MATH 624
Michelle Chung
HW 2 from Section 2.2

4. The following exercises lead to an estimator for the condition number of a nonsingular matrix A.

a) Let v be an eigenvector for A with eigenvalue λ. Prove that $||Av|| = |\lambda|||v||$. Prove that $|\lambda| \leq ||A||$.

The operator norm is defined in the real vector space such that $||\lambda v|| = |\lambda|||v||$.
If v is an eigenvector for A with associated eigenvalue λ, then it must be true that $Av = \lambda v$.
Therefore, it is also true that $||Av|| = |\lambda|||v||$.

Given that $||Av|| = |\lambda|||v||$, $||\lambda||v \leq c||v||$.
While both λ and $||A||$ are bounded above by the constant c, c is limsup$||A||$.
Therefore, $\lambda \leq ||A||$, where $0 \leq \lambda \leq c$.

b) Prove that λ is an eigenvalue for A if and only if $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}.

$Av = \lambda v$
$A^{-1}(Av) = A^{-1}(\lambda v)$
$v = \lambda A^{-1}v$
$\frac{1}{\lambda}v = A^{-1}v$

c) Let $\mu(A)$ be the absolute value of the smallest eigenvalue of A. Prove that $\sigma(A) \leq ||A||$ and $1/\mu(A) \leq ||A^{-1}||$.

In part a, we proved that $||Av|| = |\lambda|||v||$. We also know that the operator norm is defined in the real vector space such that $||Av|| \leq c||v||$. If we consider each eigenvalue for A, $|\lambda|||v|| \leq c||v||$. In other words, the maximum eigenvalue is bounded by c, so $\sigma(A)$ can never be larger than $||A||$.

d) Prove that $\sigma(A)/\mu(A) \leq C$, the condition number of A.

Let e be the relative error in $Ax = b$. The condition number is defined as the maximum ratio of relative error in x to relative error in b:

$$\frac{\frac{||A^{-1}e||}{||A^{-1}b||}}{\frac{||e||}{||b||}} = \left(\frac{||A^{-1}e||}{||e||}\right) \cdot \left(\frac{||b||}{||A^{-1}b||}\right) \tag{1}$$

$$= max\left\{\left(\frac{||A^{-1}e||}{||e||}\right) \cdot \left(\frac{||b||}{||A^{-1}b||}\right)\right\} \tag{2}$$

$$= \max\left\{\frac{||A^{-1}e||}{||e||}\right\} \cdot \max\left\{\frac{||b||}{||A^{-1}b||}\right\} \tag{3}$$

$$= \max\left\{\frac{||A^{-1}e||}{||e||}\right\} \cdot \max\left\{\frac{||Ax||}{||x||}\right\} \tag{4}$$

$$= ||A^{-1}|| \cdot ||A|| = C \tag{5}$$

$\sigma(A) \leq ||A||$ and $1/\mu(A) \leq ||A^{-1}||$, so $\sigma(A)/\mu(A) \leq C$.

6. Repeat Exercise 5 of Section 2.1. But now execute LU decomposition. Look at the condition number as a function of the matrix size. What is happening? Are these matrices nonsingular? Do they have a zero eigenvalue?

The condition number is decreasing as the size of the matrix decreases. Yes, these matrices are nonsingular because tridiagonal matrices with non-zero diagonal values are invertible. Because all of the matrices have non-zero values in the diagonal, there is no zero eigenvalue.

8. Prove Theorem 2.2.6 as follows. Suppose that A is an $n \times m$ real symmetric with orthonormal eigenvectors $v_1 \ldots v_n$.

a) Let v be a unit vector with $v = \sum \xi_i v_i$ where the v_i are orthonormal eigenvectors. Prove that $\sum \xi_i^2 = 1$.

If v is a unit vector, we know that $||v|| = ||v||^2 = 1$.
$||v|| = \sqrt{\sum(\xi_i v_i)^2}$.
$||v||^2 = \sum(\xi_i)^2(v_i)^2$ and $v_i^2 = 1$
So $||v||^2 = \sum(\xi_i)^2$.

b) Prove that $\sigma(A) \geq ||A||$, conclude that $\sigma(A) = ||A||$.

Let $\sigma(A) = \lambda$. Assume $||A|| < \lambda$.

$$||A|| < \lambda \tag{6}$$
$$||A||||v|| < \lambda||v|| \tag{7}$$
$$|\lambda|||v|| < \lambda||v|| \tag{8}$$
$$\lambda||v|| < \lambda||v|| \quad \text{, a contradiction.} \tag{9}$$

However, we can conclude that $\lambda = ||A||$.

c) Prove that λ is an eigenvalue for A if and only if λ^k is an eigenvalue for A^k.

Assume A is an $n \times n$ symmetric matrix. A is diagonalizable. Suppose we diagonalize A. There are exactly n eigenvalues in the main diagonal. In a diagonal matrix, it is true that for $\forall k$, A^k will result in $a_{ii}^k = \lambda^k$.

d) Prove that $||A^k|| = ||A||^k$

Similar proof from part c.

Prove for any square matrix A and for any $k > 0$, $\sigma(A) \leq ||A^k||^{\frac{1}{k}}$.

$$||A^k|| = ||A||^k \tag{10}$$
$$(||A||^k)^{\frac{1}{k}} = ||A|| \tag{11}$$
$$\sigma(A) \leq ||A|| \tag{12}$$

True: $||A|| = \sqrt{\max(\lambda) \text{ of } A^T A}$.