SQUARE-ROOT LAWS FOR FIRE COMPANY TRAVEL DISTANCES

PETER KOLESAR, EDWARD H. BLUM

R-895-NYC
JUNE 1975

THE
NEW YORK CITY
RAND
INSTITUTE

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SANTA MONICA, CA. 90406

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PREFACE

The work reported here--part of The New York City-Rand Institute's research on urban fire protection--was sponsored by and carried out jointly with the Fire Department of the City of New York. The research program has been broadly conceived. It includes analysis and design of communications systems, analysis of fire prevention activities, and analysis of the deployment and effectiveness of fire-fighting resources. The mathematical models were developed during research on the last of these topics. Several publications documenting the work of the fire project are referenced at the end of the report.

SUMMARY

When fire, police, or ambulance units respond to incidents where lives or property are at risk, seconds count. Once an emergency service receives a call for help, the time for personnel to reach the incident--the travel time--is one of the most important performance variables the service itself can affect.

One of the most direct influences on response time is the response or travel distance. Our analysis, which focuses on fire companies, shows that the average response distance in a region is inversely proportional to the square root of the number of locations per area from which fire companies are available to respond. The square-root law predicts that

$$ED_i = k_i \sqrt{\frac{A}{n - \lambda ES}}$$

where ED_i is the long run expected response distance of the *ith* closest fire company, A is the area of the region under consideration, λ is the expected number of alarms received per hour, ES is the expected total "service" time spent by all fire companies that respond to and work at an alarm per alarm, k_i is a constant of proportionality that can be determined empirically, and n is the number of fire companies assigned to the region. Combined with empirical data relating travel times to distances, this relation provides an easy-to-use formula relating average travel times in a region to key operational and planning variables: the region's area, its alarm rate, the number of units stationed there, service times, and travel velocities.

Termed the "square-root law" because of its mathematical form, the response distance relation stems directly from dimensional analysis. In addition, it can be motivated by many simple mathematical models of fire company and fire location, which show how the constants of proportionality

in the law depend on location patterns.

Extensive data from Institute simulation of New York City Fire Department operations confirm this relation in detail for application to the deployment of fire-fighting units. This evidence shows that the law is robust--that it applies even in complex, realistic situations where none of the sufficient conditions assumed in the theoretical models used to derive it apply. For example, these simulations show that it holds well even when the number of units busy in a region varies rapidly with time.

A similar inverse square-root relation also appears to hold for maximum response distances and for the response distance probability distribution. For these cases, however, the evidence is less extensive and the relation appears less robust. Evidence is strong for the second moment of the distribution: simulation experiments confirm that the standard deviation of response distance also follows a square-root law.

The simulation runs also show quantitatively where deviations from the square-root law become large enough that alternative formulations, including more detailed models, should be used instead.

The results of an experiment in which fire companies responding to alarms recorded travel times and distances yielded a simple time-distance function [24].* When this function is combined with the square-root law for distances, a power model for average travel times results [26].

This average travel time model addresses basic allocation questions faced by all fire departments. We illustrate how the model can be used to answer the following specific questions:

(1) Given an allocation of fire-fighting units to an area, what are the resulting average travel times?

*Numbers in brackets refer to references listed at the end of this report.

(2) What number of fire-fighting units is required in an area in order to achieve a desired average travel time?

(3) How should a fixed number of fire-fighting units be distributed across several areas to equalize average travel times or to achieve an overall minimum average travel time?

Designed to provide a quick and inexpensive first approximation with data that are easily estimated, the model has been used extensively for operational and capital-project planning and to estimate the value of deployment improvements. When used in major policy applications and in formulating programs, it is supplemented by more detailed calculations, including the fire department computer simulation.

ACKNOWLEDGMENTS

With closely knit team efforts, such as The New York City-Rand Institute's Fire Project, it is extremely difficult to separate responsibility or to give complete credit to all who contribute. Although it bears our names, this work is very much a joint product of many members of the Fire Project. We would like to note especially Grace Carter, who contributed the simulation models and ran the necessary experiments. Deputy Assistant Chief Homer G. Bishop of the New York City Fire Department worked closely with us on applications and organized and supervised the collection of response-time data described in Section IV. Discussions with Warren Walker and Edward Ignall of The New York City-Rand Institute, and with Mike Florian and the late Pierre Robilliard of the Université de Montréal, were helpful in many ways.

The work of Peter Kolesar was partially supported by the National Research Council of Canada while he was visiting professor at the Université de Montréal during 1971.

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I. INTRODUCTION

Travel time, the elapsed time between the time when fire companies are dispatched and their arrival at the scene, is an important measure of the quality and effectiveness of the fire protection service. Standards for fire protection set by insurance companies or advisory bodies such as the National Fire Protection Association often specifically include travel time bench marks. The fundamental goal of a fire department is to protect against loss of life and property from fire. However, because it is so difficult to estimate directly the lives saved or property damage avoided when a fire department moves a fire house location, changes the number of units on duty, or installs a new dispatching system, etc., travel time is often substituted as a standard of performance in evaluating such changes.

We therefore assume that reducing travel time improves service, although at present the relationship is not quantified. No one now knows how many lives or homes would be saved by a given reduction in travel time.* This report is one of a series in which we develop some simple mathematical models that can be used to predict fire company travel times using only a few easily measured characteristics of the region being studied [24, 26]. Here our focus is on predicting average travel distances.

We present a relationship called the square-root law for average response distances that states that the average response distance is

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*A considerable part of the current fire research at the New York City-Rand Institute is directed to this problem. References [18] and [21] discuss past attempts at evaluating the utility of fire engine travel times. A key problem, of course, is the fact that fire departments now can respond only after an unknown and highly variable delay in fire detection and reporting. It is clear that this interval is often orders of magnitude greater than the time it takes the fire department's units to travel to the scene. The lower the travel time the lower the loss, but the absolute levels of loss obviously depend as much on the detection and reporting delay on the response and subsequent fire-fighting performance of the fire department. All parts of the total system thus need to be examined and improved [3]; see, for example, [14] for work on the detection and warning problem.

inversely proportional to the square root of the number of locations from which fire companies are available to respond. In [24], Kolesar and Walker analyze empirical fire company travel times and response distances and formulate and validate a simple mathematical model linking the two. Finally, in [26], Kolesar combines these two relationships to obtain a general power law for average fire company travel times.

The square-root law for average distances is very robust as well as simple to use. It can be derived theoretically under various mathematical assumptions and it has been validated empirically under real-world conditions in several cities. The travel time model resulting from its combination with time-distance estimation has been used in analyzing important problems for the Fire Departments of New York City, Yonkers [16], Trenton [17], Wilmington [37], and other cities. Our continuing experience indicates the model's applicability to other cities and other emergency services. Indeed, square-root models of police patrol car travel distances have been derived by Larson [27]. Brian Whitworth [38] has indicated their applicability to other problems, including the estimation of average travel distances to solid refuse disposal sites, while in Sweden they have been used in the analysis of highway maintenance problems.

Our results enable us to predict average response distances and travel times given only the following readily measurable parameters of the region of interest:

- the number of active locations for fire companies;
- the geographical area being covered;
- the rate at which alarms are generated;
- the expected time required to service (extinguish, etc.) each fire;

and, lastly, a constant of proportionality depending upon, but rather insensitive to, the geometry of the region (mainly the street patterns and relative locations of fire alarms and fire stations).

For a manager or fire officer charged with locating and allocating fire companies, devising response policies, etc., the type of mathematical model described here provides a basic tool for quickly appraising some important consequences of those managerial decisions.

As with all simple models, the predictions so produced are approximations, which are useful in narrowing the range of policies to be considered. Where necessary, the model's results can later be refined or checked in detail by other models, such as simulation. In the following section, we discuss some of the model's uses. The examples are not hypothetical; the model has been applied in the manner indicated by the New York City Fire Department since 1970, and policy changes suggested by it have been implemented. It is currently being applied in a similar fashion in other cities as well.

APPLICATIONS

The square-root law can be used in a wide variety of circumstances where estimates of fire company travel times are needed. Its most basic use is to estimate the consequences of alternative allocations of fire-fighting units (each unit consisting of pumper or ladder together with its complement of men and officers) to regions of a city. For example:

- Fire Department personnel analyzed the fire hazards that exist in New York City and found that, on the basis of these hazards, there were seven distinct "hazard types." The city was then divided into 21 separate fire planning districts, each identified as being one of the seven hazard types. In making this division, regions of a given hazard type were chosen to be roughly homogeneous with respect to alarm rate, building construction, population density, etc. Consequently, the Fire Department felt that travel times in regions of the same hazard type should be about the same. The square-root law was used to estimate the average travel times of the first arriving engine and ladder to fires in each hazard region given the then current allocation of companies. The results showed significant imbalances and suggested potential reallocations of companies between hazard regions of the same type; i.e., from regions with lower travel times to regions with higher travel times. The model was also used to evaluate the effect of adding or subtracting units from particular hazard regions. Using this analysis in conjunction with more detailed studies of each affected area, the Fire Department in 1973 eliminated six fire companies and permanently relocated seven others [20].

- Because the hourly alarm rate varies significantly, improvements in response time and a smoothing of fire company workload can be achieved by varying the number of units on duty at different times of day. Using predictions of alarm rate by time of day, the square-root law has been used to evaluate the travel time effects of such changes within each hazard region.

- The square-root law also has been an essential tool for long-range planning. Using forecasts of total alarm rates, the percentage of false alarms, and the percentage of structural fires by hazard region for 10 years into the future, the square-root law has been used to estimate the number of fire units needed at various times in the future to maintain desired average travel times.

- An important Fire Department deployment decision is determining how many units to dispatch to an incoming alarm. Sending more units to a given alarm results in a decrease in the number of fire companies available to respond to a new alarm should one occur soon after the dispatch. In order to evaluate various dispatching rules, the square-root law has been used to translate the concomitant changes in average fire company availability into changes in average fire company travel times. This analysis played an important role in designing a new "adaptive response" policy, which was implemented in New York City in November 1972 [20, 35].

We must emphasize that in focusing here on average response distances and travel times, and in dealing with simple analytical models, we necessarily simplify many elements of the real fire-fighting situation. In analyzing real-world problems, we cannot and do not ignore the complexities of the environment or the appropriateness of other criteria. Nevertheless, the square-root law, when used as a rule of thumb, has clarified many managerial issues. For discussion of the complex environment of urban fire deployment and of the research program, of which development and use of this model is only one part, see [3], [4], [5], [30], and [31].

II. THE SQUARE-ROOT LAW FOR RESPONSE DISTANCES: SOME THEORETICAL MODELS

In its simplest form, the square-root law proposed states that $E(D_1)$, the expected distance between the points at which fires occur and the location of the closest available fire company, is given by

$$E(D_1) = k_1\sqrt{A/N}, \qquad (1)$$

where k_1 is a constant of proportionality, A is the area of the region, and N is the number of fire houses having companies available to respond.*

In addition, $E(D_m)$, the expected distance from fire locations to the *mth* closest available fire company follows the relation

$$E(D_m) = k_m\sqrt{A/N}. \qquad (1a)$$

Indeed, as m increases, $E(D_m)$ converges from below to (and for $m > 3$ can be closely approximated by):

$$E(D_m) = \sqrt{mA/N\alpha}\ ,$$

where $\alpha = \pi$ for the Euclidean metric and $\alpha = 2$ for the right-angle (Manhattan) metric.

For regions where fire company locations are distributed quite homogeneously (e.g., a large mostly medium-density residential areas), the *maximum* (first-due) response distance S follows the law:

$$S = k_s\sqrt{A/N}. \qquad (1b)$$

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*Throughout this report we will be using the terms fire company, fire engine, and ladder. A fire company consists of an apparatus, its complement of men and an officer. If the apparatus is an engine (pumper), the primary function is to put water on the fire, if a ladder, the primary function is to gain access to the fire, ventilate it, and make rescues. The square root law will usually be applied separately for engines and ladders, but may be applied to both together. We will use fire company to refer to the category of units for which the analysis is being performed (engines, ladders, or both).

In such regions, the probability distribution of response distance itself follows the relation

$$P\ [D \leq uS] = F(u), \qquad (1c)$$

where u is a dimensionless distance parameter, $0 \leq u \leq 1$, $F(u)$ is normalized such that $F(1) = 1$, and S follows (1b). From (1c) it is apparent that

$$E(D^n) = \int_0^1 (uS)^n dF(u) = [k_S^n \int_0^1 u^n dF(u)]\ (A/N)^{n/2},$$

so that $u_n(D) \sim (\sqrt{A/N})^n$, where $u_n(D)$ is the nth central moment of the response distance probability distribution. In particular, Var (D) = $u_2(D) \sim A/N$.

