Basis and Dimension

Linearly dependence: Let V be a vector space over a field K. The vectors $v_1, v_2, \dots, v_m \in V$ are said to be linearly dependent over K if there exist scalars $a_1, a_2, \dots, a_m \in K$, not all of them 0, such that $a_1v_1 + a_2v_2 + \dots + a_mv_m = 0$.

That is, if $a_1v_1 + a_2v_2 + \dots + a_mv_m = 0$ when at least one of the a's is not 0, then the vectors are linearly dependent.

* Show that the vectors $u = (1, -1, 0)$, $v = (1, 3, -1)$ and $w = (5, 3, -2)$ are linearly dependent.

Sol^n: $xu + yv + zw = 0$ where x, y, z unknown scalars

$x(1,-1,0) + y(1,3,-1) + z(5,3,-2) = (0,0,0)$

$$\Rightarrow \begin{array}{l} x + y + 5z = 0 \\ -x + 3y + 3z = 0 \\ -y - 2z = 0 \end{array} \Rightarrow \begin{array}{l} x + y + 5z = 0 \\ 4y + 8z = 0 \\ y + 2z = 0 \end{array} \Rightarrow \begin{array}{l} x + y + 5z = 0 \\ y + 2z = 0 \end{array}$$

Here z is a free variable.

Set $z = 1$, we obtain, $y = -2$ and $x = -3$

Problem: Determine whether or not the following vectors in $\mathbb{R}^3$ are linearly dependent:

(i) $(1,-2,1), (2,1,-1), (7,-4,1)$

(ii) $(1,-3,7), (2,0,-6), (3,-1,-1), (2,4,-5)$

(iii) $(1,2,-3), (1,-3,2), (2,-1,5)$

(iv) $(2,-3,7), (0,0,0), (3,-1,-4)$.

** Sol^n

Note:
1. Two vectors u and v are dependent iff one is a multiple of the other.
2. If one of the vectors $v_1, \dots, v_m$ say $v_1 = 0$, then the vectors must be dependent.
3. If two of the vectors $v_1, \dots, v_m$ are equal, say $v_1 = v_2$, then the vectors are dependent.

Basis and Dimension

Linearly dependence: Let V be a vector space over a field K. The vectors $v_1, v_2, \ldots, v_m \in V$ are said to be linearly dependent over K if there exist scalars $a_1, a_2, \ldots, a_m \in K$, not all of them 0, such that $a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0$.

That is, if $a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$ when at least one of the a's is not 0, then the vectors are linearly dependent.

* Show that the vectors $u = (1,-1,0)$, $v = (1,3,-1)$ and $w = (5,3,-2)$ are linearly dependent.

Sol^n: $xu + yv + zw = 0$ where x, y, z unknown scalars

$x(1,-1,0) + y(1,3,-1) + z(5,3,-2) = (0,0,0)$

$$\Rightarrow \begin{array}{l} x+y+5z=0 \\ -x+3y+3z=0 \\ -y-2z=0 \end{array} \Rightarrow \begin{array}{l} x+y+5z=0 \\ 4y+8z=0 \\ y+2z=0 \end{array} \Rightarrow \begin{array}{l} x+y+5z=0 \\ y+2z=0 \end{array}$$

Here z is a free variable.

Set $z=1$, we obtain, $y=-2$ and $x=-3$

Problem: Determine whether or not the following vectors in $\mathbb{R}^3$ are linearly dependent:

(i) $(1,-2,1)$, $(2,1,-1)$, $(7,-4,1)$

(ii) $(1,-3,7)$, $(2,0,-6)$, $(3,-1,-1)$, $(2,4,-5)$

(iii) $(1,2,-3)$, $(1,-3,2)$, $(2,-1,5)$

(iv) $(2,-3,7)$, $(0,0,0)$, $(3,-1,-4)$.

** Sol^n

Note: ① Two vectors u and v are dependent iff one is a multiple of the other.

② If one of the vectors $v_1, \ldots, v_m$ say $v_1 = 0$, then the vectors must be dependent.

③ If two of the vectors $v_1, \ldots, v_m$ are equal, say $v_1 = v_2$, then the vectors are dependent.

