Systems of linear equations

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1 Linear System of Equations

Consider a system of 3 linear equations for simplicity:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}$$

Matrix form is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Concise form: $Ax = b$
Where A is the coefficient matrix, x is the vector of unknowns, and b is the right-hand side vector

1.1 2 by 2 matrix

For the first example, consider the following system with two equations in two unknowns:

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

and visualize it using the matrix picture, row picture, and column picture.

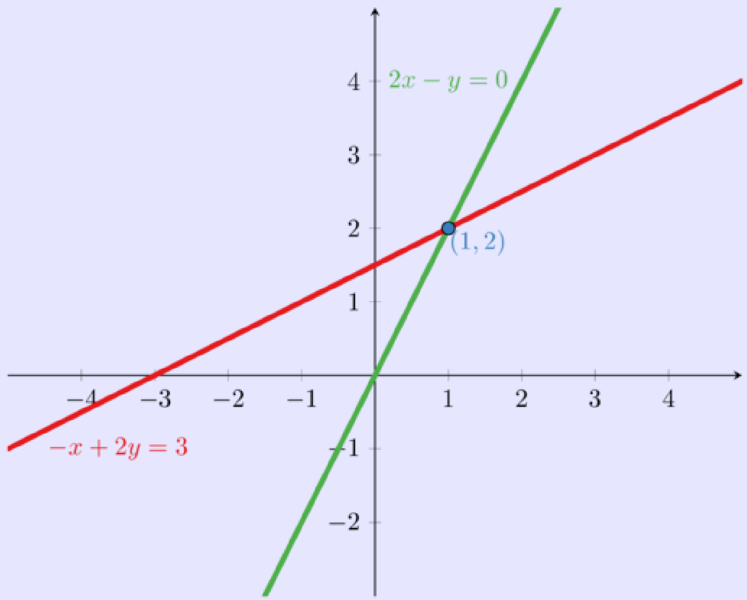
1.1.1 Matrix picture

To get the matrix picture, express the equations in this 2-by-2 linear system in the form $Ax = b$:

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{b}$$

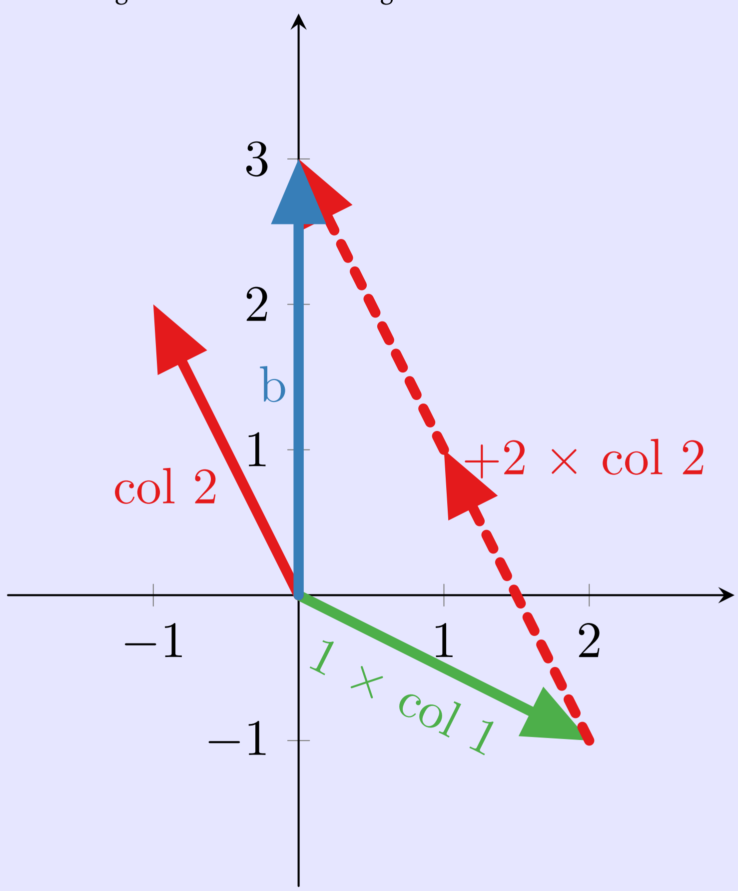
1.1.2 Row picture

To get the row picture, we plot all the points that satisfy each equation in this 2-by-2 linear system:



