Introduction to Algorithms

Subtext of lecture

Steps to developing a usable algorithm.

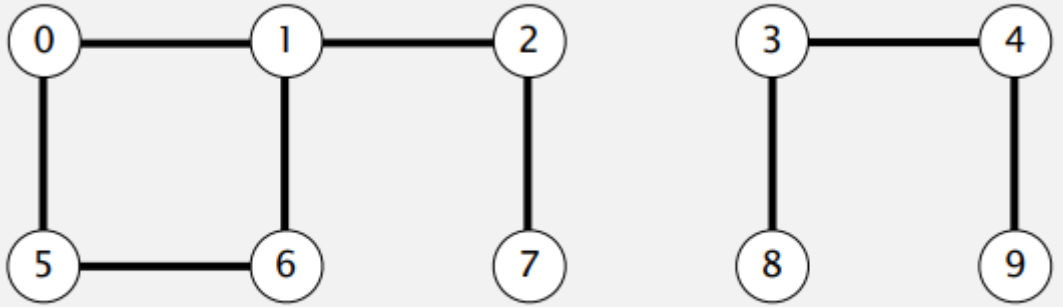
- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

Dynamic Connectivity

Given a set of N objects.

- Union command: connect two objects.
- Find/connected query: is there a path connecting the two objects?

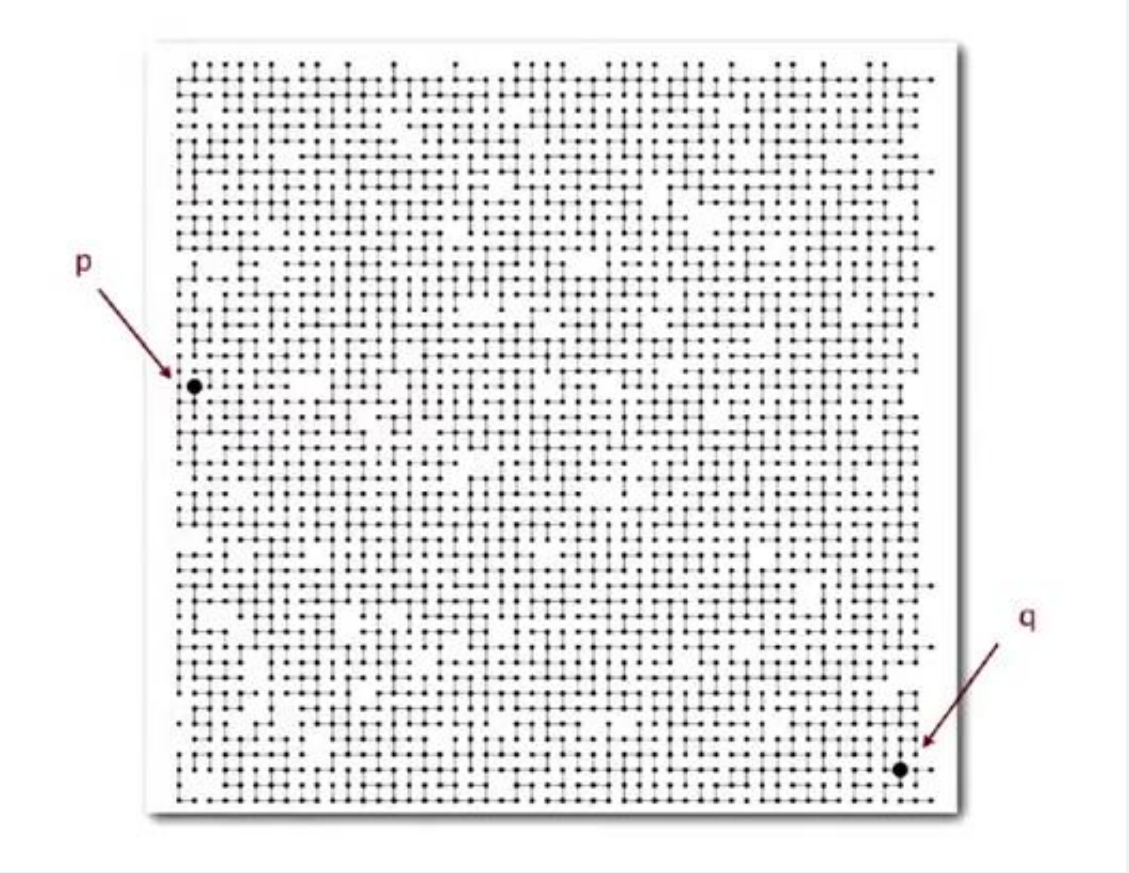
```
union(5, 0)
union(7, 2)
union(6, 1)
union(1, 0)
```