(i) $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 7 & -4 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 10 & -6 \end{pmatrix}$ to $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

Since the echelon matrix has a zero row, the vectors are dependent

(ii) ~~Yes~~, Since any four (or more) vectors in $\mathbb{R}^3$ are dependent

(iii) $\begin{pmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 2 & -1 & 5 \end{pmatrix}$ to $\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & -5 & 11 \end{pmatrix}$ to $\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

Since the echelon matrix has no zero rows, so the vectors are independent.

(iv) Since $0 = (0, 0, 0)$ is one of the vectors are dependent.

Linearly independent: Let V be a vector space over a field K. The vectors $v_1, \dots, v_m \in V$ are said to be linearly independent over K, if there exist scalars $a_1, \dots, a_m \in K$, all of them 0 such that $a_1 v_1 + a_2 v_2 + \dots + a_m v_m = 0$.

That is,

$$a_1 v_1 + a_2 v_2 + \dots + a_m v_m = 0 \text{ only if } a_1 = 0, \dots, a_m = 0$$

then the vectors are linearly independent.

* Show that the vectors $u = (6, 2, 3, 4)$, $v = (0, 5, -3, 1)$ and $w = (0, 0, 7, -2)$ are linearly independent.

Sol^n: Suppose $xu + yv + zw = 0$ where x, y, z are unknown scalars

$$x(6,2,3,4) + y(0,5,-3,1) + z(0,0,7,-2) = (0,0,0,0)$$

$$\begin{aligned} 6x &= 0 \\ 2x + 5y &= 0 \\ 3x - 3y + 7z &= 0 \\ 4x + y - 2z &= 0 \end{aligned} \implies x = 0,\ y = 0 \text{ and } z = 0$$

$xu + yv + zw = 0$ implies $x = 0$, $y = 0$ and $z = 0$

Hence u, v and w are independent.

Observe that the vectors in the above problem form a matrix in echelon form:

$$\begin{pmatrix} 6 & 2 & 3 & 4 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 7 & -2 \end{pmatrix}.$$

Thus we have shown that the nonzero rows of the above echelon matrix are independent.

* Let u, v and w be independent vectors. Show that $u+v$, $u-v$ and $u-2v+w$ are also independent.

$$x(u+v)+y(u-v)+z(u-2v+w)=0 \quad \text{where } x, y, z \text{ scalars}$$

$$\text{or } (x+y+z)u+(x-y-2z)v+zw=0.$$

But u, v and w are linearly independent; hence the coefficients in the above relation are each 0:

$$x+y+z=0$$
$$x-y-2z=0$$
$$z=0$$

$\therefore x=0, y=0, z=0$. Thus $u+v$, $u-v$ and $u-2v+w$ are independent

* Let $v_1, v_2, \ldots, v_m$ be independent vectors and suppose u is a linear combination of the v_i, say $u=a_1v_1+a_2v_2+\ldots+a_mv_m$ where the a_i are scalars. Show that the above representation of u is unique.

Proof: Suppose $u=b_1v_1+b_2v_2+\ldots+b_mv_m$ where the b_i are scalars

$$0=u-u=(a_1-b_1)v_1+(a_2-b_2)v_2+\cdots+(a_m-b_m)v_m.$$

But v_i are independent, so

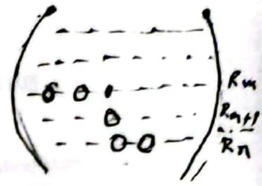
$$a_1-b_1=0, \quad a_2-b_2=0, \ldots, a_m-b_m=0$$

$$\text{or } a_1=b_1, \quad a_2=b_2, \ldots, a_m=b_m$$

* Suppose $\{v_1, v_2, \ldots, v_m\}$ is independent, but $\{v_1, v_2, \ldots, v_m, w\}$ is dependent. Show that w is a linear combination of the v_i;

Proof:- suppose $a_1v_1+a_2v_2+\ldots+a_mv_m+bw=0$.

