MASD 2022, Assignment 6

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Guidelines for the assignment.

- This is a group assignment. Hand-in in groups of 2 or 3 latest 26/10/2022 at 21:59. One submission per group and remember to include the name of all group members.
- The assignment report must be uploaded in PDF format, we strongly recommend the use of LaTeX to create the PDF.
- Please pay careful attention to the plagiarism rules, see https://absalon.ku.dk/courses/61325/pages/course-information.
- For all questions, you need to justify your answers with mathematical symbols and/or sentences rather than only reporting the final answer.

Exercise 1 (Discrete Random Variables)

Two fair dice with faces labeled $1, 2, 3, 4, 5, 6$ are thrown.

a. What is the probability mass function of the sum of the values?

b. What is the probability mass function of the absolute difference between the values?

Exercise 2 (From Inference to Prediction)

A coin is tossed n times, obtaining a sequence of heads and tails. We assume that the coin has a probability b of coming up heads. If h heads occurred in n tosses,

a. use Bayes' rule

$$f_{B|H}(b|h;n) = \frac{p_{H|B}(h|b;n) f_B(b)}{\int_{-\infty}^{\infty} p_{H|B}(h|b;n) f_B(b) db} \tag{1}$$

to obtain the probability distribution of b after seeing this data. Assume total ignorance over the values of b prior to seeing the data. You might find the following integral useful.

$$\int_0^1 x^\alpha (1-x)^\beta dx = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} \quad \text{for } \alpha, \beta \in \mathbb{N}$$

b. For the following values, plot the probability distribution you have obtained in part (a).

i. $n = 3$ and $h = 0$;
ii. $n = 3$ and $h = 2$;
iii. $n = 10$ and $h = 3$;
iv. $n = 300$ and $h = 29$.

Draw all four graphs into one figure. Use your preferred visualization tool and include the figure in the report but no code upload is necessary. Describe the graphs. In particular, mention how their shape changes as n grows larger, and where do the curves peak in terms of h and n? Do your observations make intuitive sense?

c. What is the probability that the next coin toss is heads as a function of h and n? Hint: Your answer might initially be a function b, denoted by $g(b)$. You would then need to compute the expectation of $g(b)$ over the probability distribution you have derived in part (a), i.e.

$$E[g(b)] = \int_{-\infty}^{\infty} g(b) f_{B|H}(b|h;n) db \tag{2}$$

You can check your final answer by looking at this video.

Exercise 3 (Cumulative Distribution Function)

What is the expectation of a random variable whose cumulative distribution function is as follows:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 0.5 \\ x/2 & 0.5 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$