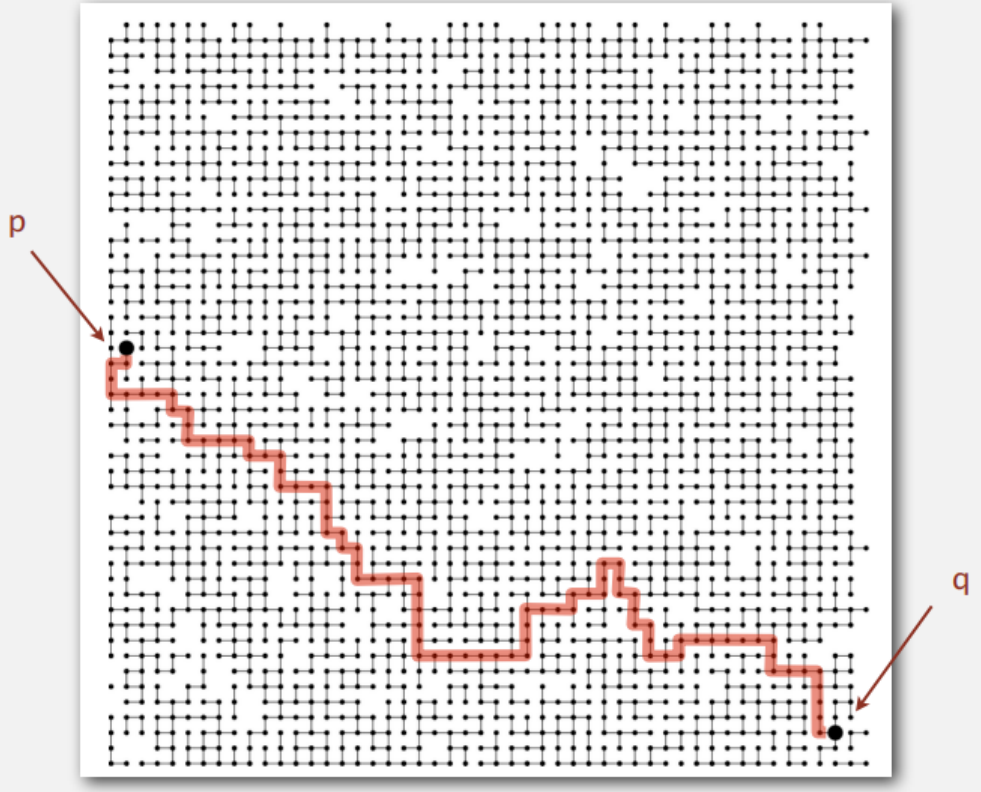



```
connected(0, 7)
connected(8, 9)
```

Dynamic Connectivity

Q. Is there a path connecting *p* and *q*?





Dynamic Connectivity

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in Fortran program.
- Metallic sites in a composite system.

Dynamic Connectivity

We assume "is connected to" is an equivalence relation:

- Reflexive: *p* is connected to *p*.
- Symmetric: if *p* is connected to *q*, then *q* is connected to *p*.
- Transitive: if *p* is connected to *q* and *q* is connected to *r*, then *p* is connected to *r*.

Dynamic Connectivity

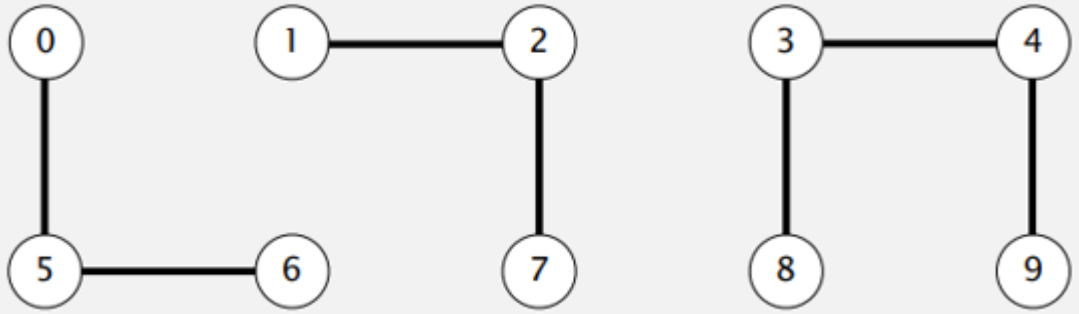
Goal. Design efficient data structure for union-find.

