Complex Analytic Functions

Hellen Zhao, hz2843

This article assumes basic knowledge of the complex plane, a field containing all values $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. Additionally, the reader should have a background in multivariate calculus, specifically with series, integration along curves, the Fundamental Theorem of Calculus, and Green's Theorem. The goal is to provide an introduction to complex analysis, which studies functions on complex numbers. Similar to regular calculus, we will define differentiation and integration of complex-valued functions, while noting the differences with real functions. We will then prove Cauchy's Integral Theorem and Formula, and finally show that complex differentiable functions can be expressed as a power series.

1 Calculus in the Complex Plane

First, we will define the notion of limits in $\mathbb{C}$, which will use absolute value as its norm.

Definition 1.1 (Absolute Value)**.** We define the *absolute value* of a complex number $z \in \mathbb{C}$ as its distance from the origin $0 + 0i$. If $z = x + iy$ where $x, y \in \mathbb{R}$, then

$$|z| = \sqrt{x^2 + y^2}.$$

Note that absolute value in the complex plane satisfies the properties of a norm. We have that $|z| = 0 \iff z = 0 + 0i$, that $|zw| = |z|\,|w|$, and triangle inequality holds:

Lemma 1.1 (Triangle Inequality)**.** *For all $z, w \in \mathbb{C}$, we have*

$$|z + w| \leq |z| + |w|.$$

Proof. Note that for any $z = a + bi$, the absolute value of z is

$$|z| = \sqrt{a^2 + b^2} = \left|\left|\begin{pmatrix} a \\ b \end{pmatrix}\right|\right|_2.$$

So then if $z = a + bi$ and $w = c + di$, we have

$$|z + w| = \left|\left|\begin{pmatrix} a + c \\ b + d \end{pmatrix}\right|\right|_2.$$

We have already proved the triangle inequality for the Euclidean norm on $\mathbb{R}^n$, so we know

$$\left|\left|\begin{pmatrix} a + c \\ b + d \end{pmatrix}\right|\right|_2 \leq \left|\left|\begin{pmatrix} a \\ b \end{pmatrix}\right|\right|_2 + \left|\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|\right|_2.$$

It follows then that

$$|z + w| \leq |z| + |w|.$$

$\square$

This allows us to define limits the same way we do in single variable calculus:

Definition 1.2 (Limits)**.** For $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is open, we say $\lim_{z\to z_0} f(z) = c$ exists if for all real $\epsilon > 0$, there exists real $\delta > 0$ so that

$$|z - z_0| < \delta \implies |f(z) - c| < \epsilon.$$

Remark. We saw above that a complex number shares some properties with its 2D vector representation, such as with absolute value and the euclidean norm. This is not to say that $\mathbb{C}$, a field, behaves the same as $\mathbb{R}^2$, a 2-dimensional vector space. The conjugate plays a special role in this, as $z\bar{z} = |z|^2$. So unlike in $\mathbb{R}^2$, division by a complex number makes sense because we can multiply by its conjugate to "get rid" of i and essentially turn the denominator into a scalar value. This turns out to be very useful in understanding differentiation, as it allows us to use the single-variable definition of the derivative.

2 Complex Differentiation

We are now ready to define differentiation in the complex plane.

Definition 2.1 (Piecewise C^1)**.** A continuous piecewise-C^1 function is a function that can be written as a sequence of C^1 curves, with the starting point of the next curve being the endpoint of the previous curve.

Example. An example of a continuous piecewise-C^1 function is $f : \mathbb{R} \to \mathbb{C}$, where

$$f(x) = \begin{cases} x^2 & x < 0 \\ xi & x \geq 0. \end{cases}$$

The function f is continuous, and both its pieces are C^1, but f is not C^1.

Definition 2.2 (Path-connected)**.** A subset $U \subset \mathbb{C}$ is *path-connected* if for any $x, y \in U$, there exists some continuous piecewise C^1 map $\gamma : [0,1] \to U$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example. The subset $U = \mathbb{C} \setminus \{0\}$ is an example of a path-connected set. For any $x, y \in U$, if the straight line from x to y does not go through the origin, then we are good. Otherwise follow a semicircle of radius $|x|$ around the origin, then follow a straight path to y.

On the other hand, the set $U = \{a + bi \mid a, b \in \mathbb{R}, a < 2 \text{ or } a > 4\}$ would not be path-connected because there is no continuous curve from say $1 + 2i$ to $5 + 3i$ that stays within U.

Remark. In the end, we'll want to integrate along these curves, so we'll want some amount of differentiability.

Definition 2.3 (Domain)**.** A non-empty open path-connected subset $U \subset \mathbb{C}$ is called a *domain.*

Definition 2.4 (Complex Differentiability)**.** A function $f : U \to \mathbb{C}$ that maps from a domain $U \subset \mathbb{C}$ to the complex plane $\mathbb{C}$ is differentiable at $w \in U$ if

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w} = f'(w)$$

exists.