1.1.3 Column picture

To get the column picture, we plot the columns of the coefficient matrix for this 2-by-2 system and combine them in right amounts to get the vector on the right-hand side:



1.2 3-by-3 equations

For the second example, consider the following system with three equations in three unknowns:

$$\begin{cases} 2x - y \phantom{{}-z} = 0 \\ -x + 2y - z = -1 \\ - 3y + 4z = 4 \end{cases}$$

A technique used at MIT curriculum is to visualize the system using the matrix picture, row picture and column picture.

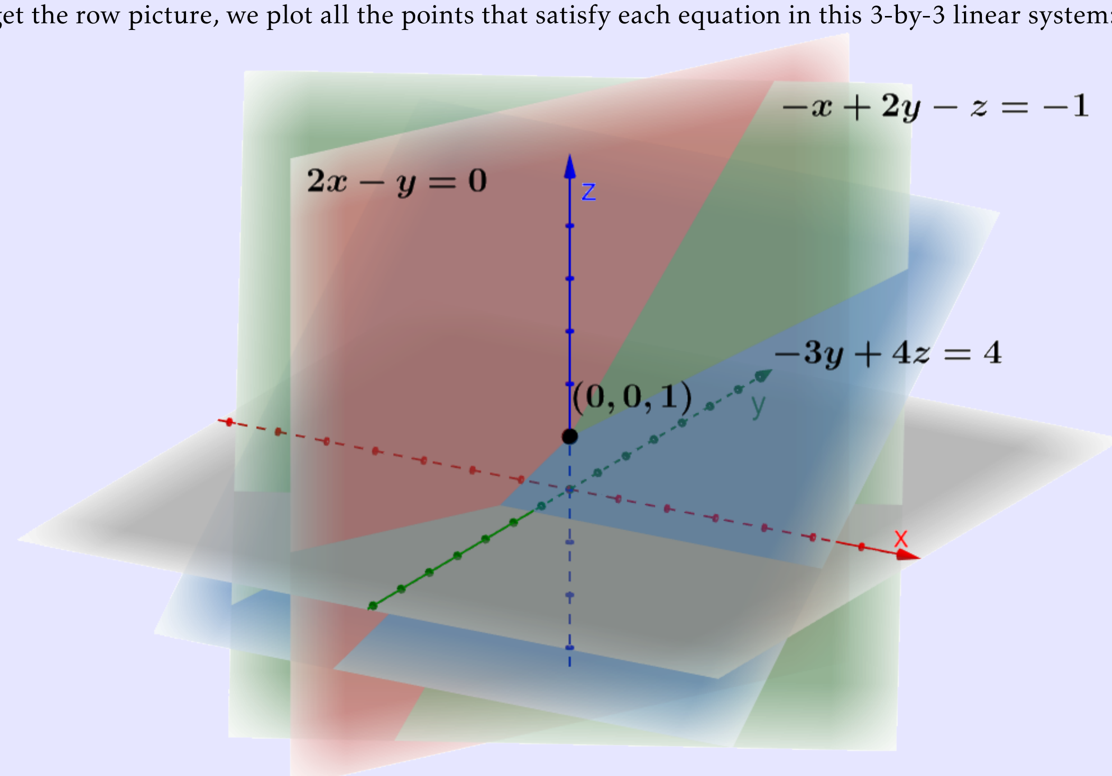
1.2.1 Matrix picture

To get the matrix picture, express the equations in this 3-by-3 linear system in the form $Ax = b$:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