- Number of objects *N* can be huge.
- Number of operations *M* can be huge.
- Find queries and union commands may be intermixed.

Dynamic Connectivity

public class UF	
UF(int N)	*initialize union-find data structure with N objects (0 to N – 1)*
void union(int p, int q)	*add connection between p and q*
boolean connected(int p, int q)	*are p and q in the same component?*
int find(int p)	*component identifier for p (0 to N – 1)*
int count()	*number of components*

Your suggestions?



Union(0, 5)
Union(5, 6)
Union(1, 2)
Union(2, 7)
Union(8, 3)
Union(3, 4)
Union(4, 9)

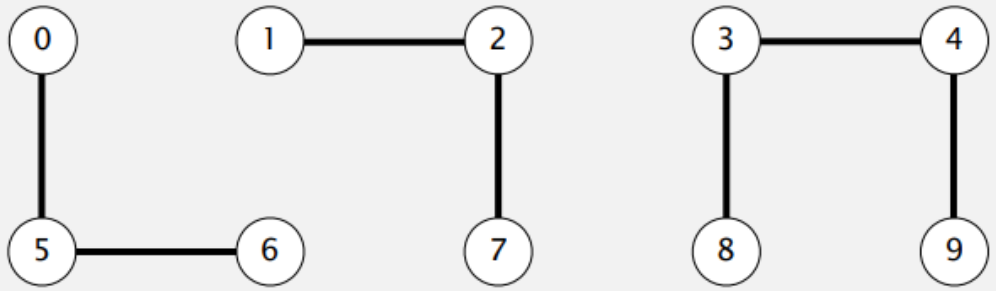
Quick Find

Data structure.

- Integer array *id[]* of length *N*.
- Interpretation: *p* and *q* are connected if and only if they have the same id.

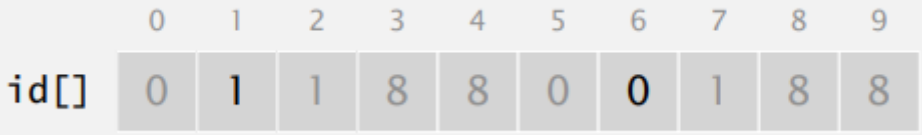
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected
1, 2, and 7 are connected
3, 4, 8, and 9 are connected



Quick Find

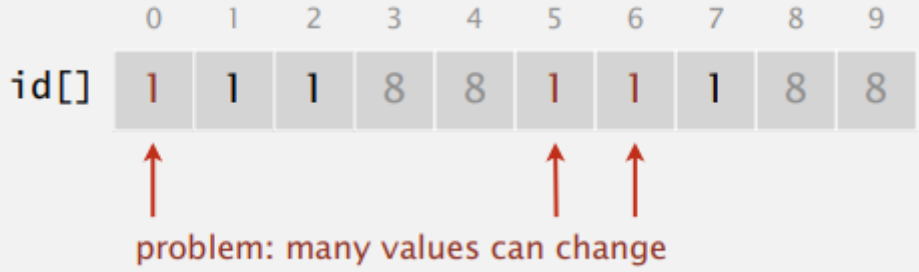
p and *q* are connected if and only if they have the same id.



Find. Check if *p* and *q* have the same id.

id[6] = 0; id[1] = 1
6 and 1 are not connected

Union. To merge components containing *p* and *q*, change all entries whose id equals *id[p]* to *id[q]*.



union(6, 1)

Quick Find Step-by-Step

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick Find Effectiveness

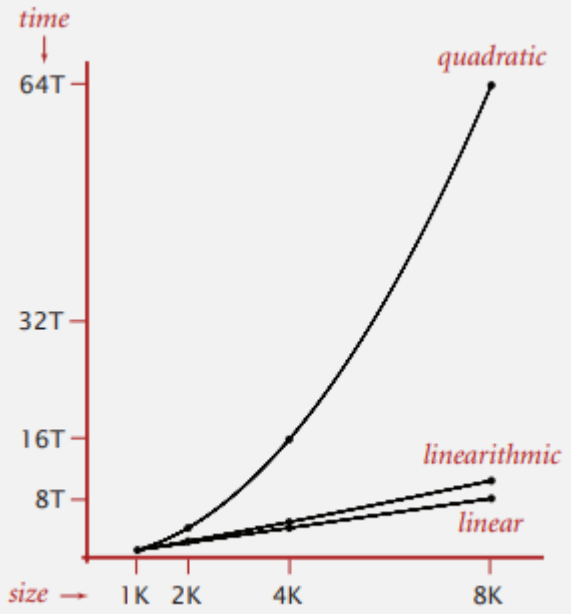
Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	N	N	1

order of growth of number of array accesses

Union is too expensive. It takes N^2 array accesses to process a sequence of N union commands on N objects.

