Foundations of modern finance cheat-sheet
by Oleg L.,

1 Foundations of Modern Finance I

Present value of annuities and perpetuities

Perpetuity: $\frac{CF}{r}$.

Growing perpetuity: $\frac{CF}{r-g}$.

Annuity: $\frac{CF}{r}\cdot\left(1-\frac{1}{(1+r)^n}\right)$.

Annuity with growth:
$\frac{CF}{r-g}\cdot\left(1-\frac{(1+g)^n}{(1+r)^n}\right)$,
where r rate of return, g growth rate, n compounding periods

Arrow-Debrew securities. State-space model

$\phi_1\dots\phi_n$ state prices
$p_1\dots p_n$ state probabilities, where $\sum p_i = 1$
$X_1\dots X_n$ state payouts

$P=\sum\phi_i\cdot X_i = \frac{E(P)}{(1+\bar{r})}$,

$\bar{r} = \frac{E(P)-P}{P} = \frac{\sum p_i\cdot X_i}{P} - 1$,

$E(P)=\sum p_i\cdot X_i = P\cdot(1+\bar{r})$,
where P is price, $E(P)$ is expected payout, $\bar{r}$ is expected return.

Discounted cash flow and rate of return

r is rate of return, r_f is risk-free rate of return, $r-r_f$ is excess return

$r = \frac{D_1+P_1-P_0}{P_0} = \frac{D_1+P_1}{P_0}-1$,

$P_0 = \frac{D_1+P_1}{1+r}$

With g as growth rate,

$P=\frac{D}{r-g}, g = \frac{D_1}{D_0}-1$

$\bar{r} = E(r)$, expected return, $\pi = \bar{r} - r_f$ is risk premium

Relation between real and nominal cash flows

$r_{real} = \frac{1+r_{nominal}}{1+inflation}-1$

For nominal flow,
$CF\cdot(1+r_{real})\cdot(1+inflation)$
For real flow,
$CF\cdot(1+r_{real})$

Accounting

$I_t = EPS_t\times b$, where b - plowback rate
$EPS_{t+1} = EPS_t + I_t\times ROI_t$
$BVPS_{t+1} = BVPS_t + I_{t+1}$
$D_t = EPS_t\times(1-b_t)$

Growth rate $g=\frac{EPS_{t+1}}{EPS_t}-1$

With growth $P_0=\frac{D}{r-g}$

Without growth $P_0^{nogrowth} = \frac{D}{r}$, where $g=0$ and $b=0$

Growth opportunity $PVGO = P_0 - P_0^{nogrowth}$

Horizon value estimation: $PV(Freecashflow)$+ P/E ratio or P/B ratio or DCF

Risk

Expected utility
$E[u(x)] = \sum p_i\cdot u(P_i)$,
where p_i is probability, P_i is payout
Expected payoff
$E(P)=\sum p_i\cdot P_i$
Relative risk aversion
$RRA(W) = -\frac{W\cdot u''(W)}{u'(W)}$

Certainty equivalent $CE = u^{-1}(E(u(x)))$
π - sure loss, risk premium, W is investment amount, so
$E(u(W\cdot(1+x))) = u(W\cdot(1-\pi))$
or
$CE = W\cdot(1-\pi), \pi = 1-\frac{CE}{W}$

$$\begin{cases}+x\%, p_1\\ -x\%, p_2\end{cases}$$

$E(u(W\cdot(1+x))) = \sum p_i\cdot u(W\cdot(1+x))$

Interest rate conversion EAR/APR

T - compounding interval, fraction yearly: $T=1$
monthly: $T=\frac{1}{12}$
daily: $T=\frac{1}{365}$
P - principal n - number of payment periods, so period payment M is

$M = P\cdot\frac{APR\cdot(1+APR\cdot T)^n}{(1+APR\cdot T)^n-1}$

$\lim_{T\to0} 1+EAR = e^{APR}$, so
$APR = \ln(1+EAR)$

$APR = \frac{(1+EAR)^T-1}{T}$

$1+EAR = (1+T\cdot APR)^{\frac{1}{T}}$

Duration

Discount bond price $B_t=(1+y)^{-t}$, discounted bond duration is t, so modified duration is $MD=\frac{t}{1+y}$

Macaulay duration is
$D=\frac{1}{B}\cdot\sum_t\frac{CF_t}{(1+y)^t}\cdot t$

Modified duration is $MD=\frac{D}{1+y}$

Modified duration for perpetuity is $MD=\frac{1}{y}$, so Macaulay duration is $D = MD\cdot(1+y) = \frac{1+y}{y}$

