Laplace equation is denoted as

$$\Delta u = 0$$

We are looking for a solution that depends on $|x| = r$.

therefore: let $u(x) = v(|x|)$

We have: $\dfrac{du(x)}{dx} = v'(r)\,\dfrac{x}{r}$ $\qquad \left(|x|' = \dfrac{x}{|x|} \right)$

$$\Rightarrow \frac{d^2 u(x)}{dx^2} = v''\big(\overset{r}{\cancel{|x|}}\big)\,\frac{x}{r} \times \frac{x}{r} + \left(\frac{r - x^2/r}{r^2} \right) v'(r)$$

$$\Rightarrow \frac{d^2 u(x)}{dx^2} = v''(r)\,\frac{x^2}{r^2} + \frac{1}{r}\,v'(r) - \frac{x^2}{r^3}\,v'(r)$$

Laplacian operator:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{d^2 u}{dy}$$

knowing $\dfrac{d^2 u(x)}{dr} = v''(r)$

$$\Delta u = v''(r)\,\frac{x^2}{r^2} + \frac{1}{r}\,v'(r) - \frac{x^2}{r^3}\,v'(r) + v''(r) = 0$$

$$\Rightarrow v''(r)\left(1 + \frac{x^2}{r^2}\right) + v'(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right) = 0$$

$$\Rightarrow v''(r) + \frac{1}{r}\,v'(r) = 0$$ $\leftarrow$ ODE

We continue solving the obtained ODE as follows:

$$\frac{v''(r)}{v'(r)} = -\frac{1}{r} \Rightarrow \ln(v'(r)) = -(n-1)\ln(r) + c_1$$

$\underline{n > 2}$

$$\Rightarrow \ln(v'(r)) = \ln(r^{1-n}) + c_1$$

$$\Rightarrow v'(r) = c\, r^{(1-n)}$$

$$\Rightarrow v(r) = c\, r^{2-n} + b$$

$$\Rightarrow \boxed{v(r) = \frac{c}{r^{n-2}} + b}$$

$\underline{n = 2}$

$$\ln(v'(r)) = -\ln(r) + c_1$$

$$\Rightarrow \ln(v'(r)) + \ln(r) = c_1$$

$$\Rightarrow \ln(v'(r)\, r) = c_1$$

$$\Rightarrow v'(r)\, r = c$$

$$\Rightarrow v'(r) = \frac{c}{r}$$

$$\Rightarrow v(r) = c\ln(r) + b$$

$$\Rightarrow \boxed{v(r) = c\ln(r) + b}$$

$$n = 0 \Rightarrow v'(r) = c \Rightarrow \boxed{v(r) = cr}$$

therefore

$$v(r) = \begin{cases} cr & n = 1 \\ c\ln(r) + b & n = 2 \\ \dfrac{c}{r^{n-2}} + b & n \geq 3 \end{cases}$$

Green's theorem

the divergence theorem states that the surface integral of a vector field over a closed surface, which is called the flux through the surface, is equal to the volume integral of the divergence over the region inside the surface.

Divergence theorem:

$$\iint_S \vec{F}.\hat{n}.dS = \iiint_D \operatorname{div} \vec{F}\, dV \quad (1)$$

If we let $\vec{F} = v\vec{\nabla} u$

$$(1) \Rightarrow \iint_S (v\vec{\nabla} u).n\, dS = \iiint_D \dot{\nabla}(v\vec{\nabla} u) \qquad \left[\vec{\nabla}(v\vec{\nabla}u) = \vec{\nabla}v.\vec{\nabla}u + v\nabla^2 u\right]$$

$$\Rightarrow \iint_S v\frac{\partial u}{\partial n}\, dS = \iiint_D (\vec{\nabla}v.\vec{\nabla}u)\, dV + \iiint_D (v\nabla^2 u)\, dV$$

We therefore obtain Green's first identity:

$$\iint_S v\frac{\partial u}{\partial n}\, dS = \iiint_D (\vec{\nabla}v.\vec{\nabla}u)\, dV + \iiint (v\nabla^2 u)\, dV \quad - G1$$

---

u and v are arbitrary chosen function, therefore:

We can similarly impose $\vec{F} = u\vec{\nabla}v$. to obtain

$$\iint_S \left(u\frac{\partial v}{\partial n}\right) dS = \iiint_D (\vec{\nabla}u.\vec{\nabla}v)\, dV + \iiint_D (u\nabla^2 v)\, dV \quad (2)$$

substracting (1) from (2) yields:

$$\iint_S v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\, dS = \iiint_D (v\nabla^2 u - u\nabla^2 v)\, dV \quad \Big]\ \text{Green's second identity.}$$

Now that we derived Green's identities:

Let's say we want to solve for Poisson equation: (3D space)

$$\nabla^2 u = f(x, y, z)$$

Let's find a solution to our PDE:

the boundary condition on this PDE is a Dirichlet B.C

[this boundary is specified on the boundary of Domain D → on S] $u = h(x, y, z)$

~ ~ ~

Green's identities: setting $v = G$

$$\iint_S G \frac{\partial u}{\partial n} dS = \iiint_D (\vec{\nabla} G \cdot \vec{\nabla} u) dV + \iiint_D (G \nabla^2 u) dV \quad -(1)$$

$$\iint_S \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}\right) dS = \iiint_D (G \nabla^2 u - u \nabla^2 G) dV \quad (2)$$

Green's function solving this PDE problem:

$$\nabla^2 G = \delta(p-x, q-y, s-z) \leftarrow \text{on domain } D$$

$$G = 0 \quad \text{on domain } S$$

1) - this serves to break down f into a bunch of impulses

2) - solving for G as a result of these impulses.