Let us illustrate these relations' origin and use, building on a particularly straightforward example: Consider an idealized city, with just one fire house located at the origin ($x = 0$, $y = 0$) of a two-dimensiona coordinate system; the city boundaries form a square with corners on the x and y axes, and its streets form a dense rectangular grid parallel to both axes. Let the city's area be A. Within the city, fires occur at random, with equal probability and severity everywhere (i.e., fire hazards and incidence are homogeneous). See Figure 1 for a schematic representation.

In this example, assume that the fire company is available to respond when an alarm is received (we will consider later the effects of unavailability). Then the maximum response distance in the city is simply the rectangular distance from the center to one of the corners: $S = \sqrt{A/2}$. The probability that the response distance is less than some fraction, u, of S is simply the area of the square with side u:

$$P[D \leq uS] = u^2, \qquad 0 \leq u \leq 1. \qquad (2a)$$

The probability density is then 2u so that the average response distance is simply

$$E(D) = S \int_0^1 u(2u)du = \frac{2}{3} S = \frac{\sqrt{2}}{3} \sqrt{A}.$$

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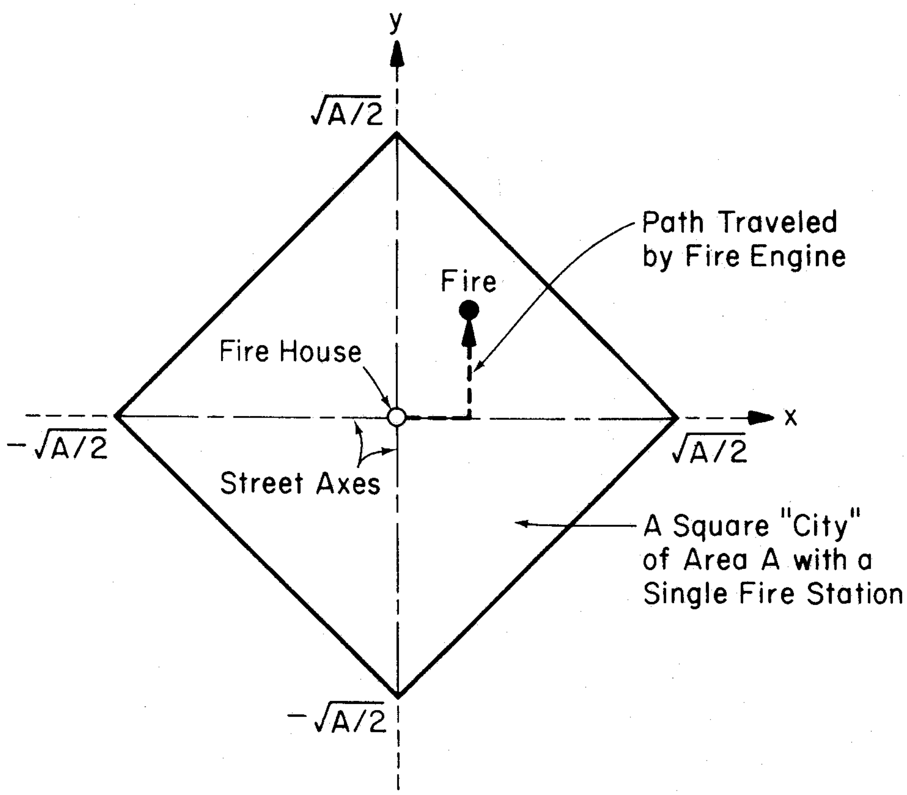

Figure 1. An idealized city with one fire house.

Now suppose that our idealized city develops in such a way that fire hazards and incidence remain homogeneous--but rise to the point where a total of N fire stations are now needed. If N were four and the city placed all stations in the locations that minimize response distances, the new configuration would be that depicted in Figure 2. Now the city is divided into four districts, each with one-fourth the total city area, each with boundaries one-half the length of the total city boundary. Indeed, all distances in the new districts are half the corresponding distances in the city as a whole. This ratio simply reflects the basic dimensional relations:

$$\text{Area} \sim (\text{Distance})^2,$$

$$\text{Distance} \sim \sqrt{\text{Area}}.$$

It is clear that for values of N such as 4, 16, 64,..., for which the region can be partitioned into squares of equal size, the maximum response distance S is given by

$$S = (1/\sqrt{2})\ \sqrt{A/N}. \qquad (2b)$$

Generalizing from this very special example, and using the dimensional relations cited above, let us argue that distances within the region served by each firehouse are proportional to the area of that region, and the area served by each fire house decreases in proportion to the number of fire houses:

$$\text{Intraregion Distance} \sim \sqrt{\text{Regional Area}},$$

$$\text{Regional Area} \sim \text{Total Area/Number of regions}.$$

It is plausible, therefore, to hypothesize the inverse square-root law--equation (1), with its corollaries.

In the section that follows, we demonstrate the robustness of the relationship, examining varied mathematical models of response distances in some detail. We will consider finite and infinite regions, travel along right angle street networks, travel as the crow flies, etc. Each

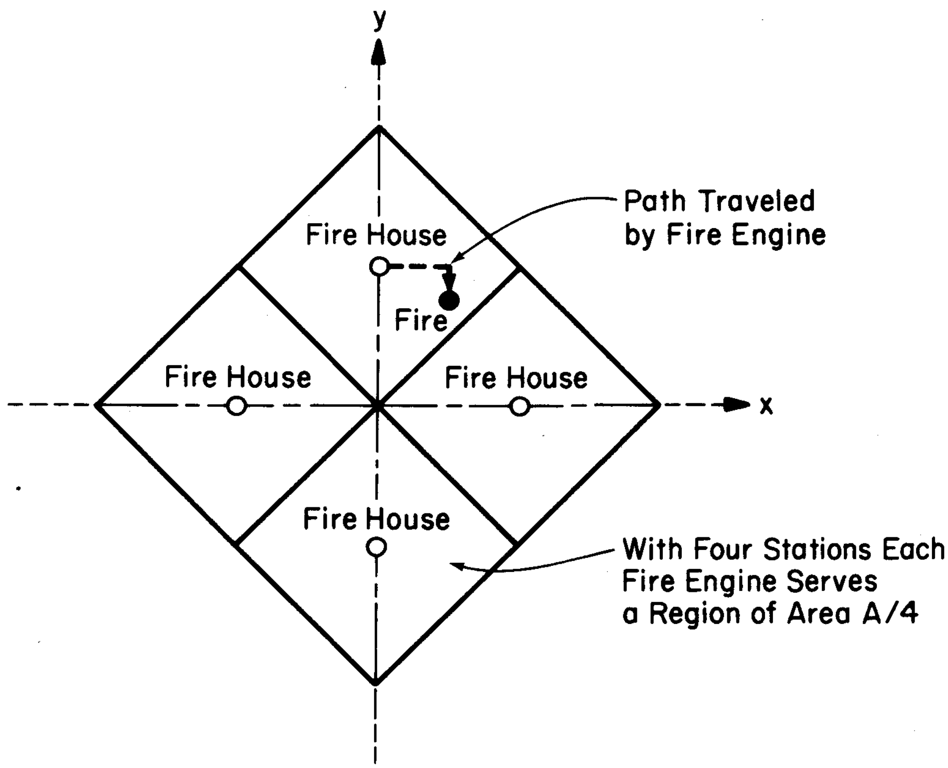

Figure 2. An idealized city with four fire houses.

model yields the inverse square-root form, differing only in the constant of proportionality k. From a mathematical point of view, nothing presented here is new. The results discussed are standard fare in probability theory and have long been used by astronomers, chemists, geographers, and others--see, for example, [13], [15], and [22].

STATIC MODELS FOR EXPECTED RESPONSE DISTANCES

Consider first a probabilistic but static situation. To keep the discussion simple, suppose that fire companies are located, one to a fire house, in a large region completely at random--i.e., according to the two-dimensional Poisson process. Under this assumption the probability of exactly j emergency units being located in a region of area A, regardless of the shape or location of the region is

$$\frac{e^{-\rho A}(\rho A)^j}{j!} \qquad j = 0, 1, 2, \ldots,$$

where the parameter ρ is the expected number of fire companies per unit area--i.e., the density of fire companies.

We further assume that the incidence of fires is spatially homogeneous throughout the region, and we focus attention on the random variable D_1, the Euclidean distance from an arbitrary point to the nearest fire company. We call D_1 the closest company response distance. D_1 is greater than r if, and only if, there are no fire companies within a circle of radius r centered at the point in question. Thus,

$$P[D_1 > r] = e^{-\rho\pi r^2}, \quad r \geq 0.$$

Of particular interest here is the expected response distance, which we obtain directly by integration as $ED_1 = 1/2\sqrt{\rho}$. The parameter ρ is the spatial density of fire companies, and in the limit for an extremely large area A, $\rho = N/A$ with probability one. With this as motivation, we can write as an approximation, if A is large, $ED_1 = (2\sqrt{N/A})^{-1}$. Thus, the expected response distance to the closest company is inversely proportional to the square root of the number of companies stationed in the region.

In a similar way, defining D_i as the distance to the ith closest company, we obtain

$$P[D_i > r] = \sum_{j=0}^{i-1} \frac{e^{-\rho\pi r^2} \quad (\rho\pi r^2)^j}{j!} , \quad r \geq 0, \ i = 0, 1, 2, \ldots .$$

ED_i is a standard integral in probability theory, and a simple recursion for calculating ED_i is

$$ED_i = \frac{1}{2\sqrt{\rho}} \sum_{j=0}^{i=1} t_j, \quad i = 1, 2, \ldots,$$

where $t_0 = 1$ and $t_j = \left(\frac{2j - 1}{2j}\right) \; t_{j-1}, \quad j = 1, 2, \ldots .$

The variance of D_i can be obtained by observing from the above formulae that $2\pi\rho D_i^2$ has a chi-square distribution with 2k degrees of freedom; hence $ED_i^2 = i/\pi\rho$. Thus, Var $D_1 = (4 - \pi)/4\pi\rho$, and Var $D_2 = (32 - 9\pi)/16\pi\rho$. Notice that, as dimensional analysis suggests, the variances are inversely proportional to the density of fire companies.

The form of these relationships does not depend upon the details of the particular model we have analyzed. Similar analyses of other models give the same structural relation between expected response distance and the number of available units. The results for some other models are summarized in Table 1. Since in fire operations two or more companies of the same type are frequently dispatched to each alarm, we present both ED_1, the expected distance of the closest unit, and ED_2, the expected distance of the second closest unit. The table notes results for the following models:

(1) Fire Companies Located at Random--Euclidean Metric. This model is analyzed above.

(2) Fire Companies Located at Random--Right Angle Metric. The analysis is similar to the analysis for (1), except distance from x to y is $|x_1 - y_1| + |x_2 - y_2|$, so that $P(D_1 > r) = e^{-2\rho r^2}$.

(3) Fire Companies Located Uniformly and Optimally--Euclidean Metric. The units are located on a regular grid so that the region for which each unit is closest is a hexagon. Calculations of expected response distance within a typical hexagon may be done by direct integration.

(4) Fire Companies Located Uniformly and Optimally--Right Angle or Manhattan Metric. The analysis is the same as for the example developed at the beginning of this section.

Table 1

EXPECTED RESPONSE DISTANCES[a]
IN THE PLANE WHEN ALARMS ARE SPATIALLY UNIFORM

Metric		Expected Distances When Companies Are Randomly Located	Expected Distances When Companies Are Uniformly And Optimally Located
Euclidean	ED_1	0.5000	0.3772
	ED_2	0.7500	0.7287
Right Angle	ED_1	0.6267	0.4714
	ED_2	0.9400	0.9428

[a]The tabular value divided by the square root of the density of companies.

The numerical values in Table 1 illustrate several points. First, they suggest that--at least for regions with homogeneous fire hazards and alarm rates--the potential improvement in average response distance that can be achieved by "optimal" fire house location or dynamic relocation of units during busy periods is limited (less than 25 percent, the difference between completely random and optimal location cases in Table 1). In practice, since units are not likely to be randomly located initially, realizable gains for homogeneous regions are likely to be much less. (See [28] for elaboration of this point, and see [23] for details of real-time fire company relocation.)

There are, of course, potentially greater gains from good location when hazards and alarm rates are not spatially homogeneous. Then, placing fire houses better with respect to the hazards or problems requiring protection can prove more significant.