What does N^2 mean?



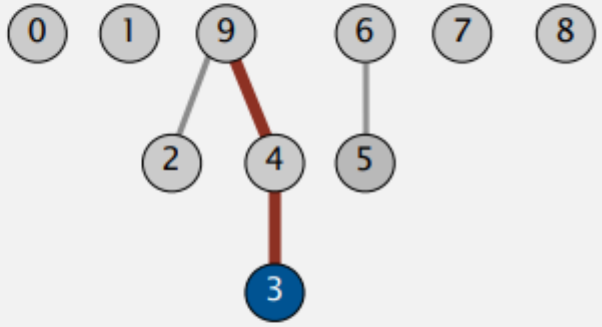
Quick Union

Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



root of 3 is 9

Quick Union

Data structure.

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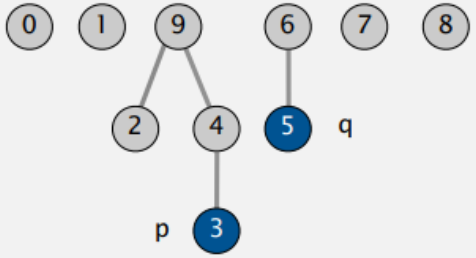
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



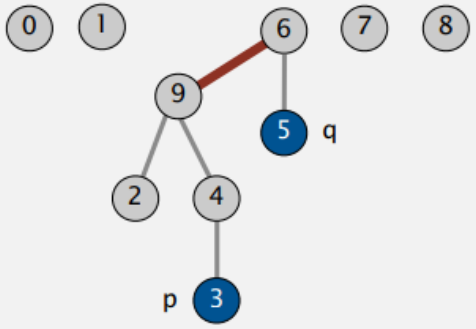
root of 3 is 9
root of 5 is 6
3 and 5 are not connected

Find. Check if `p` and `q` have the same root.

Union. To merge components containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes



Quick Union Step-by-Step

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick Union Effectiveness

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	N	N	1
quick-union	N	N †	N

← worst case

† includes cost of finding roots

Quick-find defect.

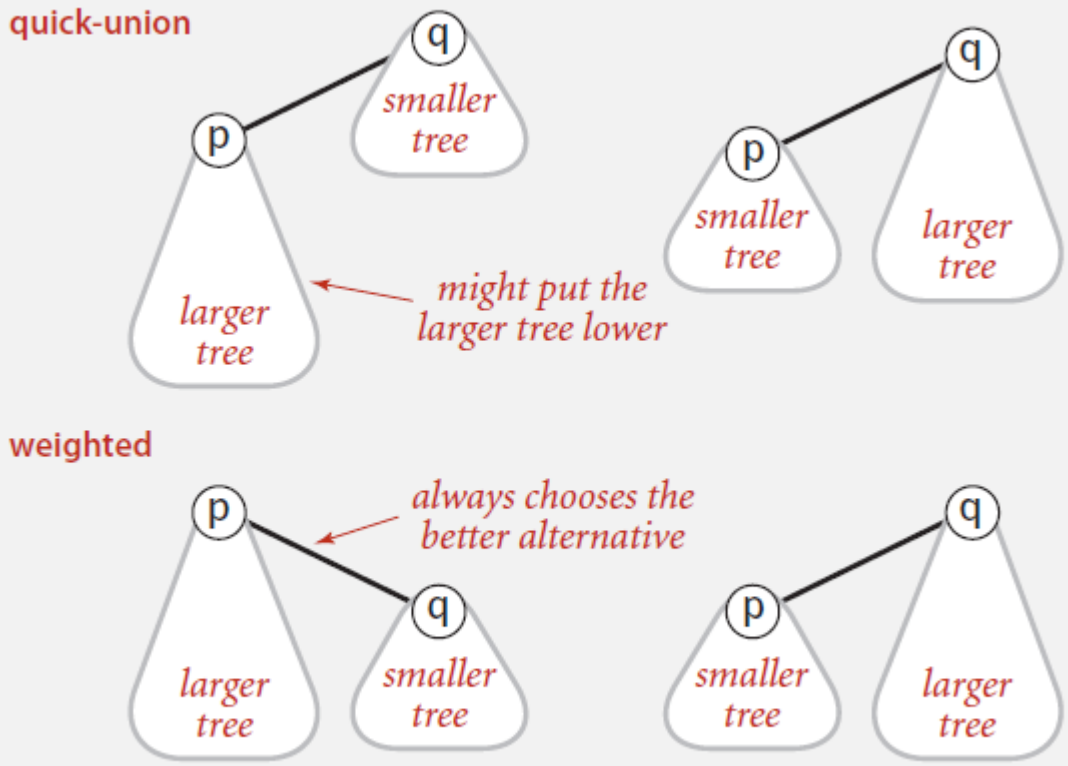
- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

