Linear Programming

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Abstract

As a branch of operations research, linear programming is an important method to assist people in scientific management. After transforming practical problems into mathematical models by using mathematical language, linear programming is combined with certain algorithms to calculate the models, which provides a scientific basis for formulating optimal schemes. The key to solving the problem is to establish a linear programming model in line with the actual situation. This paper mainly introduces two basic solutions to linear programming: the graphic method and the simplex method.

Key words:linear programming; Graphical method; Simplex method

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Chapter 1

Introduction

Linear programming, which is a mathematical method to assist people in scientific management, is an important branch of Operations Research. In economic management, transportation, industrial production, and other economic activities, people's demand is always to increase the economic effects as much as possible, there are two ways to achieve this goal: one is to improve technology, such as using new equipment or looking for new raw materials; the other is to improve production planning through use limited resources effectively. Linear programming is a way to study how to make the best economic results to make optimal choices based on the second approach. In other words, linear programming is to find the extreme value of linear objective functions under linear constraints.

As early as 1939, Soviet mathematician $LeonidKantorovich$ proposed the problem of linear programming, but it did not attract attention, unfortunately. Then in 1947, American mathematician $GeorgeB.Dantzig$ invented the simplex method that for the first time efficiently tackled the linear programming problem in most cases. In 1951, American economist $T.C.Koopmans$ applied linear programming to the economic field, for which he and $Kantorovich$ won the Nobel Prize in Economics in 1975.

Linear programming is not only a mathematical theory, it has become an important tool to help managers make scientific decisions in modern management. With the rapid development of computers, the computing ability of linear programming has been rapidly improved, so it has become more widely used.

Chapter 2

Implementation

2.1 Establishment of Model

Generally, there are three steps to establish the corresponding mathematical model according to the actual problem:

(1) Identify the decision variables according to the factors that will affect the goal to be achieved. In other words, the solution for each problem can be represented by a set of decision variables $(x_1, x_2, ..., x_n)$, while the values of this set of unknowns represent a specific solution. In general, these values are non-negative;

(2) Determine the objective function, which depends on the functional relationship between decision variables and the goal;

(3) Determine the constraint conditions, that is, the constraints on the decision variables.

The formal characteristics of linear programming problems are made up of the above three elements.

Since linear programming is used to study how to maximize the utilization of resources, the following three conditions should be met when establishing the model:[1]

(1) Optimization condition: the objective to be achieved can be described by a linear objective function. According to different practical problems, the objective function is required to be maximized or minimized;

(2) Limit condition: the constraints that the objective needs to be satisfied, which can be expressed by linear equations or inequalities of decision variables;

(3) Selection condition: there exist multiple options for decision-makers to refer to and decide under known conditions, to find the optimal solution.

The general mathematical model of the linear programming problem is shown below:

Decision variables: $X = (x_1, x_2, ..., x_n)^T$,

Objective function: $max(min)Z = c_1x_1 + c_2x_2 + ... + c_nx_n$,

$$\text{Constraint conditions: } \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq (=, \geq) b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq (=, \geq) b_2, \\ ......, \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq (=, \geq) b_m, \\ x_1, x_2, ..., x_n \geq 0. \end{cases}$$

Because objective function and constraint condition are different in content and form, linear programming problem has many expressions. However, this diversity brings inconvenience to the solution. Therefore, the following standard form is stipulated for the sake of discussion and the development of a unified algorithm:

$$maxZ = c_1x_1 + c_2x_2 + ... + c_nx_n$$
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2, \\ ......, \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m, \\ x_1, x_2, ..., x_n \geq 0. \end{cases}$$

There are three basic cases for turning non-standard forms into standard:

(1) the objective function is to find the minimum value, that is, $minZ = CX$, so let $Z' = -CX$, then the function will become: $maxZ' = -CX$;

(2) the constant term b_i is less than 0, then multiply both sides of the formula by a negative one;

(3) when the constraint conditions are inequalities, slack variables are added to transform them into equalities.

It can be seen that the standard form has the following characteristics:

(1) the objective function is to find the maximum value;

(2) all the variables are greater than or equal to 0;

(3) constraints are all equalities;

(4) the constant terms are non-negative.