However, as the table shows, major gains, if any, are likely to be realized only for the closest unit. Positioning (in a homogeneous region) has only a minor impact on the average distance to the second-closest, third-closest, and subsequent units. Indeed, as noted earlier, straightforward geometry shows that, as i increases, $E(D_i)$ converges from below to $\sqrt{i/\alpha\rho}$ for every arrangement of units having a uniform density of units $\rho = N/A$. Here $\alpha = \pi$ for the Euclidean metric and $\alpha = 2$ for the right-angle metric. Note that, even for $i = 2$, the values in the table are not far from this upper bound, $E(D_2) \leq 0.7979$ for the Euclidean metric and $E(D_2) \leq 1.000$ for the right-angle metric.

In developing these examples, from which the square-root law emerges as a natural consequence of basic dimensional relations, we have used several conditions:

- Homogeneous distribution of hazards, alarms, and fire houses in space throughout the regions of interest;

- Measurement of distances according to continuous and uniform metrics;
- Insignificant boundary effects;
- Units always available to respond.

In the real world, or course, none of these conditions applies strictly. Hazards, alarm rates, and fire house locations are not homogeneous; alarm rates, for example, can vary by an order of magnitude in a few miles (see Figure 3, for example); and both hazards and fire houses (which are located to be near hazards) tend to be concentrated in relatively small areas. Demands also vary with time (see Figure 4). Actual travel patterns v[illegible] city geometry, and must contend with discontinuities induced by ri[illegible] bridges, expressways, and the like. It is unlikely that they produ[illegible] tances according to any metric. Real regions, lacking the symmetry [illegible] infinite extensibility of our examples, will have boundary effects. And in regions having high alarm rates, closest units may be busy and thus not always available to respond at the instant new alarms are received.

A key question then is: Which of these conditions are necessary, if any, and how necessary are they? We must see to what extent the square-root relation applies when these conditions are relaxed or violated.

In the next section, we examine the effects of unavailability, looking at the effect on the square-root relation of dynamic changes in the number of fire houses from which units are immediately available to respond. In Chapter III, we use detailed simulation models to examine the effects of inhomogeneity in demand for protection and in fire house location, and to test more extensively the effects of unavailability. Then we look briefly at boundary effects. Finally, in Chapters VI and VII, we outline some applications of the square-root law and note practical requirements and

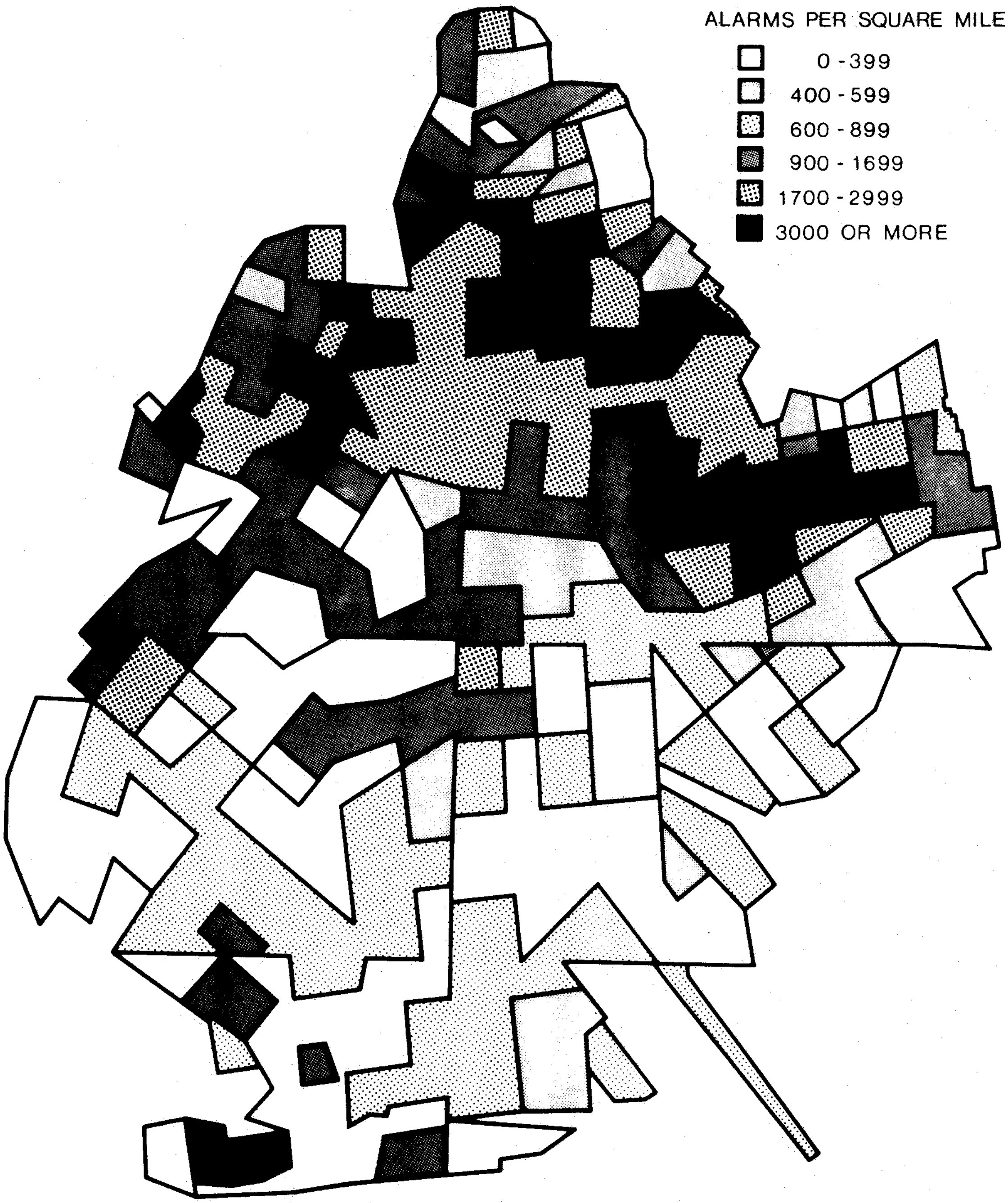

Figure 3. Fire alarm density in Brooklyn.

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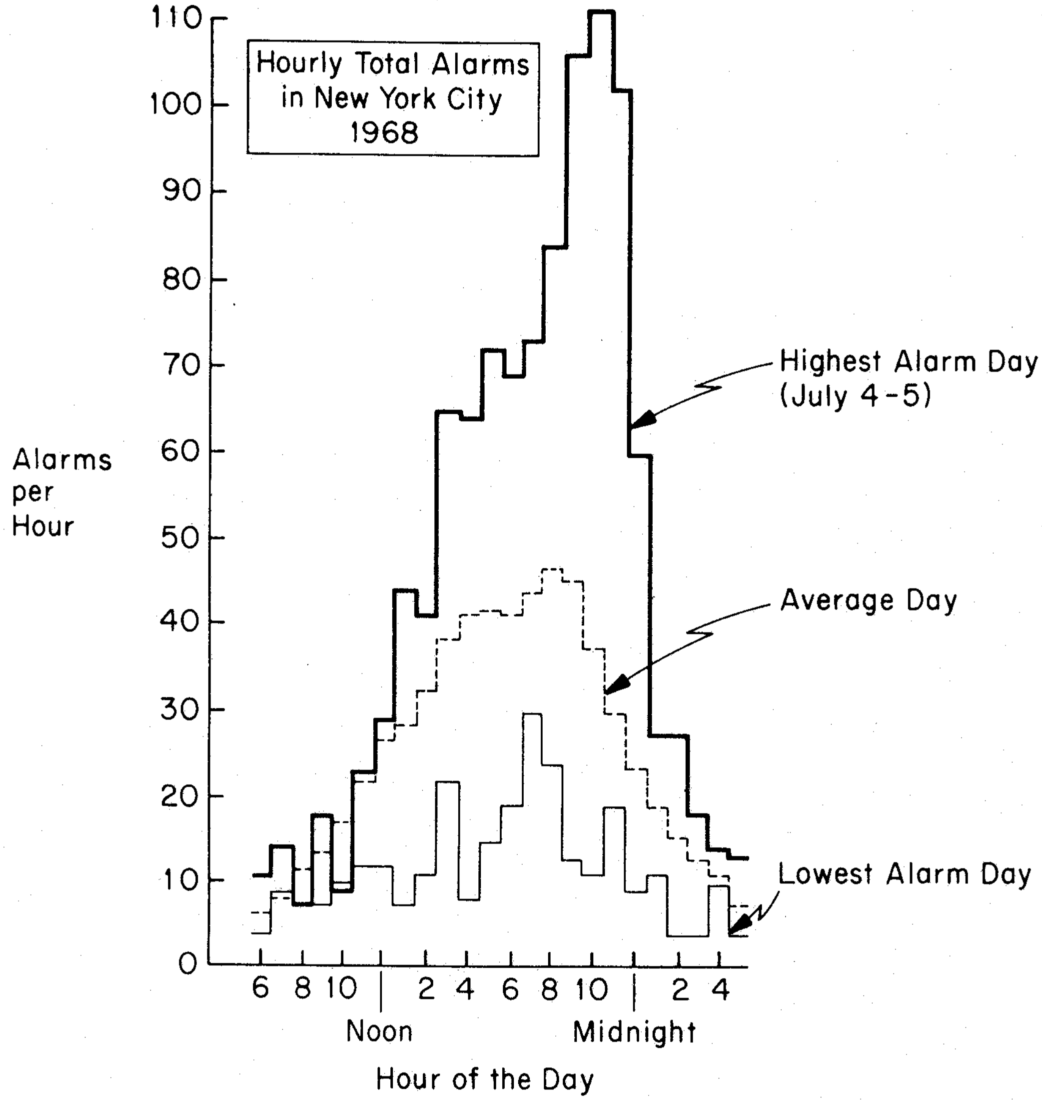

Figure 4. Hourly fire alarm distribution in New York City.

methods for parameter estimation.

A DYNAMIC MODEL FOR LONG-RUN EXPECTED RESPONSE DISTANCE

The models examined in the preceding section are probabilistic but static. They presume that all the fire companies under consideration are in quarters and are available to be dispatched to an alarm. The analysis revealed that the expected response distances to an alarm are inversely proportional to the number of companies available at the instant the alarm is received. Now we take a more dynamic, long-run view and ask how the response distance averaged over time depends on the number of companies stationed in the region. Our discussion will be heuristic. Consider the hypothesis that, when there is a low probability that all fire companies are busy, the expected response distance averaged over all alarms during a long period will be inversely proportional to the square root of the expected number of available companies. For the moment consider only an idealized mathematical model; simulation data testing this hypothesis are presented in the following section.

Suppose that fires and other alarms occur according to a random process in a region of area = 1. In addition, the number of fire companies dispatched to and working at alarms also are stochastic processes with known probability distributions. Under such a regime, suppose the p_j is the long-run, or stationary, probability that exactly j of the n companies assigned to the region are available for dispatch. The results of the preceding section indicated that the expected response distance when j companies are available is $k_j^{-1/2}$, as long as j is greater than zero. We assume that P_0 is zero, hence the long-run expected response distance, ED, is

$$ED = k \sum_{j=1}^{n} p_j \, j^{-1/2}. \qquad (3)$$

Let EN denote the long-run expected number of available companies:

$$EN = \sum_{j=0}^{n} j\, p_j .$$

Then, since $j^{-1/2}$ is a convex function, we have by Jensen's inequality ([14], vol. II, p. 153):

$$ED > k(EN)^{-1/2}.$$

We propose to use the right-hand side of the above expression as an approximation to ED, and have done calculations indicating that indeed when p_0 is "close to" zero the approximation is a good one--within 5 percent of the true value.

In most real-world situations we have encountered, EN itself can be computed by a simple approximation. Let λ denote the alarm rate, that is, the long-run average number of alarms received per hour, and ES denote the expected total fire company hours of service time required to extinguish an alarm, etc. Then, as is typical in many such "queueing" situations, we have approximately

$$EN = \lambda ES, \tag{4}$$

and, consequently, we take as an approximation for ED,

$$ED = k(n - \lambda ES)^{-1/2}. \tag{5}$$

It is reasonable to ask why we use an approximation in the first place. Why not use (3)? One answer is that it is rare that the stationary probabilities p_j are known or estimable. Even in the rare instances where they are, the resulting equation is difficult to use for the purposes we have in mind. We can demonstrate this by considering a greatly simplified model in which alarm incidence is geographically homogeneous in the region and alarms occur in time according to a Poisson process with alarm rate λ.

Suppose also that as long as there are any companies available, one is dispatched to each alarm and the total service time for each alarm is an exponentially distributed random variable with mean $1/\mu$. Service times are assumed to be mutually independent, and independent of the state of the system. Alarms that occur with all n units busy are handled by special procedures, such as calling in units from outside the region, and will be regarded in this simple treatment as "lost calls." The fire companies in the region do not respond to alarms outside the region.