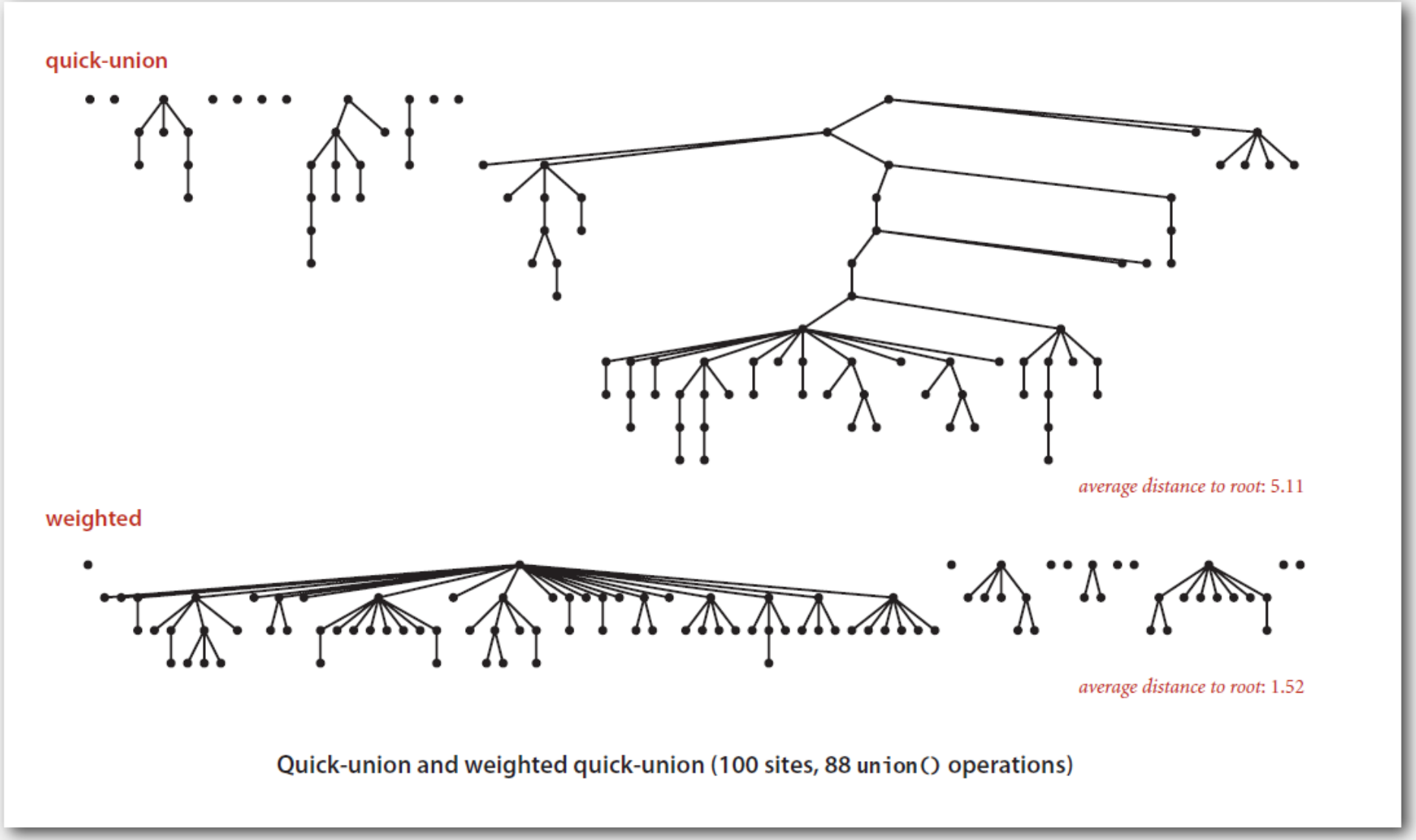
- Trees can get tall.
- Find too expensive (could be N array accesses).

Weighted quick-union

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.



