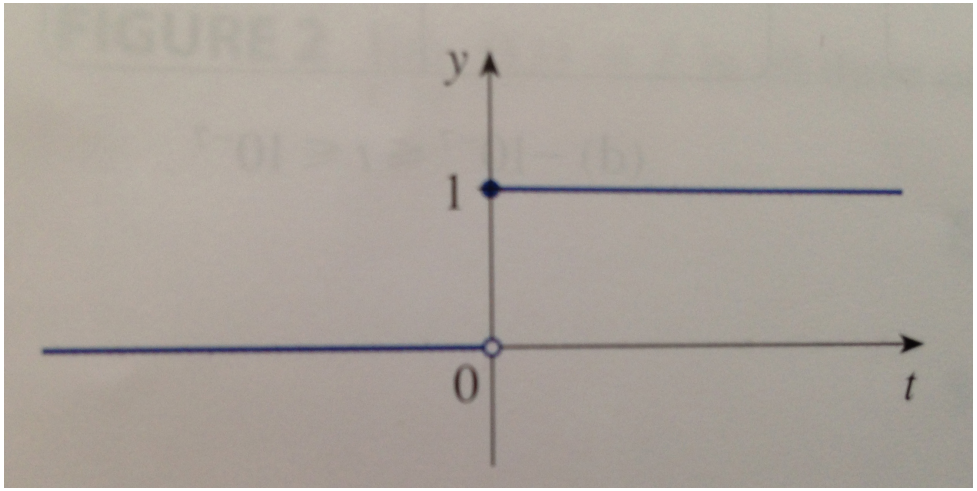
One-Sided Limits

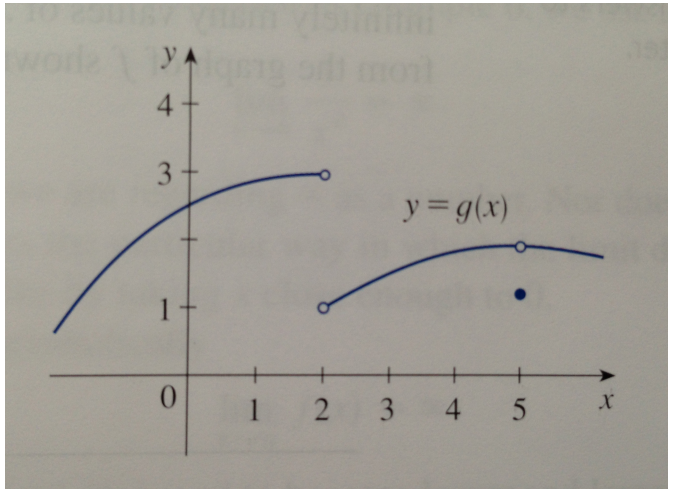




- What is the limit of Heaviside function (left) at $x = 0$? **Take a poll here**

One-Sided Limits





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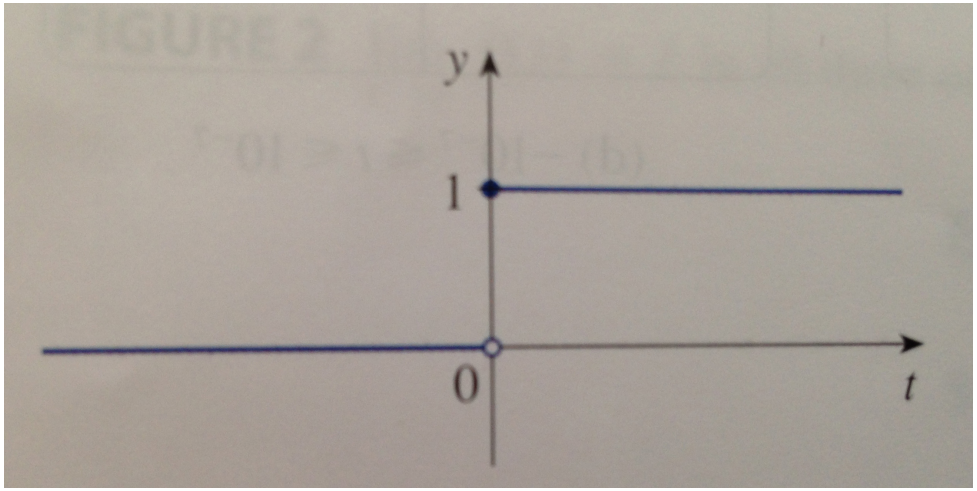

