On Mathematical modeling of erosion and sedimentation

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Erosion in the enviromental context: Illustrations

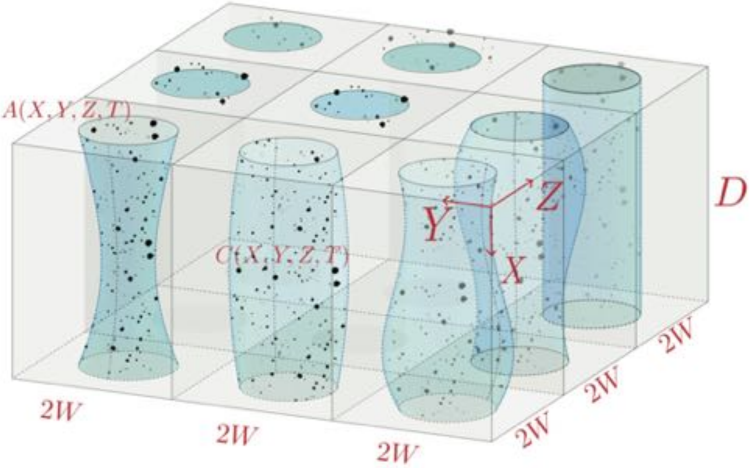



Different approaches in the litterature

- Computational Fluid Dynamics (**CFD**) tool.
  - CFD simulations are relatively inexpensive.
  - CFD simulations encouters limitation errors.
  - The accuracy of the simulation is only as good as the initial/boundary conditions provided for numerical model.
- Lattice-Boltzmann Methods (**LBM**).
  - LBM solved locally so it is easy to break the problem into calculations that can be done in parallel by multiple computer processors.
  - LBM is expensive to implement and requires high computation power.
- Pore network models (**PNM**).
  - PNM simplifies the system's geometry into pore spaces interconnected via the according inlet structure.
  - this technique achieves high-resolution accuracy running on small scale domains using,comparatively, lower computing costs.

Channels network

- Schematic showing a structure and channels with the pore radii $A(X, Y, Z, T)$ and small-particle concentration $C(X, Y, Z, T)$.



Governing equations (flow)

Our model based on the coupling of a Navier Stokes hydraulic model and an advection-diffusion solid transport model.

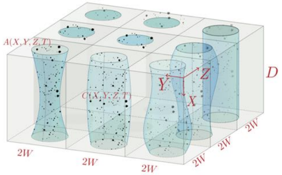
- Stokes Equation modeling the flow behavior:

$$\nabla P = \mu \nabla^2 \mathbf{U}_\mathrm{p}, \quad \nabla \cdot \mathbf{U}_\mathrm{p} = 0, \quad 0 \le X \le D, \quad 0 \le R \le A,$$

  where P and $\mathbf{U}_\mathrm{p}$ are the pressure and flow channel velocity, respectively.

- The Stokes Equation are subjected to no penetration and no slip boundary conditions

$$\mathbf{U}_\mathrm{p} \cdot \mathbf{n} = \mathbf{U}_\mathrm{p} \cdot \mathbf{t} = 0 \quad \text{at} \quad R = A,$$



where $\mathbf{n}$ and $\mathbf{t}$ are the unit normal and tangent to the channel wall respectively.

- The B.C. on the pressure $P(X,T)$ within the membrane are:

$$P|_{X=0} = P_0, \quad P|_{X=D} = 0.$$

- The total shear stress Σ at the channel walls, exerted by the flow is given by

$$\Sigma = \mu \left(\nabla \mathbf{U}_\mathrm{p} + \nabla \mathbf{U}_\mathrm{p}{}^T \right) \mathbf{n} \cdot \mathbf{t} \Big|_{R=A(X,T)} .$$

Governing equations (eroded particles)

- The full advection-diffusion equation for the eroded particles $C(R, X, T)$ is:

$$\frac{\partial C}{\partial T} + \nabla \cdot \mathbf{Q_c} = 0, \quad \mathbf{Q_c} = -\Xi \nabla C + \mathbf{u}_\mathrm{p} C,$$

where $\mathbf{Q_c}$ is the flux of total particles, Ξ is the diffusion coefficient of particles. The boundary conditions are:

$$C(R, 0, T) = C_0, \quad \left.\frac{\partial C}{\partial X}\right|_{X=D} = 0, \quad \mathbf{Q_c} \cdot \mathbf{n} = -\frac{\Lambda}{2} C \quad \text{at} \quad R = A,$$

where Λ is the attraction coefficient between the channel wall and particles.