If $b=0$, then one of the a_i is not zero and $a_1v_1+\ldots+a_mv_m=0$. But $\{v_1,\ldots,v_m\}$ is independent, so $a_1=0,\ldots,a_m=0$. Hence $b\neq 0$ and



$$R_m = a_{m+1} R_{m+1} + a_{m+2} R_{m+2} + \cdots + a_n R_n \longrightarrow (1)$$

Suppose the kth component of R_m is its first nonzero entry. Then, since the matrix is in echelon form, the kth components of $R_{m+1}, \ldots, R_n$ are all 0 and so the kth component of (1) is $a_{m+1} \cdot 0 + a_{m+2} \cdot 0 + \cdots + a_n \cdot 0 = 0$. But this contradicts the assumption that the kth component of R_m is not 0.
Hence $R_1, \ldots, R_n$ are linearly independent.

Basis and Dimension:
A vector space V is said to be of finite dimension n or to be n-dimensional if there exist linearly independent vectors $e_1, e_2, \ldots, e_n$ which span V.
Then the sequence $\{e_1, e_2, \ldots, e_n\}$ is called a basis of V. $\dim V = n$.

Examples: (1) Let K be any field. Consider the vector space K^n which consists of n-tuples of elements of K.
The vectors
$$e_1 = (1, 0, 0, \ldots, 0, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0, 0)$$
$$\cdots\cdots\cdots$$
$$e_n = (0, 0, 0, \ldots, 0, 1)$$
form a basis, called the usual basis of K^n and $\dim K^n = n$

(2) $\mathbb{R} \to$ field, $\mathbb{R}^3 \to$ vector space
The vectors
$$e_1 = (1, 0, 0)$$
$$e_2 = (0, 1, 0)$$
$$e_3 = (0, 0, 1)$$
form a basis, called the usual basis of $\mathbb{R}^3$ and $\dim \mathbb{R}^3 = 3$

Dr. Shiuly Akhter
Associate Professor
Department of Mathematics
University of Rajshahi
Rajshahi - 6205, Bangladesh

Phone : (Off.) +88-0721-711108
Cell : +88-01716-697645
Fax : +88-0721-750064
E-mail : shiuly_math_ru@yahoo.com
Web : www.ru.ac.bd/mathematics

Date ....

Problem: Determine whether or not the following form a basis for the vector space $\mathbb{R}^3$:

(i) $(1,1,1)$ and $(1,-1,5)$

(ii) $(1,2,3)$, $(1,0,-1)$, $(3,-1,0)$ and $(2,1,-2)$

(iii) $(1,1,1)$, $(1,2,3)$ and $(2,-1,1)$

Solⁿ: (i) and (ii) no; for a basis of $\mathbb{R}^3$ must contain exactly 3 elements, since $\mathbb{R}^3$ is of dimension 3.

(iii) The vectors form a basis if and only if they are independent. Thus form the matrix whose rows are the given vectors and row reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

The echelon matrix has no zero rows. Hence the three vectors are independent and so form a basis for $\mathbb{R}^3$.

Note: ① Any nonzero vector v is, by itself, independent; for

$kv = 0$, $v \neq 0$ implies $k = 0$.

② The set $\{v_1, \ldots, v_m\}$ is called a dependent or independent set according as the vectors $v_1, \ldots, v_m$ are dependent or independent. Also the empty set ϕ to be independent.

③ A set which contains a dependent subset is itself dependent.

Problem:- Let V be the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$. Determine whether $u, v, w \in V$ are independent or dependent where :

$$u = t^3 - 3t^2 + 5t + 1, \quad v = t^3 - t^2 + 8t + 2, \quad w = 2t^3 - 4t^2 + 9t + 5.$$

Solⁿ: Set a linear combination of the polynomials u, v and w equal to the zero polynomial; that is

$xu + yv + zw = 0$ where x, y, z are unknown scalars

$$\therefore x(t^3 - 3t^2 + 5t + 1) + y(t^3 - t^2 + 8t + 2) + z(2t^3 - 4t^2 + 9t + 5) = 0$$

$$\text{or } (x + y + 2z)t^3 + (-3x - y - 4z)t^2 + (5x + 8y + 9z)t + (x + 2y + 5z) = 0$$

The coefficients of the powers of t must each be 0 :

$$\begin{aligned} x + y + 2z &= 0 \\ -3x - y - 4z &= 0 \\ 5x + 8y + 9z &= 0 \\ x + 2y + 5z &= 0 \end{aligned}$$

Solving the above homogeneous system, we obtain only the zero solution: $x = 0, y = 0, z = 0$.