The system described is the birth and death process usually called the M/M/n system with losses. Well known equations give the p_j, etc. [11], and equation (3) becomes

$$ED = \left\{\sum_{j=0}^{n} \frac{(\lambda/\mu)^j}{j!}\right\}^{-1} \left[k \sum_{j=0}^{n-1} \frac{(\lambda/\mu)^j}{j!} \ (n - j)^{-1/2}\right] . \qquad (6)$$

This is about the simplest mathematical model possible of the alarm-receipt-fire-company-dispatch process, yet (6) is an extremely awkward function of n as compared to (5). A somewhat more realistic queueing model for fire company availability is given in [10], and the resulting version of (3) for this model would be even more complicated--too complicated for use as a rule of thumb.

III. TESTING THE SQUARE-ROOT MODEL WITH SIMULATION AND EMPIRICAL DATA

As noted in the preceding Section, the mathematical models used to motivate the inverse square-root law for response distances are greatly simplified. We have remarked that the conditions assumed are often not met in practice. In addition, complications abound: the city is of finite size and irregular shape; the distribution of units is not homogeneous; several companies (in varying numbers) are dispatched to each alarm; in the event of a very serious fire, companies from remote regions may be relocated into the depleted area [23]; response routes must follow actual street patterns with one-way streets, obstacles, etc. *The question to be answered here is, notwithstanding such complications, does the inverse square-root law give a useful approximation for real response distances?* We examine this question now, testing the validity of the law using both simulation and empirical data.

Two relationships will be examined. To be precise in their specifications, we adapt the notation of the last section. Let n denote the number of fire houses (locations from which units can respond) in the region in question. Suppose that we observe a sequence of fire alarms labeled $i = 1, 2, \ldots$. Let τ_i denote the time (epoch) of the *ith* alarm, let D_i denote the response distance of the house with an available fire company closest to that alarm, and let $N(\tau_i)$ denote the number of fire houses with fire companies available to respond at τ_i, the instant of the alarm. We are concerned with:

- *The relation between* ED_i *and* $N(\tau_i)$. We suppose that n, the number of fire houses located in the region, is fixed and, thus, $N(\tau_i)$ can vary between 0 and n. We shall be interested in determining if the relation between expected response distance and the number of houses with companies available when the alarm comes in is given by the square-root law; that is, if

$$ED_i = k \sqrt{\frac{A}{N(\tau_i)}}\,.$$

- *The relation between* ED_i *and* n. Now suppose that management changes n, the number of fire houses, and we are interested in the effect on the expected response distance to alarms. The argument of the previous section supports the hypothesis that

$$ED_i(n) = k \sqrt{\frac{A}{EN(\tau_i)}}$$

$$= k \sqrt{\frac{A}{n - E(\text{number of empty houses})}}.$$

Since n is a major policy variable that management can control, this relation is of more general interest than that between ED_i and $N(\tau_i)$.

Collecting empirical data to test these relationships is a formidable task at best. Consider the testing of the first relationship. Data could be gathered by posting an observer in the dispatching office who would note and record the number of companies available to respond at the instants when alarms are dispatched. Simultaneously, each responding unit could note and record odometer readings before and after each response. By assembling this information, we could produce a set of data $(D_1, N(\tau_1))$, $(D_2, N(\tau_2))$, ..., etc. This is, however, often easier said than done.

For example, few fire companies in New York City are equipped with odometers that give readings for distances less than miles. And during busy times, fire companies are dispatched on the fly while they are returning from an earlier call before arriving at their home station. These details complicate measurement problems.

Formal validation of the second relationship with empirical data is literally impossible, except on the rare occasions when units are added or removed. One would have to tinker with the Fire Department, operating with different numbers of units assigned to the experimental region. For example, at the time this work was being done, there were 37 fire engines (pumpers) and 24 ladder trucks assigned to the Bronx. One cannot imagine the Fire Department operating for a month with 30 engines, or perhaps with 40 engines, simply to test the validity of the model. Cost and risk preclude such experiments. As a result, we have relied on less direct measurements based on detailed computer simulations and on some historical data.

The simulation studies were undertaken using a very large-scale detailed computer simulation model of New York City fire-fighting operations that had already been developed by our research team. Descriptions of the model itself and of other applications of it may be found in [5], [6], and [7].

For the purposes of this part of the analysis, seven simulations were done of fire-fighting operations in the Bronx. The conditions simulated varied over a broad range. Alarm rates were varied from 5 to 30 alarms per hour, and the number of active ladder companies were varied from 12 to 31. In each simulation, the number of engine companies was kept at 37. The locations of the engine companies were the actual locations of existing companies. For the ladder companies, we chose locations in the following way: In simulations with fewer than the existing 24 ladder companies, we used a subset of the existing locations that gave (intuitively) a good geographical distribution of ladders throughout the Bronx. for simulations with additional ladder companies, we chose new locations that appeared to be good spots at which to add new companies. These locations were selected on the basis of earlier analysis that was done to select actual new locations.

Each simulation consisted of an extended time period during which the alarm rate and number of active units were unchanged. The simulation durations were chosen so that, in each case, about 3,500 alarms were handled. This sample size was selected after statistical analysis of the random variation in simulation output statistics. The results produced should be interpreted so estimates of performance of "steady state" behavior under the conditions simulated. Table 2 gives a brief listing of the simulations carried out. In addition to these simulations, we had available data from other simulation runs previously carried out for different purposes.

Table 2

LIST OF CONDITIONS SIMULATED

Simulation Number	Alarm Rate (Alarms/Hour)	Number of active Ladders	Number of active Engines
1	30	31	37
2	5	31	37
3	5	24	37
4	10	20	37
5	30	20	37
6	5	12	37
7	10	12	37

SIMULATION RESULTS

First, we consider the validation of the relation between ED_i and $N(\tau_i)$. The simulation program recorded the response distance and the number of companies available at the instant of dispatch for each alarm. These data were accumulated separately for two regions of the Bronx; these were, approximately, the south Bronx and the rest of the Bronx. The data were collected for the closest engines and ladders to each alarm, as well as for the second and third closest units for those alarms to which such units were dispatched.

In order to analyze these data and determine if the square-root model was appropriate, we graphed average response distance vs. the number of units available for each individual set of data. (By an individual set of data we mean, for example, data for second closest ladders in the north Bronx from the simulation run with 12 active ladder companies and an alarm rate of 10 alarms per hour, etc.) In addition, for each set of data we used the method of least squares to determine the parameters of two response distance models:

$$D = k(A/N)^{-1/2} \qquad (7)$$

$$D = \alpha(A/N)^{\beta}. \qquad (8)$$

We were concerned with how well these models fit the data. If square-root relations indeed hold, relations (7) and (8) should both fit well and estimates of β should be "close" to -1/2. Measuring how close β is to -0.5 is not straightforward, since the simulation data do not satisfy the conditions requisite for classical statistical analysis. For example, the observations are not independent, and the square-root relation itself implies unequal variances. Examination of the sum of squared errors indicates, of course, that (8) fits better than (7), but the difference between the models is small, and an "eyeball" check of the graphs reveals little difference between (7) and (8).

Temporarily setting aside our reservations about standard statistical tests, we calculated approximately 95 percent confidence contours for α and β. In each case, these included $\beta = 1/2$, indicating that an inverse square-root law gives a good fit. These data are typical of our other results. They indicate that the inverse square-root relationship between the average (and standard deviation of) response time and N holds well, even in realistic, complex situations.

This analysis was done for closest and second closest engines and ladders, for the north and south Bronx, for each simulation--a total of 28 cases. In each case, the fit of the square root model was good, and the parameters of the more general model were close to those of the square-root model. Table 3 gives a summary of some of the square root model fits for individual simulations.

Table 3

SQUARE-ROOT MODEL FITS FOR INDIVIDUAL REGRESSIONS

Simulation Number	Region	Closest Engine $\hat{k}$	Closest Engine R^2	Closest Ladder $\hat{k}$	Closest Ladder R^2
7	North Bronx	.598	.31	.570	--
	South Bronx	.551	.54	.525	.72
2	North Bronx	.620	.80	.602	.35
	South Bronx	.601	.94	.643	.89
3	North Bronx	.612	.54	.586	.60
	South Bronx	.555	.65	.527	.88
4	North Bronx	.617	.59	.609	.76
	South Bronx	.591	.93	.568	.97

$\hat{k}$ is the estimated value of the square-root law parameter.

R^2 is the proportion of the variance explained by the model.

There is a general consistency between the square-root law parameters for engines and ladders and between results from the north and south Bronx. (The estimates of the square-root constant appear to be slightly higher for the north Bronx and for engines.) Because of this overall consistency, we then repeated the analysis with the data grouped from various simulations and for engines and ladders. This grouping yields parameter estimates in which slight differences due to the geography of the regions and to the particular company locations are averaged out, and should for this reason be more appropriate for use in still other regions of the City. Figures 5 through 8 display the results of these groupings, and Table 4 gives a summary of the results. We note that the R^2 values for the square-root model are quite high, that the graphs show a close correspondence between the fitted functions and the data, and that one can hardly see any difference between the curves for the exponential model and the square-root model.

Data from previous simulations which had been made for other purposes were analyzed in the same way and yielded similar results.* On the basis of these results, we conclude that the data confirm the validity of the square-root law between average response distance and the number of units available at the moment of dispatch. We also observe that, although there appear to be differences in the parameter values for engines and ladders (due perhaps to the difference in the locations), and differences in the parameter values between the regions (due again perhaps to location differences as well as boundary differences), they are not so great that a single parameter value cannot generally be used. This point is important because our goal is to apply this model to other regions in which simulation or other experiments have not been carried out at all.

We now turn attention to validation of the relationship between long-run average response distance, ED, and n, and the number of companies assigned to the region. The data and analyses just discussed indicate that the square-root model describes the relationship between average response distance and the number of units available when an alarm occurs. But this does not assure that a square-root law describes the relationship between

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*Some of these results appear in [25].

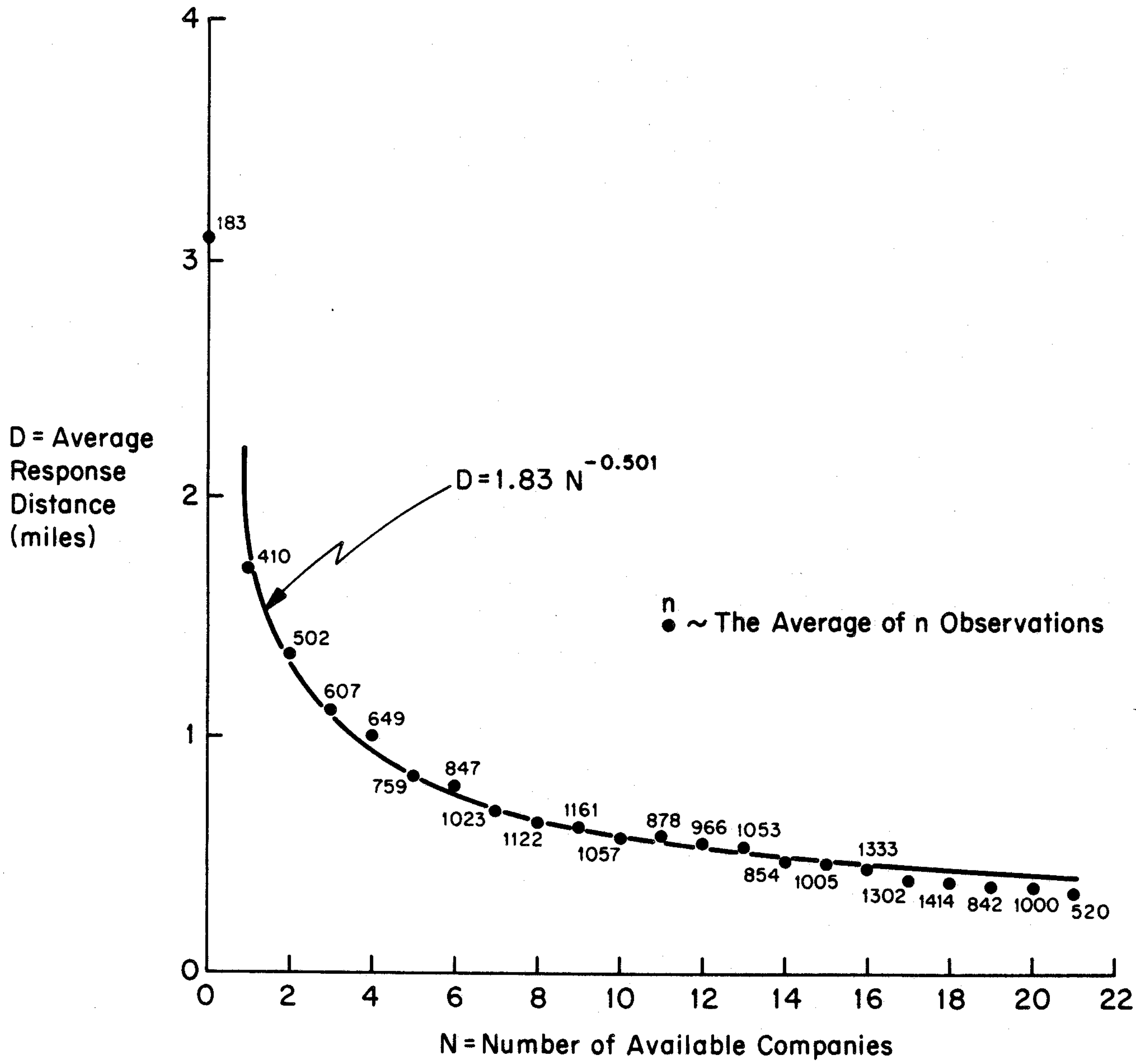

Figure 5. Simulated average response distances vs. the number of available companies (closest engine and ladder companies in the south Bronx).