Erosion model

- The proposed erosion law is of the form of threshold laws:
  - If the tangential shield stress is greater than a critical stress Σ_e, the interface erosion occurs according to the forces exerted by the fluid. The wall cannot withstand the stress exerted and greater than its optimal resistance.
  - Sedimentation occurs when the tangential stress at a given time is lower than the wall's critical shield stress Σ_s.
- Therefore, the suggested model for erosion and sedimentation follows

$$V_n = \begin{cases} -B_s C(\Sigma_s - \Sigma) & \text{if} \quad \Sigma < \Sigma_s, \\ 0 & \text{if} \quad \Sigma_s \leq \Sigma < \Sigma_e, \\ B_e(\Sigma - \Sigma_e) & \text{if} \quad \Sigma \geq \Sigma_e, \end{cases}$$

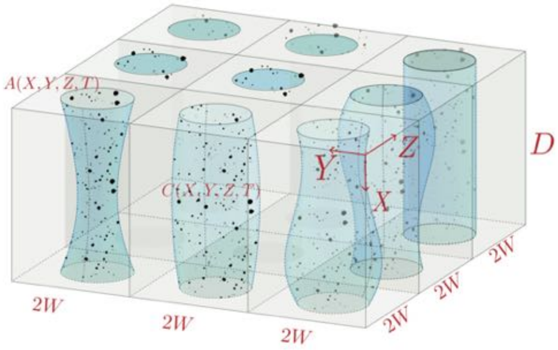
  where B_s and B_e are the sedimentation and erosion coefficient respectively, depending on the material of the channel.
- The channel wall normal velocity relates to the channel radius by

$$V_n = \frac{\partial A}{\partial T}\left(1 + \left(\frac{\partial A}{\partial X}\right)^2\right)^{-1/2}.$$

Model formulation

We make the following assumptions:

- Small aspect ratio (i.e. $\epsilon = W/D \ll 1$): the radial component of the flow velocity is negligible compared to the axial one.
- No angular dependencies: Axisymmetric geometry.
- The erosion causes a regression of the interface in the direction of $\vec{n}$ causing a reconfiguration of the geometry.



Simulation assumptions

- We use the following scalings to nondimensionalize our models

$$\mathbf{U}_\mathrm{p} = \frac{P_0 W^2}{\mu D}\mathbf{u}_\mathrm{p}, \quad X = Dx, \quad (A, R) = W(a, r), \quad P^* = P_0 p^*,$$

$$C^* = C_0 c^*, \quad (\Sigma, \Sigma_s, \Sigma_e) = \frac{WP_0}{D}(\tau, \tau_s, \tau_e), \quad T = \frac{D}{B_s C_0 P_0} t, \quad \epsilon = \frac{W}{D},$$

where $\mathbf{u}_\mathrm{p} = (\epsilon v_\mathrm{p}, 0, u_\mathrm{p})$ is the dimensionless channel velocity.

- Solving for the pore velocity yield the leading orders

$$u_\mathrm{p}(r, x, t) = \frac{-1}{4a^4 \int_0^1 \frac{dx'}{a^4(x', t)}}(r^2 - a^2).$$

The wall shear stress is given by:

$$\tau(x, t) = \left.\frac{\partial u_\mathrm{p}}{\partial r}\right|_{r=a(x,t)}.$$