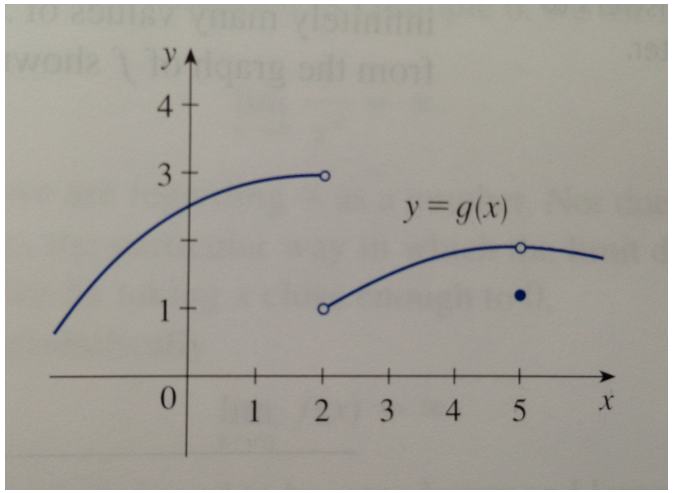

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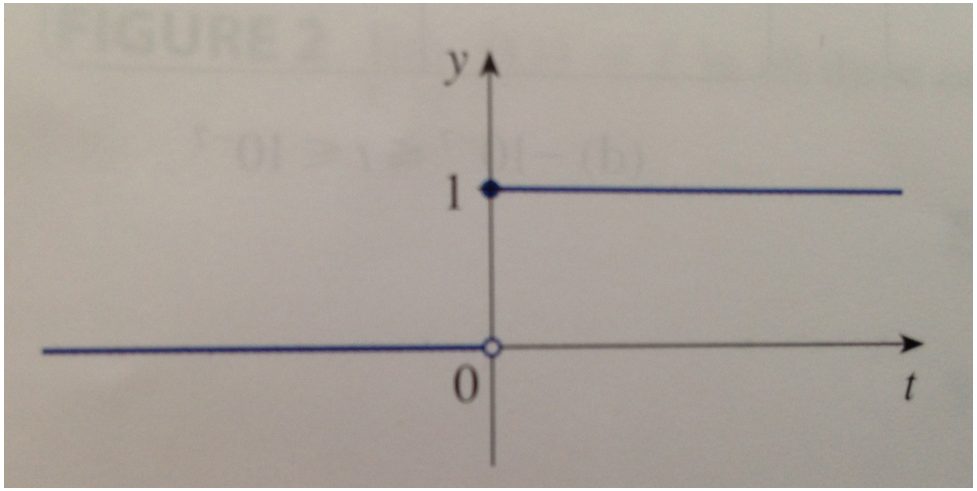

