problem 1.2

A Function $f: R^d \longrightarrow R$ is L-smooth iff it is differentiable and it's gradient is L-Lipschitz continuous (usually w.r.t norm-2):

$$\forall x_1, x_2 \in \mathbb{R}^d, \ \|\nabla f(x_2) - \nabla f(x_1)\|_2 \leq L \|x_2 - x_1\|_2$$

For all x_1, x_2, prove that

a) $f(x_2) \leq f(x_1) + \nabla f^T(x_1)(x_2 - x_1) + \frac{L}{2}\|x_2 - x_1\|_2^2$

solution:

Let define $g(t) \triangleq f(x_1 + t(x_2 - x_1))$

we know that:

$$\int_0^1 g'(t)\, dt = g(1) - g(0) = f(x_2) - f(x_1)$$

It then follows that:

$$f(x_2) - f(x_1) - \nabla f^T(x_1)(x_2 - x_1) = \int_0^1 \nabla f^T(x_1 + t(x_2 - x_1))(x_2 - x_1)\, dt - \nabla f^T(x_1)(x_2 - x_1)$$

$$\Rightarrow f(x_2) - f(x_1) - \nabla f^T(x_1)(x_2 - x_1) = \int_0^1 \left(\nabla f(x_1 + t(x_2 - x_1)) - \nabla f(x_1)\right)^T (x_2 - x_1)\, dt$$

from the cauchy-schartz inequality we can get:

$$\Rightarrow f(x_2) - f(x_1) - \nabla f^T(x_1)(x_2 - x_1) \leq \int_0^1 \|\nabla f(x_1 + t(x_2 - x_1)) - \nabla f(x_1)\|_2 \|x_2 - x_1\|_2\, dt$$