Duration based approximations

Δy is the change in the interest rate, P is the asset price.
$\Delta P = -P\times MD\times\Delta y$
Convexity CX is
$CX = \frac{1}{2}\cdot\frac{1}{P}\cdot\frac{1}{(1+y)^2}\cdot\sum_t PV(CF_t)\cdot t\cdot(t+1)$,
convexity based approximation is
$\Delta P = P\times(-MD\cdot\Delta y + CX\cdot\Delta y^2)$

Statistic

Excel functions:
Sample mean $AVG()$
Standard deviation $STDEV.S()$
Covariance:
$cov = \frac{1}{T-1}\cdot\sum(r_A-\bar{r}_A)\cdot(r_B-\bar{r}_B)$
Corellation: $corr = \frac{cov}{SD(A)\cdot SD(B)}$

Portfolio variance:
$cov_{ij} = SD_i\cdot SD_j\cdot corr_{ij}$
$Var[P_{AB}] = \sum w_i^2\cdot SD_i^2 + \sum_{i\neq j}2w_iw_j\cdot SD_iSD_j\cdot corr_{ij}$,
$Var[P] = \frac{1}{n}\cdot SD^2+\left(1-\frac{1}{n}\right)\cdot corr\cdot SD\cdot SD$,
where SD is an average standard deviation

APT

For well diversified portfolios:
$\tilde{r}_P = \bar{r}_P + \sum b_i\cdot f_i$, where $\bar{r}_P$ is expected return
$\bar{r}_P - r_f = \lambda\cdot\beta_P$, where r_f is risk free rate, λ is risk price and β_P is factor loading for single factor portfolio. Same $\bar{r}_P - r_f = \sum_i\lambda_i\cdot\beta_i$ for i factors portfolio
Return variance:
$Var(r) = \sum_i\beta_i^2\cdot Var(f_i) + Var(\epsilon)$
Covariance:
$cov(A,B) = \sum_i\beta_{i,A}\beta_{i,B}\cdot Var(f_i)$

APT in Excel

$r_i - r_f = \alpha+\beta_1(r_1-r_f)+\beta_2(r_2-r_f)+\epsilon_i$
To estimate β_1, β_2 and α (in this order):
$=LINEST(\alpha,\beta_2,\beta_1)$ (reverse order)

Capital investment

$CF = OpRev - OpEx - Tax - CapEx$
$OpProfit = OpRev - OpEx$
$Tax = \tau\cdot OpProfit - \tau\cdot Depreciation$
$CF = (1-\tau)\cdot OpProfit - CapEx + \tau\cdot Depreciation$
Work capital:
$WC = Inventory + A/R - A/P$, where A/R accounts receivable, A/P accounts payable
$CF = (1-\tau)\cdot OpProfit+\tau\cdot Depreciation-CapEx-\Delta WC$

Alternatives to NPV

Payback period
Choose S so $PB = S$, $\sum_{i=1}^S CF_i \geq -CF_0$
Discounted payback period:
$DPB = S, \sum_{i=1}^S\frac{CF_i}{(1+r)^i}\geq -CF_0$

Internal rate of return (IRR) must satisfy:
$0 = CF_0 + \sum_i\frac{CF_i}{(1+IRR)^i}$

Payback Interval:
$PI = \frac{PV}{-CF_0}$

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Forward rates

Forward interest rate between time $t-1$ and t:
$f_t = \frac{B_{t-1}}{B_t}-1 = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}}-1$
Expectation hypotesis (forward rates at time 0 are predictors of future spot rates, which is not true):
$E_0[\tilde{r}_1(t)] = \frac{(1+r_{t+1}(0))^{t+1}}{(1+r_t(0))^t}-1 = f_{t+1}$

Forward pricing

Current spot price: S_0
Spot price at maturity (random): $\tilde{S}_T$
Forward price (fixed at time 0): F_T
Forward payoff is $\tilde{S}_T - F_T$
$PV_0(\tilde{S}_T) = e^{-yT}S_0$,
where y is dividend yield
$F_T = e^{(r-y)T}S_0$, so dividend yield
$y = r - T\cdot ln\left(\frac{F_T}{S_0}\right)$
Currency forward price is:
$F_T = X_0\cdot e^{(r_{USD}-r_{CHF})\cdot T}$