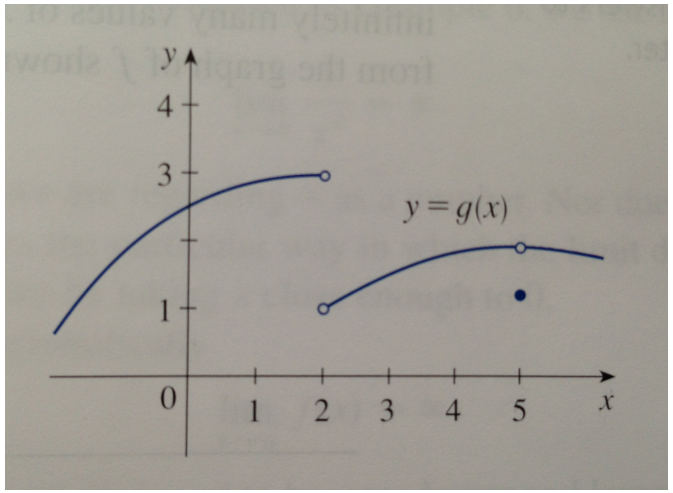

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- Let $a \in \mathbb{R}$ and $L \in \mathbb{R}$.

Left-Hand Limit - ϵ/δ Definition

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- Let f be a function defined on some interval I bounded by a on the right. Note that f can be undefined for a.

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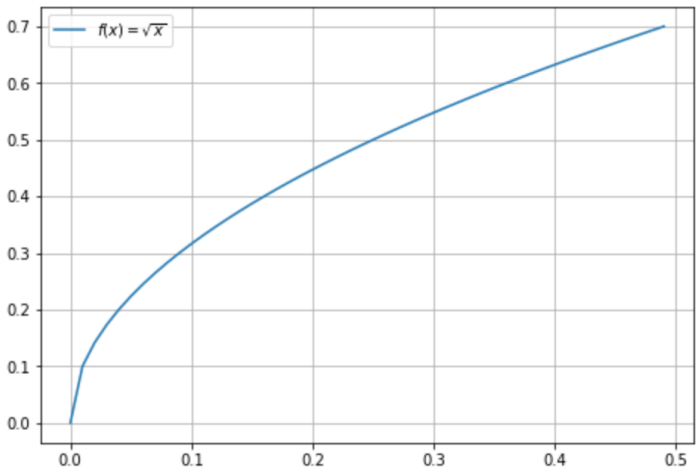
Right-Hand Limit - ϵ/δ Definition - Example



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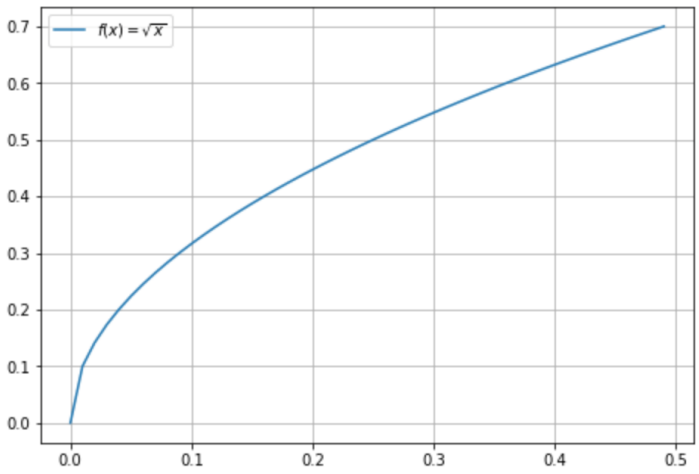

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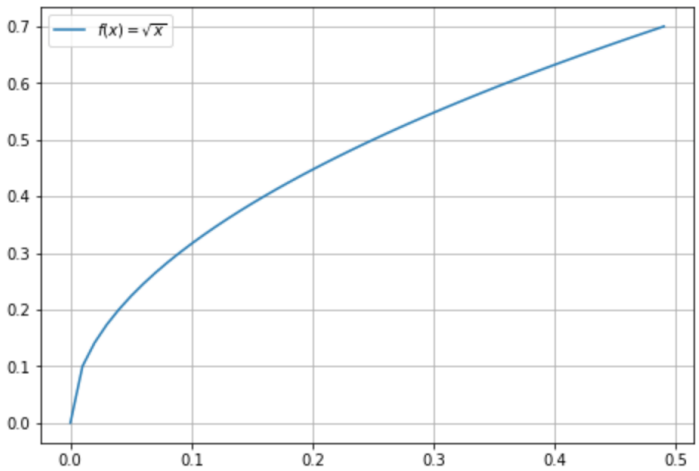

