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Euler Integral Cont'd

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OUTLINE

- **Two types of Euler Integral**
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  - *The second kind: Gamma function*
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Two types of Euler Integral

In mathematics, there are two types of Euler Integral:

- **The first kind: Beta function**

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

- **The second kind: Gamma function**

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$$

For positive integers m and n,

$$B(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}=\frac{m+n}{mn\binom{m+n}{m}}$$

$$\Gamma(n)=(n-1)!$$

Relationships between Beta function and Gamma function

$$B(x,y)=\frac{\Gamma(x)\bullet\Gamma(y)}{\Gamma(x+y)} \qquad (1)$$

To prove equation (1),

Derivation one:

We write the two factorials as

$$\Gamma(x)\bullet\Gamma(y)=\int_0^\infty u^{x-1}e^{-u}du\cdot\int_0^\infty v^{y-1}e^{-v}dv$$

$$=\int_0^\infty\int_0^\infty u^{x-1}v^{y-1}e^{-u-v}dudv$$

Relationships between Beta function and Gamma function

Changing variables by putting

$$\begin{cases} u = u(z,t) = z \cdot t \\ v = v(z,t) = z \cdot (1-t) \end{cases} \quad t \in (0,1),\ z \in (0,\infty)$$

Then, we derive the Jacobian determinant of z and t:

$$J(z,t) = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial t} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial t} \end{vmatrix} = \begin{vmatrix} t & z \\ 1-t & -z \end{vmatrix} = -z$$

Relationships between Beta function and Gamma function

Thus, $\Gamma(x)\bullet\Gamma(y)$

$$= \int_0^\infty \int_0^\infty u^{x-1} v^{y-1} e^{-u-v} du dv$$

$$= \int_0^\infty \int_0^1 (zt)^{x-1} [z(1-t)]^{y-1} e^{-zt-z(1-t)} \cdot \left| J(z,t) \right| dz dt$$

$$= \int_0^\infty \int_0^1 z^{x+y-2} t^{x-1} (1-t)^{y-1} e^{-z} \cdot z dz dt$$

$$= \int_0^\infty z^{x+y-1} e^{-z} dz \bullet \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

$$= \Gamma(x+y)\bullet \mathrm{B}(x,y)$$

That is $\mathrm{B}(x,y) = \dfrac{\Gamma(x)\bullet\Gamma(y)}{\Gamma(x+y)}$

Relationships between Beta function and Gamma function

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\bullet\Gamma(y)}{\Gamma(x+y)} \qquad (1)$$

To prove equation (1),

Derivation two:

As for Beta function $\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$

Changing variable by putting $x = \sin^2 t,\ t \in (0, \frac{\pi}{2})$

Then, $dx = 2\sin t \cos t\, dt$

$$\mathrm{B}(p,q) = \int_0^{\frac{\pi}{2}} \sin^{2(p-1)} t \cdot \cos^{2(q-1)} t \cdot 2\sin t \cos t\, dt$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2p-1} t \cdot \cos^{2q-1} t\, dt \qquad (2)$$

Relationships between Beta function and Gamma function

We write two factorials as

$$\Gamma(p)\bullet\Gamma(q)=\int_0^\infty x^{p-1}e^{-x}dx\bullet\int_0^\infty t^{q-1}e^{-t}dt$$

$$=\int_0^\infty\int_0^\infty x^{p-1}t^{q-1}e^{-x-t}dxdt$$

Changing variables as

$$\begin{cases} x=u^2 \\ t=v^2 \end{cases} \quad u\in(0,\infty),v\in(0,\infty)$$

The Jacobian determinant

$$J(x,t)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}=\begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix}=4uv$$