Futures pricing

Storage cost $Cost_t = c\cdot S_t$.
Net convenience yield $\hat{y} = y - c$, so
$H_T\approx F_T = e^{(r-\hat{y})T}S_0$
Backwardation in terms of convenience yield vs interest rate: $\hat{y} - r = y - c - r > 0$
Contango: $H_T > S_0e^{rT}$
Backwardation: $H_T < S_0e^{rT}$

Interest rate swaps

Fixed leg is paid at fixed rate r_S.
Floating leg at the end of each period t is paid as spot risk-free rate $\tilde{r}_1(t-1)$
Forward rate to future spot rate:
$PV_0[\tilde{r}_1(T)\ at\ T+1] = PV_0[f_{T+1}\ at\ T+1]$
Present value of the fixed leg: $r_S\times\sum_{u=1}^T B_u$
Present value of the floating leg of the swap: $\sum_{t=1}^T PV_0[\tilde{r}_1(t-1)\ at\ t]$

Swap rate: $r_S = \frac{\sum_{t=1}^T B_t\cdot f_t}{\sum_{u=1}^T B_u} = \sum_{t=1}^T w_t\times f_t$, where weights w_t are: $w_t = \frac{B_t}{\sum_{u=1}^T B_u}$

Alternative formula: $r_s = \frac{1-B_T}{\sum_{u=1}^T B_U}$

Options

S underlying asset price (at time 0).
S_T underlying asset price (at time T).
B price of discount bond of par \$1 and maturity T ($B\leq1$)
K strike (excercise) price.
T maturity (expriration) date.
C price of call with strike K and maturity T.
P price of put with strike K and maturity T.
European call option payoff: $CF_T = max[0,S_T-K]$
The net payoff is: $max[S_T-K,0]-C(1+r)^T$
Excercise value of a call is $S-K$.
Excercise value of a put is $K-S$.
Price bounds are: $max[S-KB,0]\leq C\leq S$
Put-Call parity: $C+BK = P+S$, where B is e^{-rT} if continious compounding is used.

Corporate securities as options

Equity (E): A call option on firm's assets (A) with K equal to its bond's redemption value.
Debt (D): A portfolio of firm's assets (A) and a short position in the call with K equal to bond's face value F.
Warrant: Call option on firm's stock, with stock dilution as a result of excercise.
Convertible bond: A portfolio of straight bonds and a call on the firm's

stock with K related to the conversion ratio.
Callable bond: A portfolio combining straight bonds and a short position in a call on these bonds.
$A = D + E => D = A - E$
$E \equiv max[0, A - F]$
$D = A - E = A - max[0, A - F]$

Binomial pricing: single period

r is the interest rate
Stock price change:

$$S_0 \to S_u = uS_0, \quad S_0 \to S_d = dS_0$$

Riskless bond price change:

$$B_0 \to B_0(1+r), \quad B_0 \to B_0(1+r)$$

Need to solve system of qquations:
$S_u \cdot a + (1+r) \cdot b = C_u$
$S_d \cdot a + (1+r) \cdot b = C_d$,
where a amount of stock shares (option's delta), b dollars invested into riskless bond B, C_u is payoff in up state, C_d is payoff in down state, so current market value of the call option is: $C_0 = S_0 \cdot a + b$. Alternative notation:
$\delta u S_0 + b(1+r) = C_u$
$\delta d S_0 + b(1+r) = C_d$
where unique solutions is:
$\delta = \frac{C_u - C_d}{(u-d)S_0}$, $b = \frac{1}{1+r} \cdot \frac{uC_d - dC_u}{(u-d)}$, so
$C_0 = \delta S_0 + b = \frac{C_u - C_d}{u-d} + \frac{1}{1+r} \cdot \frac{uC_d - dC_u}{(u-d)}$

Risk-neutral probability

$q_u = \frac{(1+r) - d}{u-d}$, $q_d = \frac{u - (1+r)}{u-d}$
$C_0 = \frac{q_u C_u + q_d C_d}{1+r} = \frac{E^Q[C_T]}{1+r}$,
where $E^Q[\cdot]$ is expectation under risk-neutral probability $Q = (q, 1-q)$.