- Show that $\lim_{x \to 0^+} \sqrt{x} = 0$ using ϵ/δ definition.
- $a = 0$ and $L = 0$.
- Pick any $\epsilon > 0$. We are looking for $\delta > 0$ such that

$$\begin{aligned} &\text{if } 0 < x < \delta \text{ then } |\sqrt{x} - 0| < \epsilon && \iff \\ &\text{if } 0 < x < \delta \text{ then } \sqrt{x} < \epsilon && \iff \\ &\text{if } 0 < x < \delta \text{ then } x < \epsilon^2 \end{aligned}$$

Let $\delta = \epsilon^2$. If $0 < x < \delta$ then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon$$

Infinite Limits

x	$\frac{1}{x^2}$
± 1	1
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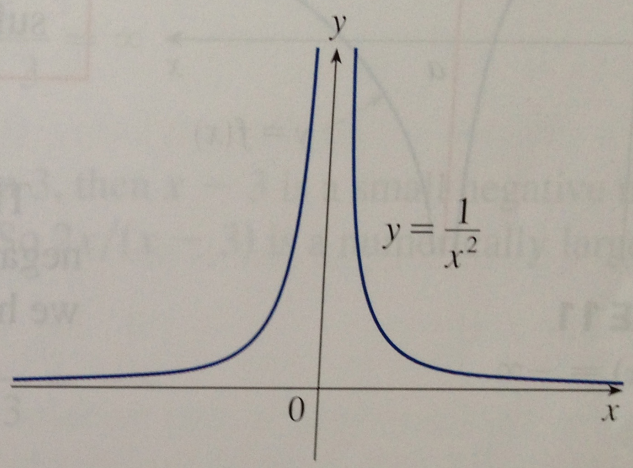

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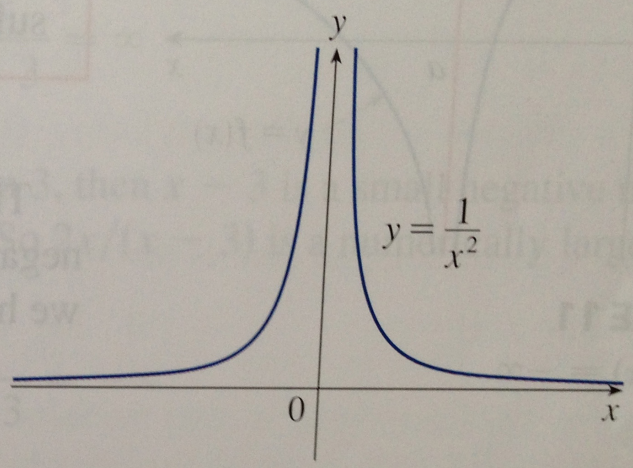

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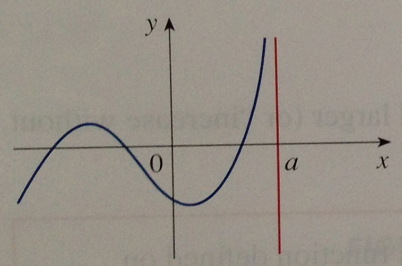

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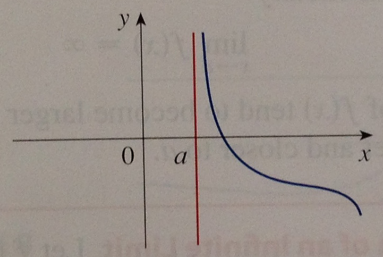
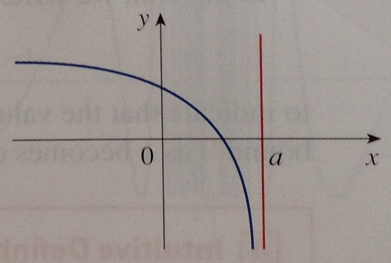
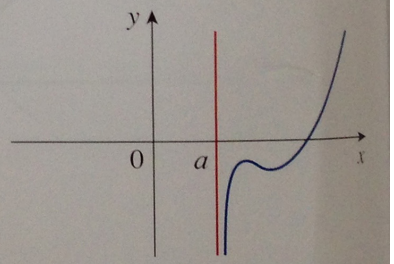