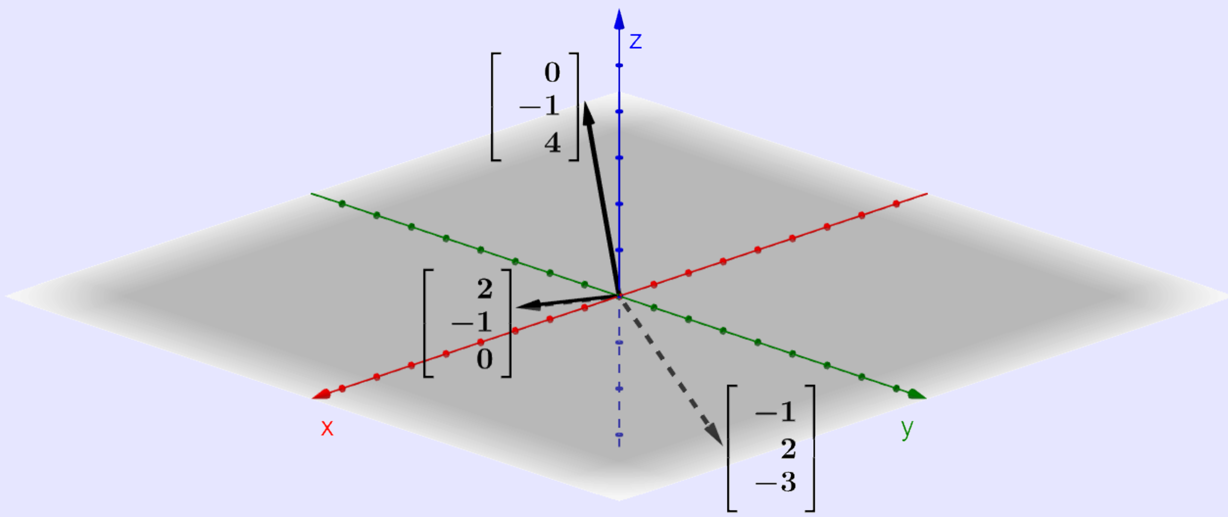
1.2.2 Row picture

To get the row picture, we plot all the points that satisfy each equation in this 3-by-3 linear system:



1.2.3 Column picture

To get the column picture, we plot the columns of the coefficient matrix for this 3-by-3 system and combine them in right amounts to get the vector on the right-hand side:



Notice that in this case, the third column of the coefficient matrix A is same as the right-hand side vector b.

1.2.4 Linear combination idea

The column picture shown for the 3-by-3 case is a graphical representation of the linear combination of left-hand side vectors (with $x = 0$, $y = 0$, and $z = 1$) in the following equation:

$$x \times \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \times \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \times \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

1.3 3-by-3 equations with same A but different b

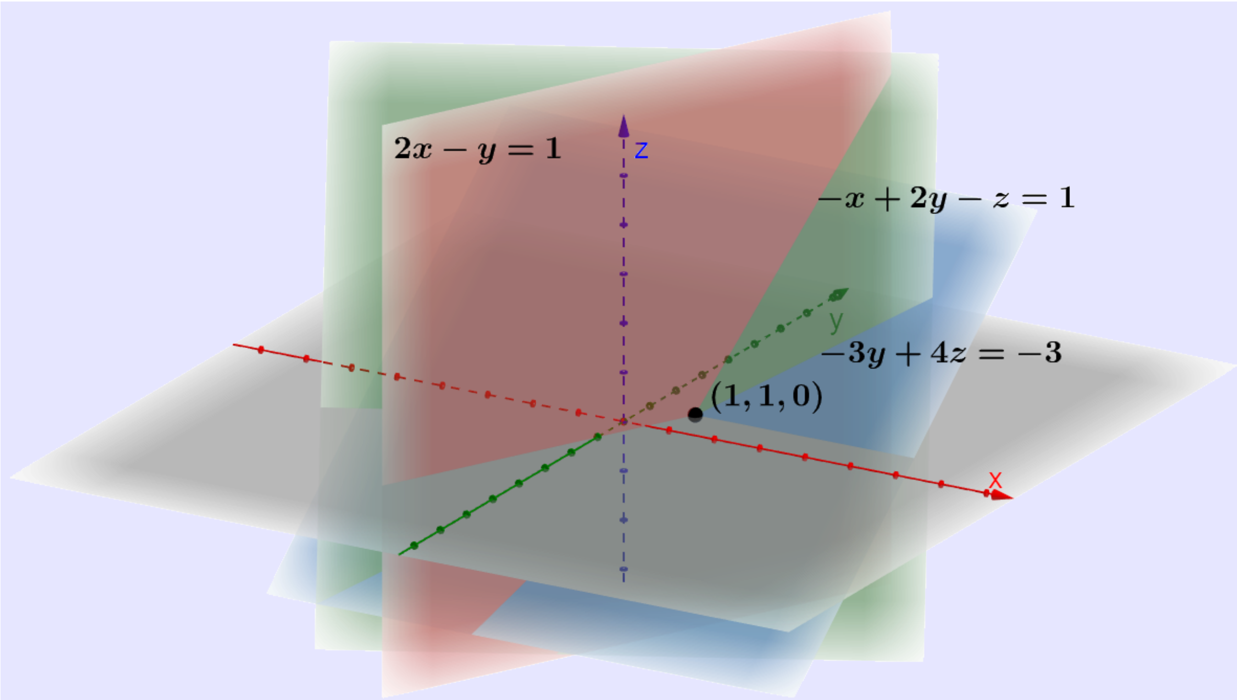
Let's replace the right-hand side vector b in the previous 3-by-3 example with the sum of the first and the second columns of the coefficient matrix A to get the following matrix picture:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