Hence u, v and w are independent.

~~so $w = b^{-1}(-a_1 v_1)$~~

so $w = -b^{-1}a_1 v_1 - \cdots - b^{-1}a_m v_m$

That is, w is a linear combination of the v_i.

Lemma 5.2: The nonzero vectors $v_1, \ldots, v_m$ are linearly dependent iff one of them, say v_i is a linear combination of the preceding vectors: $v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1}$.

Proof:– Suppose $v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1}$.

Then $a_1 v_1 + \cdots + a_{i-1} v_{i-1} - v_i + 0 v_{i+1} + \cdots + 0 v_m = 0$ and the coefficient of v_i is not 0.

Hence the vectors $v_1, \ldots, v_m$ are linearly dependent.

Conversely, suppose the nonzero vectors $v_1, \ldots, v_m$ are linearly dependent. Then there exist scalars $a_1, \ldots, a_m$, not all of them 0 such that $a_1 v_1 + \cdots + a_m v_m = 0$.

Let k be the largest integer such that $a_k \neq 0$.

Then $a_1 v_1 + \cdots + a_k v_k + 0 v_{k+1} + \cdots + 0 v_m = 0$

or $a_1 v_1 + \cdots + a_k v_k = 0$.

Suppose $k = 1$, then $a_1 v_1 = 0$, $a_1 \neq 0$ and so $v_1 = 0$. But $v_1, \ldots, v_m$ are nonzero vectors; hence $k > 1$ and

$$v_k = -a_k^{-1} a_1 v_1 - \cdots - a_k^{-1} a_{k-1} v_{k-1}.$$

That is, v_k is a linear combination of the preceding vectors.

Theorem 5.1: The nonzero rows $R_1, \ldots, R_n$ of a matrix in echelon form are linearly independent.

Proof:– Suppose $\{R_n, R_{n-1}, \ldots, R_1\}$ is dependent. Then one of of the rows, say R_m, is a linear combination of the preceding rows:

** Theorem 5.3 : Let V be a finite dimensional vector space. Then every basis of V has the same number of elements.

Lemma 5.4: suppose the set $\{v_1, v_2, \dots, v_n\}$ generates a vector-space V. If $\{w_1, \dots, w_m\}$ is linearly independent, then $m \leq n$ and V is generated by a set of the form $\{w_1, \dots, w_m, v_1, \dots, v_{n-m}\}$.

Definition:- Suppose S is a subset of a vector space V. We call $\{v_1, \dots, v_m\}$ a maximal independent subset of S if:

(i) it is an independent subset of S and

(ii) $\{v_1, \dots, v_m, w\}$ is dependent for any $w \in S$.

Theorem: Suppose S generates V and $\{v_1, \dots, v_m\}$ is a maximal independent subset of S.

Then $\{v_1, \dots, v_m\}$ is a basis of V.

** Theorem 5.6: Let V be of finite dimension n. Then: (i) Any set of $n+1$ or more vectors is linearly dependent

(ii) Any linearly independent set is part of a basis, i.e. can be extended to a basis.

(iii) A linearly independent set with n elements is a basis.

Example: The four vectors in $\mathbb{R}^3$,

$$(1, 5, -6), (2, 1, 8), (3, -1, 4) \text{ and } (2, 1, 1)$$

must be linearly dependent since they come from a vector space of dimension 3.

Theorem 5.3: Let V be a finite dimensional vector space. Then every basis of V has the same number of vectors.

Proof:- suppose $\{e_1, e_2, \cdots, e_n\}$ is a basis of V and suppose $\{f_1, f_2, \cdots\}$ is another basis of V. Since $\{e_1, e_2, \cdots, e_n\}$ generates V, the basis $\{f_1, f_2, \cdots\}$ must contain n or less vectors or else. If the basis $\{f_1, f_2, \cdots\}$ contains more than n vectors then it is dependent.

One the other hand, if the basis $\{f_1, f_2, \cdots\}$ contains less than n vectors, then $\{e_1, e_2, \cdots, e_n\}$ is dependent,

Thus the basis $\{f_1, f_2, \cdots\}$ contains exactly n vectors.

Therefore every basis of a finite dimensional vector space V has the same number of vectors.