(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



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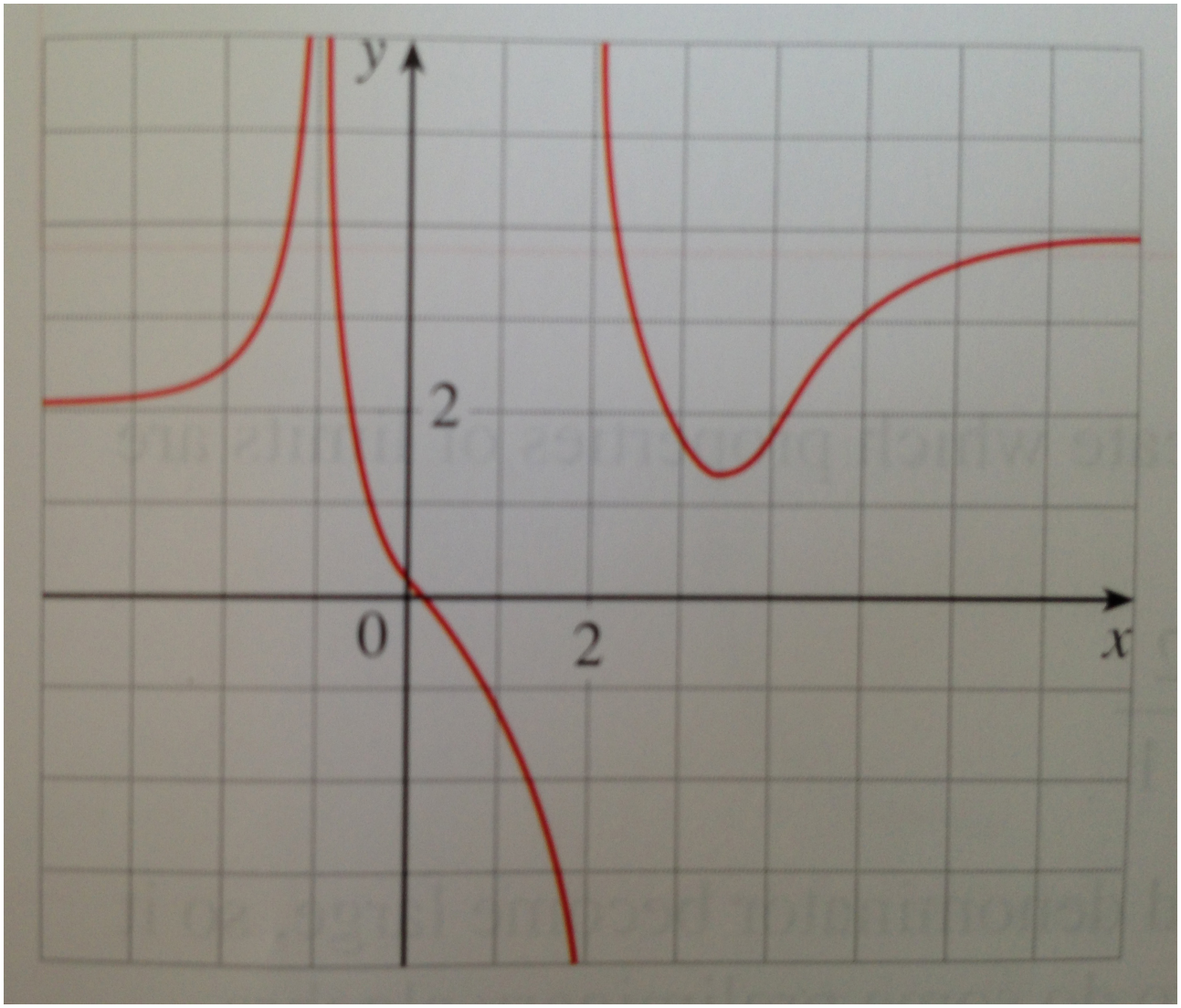