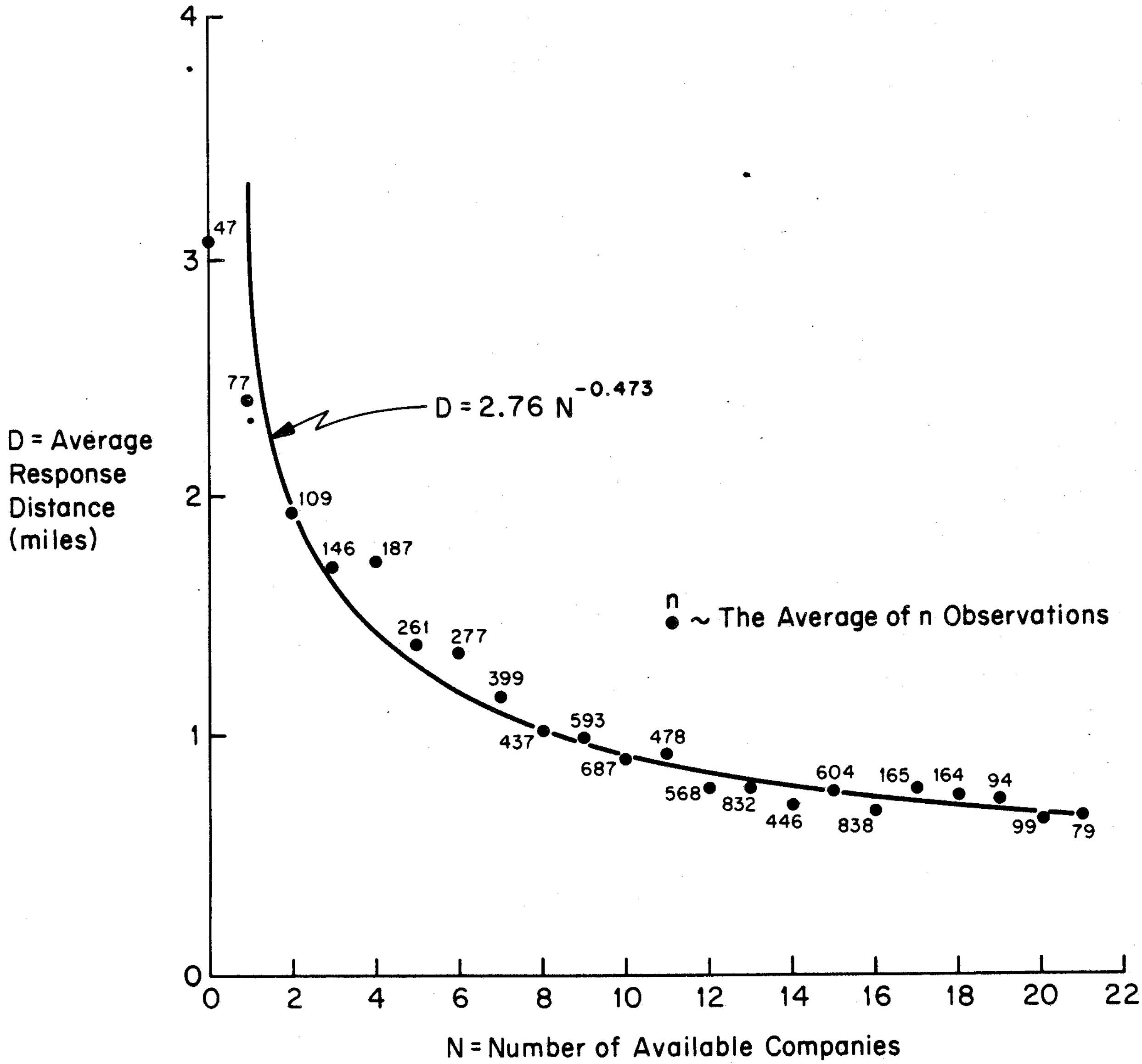

Figure 6. Simulated average response distances vs. the number of available companies (closest engine and ladder companies in the north Bronx).

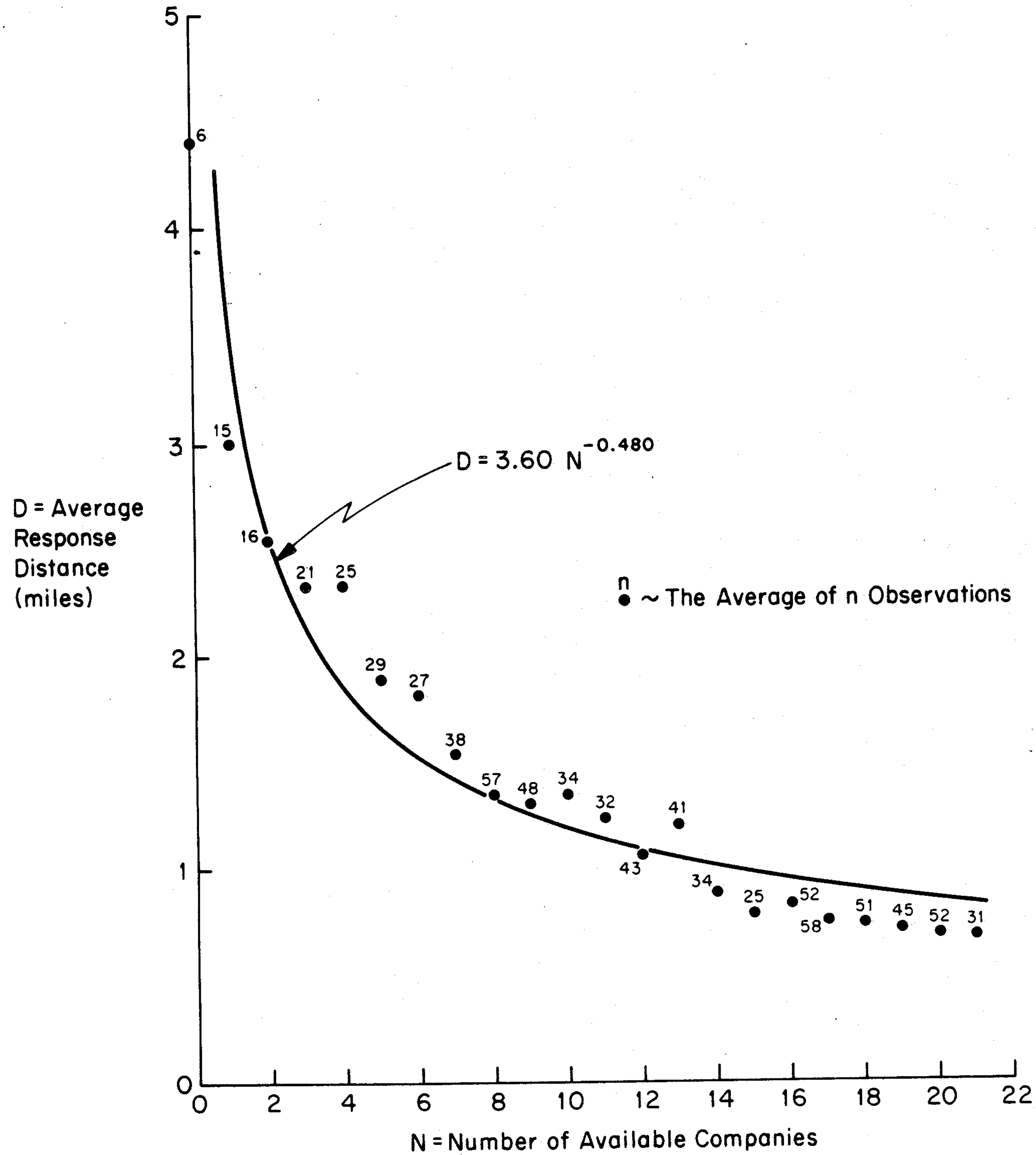

Figure 7. Simulated average response distances vs. the number of available companies (second closest engine and ladder companies in the south Bronx).

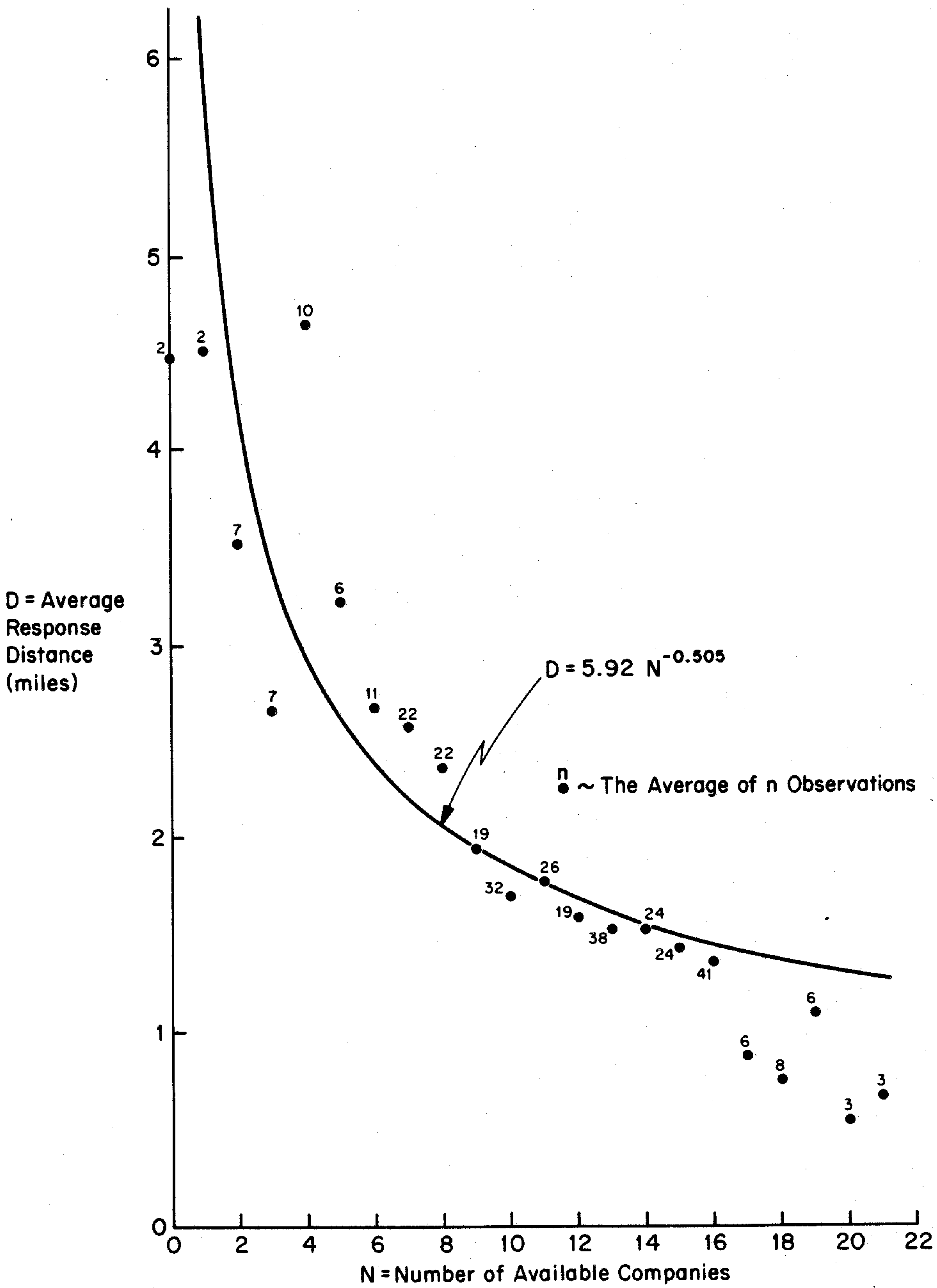

Figure 8. Simulated average response distances vs. the number of available companies (second closest engine and ladder companies in the north Bronx).