Theorem 2.1 (Giving the different rules)**.**

Proof. Note that our definition of differentiability looks quite similar to that for differentiation in the reals, except our domain is different. But the basic properties of our chosen norm (complex absolute value) are the same as the norms used in real-valued differentiation. Since the proofs for the usual rules for differentiation (sum, product, quotient, chain rule) only use norm properties, the same proofs apply in the complex plane. So we have a method for computation of derivatives just like we do in the reals. $\square$

Still, we would sometimes like a stronger condition for differentiability not just at a single point w, but for the points around w, or even the entire domain U. As a consequence, this would also let us represent our function locally as a convergent series.

Definition 2.5 (Holomorphic)**.** A function $f : U \to \mathbb{C}$ is *holomorphic* at $w \in U$ if it is complex differentiable at every point in U

Definition 2.6 (Entire)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is defined on all of $\mathbb{C}$ and is holomorphic on $\mathbb{C}$.

Remark. Many authors actually interchange the words 'analytic' and 'holomorphic'. On the other hand, some differentiate between the two words, using analytic to mean a function that is given by a convergent power series. Later we shall prove holomorphic functions are in fact analytic, so until then we shall treat the two terms differently.

We now have some understanding of derivatives in the complex plane, but it would be nice if we could relate this to differentiability in the reals, which we are more familiar with.

Proposition 2.1 (Matrix Representation)**.** The matrix corresponding to multiplication by $x + yi$ is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Proof. We can treat the real and imaginary parts of a complex number separately, so $z = x + yi$ is corresponds to the 2-dimensional vector (x, y), which goes x units in the real line and y units in the imaginary line.

Note that $(x + iy)(1) = x + yi$ and $(x + iy)(i) = -y + xi$. Also

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

So the matrix corresponding to multiplication by $x + yi$ is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

$\square$

Now if $f(x + iy) = a + bi$, then define the functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ so that

$$u \begin{pmatrix} x \\ y \end{pmatrix} = a, \quad v \begin{pmatrix} x \\ y \end{pmatrix} = b.$$

This way we have $f = u + iv$.

Proposition 2.2 (Jacobian Matrix)**.** We can represent $f'(w)$ as a 2×2 matrix of the partial derivatives of u and v, specifically

$$df_w = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}.$$

Proof. When

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = f'(w)$$

exists, we have a good approximation $f(w+h) = f(w) + f'(w)h$ so that the relevant linear map is $df_w(h) = f'(w)h$ given by multiplication by a certain complex number $f'(w)$.

Now, since f can also correspond to a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$, multiplication by $f'(w)$ corresponds to the Jacobian matrix

$$df_w = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Multiplying $f'(z)$ in $\mathbb{C}$ by 1 and i corresponds to multiplying the Jacobian matrix df_z by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and so we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_y \\ v_y \end{pmatrix}$$

Then $\begin{pmatrix} u_x \\ v_x \end{pmatrix}$ corresponds to $f'(z) = u_x + v_x i$ and $\begin{pmatrix} u_y \\ v_y \end{pmatrix}$ corresponds to $f'(z)i = u_y + v_y i$. Using the fact that $i(-i) = 1$, this gives us

$$f'(z) = u_x + v_x i = (u_y + v_y i)(-i) = v_y - u_y i.$$

So

$$u_x = v_y \text{ and } u_y = -v_x$$

and

$$df_w = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

$\square$

These are also known as the Cauchy-Riemann Equations:

Proposition 2.3 (Cauchy-Riemann Equations)**.** The complex-valued function f is complex differentiable at $w = c + id$ if and only if u and v are differentiable at (c,d), and

$$u_x = v_y,$$

$$u_y = -v_x.$$

Additionally, we have

$$f'(w) = u_x(c,d) + iv_x(c,d) = v_y(c,d) - iu_y(c,d).$$

Example ($f(z) = z^2$ is holomorphic)**.**
Consider any point $z = x + iy \in \mathbb{C}$. We have

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2xyi,$$

so

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = 2xy.$$

Then the Jacobian matrix is

$$\begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$

Since

$$u_x = 2x = v_y \text{ and } u_y = -2y = -v_x$$

indeed satisfy the Cauchy-Riemann equations, we conclude f is differentiable everywhere, and thus holomorphic. The derivative at $z = x + yi$ is

$$f'(x + yi) = 2x + 2yi.$$

Remark. This also follows from the definition of the derivative, as

$$\lim_{z \to w} \frac{z^2 - w^2}{z - w} = \lim_{z \to w} z + w = 2w$$

so $f'(w) = 2w$.

Example ($f(z) = \bar{z}$ is not differentiable anywhere)**.**
We have for $z = x + yi$ that $f(z) = \bar{z} = x - yi$, so

$$u(x, y) = x \text{ and } v(x, y) = -y.$$

Note that the partial derivatives don't satisfy the Cauchy-Riemann equations, as

$$u_x = 1 \neq -1 = v_y,$$

and so f is not differentiable anywhere.

Example ($f(z) = |z|^2$ is complex differentiable at $z = 0$, but not holomorphic)**.**
Consider any $z = x + yi$. Then

$$f(z) = |z|^2 = x^2 + y^2$$

so

$$u(x, y) = x^2 + y^2 \text{ and } v(x, y) = 0,$$

and so our Jacobian matrix is

$$\begin{pmatrix} 2x & 2y \\ 0 & 0 \end{pmatrix}.$$

In order to satisfy the Cauchy-Riemann equations, we must have $2x = 0$ and $2y = 0$, so f is only differentiable at $(x, y) = (0, 0)$ or $z = 0 + 0i$. Here, f is differentiable at the origin, but nowhere besides that point, and so it is not holomorphic.