2.2 Concept of the Solution

For linear programming problems, the solutions that satisfy all linear constraints are called feasible solutions, while the set composed of feasible solutions is called the feasible region. The solution satisfying the objective function is the optimal solution. In all, the ultimate goal of linear programming is to find the optimal solution.[2]

There are generally two methods to solve the problem: the graphical method and the simplex method. Both approaches are described in detail later in the essay.

2.3 Graphical Method

The graphical method is suitable for two or three variables. When there are only two variables, the rectangular coordinate system needs to be drawn, while for three decision variables, needs to draw the stereo coordinate system.[3] However, it's hard enough to draw a three-dimensional coordinate system, let alone when there are a lot of decision variables. Therefore, when the variables are greater than two, the simplex method should be used.

Although the graphic method is not the main method to solve linear programming problems, it is very suitable to explain the properties and characteristics of linear programming solutions. And it's worth mentioning that it is not necessary to convert to the standard form when using the graphical method. The implementation of the graphical method will be described by taking the linear programming problem with only two variables as an example.

2.3.1 Example

Graphical method, that is, to solve linear programming problems by making diagrams. Here, the use of the graphic method is described in detail with an example.

> Case 1. A factory plans to produce cars and motorcycles. The following table shows the required resources, available resources and product prices. So how to arrange the plan can maximize the profit?

	cars	motorcycles	available resources
Steel(ton)	2	2	16
Working Hours (hours)	5	2.5	25
Profit(euro per vehicle)	4	3	

First, suppose that the plan is to produce x_1 cars and x_2 motorcycles, and the profit is Z euro. The linear programming model obtained from the known conditions is:

$$maxZ = 4x_1 + 3x_2$$
$$\begin{cases} 2x_1 + 2x_2 \leq 16, \\ 5x_1 + 2.5x_2 \leq 30, \\ x_1, x_2 \geq 0. \end{cases}$$

Then, take x_1 as the horizontal axis, x_2 as the vertical axis, and select an appropriate coordinate length to establish the rectangular coordinate system. According to the non-negativism of variables, all feasible solutions are in the first quadrant.

The second step is to diagram the constraints. Each inequality of a constraint represents a semi-plane in the rectangular coordinate system. So after drawing the boundary of the half-plane, what should be done is to figure out which half-plane it is. Take the first constraint as an example, $2x_1 + 2x_2 \leq 16$ represents a half-plane, whose boundary is $2x_1 + 2x_2 = 16$. Combined with $x_1 \geq 0$ and $x_2 \geq 0$, a triangle ΔAOB can be gotten.

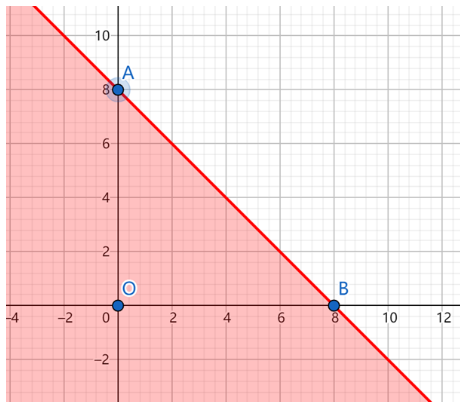

Figure 2.1: The half plane for $2x_1 + 2x_2 \leq 16$

In this triangle, the meaning of point $A(0, 8)$ is that when all resources are devoted to the production of motorcycles, the maximum number of motorcycles that can be produced is 8. The calculation is: $\quad 0 + 2x_2 \leq 16, \quad x_2 \leq 8$.[4]

What's more, the line segment AB represents the set of points corresponding to the number of cars and motorcycles when all the resources are utilized to produce, while the area of ΔAOB represents the set when resources are not fully used.

When all the half-planes are drawn, the common part is the feasible region. Within this region, each point corresponds to a feasible solution. The quadrilateral $ACDO$ shown in Figure 2.2 is the feasible region for this problem.

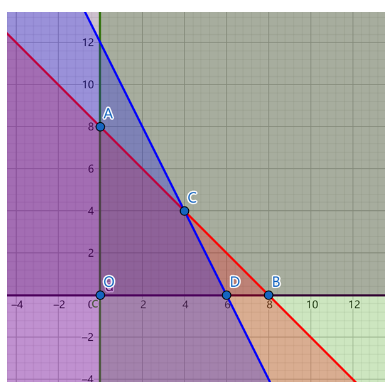

Figure 2.2: The feasible region $Quadrilateral_{ACDO}$

The third step is to draw the contour line of the objective function passing through the origin of coordinates and determine its moving direction of it. Here, when shifting the contour line in the direction of the arrow, the total profit increases. So the line should be shifted in this direction.