Weighted quick-union



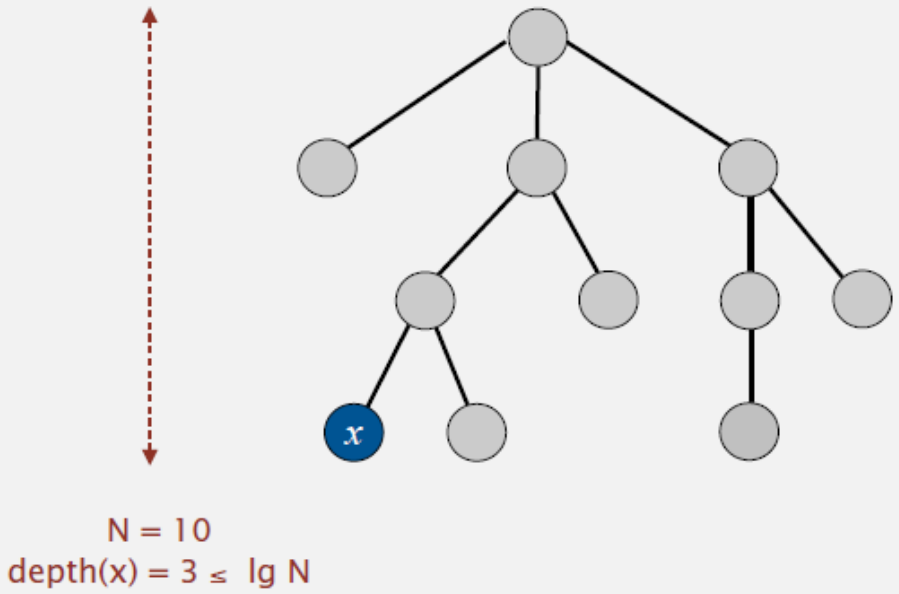
Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union

Running time.

- Find: takes time proportional to depth of *p* and *q*.
- Union: takes constant time, given roots.

Proposition. Depth of any node *x* is at most lg *N*.



Weighted quick-union

Running time.

- Find: takes time proportional to depth of *p* and *q*.
- Union: takes constant time, given roots.

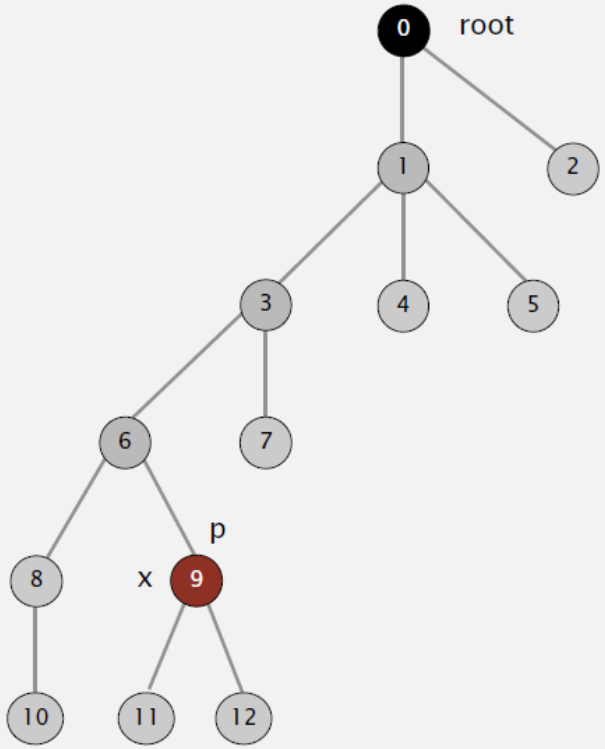
Proposition. Depth of any node *x* is at most lg *N*.

algorithm	initialize	union	connected
quick-find	N	N	1
quick-union	N	N †	N
weighted QU	N	lg N †	lg N

† includes cost of finding roots

Path compression

Quick union with path compression. Just after computing the root of *p*, set the id of each examined node to point to that root.