State prices

$\phi_u = \frac{q}{1+r}$, $\phi_d = \frac{1-q}{1+r}$
Alternatively, solve the system:

$$\begin{cases} S_0 = S_u\phi_u + S_d\phi_d \\ \frac{1}{1+r_f} = \phi_u + \phi_d \end{cases}$$

Black-Scholes-Merton formula

$C_0 = C(S_0, K, T, r, \sigma) = S_0 N(x) - Ke^{-rT}N(x - \sigma\sqrt{T})$, where x is:
$x = \frac{ln\left(\frac{S_0}{Ke^{-rT}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$

So option delta (δ) becomes $N(x) = \frac{\partial C}{\partial S}$.
$S_0 \cdot N(x)$ is the dollar mount invested into stock.
$Ke^{-rT}N(x - \sigma\sqrt{T})$ is the dollar amount borrowed.

Put-Call parity with BSM formula

$C + BK = P + S => P = C + e^{-rT}K - S$
$P = S \cdot N(x) - Ke^{-rT}N(x - \sigma\sqrt{T}) + e^{-rT}K - S$
$P = -S(1 - N(x)) + Ke^{-rT}(1 - N(x - \sigma\sqrt{T}))$

Option Greeks

Delta: $\delta = \frac{\partial C}{\partial S}$
Omega: $\Omega = \frac{\partial C}{\partial S}\frac{S}{C}$
Gamma: $\Gamma = \frac{\partial \delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}$
Theta: $\Theta = \frac{\partial C}{\partial T}$
Vega: $v = \frac{\partial C}{\partial \sigma}$

Portfolio with mean-variance preferences

NOTE: $\sigma \equiv SD$, $\sigma^2 = Variance$, $\sigma = \sqrt{Variance}$
Optimization:
(P): $Minimize_{\{w_1,\ldots,w_n\}}\sigma_P^2 = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$,
where σ_{ij} is covariance between assets, σ_P^2 is portfolio variance.
(1): $\sum_{i=1}^N w_i = 1$
(2): $\sum_{i=1}^N w_i \bar{r}_i = \bar{r}_p$,
where $\bar{r}_p$ is portfolio expected return.
Sharpe Ratio: $SR \equiv \frac{\bar{r}_p - r_f}{\sigma_p}$, the higher the better.
All portfolios on the CML (Capital Market Line), including Tangency Portfolio, have the highest Sharpe Ratio.

Tangency portfolio analytics

N risky assets, $i = 1, 2, \ldots, N$
$\bar{r}$ vector of expected returns
$\bar{x}$ vector of excess returns
$\sum$ covariance matrix
l vector of 1 of size $(N \times 1)$
w vector of portfolio weights of size $(N \times 1)$
so,
$\bar{x} = \bar{r} - r_f \cdot l$,
risk-free asset weight is $1 - w'l$
expected excess return on portfolio is $w'\bar{x}$
$w'\sum w = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sum_{ij}$ portfolio variance
$w'\bar{x} = m$ portfolio expected excess return

Solving with Langrange multipliers

$L = w'\sum w + 2\lambda(m - w'\bar{x})$
First order condition (FOC): $\frac{\partial L}{\partial w} = 0$
$2\sum w - 2\lambda\bar{x} = 0$ => Solution: $w_T = \lambda\sum^{-1}\bar{x}$
Tangency portfolio weights on risky assets sum to one: $w_T' \cdot l = 1$, so need to find:
$\lambda = \frac{1}{\bar{x}'\sum^{-1} \cdot l}$
$w_T = \frac{1}{\bar{x}'\sum^{-1} \cdot l} \cdot \sum^{-1}\bar{x}$

Asset contribution to portfolio

Portfolio return with risk-free asset:
$\tilde{r}_p = \left(1 - \sum_{i=1}^N w_i\right) r_f + \sum_{i=1}^N w_i \tilde{r}_i$
$\tilde{r}_p = r_F + \sum_{i=1}^N w_i(\tilde{r}_i - r_f)$
Expected portfolio return:
$\bar{r}_p = r_f + \sum_{i=1}^N w_i(\bar{r}_i - r_f)$
Risk premium of asset i: $\frac{\partial \bar{r}_p}{\partial w_i} = \bar{r}_i - r_f$
Variance of portfolio return:
$\sigma_p^2 = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}$
Marginal contribution of asset i to portfolio variance σ_p^2:
$\frac{\partial \sigma_p^2}{\partial w_i} = 2 \cdot \sum_{j=1}^N w_j \sigma_{ij} =$
$= 2Cov\left(\tilde{r}_i, \sum_{j=1}^N w_j \tilde{r}_j\right) =$
$= 2Cov(\tilde{r}_i, \tilde{r}_p)$ Marginal contribution of asset i to portfolio standard deviation σ_p:
$\frac{\partial \sigma_p}{\partial w_i} = \frac{\partial(\sigma_p^2)^{\frac{1}{2}}}{\partial w_i} = \frac{Cov(\tilde{r}_i, \tilde{r}_p)}{\sigma_p} = \frac{\sigma_{ip}}{\sigma_p}$
Return to risk ratio RRR_{ip}:
$RRR_{ip} = \frac{asset\ risk\ premium}{marginal\ asset\ contrib\ to\ SD} =$
$= \frac{\bar{r}_i - r_f}{(\sigma_{ip}/\sigma_p)}$
For tangency portfolio:
$RRR_{iT} = \frac{\bar{r}_i - r_f}{(\sigma_{iT}\sigma_T)} = \frac{\bar{r}_T - r_f}{\sigma_T} = SR_T$