3 Complex Integration

Just like for the reals, we should have a way to integrate in the complex plane. But what would than even mean? Unlike in single-variable calculus, there are multiple ways we could go from one point to another.

Let's first start with functions with a real domain, where we will integrate the real and imaginary parts individually.

Definition 3.1. Suppose $f : [a, b] \to \mathbb{C}$ is continuous. Write $f = u + vi$ where u outputs the real part of f and v outputs the imaginary part. Then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Similar to the reals, we also have a useful lemma for estimation.

Lemma 3.1.

$$\Big|\int_a^b f(t)dt\Big| \leq \int_a^b |f(t)|dt.$$

Proof. Note that since square root is increasing, it would suffice to prove that

$$\left|\int_a^b f(t)dt\right|^2 \leq \left(\int_a^b |f(t)|\,dt\right)^2$$

We have by definition

$$\left|\int_a^b f(t)dt\right|^2 = \left(\int_a^b u(t)dt\right)^2 + \left(\int_a^b v(t)dt\right)^2 = \int_a^b\int_a^b (u(t)u(s) + v(t)v(s))dtds.$$

On the other hand,

$$\left(\int_a^b |f(t)|dt\right)^2 = \left(\int_a^b \sqrt{u(t)^2 + v(t)^2}dt\right)^2 = \int_a^b\int_a^b \sqrt{u(t)^2 + v(t)^2}\sqrt{u(s)^2 + v(s)^2}dtds.$$

Note that for all t,

$$\sqrt{(u(t)u(s) + v(t)v(s))^2} = \sqrt{u(t)^2u(s)^2 + 2u(t)u(s)v(t)v(s) + v(t)^2v(s)^2}$$

is less than

$$\sqrt{u(t)^2 + v(t)^2}\sqrt{u(s)^2 + v(s)^2} = \sqrt{u(t)^2u(s)^2 + v(t)^2u(s)^2 + u(t)^2v(s)^2 + v(t)^2v(s)^2}.$$

This is because

$$(u(s)v(t) - u(t)v(s))^2 \geq 0$$

rearranges into

$$2u(t)u(s)v(t)v(s) \leq u(s)^2v(t)^2 + u(t)^2v(s)^2.$$

So we must have

$$\left|\int_a^b f(t)dt\right|^2 \leq \left(\int_a^b |f(t)|\,dt\right)^2.$$

Thus

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|\, dt$$

□

Now that we've considered the particular case where the domain is only real-valued, we are almost ready to integrate functions with complex domains. But first we need to address how we will perform our integration, since we don't have some one-dimensional domain to integrate along. We will define some curves called *paths* in the complex plane, on which we will be able to appropriately define an integral.

Definition 3.2 (Path)**.** A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$ where $[a, b] \subset \mathbb{R}$.

Definition 3.3 (Simple path)**.** A path $\gamma : [a, b] \to \mathbb{C}$ is *simple* if $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. So a simple path is nearly injective, except that it can intersect itself at its end points.

Definition 3.4 (Closed path)**.** A path $\gamma : [a, b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$.

Definition 3.5 (Contour)**.** A *contour* is a simple closed path that is piecewise continuously differentiable.

Example. The curve traversing the unit circle in the counterclockwise direction can be parameterized by $z(t) = e^{it}$ with $t \in [0, 2\pi]$. This is a contour since it only intersects itself at its endpoints.

Definition 3.6 (Orientation)**.** Our paths γ have a notion of order, where for $c, d \in [a, b]$, we have $c < d$ implies that $\gamma(c)$ comes before $\gamma(d)$. We can define $-\gamma$, then, as the continuous function from $[a, b] \to \mathbb{C}$ so that $-\gamma(c) = \gamma(a + b - c)$, that is $-\gamma$ travels in the opposite orientation of γ.

Definition 3.7 (Complex integration)**.** If the path $\gamma : [a, b] \to U \subset \mathbb{C}$ is C^1 and $f : U \to \mathbb{C}$ is continuous, then we can define integration of f along the path γ as

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Remark. An alternative notation for complex integration uses $dz = dx + idy$ so that

$$\int_\gamma f(z)dz = \int_\gamma (u + iv)(dx + idy).$$

If we write $f(z) = u(z) + iv(z)$ and $\gamma(t) = x(t) + iy(t)$ as 2-dimensional vectors with real and imaginary parts, this coincides with Definition 3.7:

$$\int_\gamma f(z)dz = \int_a^b [u(x(t), y(t)) + iv(x(t), y(t)](x'(t) + iy'(t))dt.$$

But $u(x(t), y(t)) + iv(x(t), y(t)) = f(\gamma(t))$ and $x'(t) + iy'(t) = \gamma'(t)$, so our equation is also

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Furthermore, our integral does not depend on how we parameterize our path, that is we can re-parameterize without changing the value of the integral:

Proposition 3.1. Let $\phi : [a', b'] \to [a, b]$ be C^1 so that $\phi(a') = a,\ \phi(b') = b$. If γ is a C^1 path and $\delta = \gamma \circ \phi$, then

$$\int_\gamma f(z)dz = \int_\delta f(z)dz.$$

Proof. The proof follows from the change of variables formula:

We have

$$\int_\delta f(z)dz = \int_{a'}^{b'} f(\delta(t))\delta'(t)dt$$

by definition, which substituting δ for $\gamma \circ \phi$ gives

$$\int_{a'}^{b'} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt.$$

If we let $u = \phi(t)$, this gives us

$$\int_a^b f(\gamma(u))\gamma'(u)du = \int_\gamma f(z)dz.$$

$\square$

Additionally, our definition for integration should match up with how we conventionally understand integration. The following properties that follow from our definition:

Proposition 3.2. Suppose $\gamma : [a, b] \to \mathbb{C}$ is C^1 and $a < u < b$. Write $\gamma|[a, u]$ for $\gamma : [a, u] \to \mathbb{C}$ and $\gamma|[u, b]$ for $\gamma : [u, b] \to \mathbb{C}$. Then

$$\int_\gamma f(z)dz = \int_{\gamma|[a,u]} f(z)dz + \int_{\gamma|[u,b]} f(z)dz.$$

So we can split up the integral by splitting up our path into pieces.

Proof. We can split our integral into its real and imaginary parts, then split those real-valued integrals into integrals on $[a, u]$ and $[u, b]$. Recombining real and imaginary parts on each subset of our interval gives us the desired result. $\square$

Proposition 3.3. Suppose $\gamma : [a, b] \to \mathbb{C}$ is C^1. If $-\gamma$ is γ with reversed orientation, then

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$$

Proof. Since $-\gamma$ gives the path from $\gamma(b)$ to $\gamma(a)$, we have

$$\int_{-\gamma} f(z)dz = \int_a^b f(\gamma(a + b - t))(-\gamma'(a + b - t))dt.$$

Substituting $a + b - t$ for u gives us

$$\int_b^a f(\gamma(u))\gamma'(u)du = -\int_\gamma f(z)dz.$$

$\square$

We have now defined $-\gamma$ as the reversed path of γ. It will be helpful also to have a way of expressing a connecting of two paths:

Definition 3.8 (Path Concatenation)**.** Say we have $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ where $\gamma_1(b) = \gamma_2(b)$. Then define $\gamma_1 * \gamma_2 : [a,c] \to \mathbb{C}$ so that

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(t) & t \in [a,b] \\ \gamma_2(t) & t \in [b,c]. \end{cases}$$

Then by Proposition 3.2, we have

$$\int_{\gamma_1 * \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

Remark. Note that by Proposition 3.1, we can re-parameterize our paths without changing our integral. The domains for γ_1, γ_2 were chosen to line up for convenience, but they could actually be anywhere so long as the endpoint of γ_1 and the startpoint of γ_2 are the same.

Proposition 3.4. Let $\text{length}(\gamma) = \int_a^b |\gamma'(t)|dt$. Then

$$\Big| \int_\gamma f(z)dz \Big| \leq \text{length}(\gamma) \sup_t |f(\gamma(t))|.$$

Proof. By definition, we have

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Taking the absolute values, we have the inequality

$$\Big| \int_a^b f(\gamma(t))\gamma'(t)dt \Big| \leq \int_a^b |f(\gamma(t))| * |\gamma'(t)|dt$$

by Lemma 3.1.

For all $t \in [a,b]$, we have $|f(\gamma(t))| \leq \sup_t |f(\gamma(t))|$, so this inequality simplifies to

$$\Big| \int_\gamma f(z)dz \Big| \leq \sup_t |f(\gamma(t))| \int_a^b |\gamma'(t)|dt = \text{length}(\gamma'(t)) \sup_t |f(\gamma(t))|,$$

and so we are done. ☐

Example (integrating $f(z) = z^2$ along the unit circle traversed counterclockwise)**.**

We can parameterize the unit circle by $\gamma : [0, 2\pi] \to \mathbb{C}$ where $\gamma(\theta) = e^{i\theta}$. Then because $\gamma'(\theta) = ie^{i\theta}$, our integral is

$$\int_\gamma f(z)dz = \int_0^{2\pi} e^{2i\theta}(ie^{i\theta})d\theta,$$

which simplifies to

$$\int_0^{2\pi} ie^{3i\theta} d\theta = \frac{ie^{3i\theta}}{3}\Big|_0^{2\pi} = 0.$$

Example (integrating $f(z) = \bar{z}$ along the straight line from 0 to $1 + i$).

We can parameterize this by $\gamma : [0, 1] \to \mathbb{C}$ where $\gamma(t) = t(1 + i)$. Then our integral is

$$\int_\gamma f(z)dz = \int_0^1 \overline{t(1+i)}(1+i)dt.$$

Simplifying the conjugate $\overline{t(1+i)} = t(1-i)$, we have

$$\int_0^1 t(1-i)(1+i)dt = \int_0^1 2tdt = t^2\Big|_0^1 = 1.$$

We now have a basic understanding of both differentiation and integration of complex-valued functions. Just like in real-valued calculus, we can relate the two ideas with the Fundamental Theorem of Calculus.