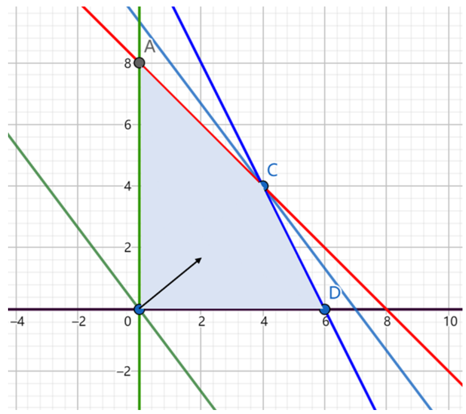

Figure 2.3: Step 3 and 4

The last step is to translate the contour to find the optimal point of the objective function in the feasible region. In the context of this problem, the profit maximizes when the isoline reaches the farthest point C in the feasible region. Therefore, C is the best point. At this point, the corresponding coordinates $x_1 = 4$ and $x_2 = 4$ are the best planning scheme and the optimal solution for this model. What's more, the optimal value of the objective function is: $maxZ = 4 \cdot 4 + 3 \cdot 4 = 28$.

2.3.2 Types of Solutions

In general, the solution to linear programming problems can be divided into the following four cases:

(1) unique solution
Case 1, which has a unique optimal solution at point C is a good example of this situation. Similarly, if the objective function is to find the minimum value, the unique optimal solution can be obtained at point O.

(2) infinite optimal solutions
Consider the following case:

$$maxZ = 5x_1 + 5x_2$$
$$\begin{cases} x_1 + x_2 \leq 3, \\ 2x_1 + x_2 \leq 4, \\ x_2 \leq 2.5, \\ x_1, x_2 \geq 0. \end{cases}$$

In this case, the feasible region is a pentagon. When the contour line of the objective function is shifted to coincide with line segment AB, it can be found that the values of all points on this line segment are equal and satisfy the goal. So, there exists an infinite number of optimal solutions.

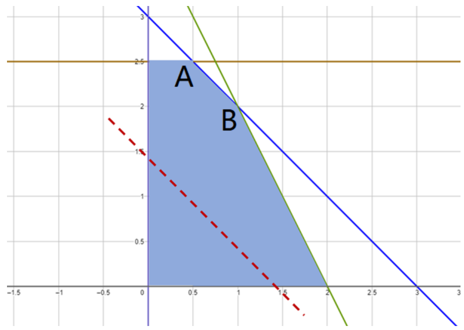

Figure 2.4: Infinite solutions

(3) unbounded solution

Consider the following case:

$$
\begin{aligned}
&maxZ = 2x_1 + 4x_2\\
&\begin{cases} -2x_1 + x_2 \leq 2, \\ 2x_1 + x_2 \geq 8, \\ x_1, x_2 \geq 0. \end{cases}
\end{aligned}
$$

It can be found that in this case, the feasible region is unbounded. And the value of Z is going to increase as x_1 and x_2 increase, so the maximum value of Z can not be found. In all, the value of the objective function can be infinite or infinitesimal when there is an unbounded feasible region.

However, it is worth noting that for this case, it may also be a unique optimal solution or infinitely optimal solution. For example, under the same constraints, the objective function is to find the minimum value.

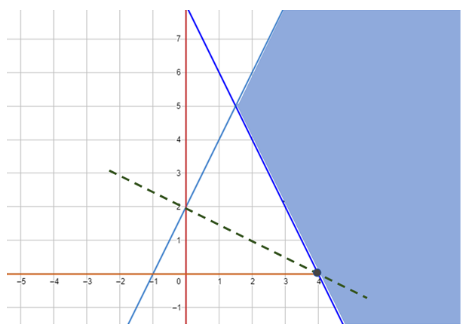

Figure 2.5: Unbounded solution

(4) no solution

Consider the following case:

$$
\begin{aligned}
&maxZ = 3x_1 + 2x_2\\
&\begin{cases} -x_1 - x_2 \geq 1, \\ x_1, x_2 \geq 0. \end{cases}
\end{aligned}
$$

It can be seen from Figure 2.6 that the feasible region constituted by constraints is an empty set, that is, there is no solution that satisfies all constraints. Therefore, there is no optimal solution.