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  Take a poll here

Limit Laws

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

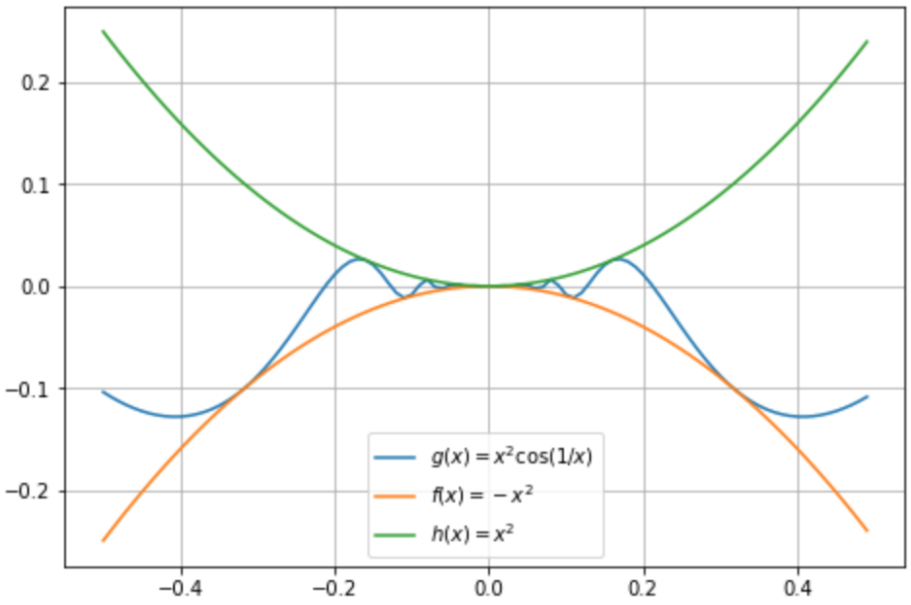
1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Sqeeze Theorem

Suppose that for all $x \in [a, b]$ (except possibly at some $c \in [a, b]$), we have

$$f(x) \leq g(x) \leq h(x) \textbf{ and } \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

Then $\lim_{x \to c} g(x) = L$.



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- Physical phenomena are usually continuous. It is important to detect discontinuities.