Table 4

FITS OF SQUARE-ROOT MODEL TO GROUPED DATA

Grouping	Parameters of Exponential Model α	β	Parameters of Square-Root Model k	R^2
Closest Engines and Ladders				
North Bronx	0.57	-0.47	0.60	0.93
South Bronx	0.57	-0.50	0.57	0.97
Second Closest engines and Ladders				
North Bronx	1.23	-0.51	1.22	0.73
South Bronx	1.13	-0.48	1.16	0.88

Exponential model: $D = \alpha\sqrt{A}\ N^{\beta}$.

Square-root model: $D = k\sqrt{A/N}$.

R^2 = Sample correlation coefficient for square-root model.

long-run average response distance and the average number of companies available to respond to an alarm. On the contrary, if the square root law holds for the former, it cannot hold exactly for the latter since the inverse square root function is convex, and for a convex function $f(\cdot)$ of a random variable X, $Ef(X) > f(EX)$ (Jensen's inequality).*

Figures 9, 10, and 11 display simulated long-run average response distances for closest ladders versus average numbers of ladder units available. Recall that the simulations were run at different alarm rates, with 12 to 31 ladder companies assigned to the region.

The graphs display data for the north Bronx, the south Bronx, and the entire Bronx. Each of the plotted points represents the results of an entire simulation run. In addition to the simulation data, we have also plotted regression fits of the functions

$$\overline{D} = k/\sqrt{\text{Average number of available ladders}},$$

$$\overline{D} = \alpha(\text{Average number of available ladders})^{\beta}.$$

As before, we are concerned with how well these functions fit the data, whether β is close to $-1/2$, and whether α is close to k. We are also concerned with consistency of the results across the regions. As the figures show, the fits are good and the two functions nearly coincide. Table 5 summarizes the results of the regressions. Note the consistency with the values of k given in Tables 3 and 4. Other regressions--not shown here--were done of the standard deviation of response distance versus the average number of companies available. The theory indicates that this relationship should also be an inverse square-root function, and the analysis supports this hypothesis.

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*See Appendix A for some comparisons of empirical estimates of $E(1/\sqrt{N})$ and $\sqrt{1/E(N)}$. The errors calculated there are always small.

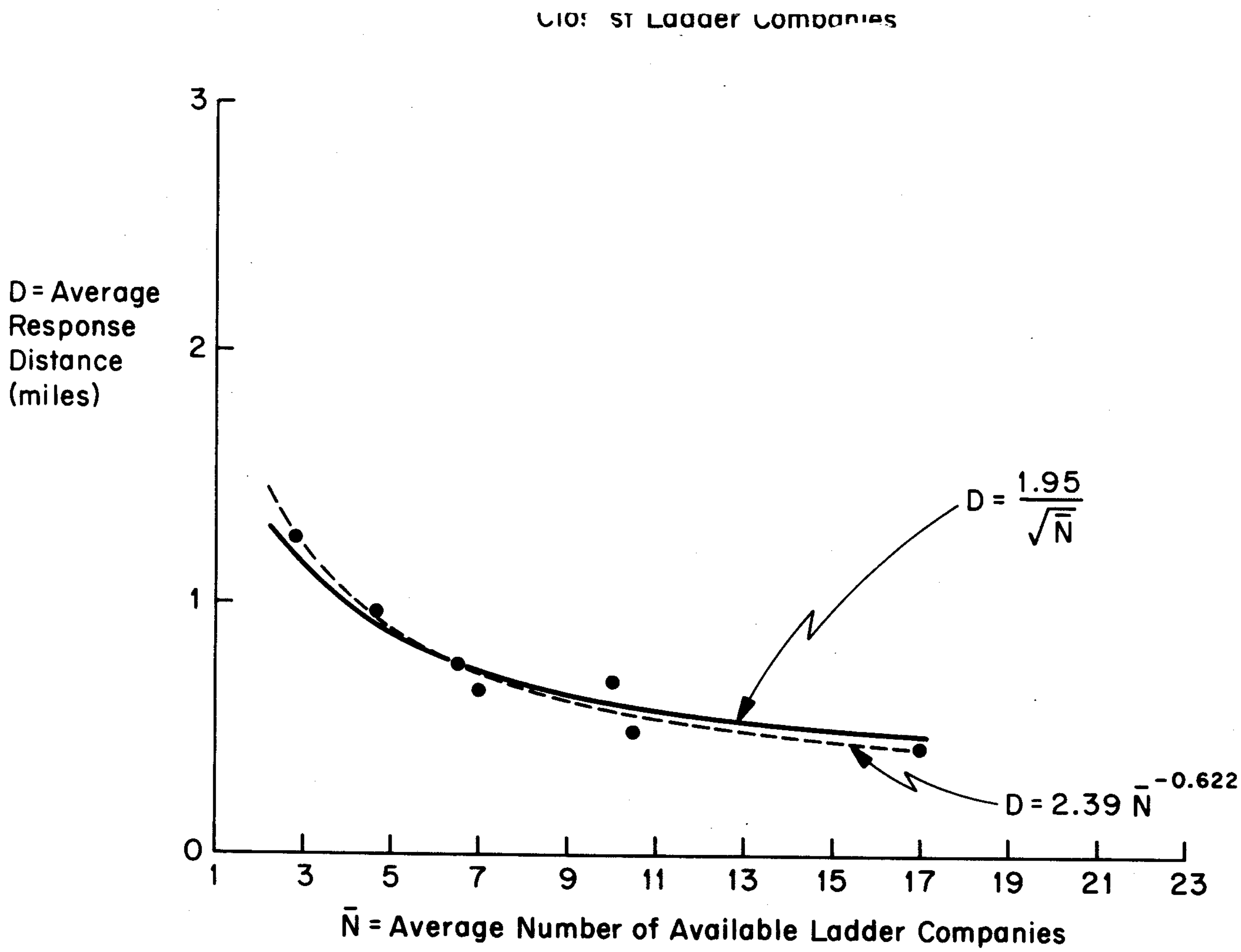

Figure 10. Simulated long-run average response distances vs. the average number of available companies (closest ladder companies in the south Bronx).

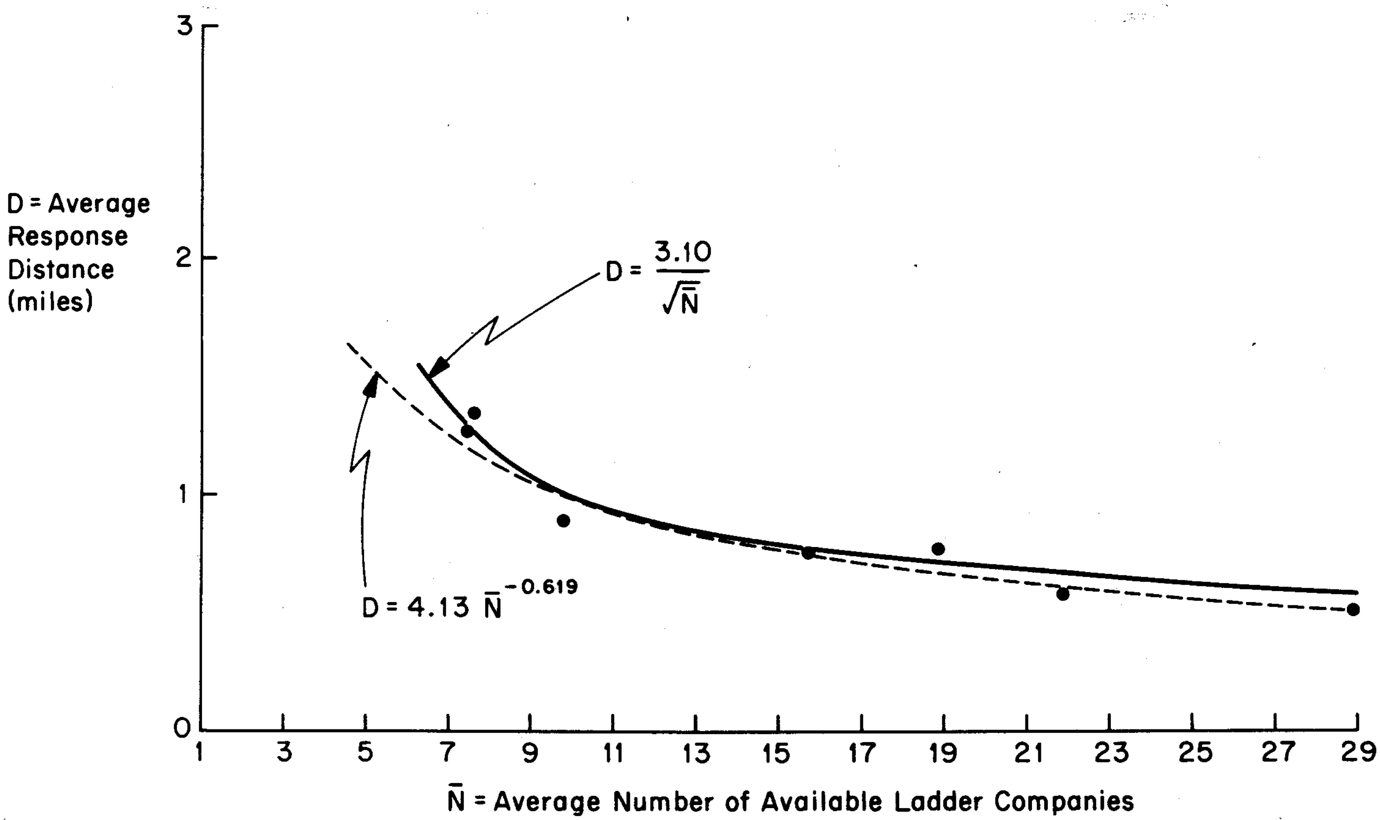

Figure 9. Simulated long-run average response distances vs. the average number of available companies (closest ladder companies in the Bronx).

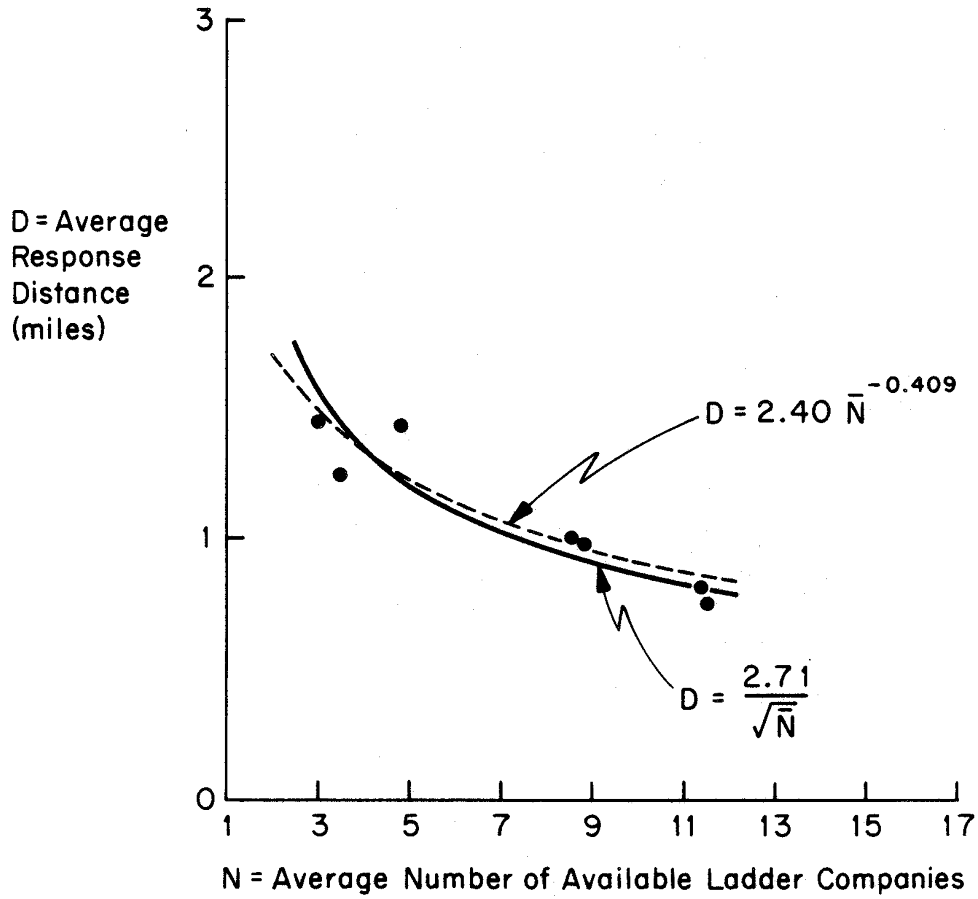

Figure 11. Simulated long-run average response distances vs. the average number of available companies (closest ladder companies in the north Bronx).