Theorem: Let W be a subspace of an n-dimensional vector space V. Then $\dim W \leq n$. In particular, if $\dim W = n$, then $W = V$

Proof:- Since V is of dimension n, so $n+1$ or more vectors are linearly dependent.

Also, since a basis of W consists of linearly independent vectors, it cannot contain more than n vectors. Accordingly, $\dim W \leq n$.

In particular, if $\{w_1, \cdots, w_n\}$ is a basis of W, then since it is an independent set with n elements so it is also a basis of V. Thus $W = V$ when $\dim W = n$.

Theorem 5.8:— Let U and W be finite dimensional subspaces of a vector space V. Then $U+W$ has finite dimension and

$$\dim(U+W) = \dim U + \dim W - \dim(U\cap W).$$

Proof:- Observe that $U\cap W$ is a subspace of both U and W. Suppose $\dim U = m$ and $\dim W = n$ and $\dim(U\cap W) = r$.

Suppose $\{v_1, \cdots, v_r\}$ is a basis of $U\cap W$. Then we can extend $\{v_1, \cdots, v_r\}$ to a basis of U and to a basis of W; say,

$\{v_1, \cdots, v_r, u_1, \cdots, u_{m-r}\}$ is a basis of U and $\{v_1, \cdots, v_r, w_1, \cdots, w_{n-r}\}$ is a basis of W.

Let $B = \{v_1, \cdots, v_r, u_1, \cdots, u_{m-r}, w_1, \cdots, w_{n-r}\}$

Here B has exactly $m+n-r$ elements. Thus the theorem is proved if we can show that B is a basis of $U+W$.

Since $\{v_i, u_j\}$ generates U and $\{v_i, w_k\}$ generates W, so the union $B = \{v_i, u_j, w_k\}$ generates $U+W$. Hence it suffices to show that B is independent.

Suppose $a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_{m-r}u_{m-r} + c_1w_1 + \cdots + c_{n-r}w_{n-r} = 0 \longrightarrow (1)$

where a_i, b_j, c_k are scalars.

Let $v = a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_{m-r}u_{m-r} \longrightarrow (2)$

Thus, by (1), also we have that

$$v = -c_1w_1 - \cdots - c_{n-r}w_{n-r} \longrightarrow (3)$$

Since $\{v_i, u_j\} \subset U$, so $v \in U$ [by (2)]

and since $\{w_k\} \subset W$, so $v \in W$ [by (3)].

Hence $v \in U\cap W$. Also since $\{v_1, \cdots, v_r\}$ is a basis of $U\cap W$ so there exist scalars $d_1, \cdots, d_r$ for which $v = d_1v_1 + \cdots + d_rv_r$

Thus by (3) we have

$$d_1v_1+\cdots+d_rv_r+c_1w_1+\cdots+c_{n-r}w_{n-r}=0 \longrightarrow (4)$$

But $\{v_i, w_k\}$ is a basis of W and so it is independent.

Hence the equation (4) gives

$$d_1=0,\ \ldots,\ d_r=0,\ c_1=0,\ \ldots,\ c_{n-r}=0.$$

Substituting $c_1=0,\ \ldots,\ c_{n-r}=0$ into (1), we obtain

$$a_1v_1+\cdots+a_rv_r+b_1u_1+\cdots+b_{m-r}u_{m-r}=0 \longrightarrow (5)$$

But $\{v_i, u_j\}$ is a basis of U and so is independent.

Hence [the equation (5) forces] $a_1=0,\ \ldots,\ a_r=0,\ b_1=0\ \ldots,\ b_{m-r}=0$

Since the equation (1) implies that a_i, b_j and c_k are all 0,

so ~~B = {v~~ $B=\{v_i, u_j, w_k\}$ is independent.

Thus $B=\{v_i, u_j, w_k\}$ is a basis of $U+W$.

That is $\dim(U+W)=m+n-r=\dim U+\dim W-\dim(U\cap W)$.

— x —

Rank of a matrix: Let A be an arbitrary $m\times n$ matrix over a field K. The row space of A is the subspace of K^n generated by its rows and the column space of A is the subspace of K^m generated by its colums. The dimensions of the row space and of the column space of A are called the row rank and the column rank of A.

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The rank of the matrix A is the common value of its row rank and column rank.