4 Fundamental Theorem of Calculus

Theorem 4.1 (Fundamental Theorem of Calculus). *Let $U \subset \mathbb{C}$ and $f : U \to \mathbb{C}$ be continuous. Assume that the antiderivative of f exists, that is there is a holomorphic function $F : U \to \mathbb{C}$ where $F'(z) = f(z)$. If the path $\gamma : [a, b] \to U$ is piecewise C^1, then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. By definition, we have

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Since $F'(z) = f(z)$, we substitute to get

$$\int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F \circ \gamma)'(t)$$

by the chain rule.

Then applying the usual Fundamental Theorem of Calculus to the real and imaginary parts separately, this gives us

$$(F \circ \gamma)(b) - (F \circ \gamma)(a) = F(\gamma(b)) - F(\gamma(a)).$$

□

This is a helpful result. We see that when f has an antiderivative, our integral does not depend on the actual path, rather it only depends on the endpoints. Furthermore, in the case that γ is a closed path, our integral vanishes. Note though, that this is only true for functions with a well-defined antiderivative. The function $f(z) = \frac{1}{z}$, for example, does not.

Example ($f(z) = \frac{1}{z}$)**.**
FTC tells us that $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ where $f(z) = \frac{1}{z}$ does not have an antiderivative. If it did, then the integral over the unit circle should be 0, as the start/end points are the same. Yet for $\gamma : [0, 2\pi] \to \mathbb{C} \setminus \{0\}$ where $\gamma(\theta) = e^{i\theta}$, we have

$$\int_\gamma f(z)dz = \int_0^{2\pi} ie^{i\theta}/e^{i\theta} d\theta = 2\pi i,$$

a contradiction.

Example (Redo $\int_\gamma z^2 dz$ along the unit circle)**.**
Note that z^2 has an antiderivative $z^3/3 = ((x^3 - 3xy^2) + (3x^2y - y^3)i)/3$. We can check this with the Cauchy-Riemann equations, as $u_x = (3x^2 - 3y^2)/3 = v_y$ and $u_y = (-6xy)/3 = -v_x$. So

$$\frac{d}{dz}(z^3/3) = x^2 - y^2 + 2xyi = z^2.$$

Since z^2 has an antiderivative and γ is closed, we have

$$\int_\gamma z^2 dz = 0.$$

So a necessary condition for the antiderivative of a function to exist is for its integral along any closed curve to be 0. This is also sufficient, which leads us to the Path Independence Lemma.

Lemma 4.2 (Path Independence Lemma)**.** *Let $f : U \to \mathbb{C}$ be a continuous function on some domain $U \in \mathbb{C}$. Then the following three statements are equivalent:*

(i) f has an antiderivative F in U.

(ii) The integral $\int_\gamma f(z)dz$ vanishes for all closed contours γ in U.

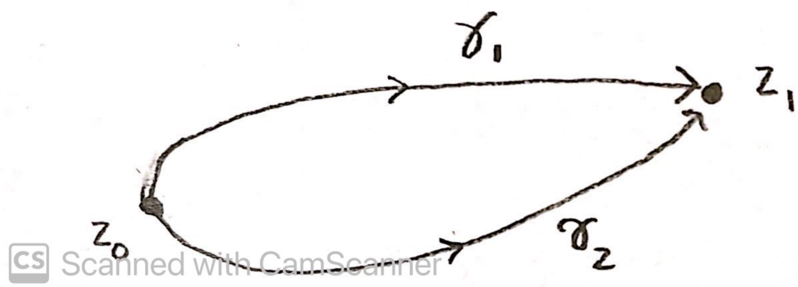
(iii) The integrals $\int_\gamma f(z)dz$ are independent of the path.

Proof. We've already seen above by the Fundamental Theorem of Calculus that (i) implies (ii) and (iii). So it would suffice to prove that (ii) and (iii) are equivalent, and that (iii) implies (i).

(ii) $\iff$ (iii):
First we will show that (iii) implies (ii). Assume that the integral $\int_\gamma f(z)dz$ is path independent. Now consider any closed path γ' that starts and ends at z_0. Because of path-independence, the integral $\int_{\gamma'} f(z)dz$ should be the same as the integral over the curve consisting of the single point z_0. Since that is 0, we must have that $\int_{\gamma'} f(z)dz = 0$.

Now to see that (ii) implies (iii), assume that $\int_\gamma f(z)dz = 0$ for any closed curve γ. Now consider the curves γ_1 and γ_2 that both start at z_0 and end at z_1. Then the concatenated path $\gamma_1 * (-\gamma_2)$, which follows the path of γ_1 (starting at z_0 and ending at z_1) then follows γ_2 in reverse (starting at z_1 and ending at z_0), should give the integral

$$\int_{\gamma_1 * (-\gamma_2)} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz.$$



But since $\gamma_1 * (-\gamma_2)$ starts and ends at the same point, it is closed, so by our initial assumption its integral vanishes. Thus

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Now all that is left is to show that (iii) implies (i), aka that if the integrals $\int_\gamma f(z)dz$ are path-independent, then f has an antiderivative.

(iii) $\implies$ (i):

Choose any $z_0 \in U$. For any $w \in U$, choose a path $\gamma_w : [0,1] \to U$ so that $\gamma_w(0) = z_0$ and $\gamma_w(1) = w$ (we know we can pick such a γ_w because domains are path-connected).