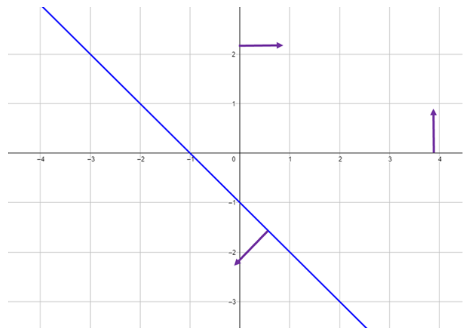

Figure 2.6: No solution

2.3.3 Summary

Since the optimal solution can always be found in the vertices of the feasible region, the solving idea of the graphical method is to compute the function value of any vertex in the feasible region, which is composed of all constraints, and then compare it with the function value of the surrounding vertices. If it is not the maximum, the comparison continues until the optimal solution is found. This is because the objective function is a linear function that can be interpreted as when the more the function moves in a particular direction, the larger or smaller the value is. In other words, when you start at any point in the feasible region, you can keep moving in that direction, then the objective function will be further optimized until reaching a vertex or an edge. The solution will be better than the starting point, so the starting point is not the optimal one. Therefore, when the function values of all vertices in the feasible region need to be calculated, if the feasible region is an N-sided shape, that is, there exist too many constraints, the amount of calculation will also increase. In this situation, using the simplex method may simplify the solution process.

The graphical method is simple and intuitive, which is convenient for beginners to explore the principle and geometric significance of linear programming. However, there exists a great limitation because it can only solve the problem with 2 or 3 variables. Therefore, the simplex method is used more in practice.

Chapter 3

simplex method

3.1 The introduction of simplex method

From linear programming, it is worth noting that some linear programming problems can be solved by plotting constraints on a graph. However, this method is only suitable for some simple linear programming problems. In some practical problems in the real world, there may be hundreds of equations and thousands of variables, which are difficult to solve graphically. Besides, although the graphical solution method can reveal the characteristics of feasible solution sets and optimal solutions to linear programming problems, unfortunately, it is not possible to know the answers to the problems by graphs as humans do when computers solve linear programming problems, hence, it is necessary to design a computer-friendly algorithm for solving linear programming problems.

Therefore, the American applied mathematician George Dantzig proposed the simplex method in 1947.[5] By using it on a computer, the algorithm can quickly find the optimal solution. Until now, more than 70 years later, as one of the most efficient algorithms, the simplex method still plays an significant role in solving linear programming problems.

3.2 The concept of the simplex method

3.2.1 Convex set

According to Jensen's inequality, the cut line over any two points on a lower convex function must be above the graph of the function between these two points.[6] In a geometric sense, some linear programming problems can be considered as convex sets and optimally solved using the relevant properties of convex sets.

In linear programming problems, the convex set will be called the feasible region, and the points composed between each linear equation will be called the vertices of the feasible domain, so the optimal solution can be found by considering the vertices of each feasible domain, i.e., the bases of the linear programming composition matrix, each of which represents a fundamental feasible solution.

3.2.2 Applicable conditions

Before using the simplex method to solve a linear programming problem, it is necessary to determine the expression format of the problem first. If it is a general form, it needs to be converted into a standard form and rewritten into a slack form for the operation.

General form to stand form

The standard form of linear programming is converted from the general form. The goal of the conversion is to convert unconstrained, equality constraints and inequality constraints into less-or-equal constraints, and to ensure each variable to a non-negative number.

For example:

$$maxZ = 3x_1 + 4x2$$
$$s.t. \begin{cases} 2x_1 + x_2 \leq & 40 \\ -x_1 - 3x_2 \geq & 30 \\ x_j(j = 1, 2) > & 0 \end{cases}$$

Convert to standard form:

$$maxZ = 3x_1 + 4x2$$
$$s.t. \begin{cases} 2x_1 + x_2 \leq & 40 \\ x_1 + 3x_2 \leq & 30 \\ x_j(j = 1, 2) > & 0 \end{cases}$$

Standard form to slack form

After transformation, linear programming problems can be described more conveniently. However, the equality constraint that is convenient to calculate is a better choice, so in the simplex method, Dantzig converts the standard form of linear programming into a slack form. The method of transformation is to convert the less-equal constraints in the standard form into equality constraints by adding non-negative slack variables.

