$$g_{x_1}(x_1) \leq \min_{\eta} g_{x_1}(x_2) + \frac{L}{2}\|\eta \nabla g_{x_1}(x_2)\|_2^2 - \eta \nabla g_{x_1}^T(x_2)\,(\nabla f(x_2) - \nabla f(x_1))$$

$$g_{x_1}(x_1) \leq \min_{\eta} g_{x_1}(x_2) + \frac{L}{2}\eta^2 \|\nabla g_{x_1}(x_2)\|_2^2 - \eta \|\nabla g_{x_1}(x_2)\|_2^2$$

The minimum Solution η to minimize this quadratic problem is:

$$L\eta^* \|\nabla g_{x_1}(x_2)\|_2^2 - \|\nabla g_{x_1}(x_2)\|_2^2 = 0$$

$$\eta^* = \frac{1}{L}$$

$\Rightarrow$ minimum Solution : $g_{x_1}(x_2) - \frac{1}{2L}\|\nabla g_{x_1}(x_2)\|_2^2$

thus from our diffinition of $g_{x_1}(x_2)$ it follows:

$$g_{x_1}(x_1) \leq g_{x_1}(x_2) - \frac{1}{2L}\|\nabla g_{x_1}(x_2)\|_2^2$$

$$0 \leq f(x_2) - f(x_1) - \nabla f^T(x_1)(x_2 - x_1) - \frac{1}{2L}\|\nabla f(x_2) - \nabla f(x_1)\|_2^2$$

$$\Rightarrow -f(x_2) \leq -f(x_1) - \nabla f^T(x_1)(x_2 - x_1) - \frac{1}{2L}\|\nabla f(x_2) - \nabla f(x_1)\|_2^2$$

$$\Rightarrow f(x_2) \geq f(x_1) + \nabla f^T(x_1)(x_2 - x_1) + \frac{1}{2L}\|\nabla f(x_2) - \nabla f(x_1)\|_2^2$$