Now define

$$F(w) = \int_{\gamma_w} f(z)dz.$$

Note that since we are assuming path-independence, this $F(w)$ is independent of our choice of γ_w.

Now we must check if F is complex differentiable. Since U is open, there exists $\epsilon > 0$ so that $B(w,\epsilon) \subset U$. Let δ_h be the radial path in $B(w,\epsilon)$ from w to $w+h$, where $|h| < \epsilon$. Now let γ be the path from z_0 to $w+h$ that follows the path of γ_w, then δ_h. Then

$$F(w+h) = \int_\gamma f(z)dz = \int_{\gamma_w} f(z)dz + \int_{\delta_h} f(z)dz.$$

Simplifying $\int_{\gamma_w} f(z)dz = F(w)$, we have

$$F(w+h) = F(w) + \int_{\delta_h} f(z)dz.$$

Furthermore, because δ_h is the radial path from w to $w+h$, we have $\delta_h(t) = w + th$ for $t \in [0,1]$ and $\delta_h'(t) = h$. So the integral of $f(w)$ along δ_h is $\int_{\delta_h} f(w)dz = \int_0^1 f(w)hdt = hf(w)$.

Then the integral of $f(z)$ over δ_h is also the integral of $f(z) - f(w)$, plus $hf(w)$. This adds the displacement of $f(z)$ as z gets farther away from w to the volume bounded by $f(w)$ from w to $w+h$. So

$$\int_{\delta_h} f(z)dz = hf(w) + \int_{\delta_h} (f(z) - f(w))dz,$$

which means

$$F(w+h) = F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w))dz,$$

or

$$\frac{F(w+h) - F(w)}{h} - f(w) = \frac{1}{h} \int_{\delta_h} (f(z) - f(w))dz.$$

Taking the absolute values, we get

$$\Big|\frac{F(w+h) - F(w)}{h} - f(w)\Big| = \frac{1}{|h|} \Big| \int_{\delta_h} (f(z) - f(w))dz\Big|,$$

which by Proposition 3.4, gives us the inequality

$$\Big|\frac{F(w+h) - F(w)}{h} - f(w)\Big| \leq \frac{1}{|h|} \text{length}(\delta_h) \sup_{\delta_h} |f(z) - f(w)|.$$

But length(δ_h) is just $|h|$, so

$$\Big|\frac{F(w+h) - F(w)}{h} - f(w)\Big| \leq \sup_{\delta_h} |f(z) - f(w)|$$

Since f is continuous, as $h \to 0$, we know $f(z) - f(w) \to 0$. So F is differentiable, with derivative $F' = f$.

□

5 Cauchy's Integral Theorem and Formula

We are now on our way to proving our original goal, that holomorphic functions are analytic. Before doing so, we will need a few more tools that will help us better understand the relation between antiderivatives, holomorphicity, and complex integration.

Lemma 5.1 (Jordan curve lemma). *Any contour in the complex plane separates the plane into two domains with the loop as common boundary: one of which is bounded and is called the interior and one of which is unbounded and is called the exterior.*

Remark. We will not prove this statement, but here is a proof that relies on ideas of compactness, continuous paths, and the Weierstrass' theorem that any bounded sequence of real numbers has a convergent subsequence.

Additionally, here is the original proof by Jordan, along with a defense for it.

Definition 5.1 (Simply-connected). A domain U is simply-connected if the interior domain of every contour in U lies wholly in U.

A disk, for example, is simply connected, but a punctured disk (which is a disk without its center point) is not.

Theorem 5.2 (Green's Theorem). *Let $U \subset \mathbb{C}$ be a simply-connected domain, and let $f : U \to \mathbb{C}$ be a C^1 holomorphic function so that*

$$f(x, y) = u(x, y)dx + v(x, y)dy.$$

Also let γ be any counterclockwise loop in U. Then the line integral of f along γ can be written as the area integral of $v_x - u_y$ on the interior of γ, that is

$$\int_\gamma u(x,y)dx + v(x,y)dy = \int_{Int(\gamma)} (v_x - u_y)dxdy.$$

This version of Green's theorem in the complex plane will help us prove Cauchy's Integral Theorem:

Theorem 5.3 (Cauchy's Integral Theorem)**.** *Let $U \subset \mathbb{C}$ be a simply-connected domain and let $f : U \to \mathbb{C}$ be a C^1 holomorphic function, then for any contour γ, the contour integral vanishes, that is*

$$\int_\gamma f(z)dz = 0.$$

As an immediate corollary of this theorem and of the path-independence lemma, we also see that a holomorphic function in a simply-connected domain has an antiderivative.

Proof. Let R be the interior region inside the curve γ. Also write $f = u + iv$. Then our integral is

$$\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma (udx - vdy) + i(vdx + udy).$$

We now apply Green's theorem to the real and imaginary pieces separately. First for the real piece, we get

$$\int_\gamma udx - vdy = \int_R (-v_x - u_y)dxdy = 0,$$

which is 0 by the Cauchy-Riemann equations, as $u_y = -v_x$ and $u_x = v_y$.

Similarly, for the imaginary piece we get

$$\int_\gamma vdx + udy = \int_R (u_x - v_y)dxdy = 0.$$

Since the real and imaginary parts vanish, our entire integral also vanishes:

$$\int_\gamma f(z)dz = 0$$

for any contour γ when f is holomorphic. □

This means that contours can be moved about and deformed without changing the result of the integral, provided that our changed contour is still in U, so that f is still holomorphic.