and recreate the row picture and the column picture.

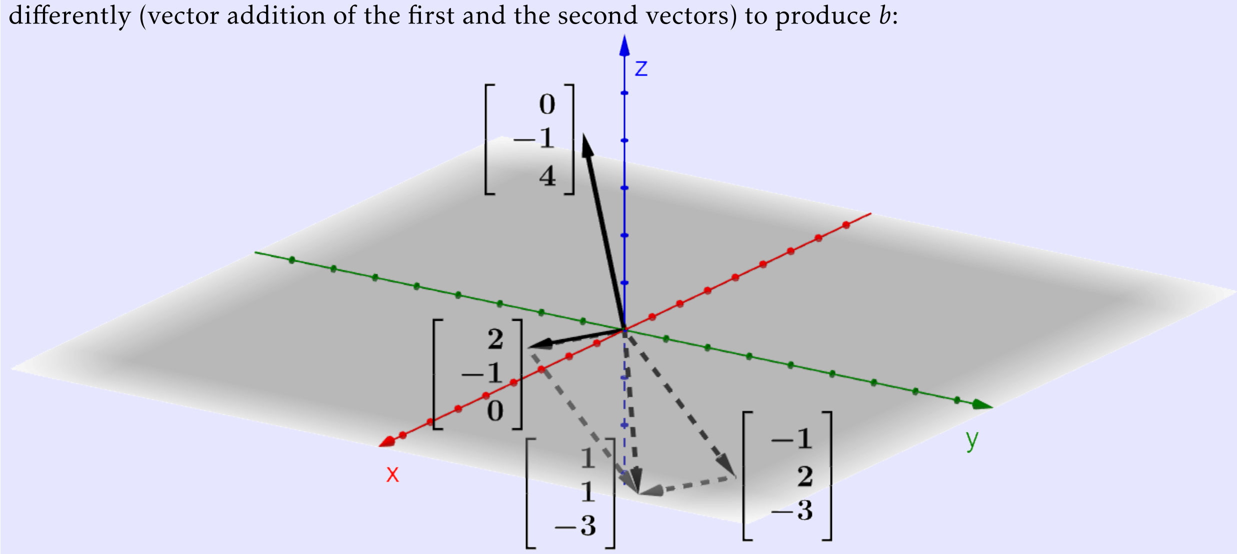
1.3.1 Row picture

The row picture for this new 3-by-3 linear system consists of three new planes that meet at the point $(1, 1, 0)$:



1.3.2 Column picture

The column picture for this new 3-by-3 system consists of the same three columns, but now they combine differently (vector addition of the first and the second vectors) to produce b:

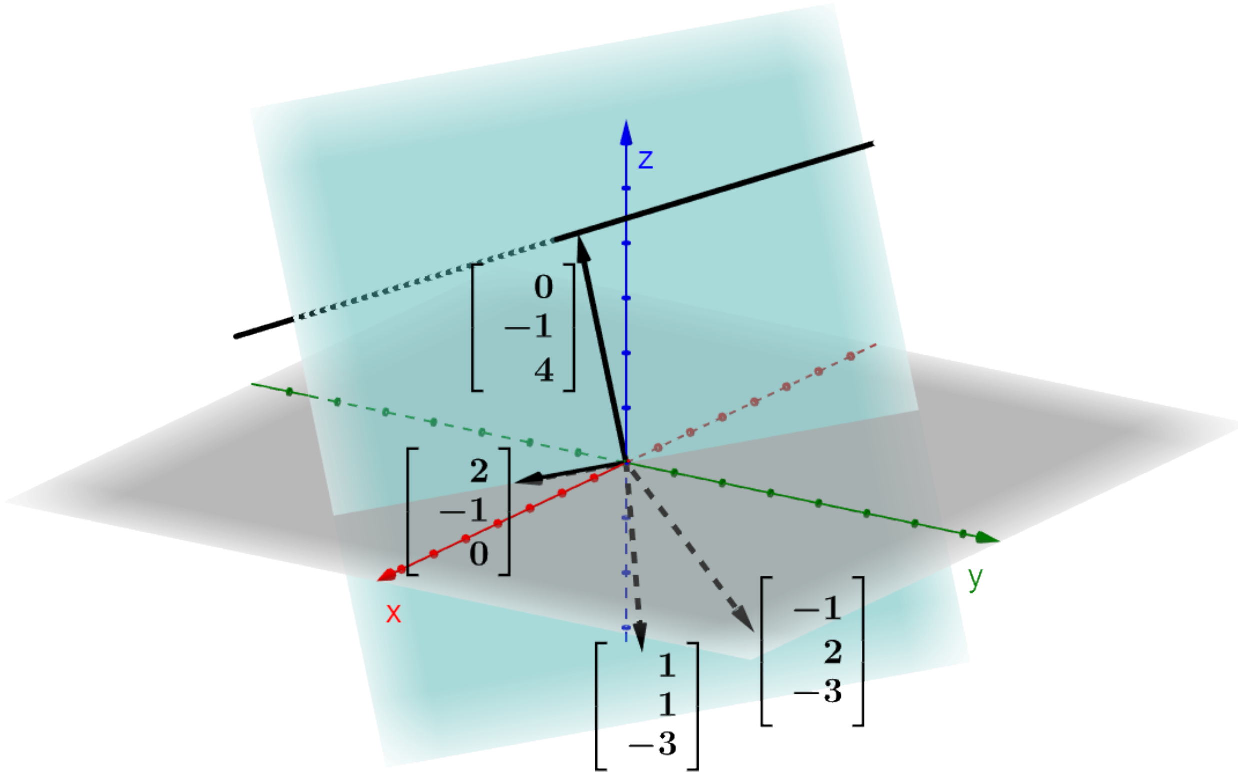


1.4 3-by-3 equations with singular A

Let's modify the original 3-by-3 linear system again, but this time change the third column of the coefficient matrix so that it is the sum of the first and the second columns:

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Now, create the column picture:



As can be seen in the picture, the three columns of the coefficient matrix lie on the same plane, and consequently all their linear combinations also lie in the same plane. So, this system can be solved only when b is in the plane and for all other cases there is no solution.

Note: The coefficient matrix A for a linear system is called singular or non-invertible if the linear system cannot be solved for every right-hand side b. If there is a solution for every b, then the matrix A is called non-singular or invertible.

Also, let's answer the two key fundamental questions for this 3-by-3 linear system with a singular A:

2 Elimination with Matrices

2.1 Overview

2.2 Gaussian elimination

Gaussian elimination is a systematic way of solving any system of linear equations. The rule of elimination simply involves selecting a pivot in each row and cleaning everything below it through row reduction.

Elimination generally succeeds by following the normal rule, but may sometimes encounter either of the two situations besides normal success:

- Normal success: Gaussian elimination succeeds normally without any row exchanges
- Temporary failure: Gaussian elimination succeeds with row exchanges
- Permanent failure: Gaussian elimination does not succeed

The following sections illustrate each of the three situations encountered by Gaussian elimination.

