Ряды

① $\sum\limits_{n=1}^{\infty} \frac{n^4}{(n!)^2}$ (Д'Аламбера)

$\lim \frac{(n+1)^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{n^n} = \lim\limits_{n\to\infty} \frac{\cancel{(n+1)}(n+1)^n \cancel{(n!)^2}}{n^n \cancel{(n!)^2}(n+1)^2} = 0 < 1$ сход.

② $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ (Коши)

$\lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{2}} = 1/2 < 1$ - сход.

③ $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n+\ln n}$ (Лейбница)

$\lim\limits_{n\to\infty} \frac{(-1)^n \to \pm 1}{n+\ln n} = 0$; $\frac{(-1)^n}{n+\ln n} > \frac{(-1)^n}{n+1+\ln(n+1)} \Rightarrow$ сход.

④ $\sum\limits_{n=1}^{\infty} \frac{3^n}{2^n}$ (Раабе)

$\lim\limits_{n\to\infty} n \cdot \left(\frac{\cancel{3^n}\,\cancel{2^n}\cdot 2}{\cancel{2^n}\,\cancel{3^n}\cdot 3} - 1\right) = \lim \left(-\frac{1}{3}n\right) = -\infty$ - расх

⑤ $f(x) = \ln(16x^2)$, $x=1$

$f'(x) = \frac{16\cdot 2x}{x^2} = \frac{2}{x}$; $f''(x) = -\frac{2}{x^2}$; $f^{(3)}(x) = \frac{4}{x^3}$; $f^{(4)}(x) = -\frac{12}{x^4}$ ...

$\ln 16 + \frac{2}{1!}(x-1) - \frac{2}{2!}(x-1)^2 + \frac{4}{3!}(x-1)^3 - \frac{12}{4!}(x-1)^2 \ldots$

$\Downarrow \frac{2}{1}$ $\quad \Downarrow \frac{2}{2}$ $\quad \Downarrow \frac{2}{3}$ $\quad \Downarrow \frac{2}{4}$

$f(x) = \sum\limits_{n=1}^{\infty} -\frac{2(-1)^n}{n}(x-1)^n + \ln 16$ — ряд Тейлора

⑥ $f(x) = x^2 \Rightarrow$ четн $x \in [-\pi, \pi]$

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} 2 a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1\cdot x^3}{\pi\cdot 3}\Big|_{-\pi}^{\pi} = \frac{1}{3\pi}(\pi^3+\pi^3) = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$\int_{-\pi}^{\pi} x^2\cos nx\, dx = \frac{1}{n}\int_{-\pi}^{\pi} x^2\, d\sin nx = \frac{x^2\sin nx}{n} - \frac{1\cdot 2}{n}\int_{-\pi}^{\pi} x\sin nx\, dx =$$

$$= \frac{x^2\sin nx}{n} + \frac{2}{n^2}\int_{-\pi}^{\pi} x\, d\cos nx = \frac{x^2\sin nx}{n} + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\int_{-\pi}^{\pi}\cos nx\, dnx =$$

$$= \frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\Big|_{-\pi}^{\pi} = \frac{2\pi\cos\pi n}{n^2} + \frac{2\pi\cos(-\pi n)}{n^2} =$$

$$= \frac{2\pi}{n^2}(\cos\pi n + \cos\pi n) = \frac{4\pi}{n^2}(-1)^n$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{8\pi}{n^2}(-1)^n\cos nx$$ — ряд Фурье

Интегралы

① $\int 2x^2 - 2x - 1 + \sin x - \cos x + \ln x + e^x\, dx =$

$= \frac{2x^3}{3} - \frac{2x^2}{2} - x - \cos x - \sin x + e^x + \int \ln x\, dx =$

$\left\{ \int \ln x\, dx = x\ln x - \int x\, d\ln x = x\ln x - \int \frac{x}{x} dx = x\ln x - x \right\}$

$= \frac{2x^3}{3} - x^2 - 2x - \cos x - \sin x + e^x + x\ln x + C$

② $\int 2x + 6xz^2 - 5x^2y - 3\ln z\, dx = x^2 + \frac{6z^2x^2}{2} - \frac{5yx^3}{3} - 3x\ln z$

③ $\int_0^{\pi} 3x^2\sin 2x\, dx = -\frac{3}{2}\int_0^{\pi} x^2\, d\cos 2x = -\frac{3}{2}\left(x^2\cos 2x - \int_0^{\pi}\cos 2x\, dx^2\right)$

$= -\frac{3}{2}\left(x^2\cos 2x - 2\int_0^{\pi} x\cos 2x\, dx\right) = -\frac{3}{2}\left(x^2\cos 2x - \frac{2}{2}\int_0^{\pi} x\, d\sin 2x\right) =$

$= -\frac{3}{2}\left(x^2\cos 2x - x\sin 2x + \frac{1}{2}\int_0^{\pi}\sin 2x\, d2x\right) = -\frac{3}{2}\left(x^2\cos 2x - x\sin 2x - \frac{\cos 2x}{2}\right)\Big|_0^{\pi}$

$= \frac{3\cos 2x}{4} - \frac{6x^2\cos 2x}{4} + \frac{6x\sin 2x}{4}\Big|_0^{\pi} = \frac{1}{4}\left((3-6x^2)\cos 2x + 6x\sin 2x\right)\Big|_0^{\pi} =$

$$= \frac{1}{4}\left((3-6\pi^2)\overset{\uparrow 1}{\cos 2\pi} + 6\pi \overset{\uparrow 0}{\sin 2\pi} - 3\overset{\uparrow 1}{\cos 0} - 0 \right) =$$

$$= \frac{1}{4}\left((3-6\pi^2) - 3 \right) = -\frac{6\pi^2}{4} = -3\pi^2/2$$

(4) $\int \frac{1}{\sqrt{x+1}}\,dx =$

$\{ t = \sqrt{x+1};\ t^2 = x+1;\ x = t^2-1;\ dx = 2t\,dt \}$

$= \int \frac{1}{\cancel{t}} 2\cancel{t}\,dt = 2t + C = 2\sqrt{x+1} + C$