For example:

$$maxZ = 3x_1 + 4x_2$$
$$s.t. \begin{cases} 2x_1 + x_2 \leq & 40 \\ x_1 + 3x_2 \leq & 30 \\ x_j(j = 1, 2) > & 0 \end{cases}$$

(i)Ensure each variables ≥ 0
(ii)Add slack slack variables x_3, x_4 to each constraints
(iii)Put x_3, x_4 into objective function

$$maxZ = 3x_1 + 4x2 + 0x_3 + 0x_4$$
$$s.t. \begin{cases} 2x_1 + x_2 + x_3 + 0x_4 = & 40 \\ -x_1 - 3x_2 + 0x_3 + x_4 = & 30 \\ x_j(j = 1, 2, 3, 4) \geq & 0 \end{cases}$$

Imagine adding n non-negative slack variables to the constraint equation so that the equation is exactly equal to 0.

3.2.3 principle

Basic feasible solution

From the slack form, it can be expressed in matrix form, that is:

$$maxZ = CX$$

$$s.t.AX = bX \geq 0$$

In here, A represents a matrix composed of different variables, and b is a vector composed of different results. Let X_B be the basis variable and X_N be the non-basic variable, that is, the newly added non-negative slack variable, the coefficient matrices corresponding to the variables X_N and X_B are the non-basic matrix N and the basic matrix B:

$$AX = \begin{pmatrix} B & N \end{pmatrix} \begin{pmatrix} X_B \\ X_N \end{pmatrix} = B$$

$$X_B = \mathbf{B}^{-1}b - \mathbf{B}^{-1}NX_N$$

In this equation, if the non-basic variable X_N equal to 0:

$$X = \begin{pmatrix} \mathbf{B}^{-1}b \\ 0 \end{pmatrix}$$

This X is called the basic feasible solution.
To the objective function:

$$maxZ = CX = \begin{pmatrix} C_B & C_N \end{pmatrix} \begin{pmatrix} \mathbf{B}^{-1}b \\ 0 \end{pmatrix} = C_B\mathbf{B}^{-1}b$$

C_B and C_N are the results corresponding to base variables and non-base variables in result C, respectively. Therefore, By converting the non-basic variables to 0, the best result of the objective function can be obtained, that is, the optimal solution.

Pivot

In order to find a better solution from the objective function, it is necessary to pivot the variables. One pivot selects a non-basic variable(input variable) and a basic variable(output variable) and then replaces the two. The linear programming before and after the rotation is equivalent.

Determine the optimal solution

After pivoting, a new equation will be obtained. At this time, it is necessary to judge whether the optimal solution in the new equation is.

From the above formula, it is obtained that $X_B = \mathbf{B}^{-1}b - \mathbf{B}^{-1}NX_N$, bring X_B into the objective function,The value of the objective function at a feasible solution is:

$$
\begin{aligned}
maxZ &= C_B(\mathbf{B}^{-1}b - \mathbf{B}^{-1}NX_N) + C_NX_N \\
&= C_B\mathbf{B}^{-1}b - (C_B\mathbf{B}^{-1}N - C_N)X_N \\
&= C_B\mathbf{B}^{-1}b + (C_N - C_B\mathbf{B}^{-1}N)X_N
\end{aligned}
$$

Due to variables are greater than or equal to 0, it is easy to prove that when $C_N - C_B\mathbf{B}^{-1}N$ is less than or equal to 0, $C_B\mathbf{B}^{-1}b$ can get the maximum value, indicating that it is no longer necessary to pivot.

Therefore, test vector $\sigma_N = C_N - C_B\mathbf{B}^{-1}N$ can be used to test whether pivot is required. When all σ are less than or equal to 0, it can be judged that the objective function obtains the maximum value.

Since the objective function is to find the maximum value of Z, we want all the tests in the test vector to be less than or equal to 0. If the objective function is to find the minimum value of Z, then we only need all the tests in the test vector to be greater than or equal to 0.

3.2.4 Different Solutions of Simplex Method

For linear programming problems:

$$maxZ = CX$$

$$s.t. AX = bX \geq 0$$

There are essentially three different solutions.