Relationships between Beta function and Gamma function

Then,

$$\Gamma(p)\bullet\Gamma(q)=\int_0^\infty\int_0^\infty u^{2(p-1)}v^{2(q-1)}e^{-u^2-v^2}\left|J(u,v)\right|dudv$$

$$=\int_0^\infty\int_0^\infty u^{2(p-1)}v^{2(q-1)}e^{-u^2-v^2}\,4uvdudv$$

$$=4\int_0^\infty\int_0^\infty u^{2p-1}v^{2q-1}e^{-u^2-v^2}dudv$$

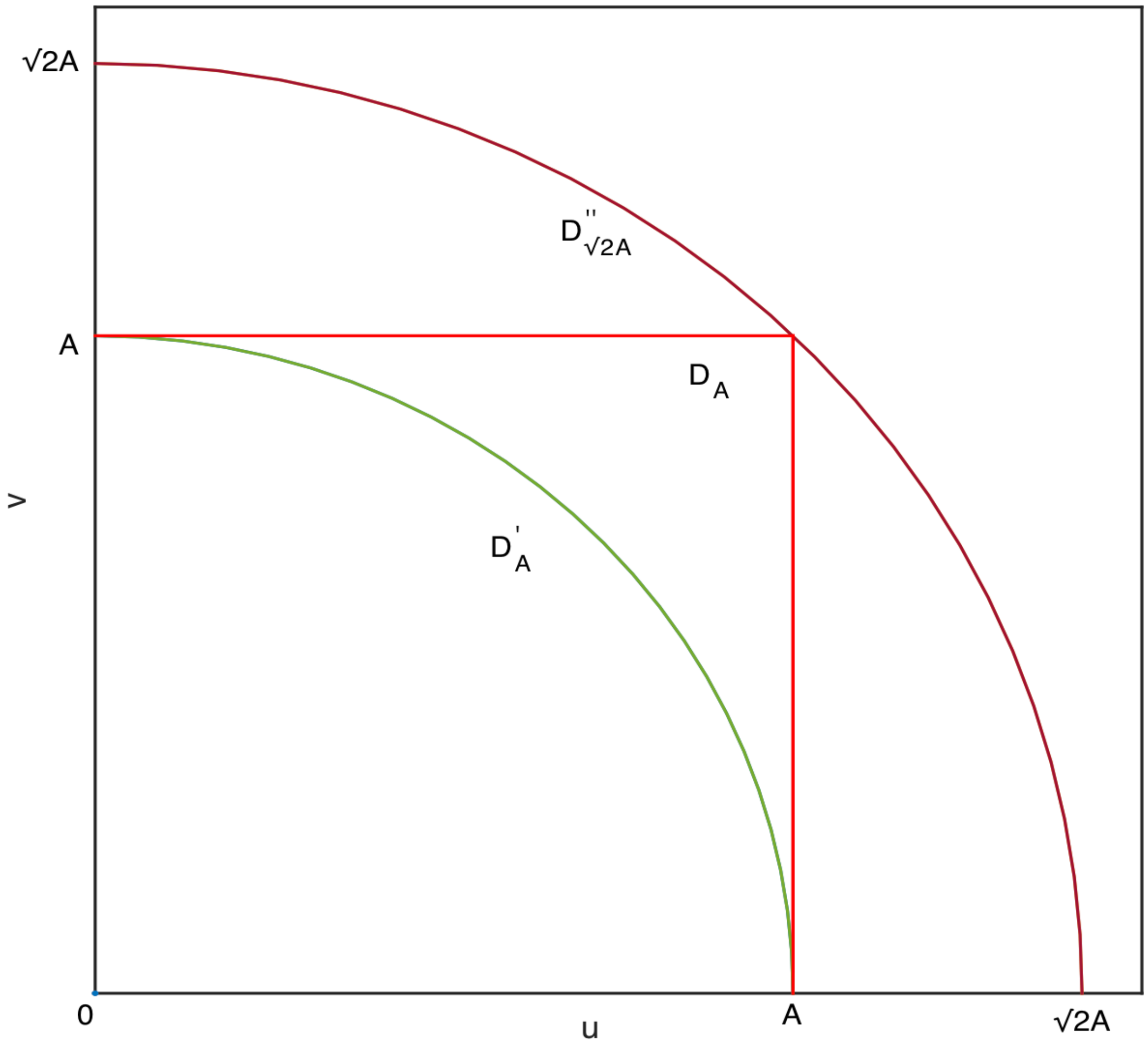
$$=4\lim_{A->\infty}\int_0^A\int_0^A u^{2p-1}v^{2q-1}e^{-u^2-v^2}dudv$$

$$=4\lim_{A->\infty}\iint_{D_A} u^{2p-1}v^{2q-1}e^{-u^2-v^2}d\sigma$$

D_A is a square with length A, $0\le u\le A,\ 0\le v.\le A$

We, then, draw two quadrants with centers at the original point, and radiuses of A and $\sqrt{2}A$, respectively.