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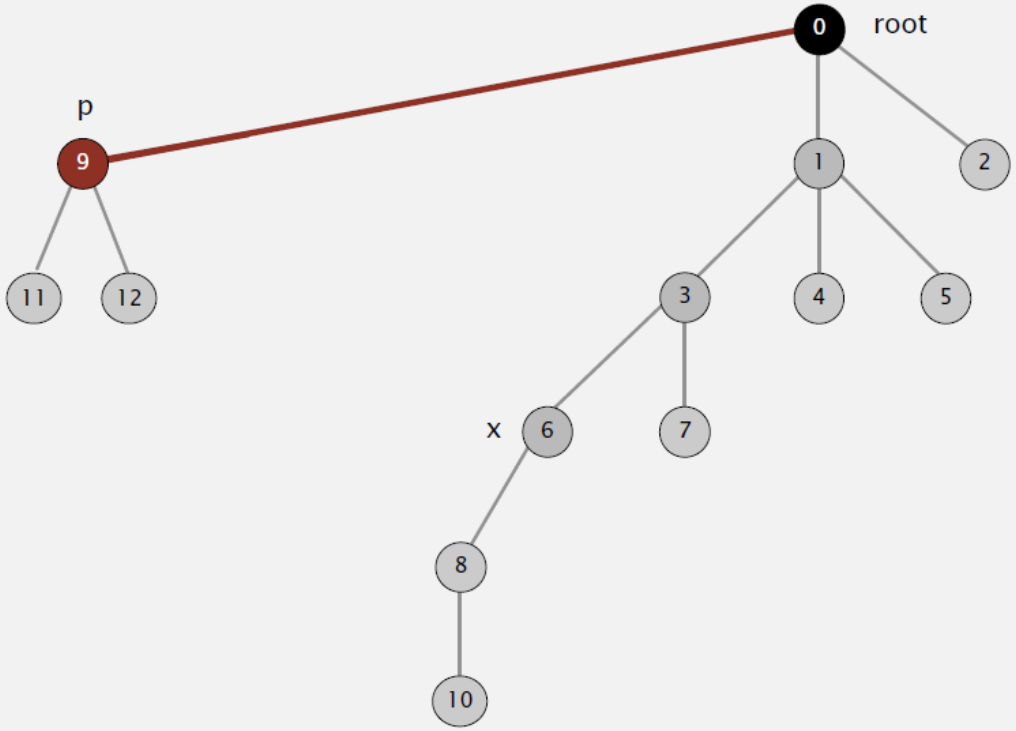

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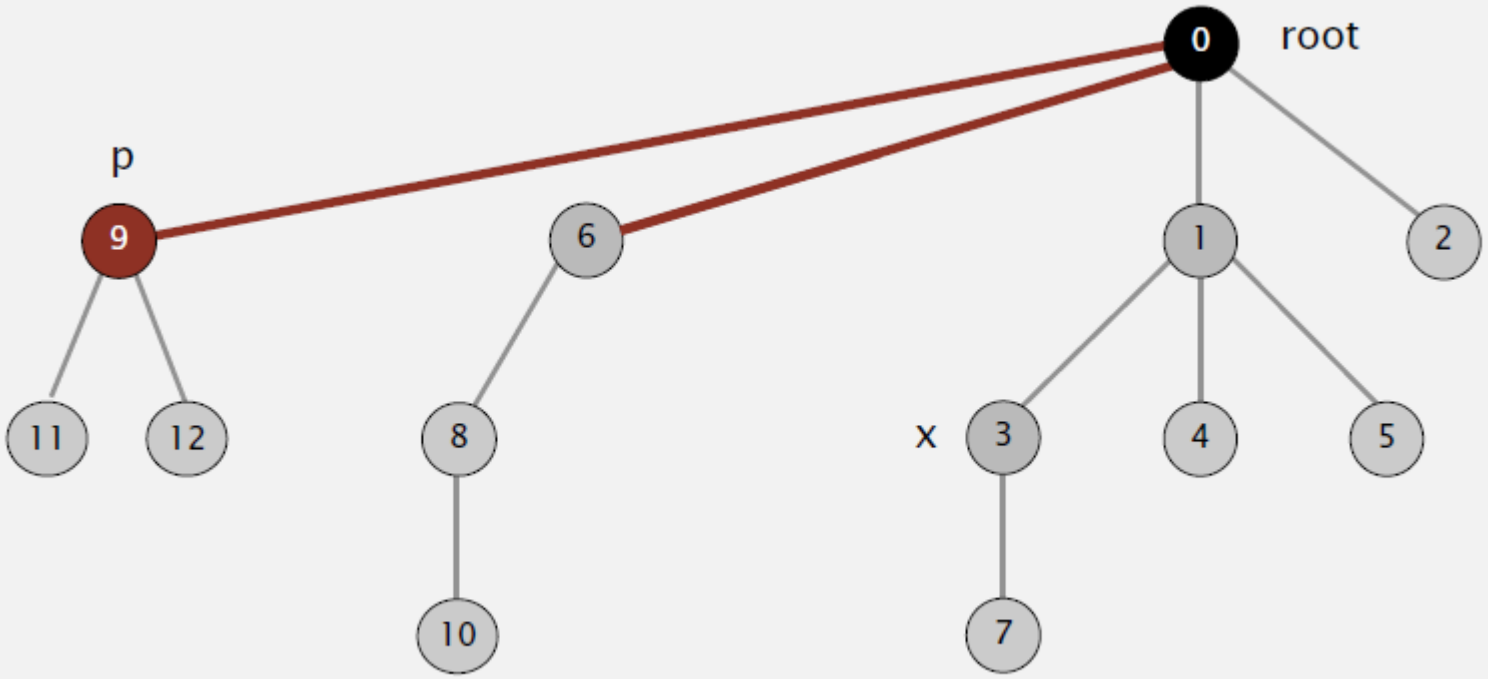