Table 5

RESULTS OF REGRESSIONS OF LONG-RUN AVERAGE RESPONSE DISTANCES FROM BRONX SIMULATIONS

Region	α	β	k	R^2
North Bronx	.50	-0.41	0.56	0.68
South Bronx	.75	-0.62	0.61	0.92
Entire Bronx	0.71	-0.62	0.54	0.86

In addition to these simulation experiments run with the detailed model of fire-fighting operations in the Bronx, we also constructed and experimented with a "mini-simulation." In this more idealized model, alarms were generated according to a homogeneous Poisson process in a unit square. Fire companies could be located as desired, and distances could be calculated according to any metric. The results of simulations carried out with several metrics and various arrangements of companies were analyzed as above, yielding similar results confirming the robustness of the square-root relations even for situations with as few as three companies in the region.

BOUNDARY EFFECTS

The results obtained from the simulations show that the square-root relation holds as a good approximation even when hazards, alarms, and unit locations are inhomogeneous, when metrics have real-world irregularities, and when units may often be unavailable to respond. In these simulations, we have not specifically isolated regions near the boundaries to check the validity of the relation there. We briefly consider boundary affects here.

Two sets of results are informative. Leamer [29] considered location problems in a finite region with homogeneous demands and distances calculated from a Euclidean metric. He considered finite regions of regular shape (equilateral triangles, squares, and circles) in which the number of facilities varied. Choosing locations to minimize $E(D_1)$, he calculated $E(D_1)$ for N = 1 through 16. The results show that even for small N, where boundary effects are greatest, the relationship between $E(D_1)$ and N could be well-approximated by an inverse square-root function.

Chaiken [8] calculated the exact distribution for travel distances in regions with randomly distributed vehicles. His results show that "the distance from points near the boundary of a finite region to the closest vehicle is large enough that the actual average distance can be substantially larger than" the region-wide estimate. He still finds the square-root relation valid, but cautions against the use of a priori theoretical estimates for the proportionality constant, which may not adequately account for effects near the boundaries. Instead, he recommends using empirical estimates for k that take into account the particular features of the region being examined.

Thus, boundary effects may be significant, but do not appear to affect the basic nature of the distance-facility density relationship.

EMPIRICAL DATA

We conclude this section by examining two sets of independent data--one a set of empirical data on fire stations and fires in Bristol, England; and the other a set of optimization and simulation results for ambulances in a suburban county near Washington, D.C.

Jane Hogg [18] examined 15 sites in Bristol at which fire stations could be located, drawing on data detailing the locations of 6,813 fires that had occurred there between 1958 and 1964. Neither the site locations nor the fires were evenly distributed spatially, both being more dense in the center of the rectangular region under study. In Hogg's analysis, nonstationarities of demand and travel velocity in time were considered, but possible unavailability of fire companies were ignored. Travel times were calculated from knowledge of the distances involved and estimates of travel speed, which recognized variations by region and time of day. We are interested in analyzing Hogg's results for travel times, even though our subject is distance, because if run lengths are long and the time-distance relation is approximately linear, her data can be used as a check on the square root model for distances.

Some of Hogg's results--replotted here in Figure 12--give average travel times as a function of N, the number of fire house locations occupied. These results were fit by least squares to the two models previously discussed:

$$T = k/\sqrt{N} \tag{9}$$

$$T = \alpha N^{\beta} \tag{10}$$

The sums of squared errors indicate that (10) fits better; the difference between (9) and (10) is small.

Berlin and Liebman [2] present results of a combined optimization-simulation study of emergency ambulance location in a suburban county near Washington, D.C., with 24 potential depot locations. Using a set-covering model, they computed the relation between the maximum response time and the number of locations occupied. Figure 13 replots their Figure 2, containing these results, showing the fitted equation

$$\text{Maximum Response Time} = K/\sqrt{\text{Number of facilities}}.$$

This relation has an empirical constant, K , which we can interpret as our k times the square root of the area of the region. It fits the results for the ten sets of "optimal" locations well. The square root law indicates the benefits that might be achieved from improving the depot locations. It shows that 24 locations, better selected, should be able

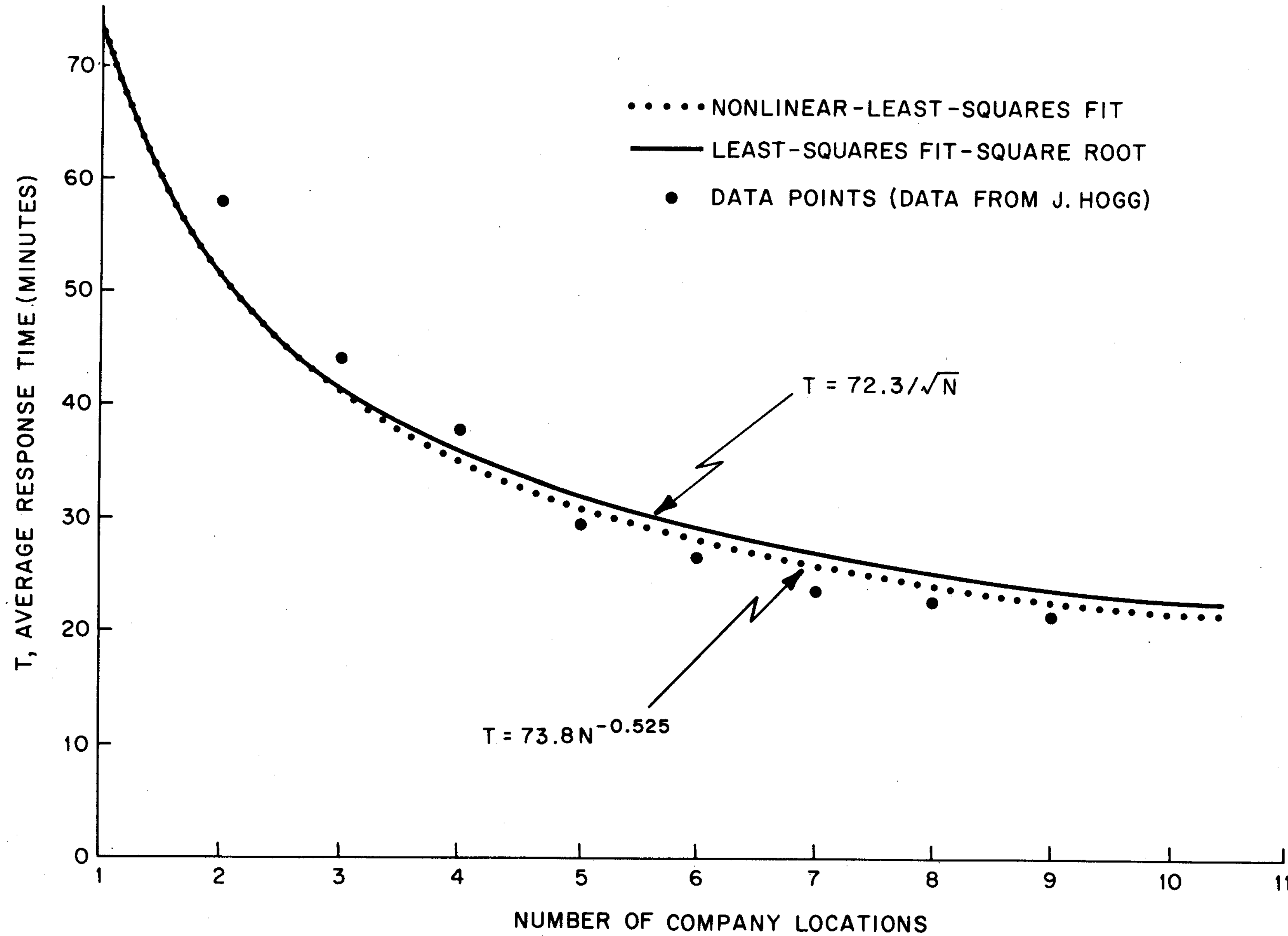

Figure 12. Average response time vs. number of fire company locations (Bristol, England).

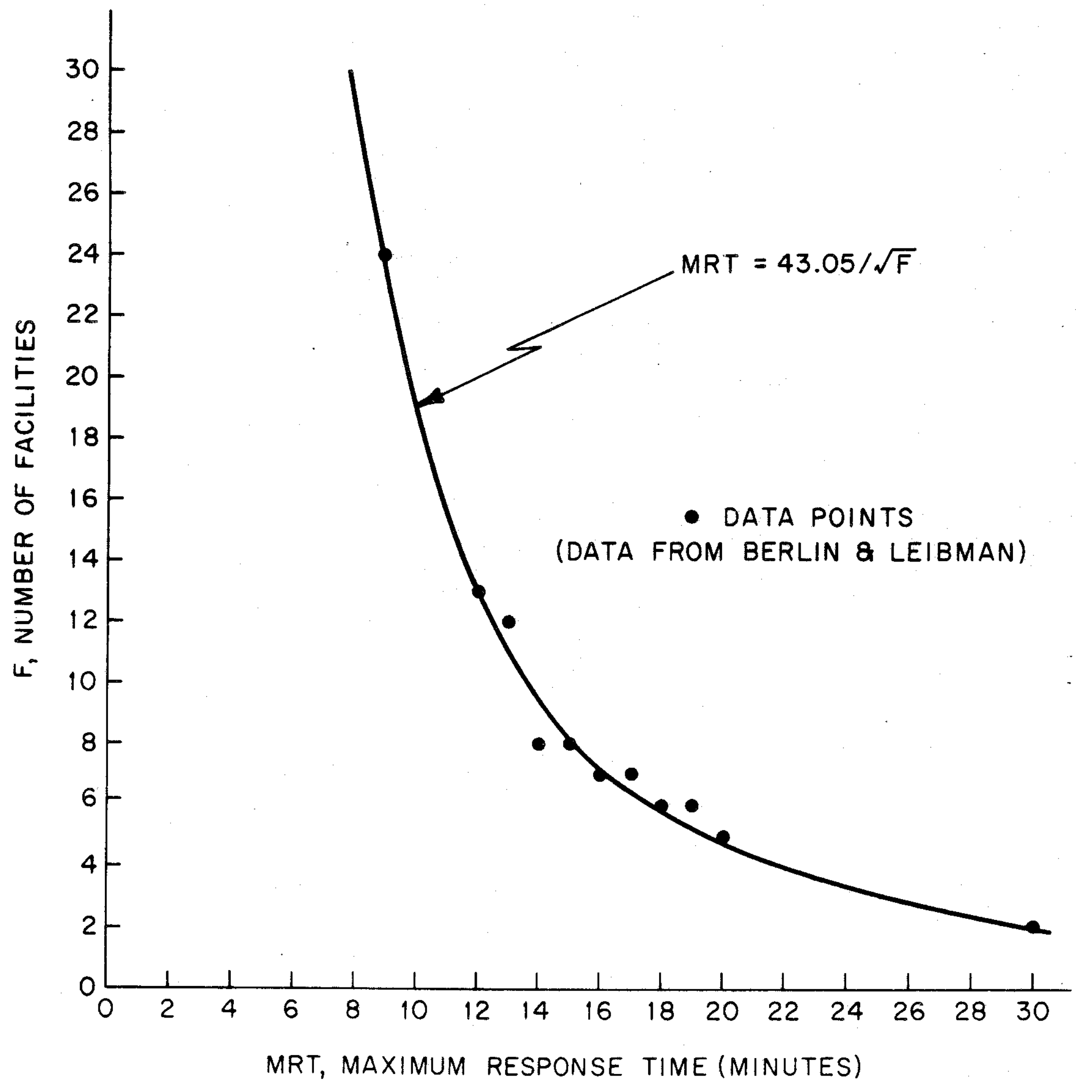

Figure 13. Maximum ambulance response time vs. number of facilities.

to yield a maximum response time of approximately 8.8 minutes, rather than the 11.7-minute minimum value cited as achieved with the current locations. Such an estimate, even though an approximation, permits one to evaluate the desirability of carrying out a siting improvement study, gives a target value for which the optimization can strive, and may also highlight the local areas in which attention to better siting could have the greatest returns. The square root law permits one to extrapolate beyond the computed values and to interpolate accurately between them, both with a minimum of effort.

