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Vector Spaces and Subspaces

Field: $(F; +, \cdot)$

$F \rightarrow$ non-empty set and $+, \cdot$ are binary operations.

Axioms for addition:

(A1) $a, b \in F \Rightarrow a+b \in F$ (Closer law)

(A2) For all $a, b, c \in F$

$(a+b)+c = (b+c)+a$ (associative law)

(A3) There exists $0 \in F$ such that $a+0 = 0+a = a$ for all $a \in F$.

[$0 \rightarrow$ additive identity]

(A4) For every $a \in F$, there exists an element $-a \in F$ such that $a+(-a) = (-a)+a = 0$

[$-a \rightarrow$ ~~inverse~~ inverse of a]

(A5) Addition is commutative: $a+b = b+a$, for all $a, b \in F$

Axioms for multiplication:

(M1) If $a, b \in F$, then $ab \in F$

(M2) $(ab) \cdot c = a \cdot (bc)$ for all $a, b, c \in F$ [Multiplication is associative]

(M3) There exists 1 in F such that $a \cdot 1 = 1 \cdot a = a$ $\forall$ $a \in F$.

[$1 \rightarrow$ multiplicative identity].

(M4) For every $a \in F$, there exists an element $\frac{1}{a} \in F$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

[$\frac{1}{a} \rightarrow$ inverse of a for multiplication]

(M5) Multiplication is commutative: $a \cdot b = b \cdot a$ for all $a, b \in F$.

Distributive law:

(i) $a(b+c) = ab + ac$ $\forall$ $a, b, c \in F$

(ii) $(a+b)c = ac + bc$ $\forall$ $a, b, c \in F$

Examples:— (i) real number ($\mathbb{R}$), (ii) complex number ($\mathbb{C}$)

2 ~~vector spaces~~:

Vector space: Let K be a field and let V be a non-empty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$, $u+v \in V$ and to any $u \in V, k \in K$, $ku \in V$. Then V is called a vector space over K if the following axioms hold:

[The elements of V are called vectors]

Axioms for addition.

(A1) $u, v, w \in V$, $(u+v)+w = u+(v+w)$

(A2) There is a vector in V, denoted by 0 for which
$u+0 = 0+u = u \quad \forall\, u \in V$ [$0 \to$ zero vector]

(A3) For each vector $u \in V$ there is a vector in V, denoted by $-u$, for which $u+(-u) = 0$. [$-u \to$ inverse of u]

(A4) $u+v = v+u \quad \forall\, u, v \in V$

Axioms for multiplication:

(M1) For any $k \in K$ and any $u, v \in V$, $k(u+v) = ku+kv$.

(M2) For any scalars $a, b \in K$ and $u \in V$, $(a+b)u = au+bu$.

(M3) For any $a, b \in K$ and any $u \in V$, $(ab)u = a(bu)$

(M4) For the unit scalar $1 \in K$, $1 \cdot u = u \quad \forall\, u \in V$.

Example: (1) Let K be an arbitrary field. The set of all n-tuples of elements of K with vector addition and scalar multiplication defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$$

and $k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$ where $a_i, b_i, k \in K$, is a vector space over K.

This space is denoted by K^n.

2 $\mathbb{R} \to$

$\mathbb{R}^2$, $\mathbb{R}^3$

ASM. 4.77 → P→85
4.55
P→83
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Theorems: 4.8, 4.9

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Subspace: Let W be a subset of a vector space V over a field K. W is called a subspace of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication on V.

Theorem: 4.2 W is a subspace of V iff
(i) W is non-empty,
(ii) $v, w \in W$ implies $v+w \in W$,
(iii) $v \in W$ implies $kv \in W$ for every $k \in K$.

Corollary: W is a subspace of V iff
(i) $0 \in W$ $(W \neq \phi)$ and $v, w \in W \Rightarrow av + bw \in W$ for every $a, b \in K$.

Proof: 4.8 [P→72]

Problem: Let $V = \mathbb{R}^3$. Show that W is a subspace of V where:
4.9
(i) $W = \{(a, b, 0) : a, b \in \mathbb{R}\}$.
(ii) $W = \{(a, b, c) : a + b + c = 0\}$.

Problem: Let $V = \mathbb{R}^3$. Show that W is not a subspace of V where:
(i) $W = \{(a, b, c) : a \geq 0\}$
(ii) $W = \{(a, b, c) : a^2 + b^2 + c^2 \leq 1\}$.

Soln: (i) $v = (1, 2, 3)$ and $k = -5 \in \mathbb{R}$. But $-5(1, 2, 3) = (-5, -10, -15) \notin W$, since $-5 < 0$. Hence W is not a subspace of V.

(ii) we $v = (1, 0, 0) \in W$ and $w = (0, 1, 0) \in W$. But
$v + w = (1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin W$, since $1^2 + 1^2 + 0^2 = 2 > 1$.
Hence W is not a subspace of V.

Problem: Determine whether or not W is a subspace of $\mathbb{R}^3$ if W consists of those vectors $(a, b, c) \in \mathbb{R}^3$ for which: (i) $a = 2b$; (ii) $a \leq b \leq c$; (iii) $ab = 0$; (iv) $a = b = c$; (v) $a = b^2$.