CAPM derivation

$MCAP_M = \sum_{i=1}^N MCAP_i$,
$w_i = \frac{MCAP_i}{MCAP_M}$
$\tilde{r}_i - r_f = \alpha_i + \beta_{iM}(\tilde{r}_M - r_f) + \tilde{\epsilon}_i$, where α is always 0, $E[\tilde{\epsilon}_i] = 0$ and $Cov[\tilde{r}_M, \tilde{\epsilon}_i] = 0$.
$\sigma(\tilde{\epsilon}_i) = SD$ measures non-systematic risk.
$SR_T = \sqrt{SR_M^2 + SR_P^2}$

Leverage

$A = E + D$, so $\beta_A = \frac{E}{E+D}\beta_E + \frac{D}{E+D}\beta_D$
$r_A = \frac{D}{E+D}r_D + \frac{E}{E+D}r_E$
$WACC = w_D r_D + w_E r_E$
If CAPM holds and debt is riskless, so $\beta_D = 0$,
$\beta_A = w_D\beta_D + (1 - w_D)\beta_E = (1 - w_D)\beta_E$
$or\ \beta_E = \frac{1}{1 - w_D}\beta_A = \left(1 + \frac{D}{E}\right)\beta_A$.
Equity cost for levered firm is:
$r_E = r_A + \frac{D}{E}(r_A - r_D)$
Unlevered firm r_E compensates for business risk. Levered firm r_E in addition compensates for finansial risk from leverage.

Default and risk premium

B zero-coupon bond face value
P zero-coupon bond current price
$E[P]$ expected payoff
y_f risk-free yield to maturity
Promised YTM: $y = \left(\frac{B}{P}\right)^{1/T} - 1$
Expected YTM: $\bar{y} = \left(\frac{E[P]}{P}\right)^{1/T} - 1$
Default premium = Promised YTM - Expected YTM
Risk premium = Expected YTM - y_f
With λ as loss rate, p as probability of default and $\bar{y}$ as yield for bonds of similar risk, promised yield y to sell at face value is:
$y = \frac{\bar{y} + p\lambda}{1 - p\lambda}$

Firm value with debt and taxes

$V_U = \frac{(1-\tau)X}{1+r_A}$
$V_L = \frac{(1-\tau)X}{1+r_A} + \frac{\tau r_D D}{1+r_D} = V_U + \frac{\tau r_D D}{1+r_D}$
X pre tax cash flow τ corporate tax
π equity tax (dividend and capital gain)
δ debt tax rate
Equity holders CF: $(1-\pi)(1-\tau)(X - r_D D)$
Debt holders CF: $(1-\delta)r_D D$
Total after tax CF:
$(1-\pi)(1-\tau)X + [(1-\delta) - (1-\pi)(1-\tau)]r_D D$
All equity firm discount rate: $(1-\pi)r_A$
For debt discout rate is: $(1-\delta)r_D$

Firm value without tax shield

$V = D + E = \sum_{s=1}^\infty \frac{(1-\tau)X_s}{(1+r_A)^S} = V_U =$
$= \sum_{s=1}^\infty \frac{(1-\tau)X_s}{(1+WACC)^s} = V_L$
$V_L = E + D = V_U + PVTS - PVDC = APV$
$WACC = \frac{D}{D+E}(1-\tau)r_D + \frac{E}{D+E}r_E$
Firm value with growth:
$V_U = \frac{(1-\tau)X_t}{r_A - g}$, where g is growth factor.