Relationships between Beta function and Gamma function



Relationships between Beta function and Gamma function

Since $u^{2p-1}v^{2q-1}e^{-u^2-v^2}$ **is non-negative, we know**

$$\iint_{D_A^{'}} u^{2p-1}v^{2q-1}e^{-u^2-v^2}d\sigma \le \iint_{D_A} u^{2p-1}v^{2q-1}e^{-u^2-v^2}d\sigma \quad (3)$$

$$\le \iint_{D_{\sqrt{2}A}^{''}} u^{2p-1}v^{2q-1}e^{-u^2-v^2}d\sigma$$

Then, as to

$$\iint_{D_A^{'}} u^{2p-1}v^{2q-1}e^{-u^2-v^2}d\sigma$$

We represent variables using polar coordinates.

Relationships between Beta function and Gamma function

Then we get

$$I_A = \int_0^{\frac{\pi}{2}} \int_0^A (r\cos\theta)^{2p-1} (r\sin\theta)^{2q-1} e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^{2p-1}\theta \sin^{2q-1}\theta\, d\theta \cdot \int_0^A r^{2p+2q-1} e^{-r^2} dr$$

$$= \frac{1}{2} \mathrm{B}(p,q) \cdot \int_0^A r^{2p+2q-1} e^{-r^2} dr$$ **using equation(2)**

$$= \frac{1}{2} \mathrm{B}(p,q) \cdot \frac{1}{2} \int_0^A r^{2(p+q-1)} e^{-r^2} dr^2$$

$$\underline{\underline{x=r^2}} \frac{1}{4} \mathrm{B}(p,q) \cdot \int_0^{A^2} x^{(p+q-1)} e^{-x} dx$$

Relationships between Beta function and Gamma function

Thus

$$\lim_{A->\infty}\iint_{D_A^{'}} u^{2p-1}v^{2q-1}e^{-u^2-v^2}d\sigma$$

$$=\frac{1}{4}\mathrm{B}(p,q)\cdot\lim_{A->\infty}\int_0^{A^2}x^{(p+q-1)}e^{-x}dx$$

$$=\frac{1}{4}\mathrm{B}(p,q)\cdot\int_0^{\infty}x^{(p+q-1)}e^{-x}dx$$

$$=\frac{1}{4}\mathrm{B}(p,q)\cdot\Gamma(p+q)$$

Relationships between Beta function and Gamma function

Similarly,

$$\lim_{A->\infty}\iint_{D''_{\sqrt{2}A}} u^{2p-1}v^{2q-1}e^{-u^2-v^2}d\sigma$$

$$=\frac{1}{4}\mathrm{B}(p,q)\cdot\lim_{A->\infty}\int_0^{2A^2} x^{p+q-1}e^{-x}dx$$

$$=\frac{1}{4}\mathrm{B}(p,q)\cdot\int_0^{\infty} x^{p+q-1}e^{-x}dx$$

$$=\frac{1}{4}\mathrm{B}(p,q)\cdot\Gamma(p+q)$$

Relationships between Beta function and Gamma function

According to the Squeeze Theorem and equation (3), we get

$$\lim_{A \to \infty} \iint_{D_A} u^{2p-1} v^{2q-1} e^{-u^2-v^2} d\sigma = \frac{1}{4} \mathrm{B}(p,q) \cdot \Gamma(p+q)$$

Thus, $\Gamma(p) \bullet \Gamma(q) = 4 \lim_{A \to \infty} \iint_{D_A} u^{2p-1} v^{2q-1} e^{-u^2-v^2} d\sigma$

$$= 4 \cdot \frac{1}{4} \mathrm{B}(p,q) \cdot \Gamma(p+q)$$

$$= \mathrm{B}(p,q) \cdot \Gamma(p+q)$$

That is $\mathrm{B}(p,q) = \dfrac{\Gamma(p) \bullet \Gamma(q)}{\Gamma(p+q)}$

Thanks