Yet another siting study, of Denver fire stations, conducted by a group at the University of Colorado, has yielded similar kinds of results that reportedly fit the square root law with R^2 approaching 0.99 [9].

IV. APPLICATION TO ALLOCATION PROBLEMS

This section illustrates the mathematical methodology employed in using a power model for expected travel times in analyzing fire engine allocation problems. Such a model is derived in [26] by combining the square-root law with the travel time vs. distance model of [24]. We formulate several allocation problems of practical interest, propose algorithms for their solution, and discuss some implications of their use. As we mentioned at the outset of this report, the model has actually been used to make such analysis for several cities including New York, Yonkers, Jersey City, and Wilmington. In New York, several significant allocation changes resulted, while in the other cities long-range allocation plans are being formulated using the model. Some of these concrete applications are discussed in [16], [17], and [37].

Consider the following allocation situation: Divide the City into m disjoint regions or neighborhoods, each of which is (roughly) homogeneous with respect to alarm rate, velocity of responding vehicles, types of fires, etc. In addition, we consider a particular time of day and season of the year during which the alarm process is stable, and we interpret the following analysis as applying to the time period under study. Each region is to be assigned a number of fire companies that, except in special situations, will serve only that region. In region i (i = 1, 2, ..., m), define:

A_i = the geographic area (square miles);

λ_i = expected alarm rate in the period of interest (alarms per hour);

ES_i = expected total time spent servicing an alarm;

n_i = the number of units assigned.

Assume that for district i the expected number of companies busy can be approximated by $\lambda_i ES_i$ and that a power law holds for expected response times--we include dispatching delays, turn-out time, etc. in the term α_i:

$$ET_i = \alpha_i + \beta_i \left(\frac{A_i}{n_i - \lambda_i ES_i} \right)^{\gamma_i} \tag{6.1}$$

where α_i, β_i, and γ_i are the power-law parameters, and depend on the street configuration, the location of houses, the travel velocities attained, the dispatching delays, etc. We discuss the estimation of these parameters in the following section of this report. Note that this function makes sense only if $n_i > \lambda_i ES_i$, that is, if there are at least as many companies in the region as are necessary, on the average, to service all alarms. Indeed, as we have already seen, the square-root approximation depends on an assumption that the probability of all companies being busy is small, and this will happen only if n_i is considerably larger than the expected number of busy units.

The conditional probability that an alarm occurring somewhere in the city is in fact in region i is given by

$$p_i = \lambda_i / \sum_{j=1}^{m} \lambda_j . \tag{6.2}$$

From this we have for the city-wide expected travel time

$$ET = \sum_{i=1}^{m} p_i \; ET_i . \tag{6.3}$$

We now state the first optimization problem.

ALLOCATING FOR MINIMUM CITY-WIDE EXPECTED TRAVEL TIME

Find integers n_1, n_2, ..., n_m that minimize ET subject to

$$n_i > \lambda_i/\mu_i, \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} n_i \leq N,$$

where N is the total number of fire engines available in the City. Notice that ET is a separable function and is convex decreasing in n_i. Hence, a simple iterative procedure of examining marginal decreases in ET will yield an optimal integer solution:

Step 1: Set $n_i = [\lambda_i ES_i] + 1$, $i = 1, 2, \ldots, m$. If $\Sigma n_i > n$ the problem is infeasible; if $\Sigma n_i = n$ the allocation is optimal; otherwise go to step 2.

Step 2: Calculate $\Delta_i = p_i[ET_i(n_i) - ET_i\ (n_i + 1)]$, $i = 1, 2, \ldots, m$. Set $n_j = n_j + 1$ for j such that $\Delta_j = \max \Delta_i$. If $\Sigma n_i = n$ the allocation is optimal; if not, repeat step 2.

One can gain some insight by relaxing our requirement that the n_i be integers, treating the problem as if the n_i were continuous variables, and solving the problem using a classical Lagrangean multipliers approach. These Lagrangean "solutions" when rounded can be used as a starting point for the exact algorithm or, if a computer is not available, as an alternative to the optimal integer solution, which can be tedious to compute. On the other hand, the unrounded solutions are themselves of some interest. Since the regions in our partitioning of the city are not truly independent--sometimes having fire engines responding across their boundaries--the fractional solutions may be indicative of how adjacent regions can "share" engines.

The continuous or Lagrangean solution is to set

$$n_i = \frac{K_i}{\left(\sum_{j=1}^{m} K_j\right) \Big/ \left(N - \sum_{j=1}^{m} \lambda_j ES_i\right)} + \lambda_i ES_i \quad i = 1, \ldots, m$$

where

$$K_i = \left(p_i \ \beta_i \ \gamma_i\right)^{\frac{1}{\gamma_i+1}} A_i^{\frac{\gamma_i-1}{\gamma_i+1}} .$$

Problem 1 can be generalized easily as follows. Let $F_i(ET_i)$ be some function of expected travel time of interest. One interpretation of theoretical interest is that $F(\cdot)$ is a utility function placing a value on expected travel time. Then, we may wish to find values $n_1, n_2, \ldots, n_m$ to

$$\text{minimize} \sum_{i=1}^{m} F_i(ET_i(n_i))$$

$$\text{subject to} \sum_{i=1}^{m} n_i = N.$$

Again, our objective is separable in n_i, and if the $F_i(\cdot)$ are convex decreasing in n_i, our marginal allocation procedure can be adapted to solve for optimal integral values of n_i. If the $F_i(\cdot)$ are differentiable, the Lagrangean method can be used. In [32], Rider discusses the latter approach.

MINIMUM COMPLEMENT OF COMPANIES (AVERAGE TRAVEL TIME CONSTRAINT)

Problem: Find integers $n_1, n_2, \ldots, n_m$ that minimize $\sum_{i=1}^{m} n_i$

subject to $ET_i \leq t_i, \quad i = 1, 2, \ldots, m.$

In this formulation, management specifies the standard or protection (in terms of average travel time) to be provided in each region, and the minimum number of companies necessary to achieve this protection is determined. The solution to this problem is to set n_i to the smallest integer larger than

$$A_i \left(\frac{\beta_i}{t_i - \alpha_i}\right)^{1/\gamma_i} + \lambda_i ES_i .$$

This model has proven quite useful in practice. Fire departments are more interested in specifying protection standards such as the t_i and finding out how to allocate resources to achieve them than in achieving some theoretical overall optimality objective that may exist only in the imagination of an analyst. In fact, the most frequent use of the power laws has been to give numerical estimates of travel time performance of allocations created by Fire Department management.

V. DATA REQUIREMENTS AND PARAMETER ESTIMATION

THE SQUARE ROOT MODEL FOR EXPECTED RESPONSE DISTANCES

The data required to use the square root model are modest. One must estimate the following for engine region: the geographic area; the average number of alarms--or calls for service--per hour; the average total service time per call; and the square root law constant of proportionality. We discuss the estimation of each parameter in turn.

The Geographic Area, A_i

If the regions conform to political entities, planning districts, or the like, the areas may be known. If not--as has been usually the case in our applications--we have measured the areas either by tracing out the region on a map with a polar planimeter or by tracing the region on quadrille paper and counting grid squares. In any event, some judgment is required in deciding how to treat large parks, cemeteries, or bodies of water included in the region. If units do not generally respond across these, they may be eliminated.

Expected Number of Alarms Per Hour, λ_i

Most fire departments maintain records from which alarm rates can be estimated by region, time of day, season, etc. The computerization of alarm records in New York City has eased our applications there.

Expected Total Service Time Per Alarm, ES_i

By the expected total service time is meant the total number of hours spent by all companies of the same type (engines or ladders) responding to and/or working at alarms, averaged over all the alarms received. This requires an averaging over many different types of alarms, from false alarms to multiple-alarm structural fires. The averaging should be done separately for engines (pumpers) and ladders.

The Square-Root Law Constant, k

The values of k appropriate to the various simple models examined in Section II provide some bounds on the values encountered in practice. Estimation of k in real cities, however, can be more difficult. One method of estimating k would be to run simulation experiments of the type described in Section III and use regression analysis on these data. Another is to collect actual response distance data and actual engine (and ladder) availability data. Letting $\overline{D}$ be the average response distance, $\overline{N}$ the average number of companies available, and A the area, estimate k by

$$k = \overline{D} \sqrt{\overline{N}/A}.$$

Such data may be difficult to obtain. As an alternative, the following simple procedure may be followed to estimate k by calibrating it to the limiting case of zero alarm rate:

(1) Select a set of alarm locations at random from the set of all alarm locations of interest. With a map, or by direct field measurement, obtain the response distance to the closest fire house for each location. Compute $\overline{D}$, the average of these closest distances.

(2) With n, the number of fire houses, and A, the area of the region, obtain an estimate of k

$$\hat{k} = \overline{D} \sqrt{n/A}.$$

This procedure can be repeated for both engine and ladder distances, and can also be used for second closest houses, third closest houses, etc., in order to estimate k_1, k_2, k_3, etc.

Our data from New York City indicate that the value of k varies little between regions and is quite close to the values presented in Section III.

APPENDIX

COMPARING $E(1/\sqrt{N})$ WITH $\sqrt{1/E(N)}$

In Section II, the important point was noted that one could satisfactorily approximate $E(1/\sqrt{N})$ by $\sqrt{1/E(N)}$ for conditions of practical interest. Now let us quantify this argument--first with straightforward analysis and then with simulation experiments.

An analytical estimate of the difference between $E(1/\sqrt{N})$ and $\sqrt{1/E(N)}$ can be derived by expanding the inverse square-root function in a Taylor series about its mean and integrating to obtain the expected value:

$$1/\sqrt{x} = 1/\sqrt{x_o} + \sum_{k=1}^{\infty} (x - x_o)^k \; (-1)^k \; \binom{2n}{n}/2^{2n} x_o^{\;-(k + 1/2)},$$

where $\binom{2n}{n}$ is the binomial coefficient $(2n)!/(n!)^2$. Then

$$E(1\sqrt{x}) \stackrel{\sim}{=} \sqrt{1/E(x)} \left[1 + \frac{3}{8} \; \frac{\sigma(x)}{E(x)}^{\;2} + \sum_{k=3}^{\infty} (-1)^k \beta_k \; \frac{\sigma(x)}{E(x)}^{\;k} / \sqrt{\pi k} \right].$$

where $\beta_n \equiv \mu_n(x)/\sigma^n(x)$, $\mu_n(x) \equiv$ *nth* central moment of the distribution of x, and we have used approximation $1/\sqrt{\pi k} \stackrel{\sim}{=} \binom{2k}{k}/2^{2k}$. For situations of practical interest, the β_k typically became small enough with increasing k that the terms in the sum decline quickly to insignificance. Then the relative error is

$$\frac{E(1\sqrt{x}) - \sqrt{1/E(x)}}{E(1/\sqrt{x})} \stackrel{\sim}{=} \frac{3}{8} \left[\frac{\sigma(x)}{E(x)}\right]^2.$$

This estimate shows that the relative error is only 6 percent even when the standard deviation of N is 40 percent of the mean. Therefore, one would expect the approximation to hold quite well, even where N varies a great deal.

This analytical estimate was confirmed empirically by comparing simulation estimates of $E(1/\sqrt{N})$ and $\sqrt{1/E(N)}$. Using the simulation experiments discussed in Section III, data were derived to estimate both quantities for the three closest engines dispatched and the two closest ladders dispatched in the northern and southern regions of the Bronx. Table A-1 displays the percentage errors thus calculated.

As the analytical estimate indicates they should be, the relative errors are larger in the south Bronx, where the higher alarm rate causes greater fluctuations in the number of units available. Yet, all the errors are less than 5 percent, indicating that the approximation should work well in real circumstances.

Table A-1

RELATIVE PERCENTAGE ERRORS IN APPROXIMATING
$E(1/\sqrt{N})$ By $\sqrt{1/E(N)}$

Region/Equipment	Simulation 1	Simulation 2
North Bronx		
1st Engine	1.63	0.85
2nd Engine	1.64	0.84
3rd Engine	1.65	0.87
1st Ladder	1.80	3.44
2nd Ladder	1.79	3.46
South Bronx		
1st Engine	2.81	3.81
2nd Engine	2.80	3.79
3rd Engine	2.79	3.78
1st Ladder	2.51	4.47
2nd Ladder	2.50	4.47

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