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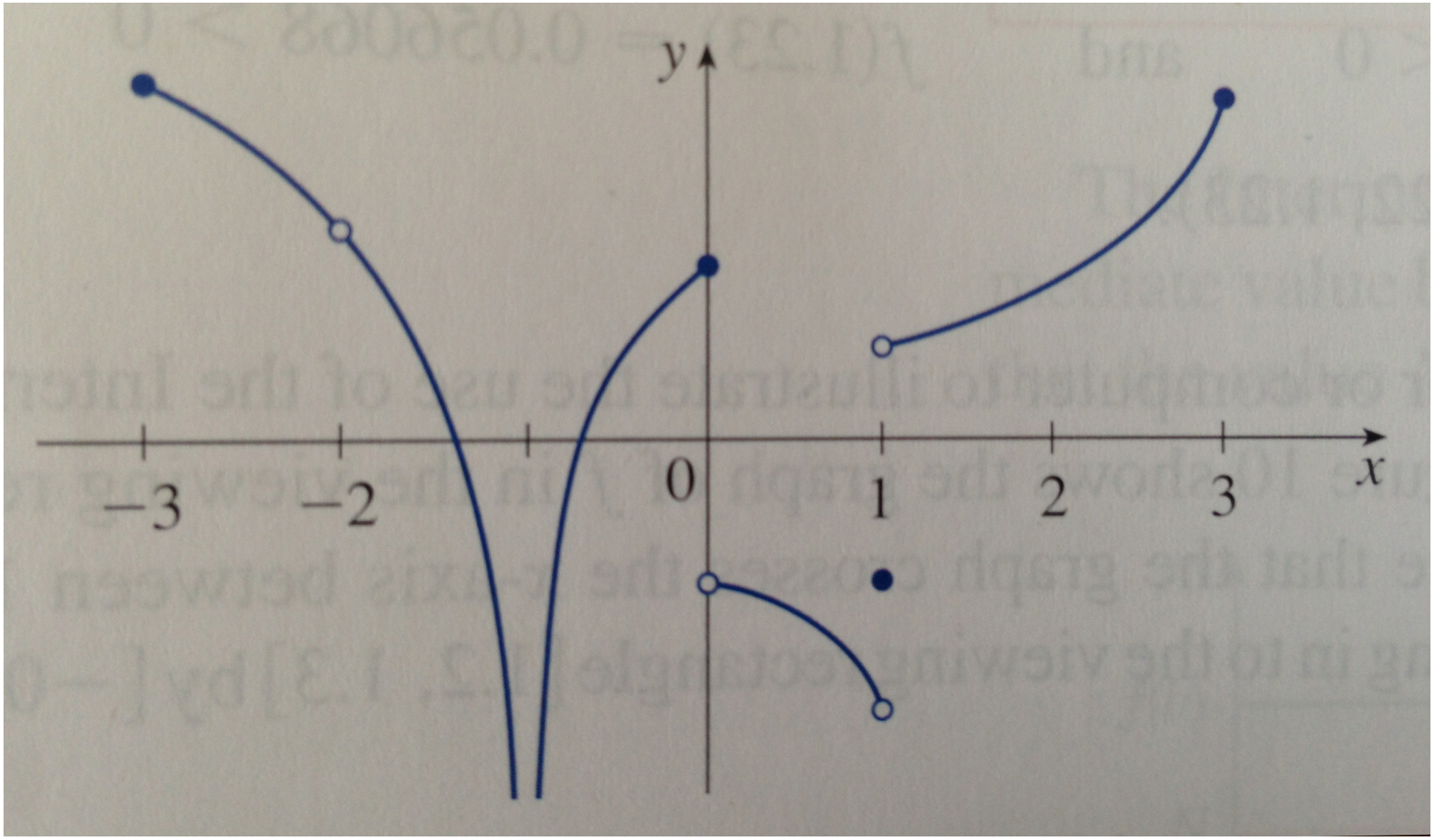