Example (Integral of $f(z) = \bar{z}$ over unit circle)**.**
We can parameterize the unit circle by $\gamma : [0, 2\pi] \to \mathbb{C}$ where $\gamma(\theta) = e^{i\theta}$. Then because $\gamma'(\theta) = ie^{i\theta}$, our integral is

$$\int_0^{2\pi} \overline{e^{i\theta}}(ie^{i\theta})d\theta = \int_0^{2\pi} i|e^{i\theta}|d\theta = \int_0^{2\pi} id\theta = 2\pi i.$$

This makes sense. We saw in an earlier example that $\bar{z}$ is not differentiable, so its integral over the unit circle is $2\pi i$ rather than 0.

Example (Punctured disk is not simply-connected)**.**
Consider the function $f : D_r(0)\backslash\{0\} \to \mathbb{C}$ where $f(z) = \frac{1}{z}$ and $D_r(0)$ is the punctured disk with radius $r > 0$ around the origin. Let us integrate this over the circle of radius $r' < r$, parameterized by $\gamma : [0, 2\pi] \to \mathbb{C}$ where $\gamma(\theta) = r'e^{i\theta}$. Because $\gamma'(\theta) = ir'e^{i\theta}$, our integral is

$$\int_0^{2\pi} \frac{ir'e^{i\theta}}{r'e^{i\theta}} d\theta = i\Big|_0^{2\pi} = 2\pi i \neq 0.$$

We can not apply Theorem 5.2 here since $\frac{1}{z}$ is not defined at $z = 0$, meaning our contour γ surrounds a "hole."

On the other hand, were we to integrate $\frac{1}{z}$ over a circle outside the origin, then our domain would be simply-connected, meaning our integrals over closed contours would vanish.

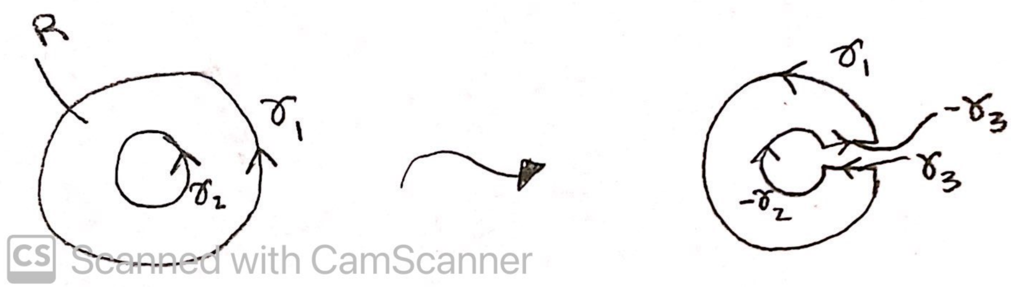
So we can think of a simply-connected domain as one without any "holes."

Although Cauchy's Theorem requires a function f be holomorphic on a simply connected domain, we can extend it into cases where our domain is not simply connected:

Proposition 5.1 (Extended Cauchy's Theorem)**.** Let R be the region between two closed contours γ_1, γ_2 oriented in the same direction. Say γ_2 lies within the interior of γ_1, that is $\text{Image}(\gamma_2) \in \text{Interior}(\gamma_1)$. If $f(z)$ is C^1 holomorphic on R then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. The proof for this cuts through γ_1 and γ_2 then connects them by γ_3 and $-\gamma_3$ (two copies of γ_3, one in each direction). Say γ_3 starts at the endpoint of γ_1 and ends at the start-point of $-\gamma_2$. Now, with γ_3 cutting between, the region enclosed by $\gamma_1 * \gamma_3 * (-\gamma_2) * (-\gamma_3)$ is simply connected.



So by Cauchy's Theorem, we have

$$\int_{\gamma_1 * \gamma_3 * (-\gamma_2) * (-\gamma_3)} f(z)dz = 0.$$

That is, by Definition 3.8,

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz - \int_{\gamma_2} f(z)dz - \int_{\gamma_3} f(z)dz = 0.$$

Since the contributions on γ_3 and $-\gamma_3$ cancel, we are left with

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = 0,$$

or

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

$\square$

So any two contours (in the same orientation) that surround a problem point/hole will evaluate to the same integral. This gives us a nice lemma:

Lemma 5.4. *If γ is a contour in the counter-clockwise direction and $z_0 \notin Image(\gamma)$, then*

$$\int_\gamma \frac{1}{z - z_0} dz = \begin{cases} 2\pi i & \text{for } z_0 \text{ in the interior of } \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In the latter case, $\frac{1}{z-z_0}$ is holomorphic everywhere but at z_0, and so it is holomorphic in the interior of γ. Then we can apply Cauchy's Integral Theorem, meaning in this case the integral evaluates to 0.