2.3 Normal success

Let's solve this linear system using Gaussian elimination and see it succeed normally:

$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ 4y + z = 2 \end{cases}$$

Step 1: Start with marking the first pivot at $(1,1)$ position in the coefficient matrix A and clean off $(2,1)$ position

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{array}{l} {}^{-3} \\ {}_{+} \\ \end{array} \xrightarrow{E_{21}} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 2: Since everything below the first pivot is now clean, mark the second pivot at $(2,2)$ and clean off $(3,2)$

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \begin{array}{l} \\ {}^{-2} \\ {}_{+} \end{array} \xrightarrow{E_{32}} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Final step: Since everything below the second pivot is now clean and there is nothing to clean below the third pivot at $(3,3)$, just mark it and call it the end

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$$

Fact 1: This matrix is called an *upper triangular matrix*, usually denoted with U

Fact 2: The product of the pivots of U gives its *determinant*, so here $det(U) = 10$

Note: E_{21} and E_{32} are *elimination matrices* that pre-multiply the matrices on the left of the arrow to produce the ones on the right. See the section on elimination matrices to learn more.

2.4 Temporary failure

Let's solve this linear system using Gaussian elimination and see it overcome temporary failure:

$$\begin{cases} x + 2y + z = 2 \\ 3x + 6y + z = 12 \\ 4y + z = 2 \end{cases}$$

Step 1: This step is same as the step in the *normal success example*

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{array}{l} {}^{-3} \\ {}_{+} \\ \end{array} \xrightarrow{E_{21}} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 2: Similar to the normal success example, entries below the first pivot are clean, but now there is an undesirable 0 at the second pivot position $(2,2)$. Let's swap the second and the third row to get a non-zero entry at $(2,2)$.

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{0} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{P_{23}} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{4} & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Final step: This step is also same as the step in the normal success example

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{4} & 1 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$$

Fact 1: $det(U) = 1 \times 4 \times -2 = -8$

Fact 2: Since the $det(U)$ is non-zero, the underlying matrix A is *invertible* or *non-singular*

2.5 Permanent failure

Let's solve this linear system using Gaussian elimination and see it fail to escape failure:

$$\begin{cases} x + 2y + \ \ z = 2 \\ 3x + 8y + \ \ z = 12 \\ \quad\ \ 4y - 4z = 2 \end{cases}$$

Step 1: This step is same as the step in the normal success example

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \begin{matrix} -3 \\ + \end{matrix} \xrightarrow{E_{21}} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{bmatrix}$$

Step 2: This step is same as the step in the normal success example

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & -4 \end{bmatrix} \begin{matrix} -2 \\ + \end{matrix} \xrightarrow{E_{32}} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Final step: Entries below the first and the second pivots are all clean, which is good, but there is a 0 at the pivot position with no row below to exchange it, so call it the end and declare elimination failure.

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

Fact 1: $det(U) = 1 \times 2 \times 0 = 0$

Fact 2: Since the $det(U)$ is zero, the underlying matrix A is *non-invertible* or *singular*

2.6 Back substitution

Back substitution is a technique used for obtaining the solution to a linear system by working bottom-up with the reduced form of the system, $Ux = c$, obtained through Gaussian elimination.

For the linear system in the normal success example, create an augmented matrix by tacking on the right-hand-side vector b onto the coefficient matrix A and repeat the Gaussian elimination steps on the augmented matrix:

Step 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \begin{matrix} -3 \\ + \end{matrix} \xrightarrow{E_{21}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

Step 2:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \begin{matrix} -2 \\ + \end{matrix} \xrightarrow{E_{32}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

The system of equations can now be rewritten as:

$$\begin{cases} x + 2y + \quad z = 2 \\ \quad\;\; 2y - 2z = 6 \\ \qquad\quad\;\; 5z = -10 \end{cases}$$

Use back-substitution to obtain $z = -2$, $y = 1$ and $x = 2$ as the solution to this linear system.

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. λ is an eigenvalue of A if and only if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an eigenvector.

Example

$$\begin{aligned} A &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A\mathbf{x} &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \end{bmatrix} \\ \lambda\mathbf{x} &= 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Thus, $\lambda = 4$ is an eigenvalue.

