① $U = 3 - 8x + 6y \quad x^2 + y^2 = 36$

$L(\lambda_1, x, y) = 3 - 8x + 6y - \lambda_1 (x^2 + y^2 - 36)$

$$\begin{cases} L'_x = -8 + 2x\lambda_1 = 0 \\ L'_y = 6 + 2y\lambda_1 = 0 \\ L'_\lambda = x^2 + y^2 - 36 = 0 \end{cases} \quad \begin{cases} x = 4/\lambda_1 \\ y = -3/\lambda_1 \\ \frac{16}{\lambda_1^2} + \frac{9}{\lambda_1^2} - 36 = 0 \end{cases}$$

$\frac{25}{\lambda_1^2} = 36 \quad \lambda_1^2 = \frac{25}{36} \quad \lambda_1 = \frac{5}{6} \quad \lambda_2 = -\frac{5}{6}$

$\left(\frac{5}{6}, \frac{4 \cdot 6}{5}, -\frac{3 \cdot 6}{5}\right); \left(-\frac{5}{6}, -\frac{4 \cdot 6}{5}, +\frac{3 \cdot 6}{5}\right)$

$L''_{xx} = 2\lambda \quad L''_{yy} = 2\lambda \quad L''_{\lambda\lambda} = 0$

$L''_{xy} = 0 \quad L''_{x\lambda} = 2x \quad L''_{y\lambda} = 2y$

$\Delta \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2\lambda & 0 \\ 2y & 0 & 2\lambda \end{pmatrix} = -8\lambda (x^2 + y^2)$

$1: -\frac{8 \cdot 5}{6} \left(\frac{24^2 + 18^2}{25}\right) < 0$ min

$2: +\frac{8 \cdot 5}{6} \left(\frac{24^2 + 18^2}{25}\right) > 0$ max

② $U = 2x^2 + 12xy + 32y^2 + 15 \quad x^2 + 16y^2 = 64$

$L(\lambda, x, y) = 2x^2 + 12xy + 32y^2 + 15 + \lambda(x^2 + 16y^2 - 64) = 0$

$$\begin{cases} L'_x = 4x + 12y + 2x\lambda = 0 \\ L'_y = 64y^2 + 12x + 32y\lambda = 0 \\ L'_\lambda = x^2 + 16y^2 - 64 = 0 \end{cases} \quad \begin{cases} x(\lambda+2) + 6y = 0 \\ 3x + 8y(\lambda+2) = 0 \\ x^2 + 16(y^2 - 4) = 0 \end{cases} \quad \begin{matrix} | \cdot 3 \\ | \cdot (\lambda+2) \end{matrix}$$

$$- \frac{\begin{matrix} 3x(\lambda+2) + 18y = 0 \\ 3x(\lambda+2) + 8y(\lambda+2)^2 = 0 \end{matrix}}{18y = 8y(\lambda+2)^2}$$

$4(\lambda+2)^2 = 9$

$(\lambda+2)^2 = 9/4 \quad \lambda + 2 = \pm 3/2$

$\lambda_1 = -1/2 \quad \lambda_2 = -7/2$

$x = -\frac{6y}{\lambda+2}$

$x_1 = -\frac{6y}{-\frac{1}{2}+2} = -\frac{6y \cdot 2}{3} = -4y$

$x_2 = -\frac{6y}{-7/2+2} = +\frac{6y \cdot 2}{3} = 4y$

$x^2 + 16(y^2 - 4) = 0 \quad 16y^2 + 16y^2 = 64$

$32y^2 = 64 \quad y^2 = 2$

$y_{1,2} = \pm\sqrt{2}$

$x_{1,2} = \pm 4\sqrt{2}$

$\lambda_1 = -1/2 \quad \lambda_2 = -7/2$

$$\begin{vmatrix} 0 & 2x & 32y \\ 2x & 4+2\lambda & 12 \\ 32y & 12 & 128y+32\lambda \end{vmatrix} = -512x^2y - 128x^2\lambda + 1536xy - 2048y^2\lambda - 4096y^2$$

$(-4\sqrt{2}, -\sqrt{2}, -7/2)$ $\quad 16384(2+\sqrt{2}) > 0$ $\quad$ max

$(4\sqrt{2}, \sqrt{2}, -7/2)$ $\quad -16384(\sqrt{2}-2) > 0$ $\quad$ max

$(-4\sqrt{2}, \sqrt{2}, -1/2)$ $\quad -16384(\sqrt{2}-1) < 0$ $\quad$ min

$(4\sqrt{2}, -\sqrt{2}, -1/2)$ $\quad 16384(-1+\sqrt{2}) < 0$ $\quad$ min

(3) $u = x^2+y^2+z^2$ $\quad \vec{c}(-9, 8, -12)$, $M(8, -12, 9)$

$|\vec{c}| = \sqrt{81+64+144} = 17$

$c_0 = \left(-\frac{9}{17}, \frac{8}{17}, -\frac{12}{17}\right)$

$u'_x = 2x \quad u'_y = 2y \quad u'_z = 2z$

$\operatorname{grad} u \Big|_{(8,-12,9)} = (16, -24, 18)$

$u'_{\vec{c}} = -\frac{9\cdot 16}{17} - \frac{8\cdot 24}{17} - \frac{12\cdot 18}{17} = \frac{-144-192-216}{17} = -\frac{552}{17}$

(4) $u = e^{x^2+y^2+z^2}$ $\quad \vec{d}(4, -13, -16)$, $L(-16, 4, -13)$

$|\vec{d}| = \sqrt{16+169+256} = 21$

$d_0 = \left(\frac{4}{21}, -\frac{13}{21}, -\frac{16}{21}\right)$

$u'_x = 2x\, e^{x^2+y^2+z^2}, \quad u'_y = 2y\, e^{x^2+y^2+z^2}, \quad u'_z = 2z\, e^{x^2+y^2+z^2}$

$\operatorname{grad} u \Big|_{(-16,4,-13)} = \left(-32e^{441}, 8e^{441}, -26e^{441}\right)$

$u'_{\vec{d}} = e^{441}\left(\frac{-32\cdot 4 - 13\cdot 8 + 26\cdot 16}{21}\right) = e^{441}\left(\frac{-128-104+416}{21}\right) = \frac{184e^{441}}{21}$