Optimal solution

If the test vector corresponding to a basic feasible solution are all less than 0, that is, $\sigma_N = C_N - C_B\mathbf{B}^{-1}N < 0$, it can be determined that the basic feasible solution is the optimal solution.

$$maxZ = 3x_1 + 4x_2$$

$$s.t.\begin{cases} 2x_1 + x_2 \leq & 40 \\ x_1 + 3x_2 \leq & 30 \\ x_j(j = 1, 2) > & 0 \end{cases}$$

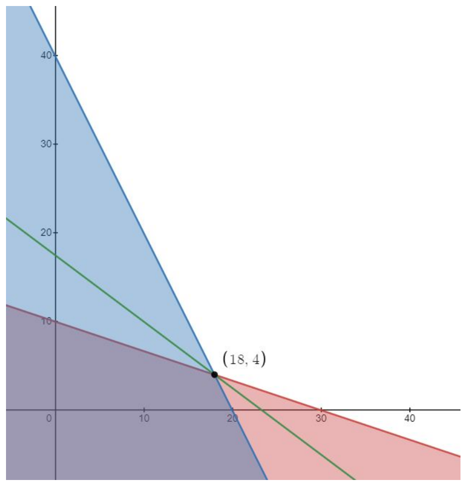

Figure 3.1: Optimal solution

It can be seen from figure 3.1 that the maximum value can be obtained when the objective function reaches the basic feasible solution (18, 4).

Alternative solution

If the test vectors corresponding to a basic feasible solution are all less than or equal to 0, that is, $\sigma_N = C_N - C_B\mathbf{B}^{-1}N \leq 0$. In this case, there is a test number σ_m in the test vector equal to 0, it can be determined that this basic feasible solution is an alternative optimal solution.

$$maxZ = 2x_1 + x_2$$

$$s.t.\begin{cases} 2x_1 + x_2 \leq & 40 \\ x_1 + 3x_2 \leq & 30 \\ x_j(j = 1, 2) > & 0 \end{cases}$$

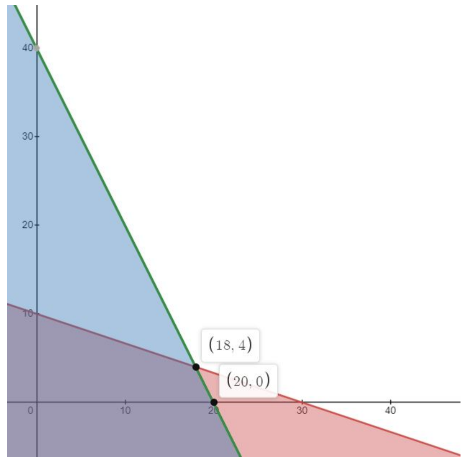

Figure 3.2: Alternative solution

It can be seen from figure 3.2 that when the test number is less than or equal to 0 and exist a test number equal to 0, the slope of the objective function and a constraint function is the same, so the optimal solution is obtained on the line segment where the objective function and the constraint function coincide All points of the optimal solution.

Unbounded

In the simplex method, if the test number of the entry variable is greater than 0, the entry variable can continue to be increased to improve the value in the objective function. Since the problem must be kept feasible during the pivot, the increase of the base variable is restricted by the constraints too, which means it cannot increase indefinitely. Therefore, the increase in the value of the objective function as the base variable is increased is bounded. If the coefficients corresponding to the base variable are all less than zero, there will be no constraints on the base variable, so the linear programming problem is unbounded. In this case, it is reflected in the simplex method that a feasible output variable cannot be found to pivot.

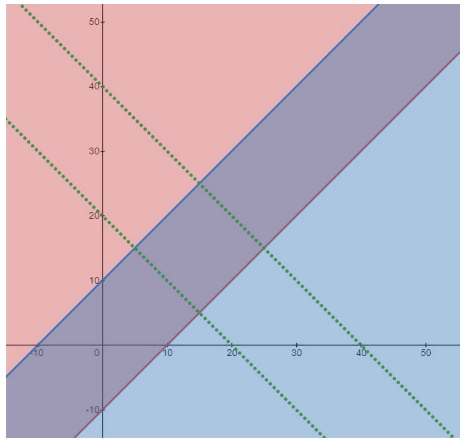

Figure 3.3: Unbounded

It can be found from figure 3.3 that the linear programming problem has no constraints on x2. So x2 can increase infinitely. Geometrically, this linear programming problem cannot form a convex set, the simplex method cannot be used to find the optimal solution, this situation is called unbounded.