3) - integrating these solutions for the impulses over the domain.

Plugging $\nabla^2 G = \delta$ into ~~cond~~ and for $\nabla^2 u = f$

$$\iint_S \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) dS = \iiint_D G f - u\, \delta(p-x, q-y, s-z)\, dV$$

$dV = dp\,dq\,ds$

$$\Rightarrow \underbrace{\iint_S \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) dS}_{\text{this term is evaluated at the boundary } S \text{ domain}} = \iiint_D G f\, dV - u(x,y,z)$$

this term is evaluated at the ~~boundary~~ S domain $\longrightarrow$ apply B.Cs

$u = h$

$G = 0$

$$\underbrace{\left[\int_{x_0-\varepsilon}^{x_0+\varepsilon} m(x)\, \delta(x-x_0) = m(x_0) \right]}_{\text{property}}$$

therefore $\boxed{u(x,y,z) = \iint_S \left(h \frac{\partial G}{\partial n} \right) \partial S + \iiint_D G f\, dV}$

Green's function $\frac{\partial G}{\partial n} = 0$?? $\quad u = \iiint_D [G f] dV$??

We have found a general solution to u on the domain D and sub-domain S. For conventional purposes, let's switch the notation:

Reformulate Green's identity for domain Ω and subdomain Γ gives

let $G = u$ and; u (previously) $= \varphi$. and $\Delta = \nabla \cdot \nabla = \nabla^2$

$$\left[u = \int_{R^N} \phi f\, dV = \int_{R^N} \phi f\, dV \right]$$

$$(1) \quad \int_\Gamma u \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial u}{\partial n}\, dS_y = \int_\Omega u \Delta \varphi - \varphi \frac{\partial u}{\partial n}\, dS_y$$

Let $\varphi(y) = \phi(x-y)$

$$(1) \Rightarrow \int_\Gamma \left[u \frac{\partial \phi(x-y)}{\partial n} - \phi(x-y) \frac{\partial u}{\partial n} \right] dS_y = \int_\Omega \left[u\, \Delta \phi(x-y) - \phi(x-y) \underset{\Delta u}{\cancel{\frac{\partial u}{\partial n}}} \right] dV$$

(1)

we have $-\Delta\phi(x) = \delta(x)$ and $\int_{\mathbb{R}^N} \rho(x)\delta(x) = \rho(0)$

$$\Rightarrow \int_{\mathbb{R}^N} \rho(x-y)\,\delta(x) = \rho(y)$$

$\cancel{(1) \Leftrightarrow \int_\Omega - u\,\delta(x-y)}$

$$(1) \Leftrightarrow \int_\Omega \left[-u(x)\,\delta(x) + \phi(x-y)\,\rho(y)\right] dV_y = \int_\Gamma \left[u \frac{\partial \phi(x-y)}{\partial m} - \phi(x-y)\frac{\partial u}{\partial m}\right] dS_y .$$

Therefore, we obtain solution $u(x)$ as an integral equation:

$$-u(x) + \int_\Omega \left[\phi(x-y)\,\rho(y)\right] dV_y = \int_\Gamma \left[u \frac{\partial \phi(x-y)}{\partial m} - \phi(x-y)\frac{\partial u}{\partial m}\right] dS_y$$

Neumann Boudary value problem:

$\frac{\partial u(x)}{\partial m} = g(x)$; $\Delta u = 0 = \rho$ (evaluated a Ω)

$$\therefore u(x) = \int_\Omega \left[\phi(x-y)\,\rho(y)\right] dV_y - \int_\Gamma \frac{\partial \phi(x-y)}{\partial m} u - \phi(x-y) g(y) \, dS_y$$

$$\Rightarrow u(x) = \int_\Gamma \phi(x-y)\,g(y)\, dS - \int_\Gamma \frac{\partial \phi(x-y)}{\partial m} u(y)\, dS_y .$$

To further evaluate this solution. Let's define the double layer potential $\bar{\bar{u}}(x)$:

$$\bar{\bar{u}}(x) = -\int_\Gamma \frac{\partial \phi(x-y)}{\partial n} u(y) \partial S_y .$$

taking the limit as $x \to x_0$ in the subdomain Γ:

$$\lim_{x \in \Omega \to x_0} \bar{\bar{u}}(x) = \frac{1}{2} h(x_0) + \bar{\bar{u}}(x_0)$$

(h is defined as g in our problem)

Also (from slide 14) taking a better approach for ϕ with $\frac{\partial \phi(x)}{\partial n} = 0$

we get $u(x) = -\int_\Gamma \phi(x-y) \sigma(y) \, d S_y$

therefore $g(x)$ takes the form

(proof Stanford potential theory)

$$g(x) = \frac{1}{2} \sigma(x) + \underbrace{\int_\Gamma \frac{\partial \phi(x-y)}{\partial n} \sigma(y)}_{\bar{\bar{u}}(y)} \, d S_y$$