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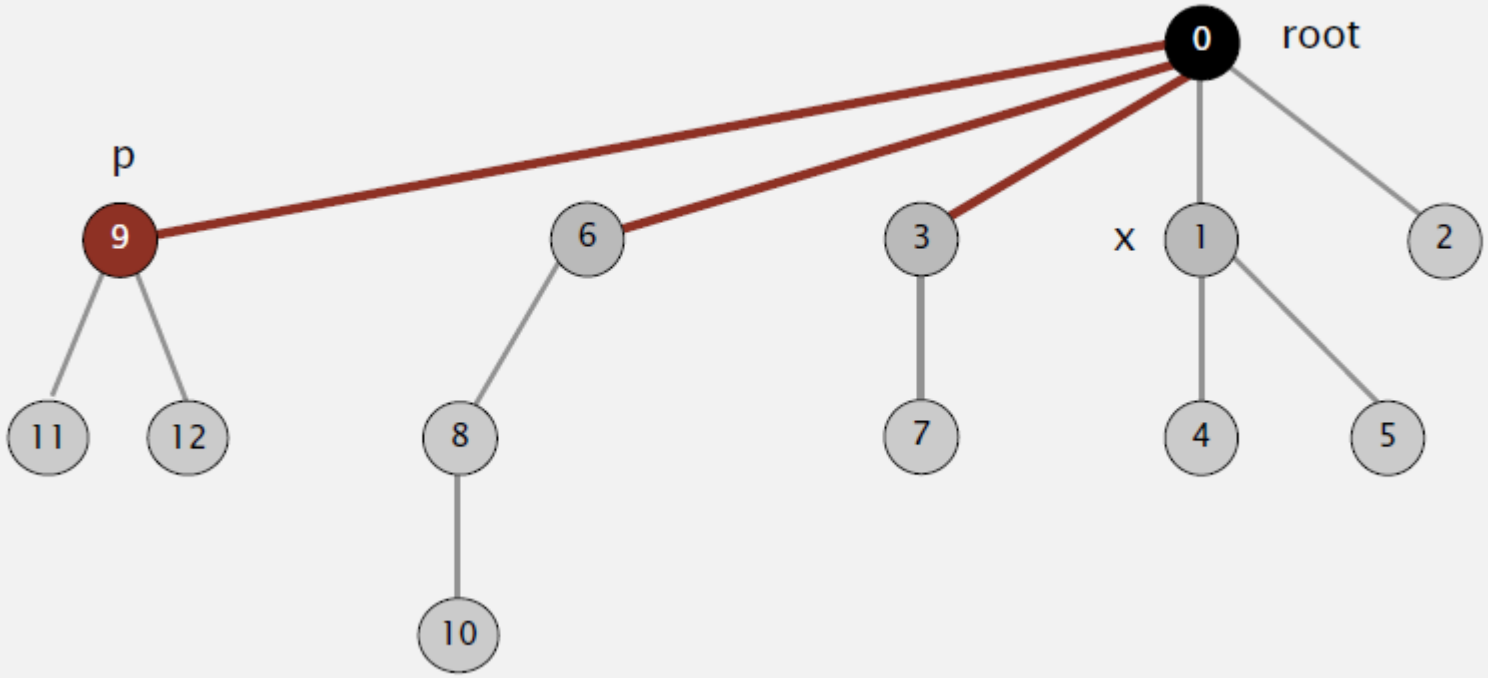

Summary

Bottom line. Weighted quick union (with path compression) makes it possible to solve problems that could not otherwise be addressed.

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

algorithm	worst-case time
quick-find	M N
quick-union	M N
weighted QU	N + M log N
QU + path compression	N + M log N
weighted QU + path compression	N + M lg* N

M union-find operations on a set of N objects