3.3 Degeneracy

In the process of pivoting using the simplex method, there is a situation where the value before pivoting and the value after pivoting are still the same. When the basic feasible solution of the linear programming problem has at least one basis variable equal to 0, this solution is called degeneracy.[7]

It is worth noting that there are redundant constraints in the linear programming problem, which may cause degeneracy.

Case with redundant constraints:

$$maxZ = 3x_1 + 4x2$$

$$s.t. \begin{cases} 2x_1 + x_2 \leq & 60 \\ 2x_1 + x_2 \leq & 40 \\ x_1 + 3x_2 \leq & 30 \\ x_j(j = 1, 2) > & 0 \end{cases}$$

Figure 3.4: Degeneracy

3.3.1 cycling

Since the objective function in the degenerate phenomenon will not find a better basic feasible solution during the iterative process of pivoting. If the basic feasible solution that has appeared reappears during the pivoting process, the iterative process will continue to loop. In this case, the optimal solution cannot be found, which causes cycling.[8]

3.3.2 Bland's rule

An effective way to avoid degeneracy is to use Bland's rule, the simplex method based on Bland's rule can stop infinite steps. When selecting substitution variables and substitution variables, always select the smallest subscript that satisfies the condition.[9]

(i)For input variable: In the objective function, select the first variable (if any) with a negative coefficient as the substitute variable.

(ii)For output variable: For all constraints, select the variable with the smallest result of the constraint function as the output variable.[10]

3.4 Example of Simplex Method

3.4.1 Solve problems using the simplex method

Suppose you are a manager of the factory, you plan to produce two different pens, both of which require machines and materials to finish. Two machines are required to produce pen A, and one machine is required to produce pen B, and the total number of machines is 40.

In addition, 1 part of the material is needed to produce pen A and 3 parts of the material are needed to produce pen B. The total number of materials is

30 parts, the profit of pen A is 3euros and the profit of pen B is 4 euros. How to maximize the profit of the factory?

Perhaps the problem is not difficult, but a bunch of words may make it confusing to understand exactly what needs to do. In this case, by using linear programming the problem can be quickly expressed mathematically so that the meaning of this problem can be clearly understood.

$$maxZ = 2x_1 + x_2$$

$$s.t. \begin{cases} 2x_1 + x_2 \leq & 40 \\ x_1 + 3x_2 \leq & 30 \\ x_j(j = 1, 2) > & 0 \end{cases}$$

Let's gain a better understanding of the simplex method by solving this practical linear programming problem. In the beginning, it is necessary to change the problem from standard form to slack form.

$$maxZ = 3x_1 + 4x2 + 0x_3 + 0x_4$$

$$s.t. \begin{cases} 2x_1 + x_2 + x_3 + 0x_4 = & 40 \\ -x_1 - 3x_2 + 0x_3 + x_4 = & 30 \\ x_j(j = 1, 2, 3, 4) \geq & 0 \end{cases}$$

Then we take out the coefficients of each variable to form a matrix, and then take out the result of each constraint function to form a vector, we can get:

First iteration

C_B	X_B	b	x_1	x_2	x_3	x_4	θ
0	X_3	40	2	1	1	0	40
0	X_4	30	1	3	0	1	10
			3	4			

Find basic feasible solution in first iteration:

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 30 \end{pmatrix}$$

Check test number:

$$\sigma_1 = 3 - ((0 * 2) - (0 * 1)) = 3$$

$$\sigma_2 = 4 - ((0 * 1) - (0 * 3)) = 4$$

$$\sigma_1, \sigma_2 > 0$$

Second iteration

C_B	X_B	b	x_1	x_2	x_3	x_4	θ
0	X_3	30	5/3	0	1	-1/3	18
4	X_4	10	1/3	1	0	1/3	30
			5/3			-4/3	

Find basic feasible solution in second iteration:

$$X_2 = \begin{pmatrix} 0 \\ 10 \\ 30 \\ 0 \end{pmatrix}$$

Check test number:

$$\sigma_1 = 3 - ((0 * \frac{5}{3}) - (4 * \frac{1}{3})) = \frac{5}{3}$$

$$\sigma_2 = 0 - ((0 * \frac{1}{3}) - (4 * \frac{1}{3})) = -\frac{4}{3}$$

$$\sigma_1 > 0, \sigma_2 < 0$$

Third iteration

C_B	X_B	b	x_1	x_2	x_3	x_4	θ
3	X_3	18	1	0	3/5	-1/5	
4	X_4	4	0	1	-1/5	2/5	
					-1	-1	

Find basic feasible solution in third iteration:

$$X_3 = \begin{pmatrix} 18 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Check test number:

$$\sigma_1 = 0 - ((3 * \frac{3}{5}) - (4 * -\frac{1}{5})) = -1$$

$$\sigma_2 = 0 - ((3 * -\frac{1}{5}) - (4 * \frac{2}{5})) = -1$$

$$\sigma_1, \sigma_2 < 0$$

Through a simple example, we can summarize the calculation conclusion of the simplex method.