Example

Show that $\lambda = 5$ is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

$$\begin{aligned} A\mathbf{x} &= 5\mathbf{x} \\ A\mathbf{x} - 5\mathbf{x} &= \mathbf{0} \\ A\mathbf{x} - 5I\mathbf{x} &= \mathbf{0} \\ (A - 5I)\mathbf{x} &= \mathbf{0} \\ \left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\mathbf{x} &= 0 \\ \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

If we solve this using an augmented matrix and Gauss-Jordan elimination, we get the following:

$$\begin{aligned} \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} &\to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 - \frac{1}{2}x_2 &= 0 \\ x_2 &= s \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \end{aligned}$$

For $\lambda = 5$, $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$, but an eigenvector can be found for any s.

Example

To find eigenvalues, $A\mathbf{x} = \lambda\mathbf{x}$ needs to have nontrivial solutions.

$$A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{0}$$
$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

In order for this to be true, the determinant of $A - \lambda I$ must be 0. For example:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$
$$det(A - \lambda I) = 0$$
$$det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$
$$det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$
$$(3-\lambda)(3-\lambda) - 1 = 0$$
$$9 - 6\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda - 4)(\lambda - 2) = 0$$
$$\lambda_1 = 4 \quad \lambda_2 = 2$$

To find eigenvectors for $\lambda_1 = 4$:

$$A\mathbf{x} = 4\mathbf{x}$$
$$(A - 4I)\mathbf{x} = 0$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$x_1 = x_2 = s$$
$$\mathbf{x} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 4$ has an infinite number of eigenvectors along $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To find eigenvectors for $\lambda_2 = 2$:

$$A\mathbf{x} = 2\mathbf{x}$$
$$(A - 2I)\mathbf{x} = 0$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$x_1 = -x_2 = -s$$
$$\mathbf{x} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$ has an infinite number of eigenvectors along $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example

Find the eigenvalues and eigenvectors of the matrix.

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \\
\det(A - \lambda I) &= 0 \\
\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix}\right) &= 0 \\
-\lambda \begin{vmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 2 & 4-\lambda \end{vmatrix} + 0 &= 0 \\
-\lambda[(-\lambda)(4-\lambda) - (-5)] - [0-2] &= 0 \\
\lambda^2(4-\lambda) - 5\lambda + 2 &= 0 \\
-\lambda^3 + 4\lambda^2 - 5\lambda + 2 &= 0 \\
\lambda^3 - 4\lambda^2 + 5\lambda - 2 &= 0 \\
(\lambda-1)^2(\lambda-2) &= 0 \\
\lambda_1 &= 1 \quad \lambda_2 = 2
\end{aligned}
$$

Each eigenvalue will have a corresponding set of eigenvectors. For $\lambda = 1$:

$$
\begin{aligned}
(A - 1I)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
x_1 - x_3 &= 0 \\
x_2 - x_3 &= 0 \\
x_1 = x_3 &= x_2 = s \\
\mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

For $\lambda = 2$:

$$
\begin{aligned}
(A - 2I)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
x_1 - \frac{1}{4}x_3 &= 0 \\
x_2 - \frac{1}{2}x_3 &= 0 \\
x_3 &= s \\
\mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}s \\ \frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ s \end{bmatrix} s
\end{aligned}
$$

Example

Find the eigenvalues and eigenvectors of the matrix.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + 2\lambda^2 = 0$$

$$\lambda^2(\lambda + 2) = 0$$

$$\lambda_1 = \lambda_2 = 0$$

$$\lambda_3 = -2$$

For $\lambda = 0$:

$$(A - 0I)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This is a two dimensional eigenspace (the eigenvalue has a geometric multiplicity of 2) spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Since the eigenvalue $\lambda = 0$ also appears twice, the eigenvalue has a algebraic multiplcity of 2. For $\lambda = -2$:

$$(A + 2I)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$x_2 - 3x_3 = 0$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ 3s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

In summary, the algebraic multiplicity of an eigenvalue is the number of times it appears as a solution to the characteristic polynomial, and the geometric multiplicity of the eigenvalue is the dimensionality of its corresponding eigenspace.