- At how many values of x is this function discontinuous? **Take a poll here**

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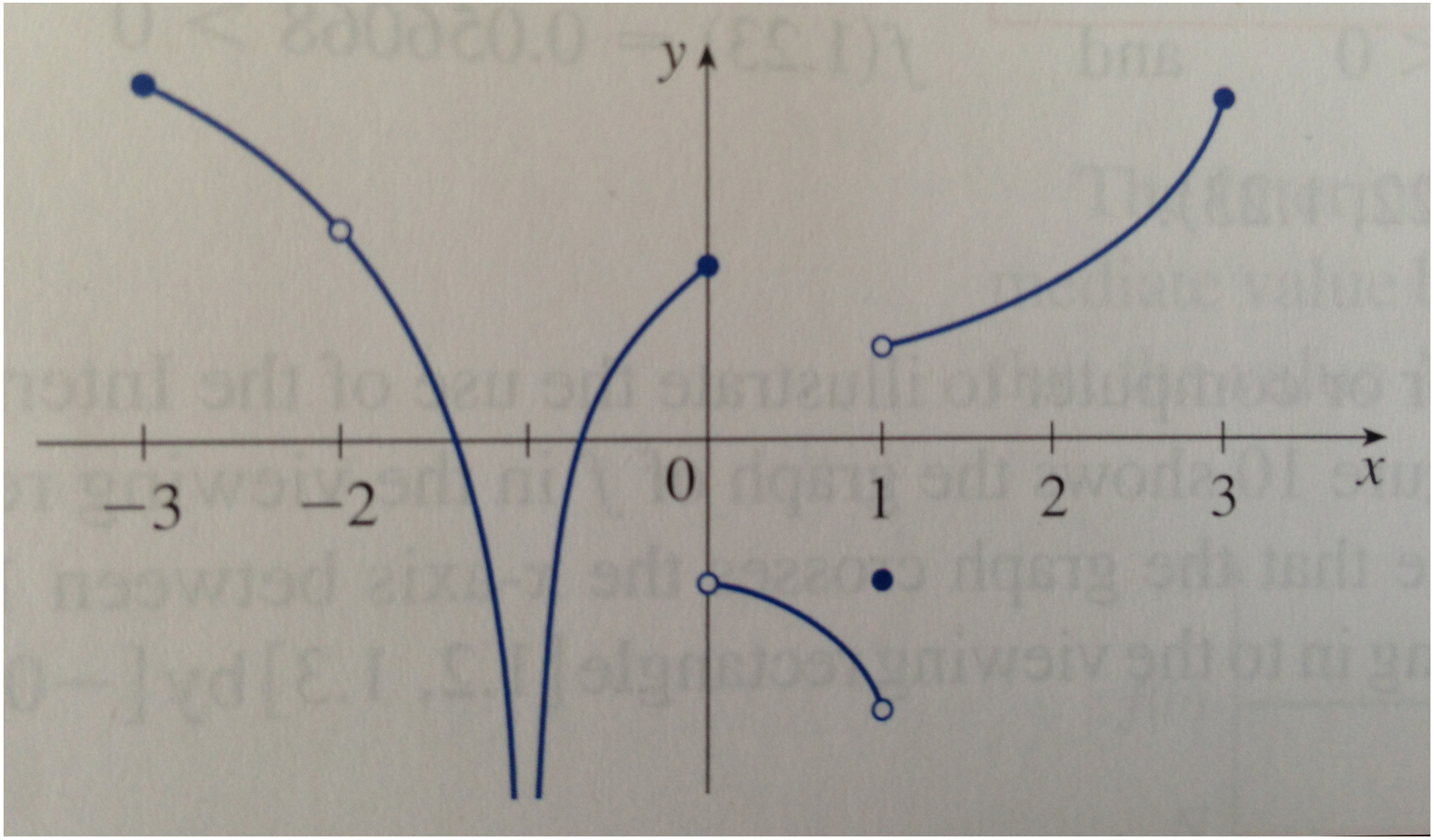

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- At which x-values is this function discontinuous and why?

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  - $|s| = |s - r + r| \leq |s - r| + |r|$ which is equivalent to $|s| - |r| \leq |s - r| = |r - s|$
  - Hence, $||r| - |s|| = \max\{|r| - |s|, |s| - |r|\} \leq |r - s|$.
- $|f(x) - f(a)| = ||x| - |a|| \leq |x| - |a| \leq |x - a|$
- Set $\delta = \epsilon$
- Consider any x satisfying $|x - a| < \delta$. Then $|f(x) - f(a)| \leq |x - a| < \delta = \epsilon$ as required.

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- Let f be a continuous function on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, $f(a) \neq f(b)$. Then there exists a number $c \in]a, b[$ such that $f(c) = N$.

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- Show that the equation $-x^3 + 4x - 1 = 0$ has a root for some $x \in]0, 1[$
- $f(x) = -x^3 + 4x - 1$ is a polynomium and therefore continuous for all $x \in \mathbb{R}$. Furthermore, $f(0) = -1$ while $f(1) = -1 + 4 - 1 = 2$.

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