(1) First check each test number σ in the test vector, if we find that the test number σ is greater than zero, it means that the solution is not optimal and there is still an opportunity for progress. It is worth noting that sometimes

the number of tests for the initial basic feasible solution is less than or equal to 0. In this situation, the initial feasible solution will be the optimal solution. From the base feasible solution, we can see that the test numbers are 3 and 4. Essentially, the test numbers are the coefficients of the objective function on x_1 and x_2 before the iterations.

(2) Choose the variable with the larger test number σ as the input variable, choose the variable with the smaller θ as the output variable, θ equals b divided by the coefficient of the input variable.

(3) The row with σ and the column θ are called pivot elements and iterate by replacing the pivot element with 1.

(4) After completing the first iteration, a new basic feasible can be derived, next check the test number σ, found that there are still σ greater than 0, so continue to iterate until all the test numbers are less than or equal to 0.

3.4.2 Using simplex method in computer

In the previous example, we summarize several steps of the simplex method which could use in the computer. With these steps, the computer can be used to compute the simplex method.

(1)check if $\sigma \leq 0$

(2)choose input variable and output variable

(3)Perform a pivot operation to find the next feasible solution

(4)Determine whether the solution is optimal or not, if yes then complete the calculation, if not then return to the first step.

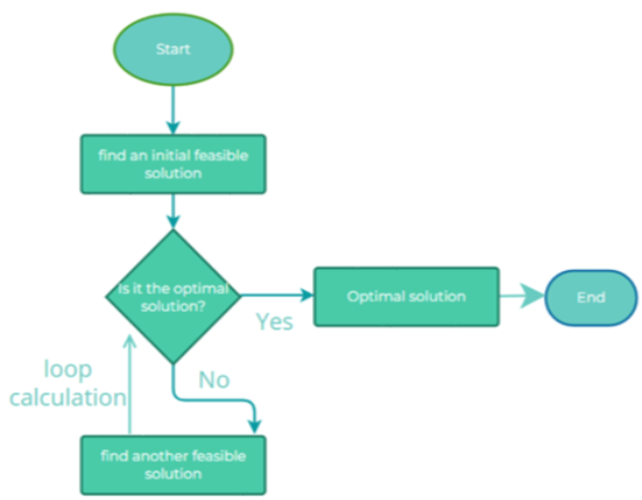

Figure 3.5: flow chart

Chapter 4

Application

Linear programming is a mathematical method that has been applied very successfully at present. It was proposed during World War II as a way to plan expenditures so as to effectively reduce the cost of the army, so it was mainly used in the military field. As a core part of operations research, linear programming is not only adaptable but also simple in computing technology. Since World War II, its application has turned to People's daily life. Nowadays, it is mainly applied to the following aspects:

(1) marketing: such as making an advertising budget and pricing new products;

(2) production planning: mainly refers to determining the cooperation of production, storage, and labor, which can adapt to the fluctuations in requirements planning, from the overall situation;[11]

(3) financial issues: mainly involving budget, loans, investment, and other issues.

... ...

In addition, it also plays an important role in food and agriculture, engineering, transportation optimization, efficient manufacturing, and energy industry these five aspects.[12]

Chapter 5

Conclusion

By building mathematical models that use constrained objective functions to find the maximum or minimum objective, linear programming has matured to face real-world problems and is widely used in many industries such as finance and computing. Managers can maximize the benefits of a limited amount of human resources and materials by using linear programming for rational integration. At the same time, the process of using linear programming to translate real-world problems into simple and understandable mathematical formulas and solve them is scientific and explanatory, allowing companies to be more efficient in their production and investment through the rational allocation of resources, increasing the efficiency of their operations and thus being more competitive than other companies. It is conceivable that the concept of linear programming will become the first choice for significant decisions in the future.

Contribution

Zhengyang Yu: chapter 1,2,4
Ruilin He: chapter 3, 5

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