On the other hand, when z_0 is in the interior of γ, let $\gamma' : [0, 2\pi] \to U$ trace out a counterclockwise circle of radius r centered around z_0, that is $\gamma(\theta) = z_0 + re^{i\theta}$. If we choose r so that Image$(\gamma') \in$ Interior(γ), then by Proposition 5.1, we have

$$\int_\gamma \frac{1}{z - z_0} dz = \int_{\gamma'} \frac{1}{z - z_0} dz,$$

which simplifies to

$$\int_0^{2\pi} \frac{1}{(z_0 + re^{i\theta}) - z_0} (ire^{i\theta}) d\theta = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = \int_0^{2\pi} i d\theta = 2\pi i.$$

$\square$

Remark. Notice here that the radius r we choose does not end up mattering, so we can make r as small as we want.

Above is a special case of the Cauchy integral formula. Now we will prove the general case.

Theorem 5.5 (Cauchy's Integral Formula)**.** *Let $f : U \to \mathbb{C}$ be C^1 holomorphic in a simply-connected domain U, and let γ be a positively-oriented (counter-clockwise) contour within U. Consider any point z_0 in the interior of γ. Then we have*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Proof. Note that by Lemma 5.4 above, we have

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z_0)}{z - z_0} dz,$$

so it would suffice to show that

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

First, by Proposition 5.1, we can compute the above integral along a small circle of radius r centered about z_0 without changing its value. So define the curve $C_r : [0,1] \to U$ so that $C_r(t) = z_0 + re^{2\pi it}$. Then

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

Note that this is independent of the radius r, and thus true for all small enough r, so we can take the limit in which the radius goes to 0, so that

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = \lim_{r\to 0} \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

At the same time, our integral is

$$\int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz = \int_0^1 \frac{f(C_r(t)) - f(z_0)}{re^{2\pi it}} (2\pi i re^{2\pi it}) dt = 2\pi i \int_0^1 (f(C_r(t)) - f(z_0)) dt.$$

Estimating this integral using Lemma 3.1, we get

$$\left| \int_0^1 (f(C_r(t)) - f(z_0)) dt \right| \le \int_0^1 |f(C_r(t)) - f(z_0)| dt \le \max_{t\in[0,1]} |f(C_r(t)) - f(z_0)|.$$

Note that the maximum of $|f(C_r(t)) - f(z_0)|$ when $t \in [0,1]$ is also the maximum of $|f(z) - f(z_0)|$ when $|z - z_0| = r$, so we rewrite this as

$$\left| \int_0^1 (f(z) - f(z_0)) dt \right| \le \max_{|z-z_0|=r} |f(z) - f(z_0)|.$$

Since f is holomorphic at z_0, it is continuous at z_0, so as $z \to z_0$, we have $f(z) \to f(z_0)$. Then as the limit of $r \to 0$, we have $|f(z) - f(z_0)|$ approach 0. So

$$\lim_{r\to 0} \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

□

This tells us that in a simply-connected domain, a holomorphic function f is determined by its behavior on the boundary. So if two holomorphic functions agree on the boundary of a simply-connected domain, they agree everywhere in the domain.

Remark. Fun fact: apparently this formula is actually very similar to the idea of 'Holographic Principle' in today's theoretical physics. The black-hole entropy formula tells us that the entropy of a 3-D object like a black hole is proportional to the area of the event horizon, a 2-D system. Similarly, the Cauchy Integral Formula is holographic in the sense that a holomorphic function (2-D) is determined by its behavior on contours (1-D).

6 Every Complex Differentiable Function is Analytic

Recall from earlier our distinction between the terms analytic and holomorphic. We know an analytic function is infinitely differentiable, or C^∞. The converse is not true in the reals though. Take

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \le 0. \end{cases}$$

for example, which has continuous derivatives of all orders, but whose Taylor series at the origin converges to $0 \neq f(x)$ when $x > 0$.

On the other hand, in the complex plane holomorphic functions are analytic, and in fact infinitely differentiable.

Theorem 6.1 (Holomorphic means analytic)**.** *Let $f : B(a,r) \to \mathbb{C}$ be C^1 holomorphic. Then f has a convergent power series representation*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n.$$

Proof. Let $\partial B(a,r)$ be the counterclockwise contour that traces the boundary of $B(a,r)$. Now consider any point w in the interior of $B(a,r)$. By Cauchy's integral formula, we have

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(a,r)} \frac{f(z)}{z-w} dz.$$

Note that

$$\frac{1}{z-w} = \frac{1}{(z-a)(1-\frac{w-a}{z-a})}.$$

Since w is the interior of $B(a,r)$ and $z \in B(a,r)$, we have $|w-a| < |z-a| = r$, so $\frac{|w-a|}{|z-a|} < 1$. Then we can use the formula for geometric series to get that

$$\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^n} = \frac{1}{1-\frac{w-a}{z-a}},$$

and

$$\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}.$$

We know that the integral of uniformly convergent series converges to the sum of the integrals (see here). Since this series is uniformly convergent for w in the interior of $B(a,r)$, we can exchange integration and summation to get

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(a,r)} \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} f(z)$$

$$= \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \int_{\partial B(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz \Big) (w-a)^n$$

$$= \sum_{n=0}^{\infty} c_n (w-a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz.$$

Since c_n does not depend on w, we have thus found a convergent power series representation of f. Even more, the formula for c_n in terms of the derivative comes for free from the derivative of a power series, that is $c_n = \frac{f^{(n)}(a)}{n!}$.

□

7 References

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