We further simplify it by using the expressions for the axial channel velocity and pressure, u_p and p respectively, to

$$\tau(x, t) = \frac{1}{2a^3(x, t) \int_0^1 \frac{dx}{a^4(x,t)}}.$$

Simulation Assumption

- At the leading order, we obtain the quasi-static nondimensional advection-diffusion equation

$$-\frac{1}{\hat{\mathrm{Pe}}}\frac{\partial^2 \bar{c}}{\partial x^2} + \left(\bar{u}_{\mathrm{p}} - \frac{2}{a\hat{\mathrm{Pe}}}a'(x)\right)\frac{\partial \bar{c}}{\partial x} + \frac{\lambda}{a}\bar{c} = 0, \quad \text{where} \quad \lambda = \frac{\Lambda \mu D^2}{P_0 W^3},$$

for the cross sectionally averaged particle concentration $\bar{c}$, subject to the boundary conditions

$$\bar{c}(0,t) = 1, \quad \left.\frac{\partial \bar{c}}{\partial x}\right|_{x=1} = 0.$$

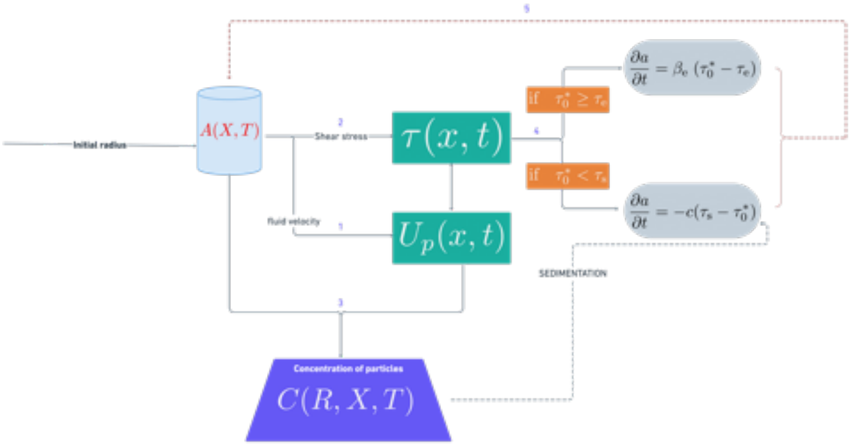
- Note that when $\hat{\mathrm{Pe}} \to \infty$ the second order derivatives trivialize resulting in a reduced **Advection-only** model
- Using our scalings, and at the leading order, we obtain the following erosion model:

$$\frac{\partial a}{\partial t} = \begin{cases} -\bar{c}(\tau_{\mathrm{s}} - \tau) & \text{if} \quad \tau < \tau_{\mathrm{s}}, \\ 0 & \text{if} \quad \tau_{\mathrm{s}} \le \tau < \tau_{\mathrm{e}}, \\ \beta_{\mathrm{e}}\,(\tau - \tau_{\mathrm{e}}) & \text{if} \quad \tau \ge \tau_{\mathrm{e}}, \end{cases} \qquad \beta_e = \frac{B_e}{B_s C_0},$$

with the initial condition

$$a(x,0) = a_0(x).$$

System diagram



System summary

An interaction between the flow and the solid channel occurs as soon as there is sediment transport. The diagram above describes these interactions in a schematic diagram:

1. The forces exerted by the flow, causing particles transport.
2. This morphological evolution causes a change in the geometry of the cross section, hence, leading to a change in the flow regime.
3. Update of the amount of sediment transported during this process.
4. The particles carried away by the flow result in a morphological change given the erosion/sedimentation, The channel's radius responds by widening or narrowing.
5. The pore's radius gets updated depending on the acting force at a given time.

Simulation: Evolution along x axis

Simulation: Parabolic initial radius

Simulation: Parabolic initial radius with higher erosion shear rate

Simulation: Parabolic initial radius with higher Peclet Number

Simulation: Trigonometric initial radius

Trigonometric initial radius with higher erosion shear rate

Trigonometric initial radius with higher Peclet Number

Conclusions

- Erosion and deposition within geological structures and porous media always lead to channelization
- the final configuration is either completely clogged or completely eroded, because of the constant pressure drop
- We also showed that the final state of a channel altered by erosion and deposition crucially depends on the balance between specific values such as: shear stress, Péclet number.

Thank You!