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A Planetary Cluster of Differential Equations

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Contents

Abstract

Put abstract here

1 Introduction

The aim of this project was to develop a code for simulating the solar system. We used the velocity Verlet algorithm to solve coupled ordinary differential equations. In the development process, we set up 48 coupled differential equations, illustrating all planets in the x-, y- and z-plane.

Initial conditions were obtained through the NASA website `http://ssd.jpl.nasa.gov/horizons.cgi#top`.

2 Method

For simplicity, we started with a system consisting of the Sun and Earth. We assumed that mass units could be obtained by using the fact that Earth's orbit is almost circular around the Sun. Using Newton's law for circular motion, we obtained the expression of force

$$F_G = \frac{M_E v^2}{r} = \frac{GM_\odot M_E}{r^2}, \tag{1}$$

where we used that

$$v^2 r = GM_\odot = \frac{4\pi^2 (AU)^3}{(yr)^2}$$

We wanted to make our solar system 3-dimensional, and obtained the expressions for the derivatives.

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_E}, \quad \frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_E}, \quad \frac{d^2z}{dt^2} = \frac{F_{G,z}}{M_E}$$

We used that $r = \sqrt{x^2 + y^2 + z^2}$, and obtained the equations

$$F_{G,x} = -\frac{GM_\odot M_E}{r^3} x, \quad F_{G,y} = -\frac{GM_\odot M_E}{r^3} y, \quad F_{G,z} = -\frac{GM_\odot M_E}{r^3} z$$

$$
\begin{aligned}
\frac{dx}{dt} = v_x \rightarrow \frac{d^2x}{dt^2} = \frac{dv_x}{dt} &= \dot{v_x} = \frac{F_{G,x}}{M_E} = -\frac{GM_\odot}{r^3}x \\
\frac{dv_y}{dt} &= \dot{v_y} = \frac{F_{G,y}}{M_E} = -\frac{GM_\odot}{r^3}y \\
\frac{dv_z}{dt} &= \dot{v_z} = \frac{F_{G,z}}{M_E} = -\frac{GM_\odot}{r^3}z
\end{aligned} \tag{2}
$$

2.1 Euler's forward algorithm

We wanted to set up Euler's forward algorithm for solving equation 2. For simplicity, we showed this for the x-direction, and copied the method for the two other directions.

$$
x = x_i, \quad v_x = v_{x,i}, \quad t = t_i = t_0 + ih
$$

We set $h = \frac{t_F - t_0}{N}$, where t_F was the final time, t_0 was the initial time, and N was the number of integration points. By Taylor expanding both x_i and $v_{x,i}$ we obtained the forward Euler algorithm.

$$
x_{i+1} = x(t_{i+1}) = x_i + hx'_i + \frac{h^2}{2!}x''_i + O(h^3) \tag{3}
$$

$$
v_{x,i+1} = v_{x,i} + hv'_{x,i} + \frac{h^2}{2!}v''_{x,i} + O(h^3) \tag{4}
$$

We truncated after the first derivative, and the algorithms looked like

$$
\begin{aligned}
x_{i+1} &= x_i + hv_{x,i} + O(h^2) \\
v_{x,i+1} &= v_{x,i} + hv'_{x,i} + O(h^2) \\
&= v_{x,i} - h\frac{4\pi^2}{r_i^3}x_i + O(h^2),
\end{aligned} \tag{5}
$$

remembering that $r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$. We did the same procedure for the y- and z-direction, and obtained the rest of the algorithms for solving the forward Euler method.

$$
\begin{aligned}
y_{i+1} &= y_i + hv_{y,i} + O(h^2) \\
v_{y,i+1} &= v_{y,i} + hv'_{y,i} + O(h^2) \\
&= v_{x,i} - h\frac{4\pi^2}{r_i^3}y_i + O(h^2)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
z_{i+1} &= z_i + hv_{z,i} + O(h^2) \\
v_{z,i+1} &= v_{z,i} + hv'_{z,i} + O(h^2) \\
&= v_{x,i} - h\frac{4\pi^2}{r_i^3} z_i + O(h^2)
\end{aligned}
\tag{7}
$$

2.2 Velocity Verlet algorithm

Verlet integration is a numerical method used to integrate Newton's equation of motion **add reference to wiki**. The velocity Verlet method is a more commonly used algorithm, where velocity and position are calculated at the same value of the time variable.

As with Euler's forward algorithm, we started by performing Taylor expansions, where the expansion of position went as equation 3. Instead of Taylor expanding the velocity, we used an expansion of the acceleration.

$$
v'_{x,i+1} = v'_{x,i} + hv''_{x,i} + O(h^2) \tag{8}
$$

We set $hv''_{x,i} \simeq v'_{x,i+1} - v'_{x,i}$, and substituted this into 8 to obtain the algorithm for velocity. Finally, we had algorithms for both position and velocity.

$$
\begin{aligned}
x_{i+1} &= x_i + hv_{x,i} + \frac{h^2}{2} + O(h^3) \\
v_{x,i+1} &= v_{x,i} + \frac{h}{2}(v'_{x,i+1} + v'_{x,i}) + O(h^3) \\
&= v_{x,i} - \frac{4\pi^2 h}{2}\Big(\frac{x_{i+1}}{r_{i+1}^3} + \frac{x_i}{r_i^3}\Big) + O(h^3)
\end{aligned}
\tag{9}
$$

By applying this in the remaining directions, we finally obtained all algorithms for solving the velocity Verlet method.

$$
\begin{aligned}
y_{i+1} &= y_i + hv_{y,i} + \frac{h^2}{2} + O(h^3) \\
v_{y,i+1} &= v_{y,i} + \frac{h}{2}(v'_{y,i+1} + v'_{y,i}) + O(h^3) \\
&= v_{y,i} - \frac{4\pi^2 h}{2}\Big(\frac{y_{i+1}}{r_{i+1}^3} + \frac{y_i}{r_i^3}\Big) + O(h^3)
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
z_{i+1} &= z_i + hv_{z,i} + \frac{h^2}{2} + O(h^3) \\
v_{z,i+1} &= v_{z,i} + \frac{h}{2}(v'_{z,i+1} + v'_{z,i}) + O(h^3) \\
&= v_{z,i} - \frac{4\pi^2 h}{2}\Big(\frac{z_{i+1}}{r^3_{i+1}} + \frac{z_i}{r^3_i}\Big) + O(h^3)
\end{aligned}
\tag{11}
$$