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Date ....

Linear Mappings

Let A and B be two sets. Suppose to each $a \in A$ there is assigned a unique element of B; the collection f, of such assignments is called a function or mapping from A into B and is written $f: A \to B$ or $A \xrightarrow{f} B$.

We write $f(a)$, for the element of B that f assigns to $a \in A$; it is called the value of f at a or the image of a under f.

The set of all images, i.e. $f(A)$ is called the image (or range) of f. Also, A is called the domain of the mapping $f: A \to B$ and B is called its co-domain.

* A mapping $f: A \to B$ is said to be one-to-one (or one-one) or injective if different elements of A have distinct images;

That is, if $a \neq a'$ implies $f(a) \neq f(a')$

or, if $f(a) = f(a')$ implies $a = a'$

* A mapping $f: A \to B$ is said to be onto or surjective if every $b \in B$ is the image of at least one $a \in A$.

** A mapping which is both one-one and onto is said to be bijective.

Linear mapping: Let V and U be vector spaces over the same field K. A mapping $F: V \to U$ is called a linear mapping (or linear transformation or vector space homomorphism) if it satisfies the following two conditions:

~~① For any $k \in K$ and any $v \in$~~

① For any $v, w \in V$, $F(v+w) = F(v) + F(w)$.

② For any $k \in K$ and any $v \in V$, $F(kv) = kF(v)$

Substituting $k=0$ into ② we obtain, $F(0)=0$.

That is, every linear mapping takes the zero vector into the zero vector.

For any scalars $a, b \in K$ and any vectors $v, w \in V$ we obtain, by applying both conditions of linearity

$$F(av+bw) = F(av) + F(bw) = aF(v) + bF(w)$$

Ex. Let $F: V \to U$ be the mapping which assigns $0 \in U$ to every $v \in V$. Then for any $v, w \in V$ and any $k \in K$,

$$F(v+w) = 0 = 0+0 = F(v) + F(w) \text{ and}$$

$$F(kv) = 0 = k \cdot 0 = kF(v).$$

Hence F is linear mapping.

We call F the zero mapping and shall usually denote it by 0.

Problem:

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spaces called

Problem: Show that the following mappings F are linear:

(i) $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x,y) = (2x-y, x)$.

(ii) $F: \mathbb{R} \to \mathbb{R}^2$ defined by $F(x) = (2x, 3x)$

Defn: A linear mapping $F: V \to U$ is called an isomorphism if it is one-to-one. The vector spaces V and U are said to be isomorphic if there is an isomorphism of V onto U.

Theorem 6.2: Let V and U be vector spaces over a field K. Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V and let $u_1, u_2, \dots, u_n$ be any arbitrary vectors in U. Then there exists a unique linear mapping $F: V \to U$ such that $F(v_1) = u_1$, $F(v_2) = u_2$, $\dots$, $F(v_n) = u_n$.

Proof: There are three steps to the proof of the theorem:

(i) Define a mapping $F: V \to U$ such that $F(v_i) = u_i$; $i = 1, 2, \dots,$

(ii) Show that F is linear. (iii) Show that F is unique.

Step (i): Let $v \in V$. Since $\{v_1, \dots, v_n\}$ is a basis of V, so there exist unique scalars $a_1, \dots, a_n \in K$ for which $v = a_1v_1 + \dots + a_nv_n$.

We define $F: V \to U$ by $F(v) = a_1u_1 + a_2u_2 + \dots + a_nu_n$.

Now, for $i = 1, 2, \dots, n$,

$$v_i = 0v_1 + \dots + 1v_i + \dots + 0v_n$$

Hence $F(v_i) = 0u_1 + \dots + 1u_i + \dots + 0u_n = u_i$.

Step (ii): suppose $v = a_1v_1 + a_2v_2 + \dots + a_nv_n$ and

$$w = b_1v_1 + b_2v_2 + \dots + b_nv_n.$$

Then $v + w = (a_1+b_1)v_1 + (a_2+b_2)v_2 + \dots + (a_n+b_n)v_n$

and for any $k \in K$, $kv = ka_1v_1 + ka_2v_2 + \dots + ka_nv_n$.

~~Then~~ By the definition of the mapping F,

$F(v) = a_1u_1 + a_2u_2 + \dots + a_nu_n$ and $F(w) = b_1u_1 + b_2u_2 + \dots + b_nu_n$

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Hence $F(v+w) = (a_1+b_1)u_1 + (a_2+b_2)u_2 + \cdots + (a_n+b_n)u_n$

$= (a_1u_1 + a_2u_2 + \cdots + a_nu_n) + (b_1u_1 + b_2u_2 + \cdots + b_nu_n)$

$= F(v) + F(w)$

and $F(kv) = k(a_1u_1 + a_2u_2 + \cdots + a_nu_n) = kF(v)$.

Therefore F is linear.

Step (iii): Suppose $G : V \to U$ is linear and $G(v_i) = u_i$, $i = 1, 2, \cdots, n$

If $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ then

$G(v) = G(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1G(v_1) + a_2G(v_2) + \cdots + a_nG(v_n)$

$= a_1u_1 + a_2u_2 + \cdots + a_nu_n = F(v)$.

Since $G(v) = F(v)$ for every $v \in V$, so $G = F$.

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Show that the following mapping F is linear //

$F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = 2x - 3y + 4z$.

Sol^n: Let $v = (a, b, c)$ and $w = (a', b', c')$.

Hence $v + w = (a+a', b+b', c+c')$ and $kv = (ka, kb, kc)$, $k \in \mathbb{R}$

$F(v) = F(a, b, c) = 2a - 3b + 4c$ and

$F(w) = F(a', b', c') = 2a' - 3b' + 4c'$

$\therefore F(v+w) = F(a+a', b+b', c+c')$

$= 2(a+a') - 3(b+b') + 4(c+c')$

$= (2a - 3b + 4c) + (2a' - 3b' + 4c') = F(v) + F(w)$

$F(kv) = F(ka, kb, kc) = 2ka - 3kb + 4kc = kF(v)$

Hence F is linear.

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Image and kernel of a linear mapping:

Let $F: V \to U$ be a linear mapping. The image of F, written $\operatorname{Im} F$, is the set of image points in U:

$$\operatorname{Im} F = \{u \in U : F(v) = u \text{ for some } v \in V\}.$$

The Kernel of F, written $\ker F$, is the set of elements in V which map into $0 \in U$:

$$\ker F = \{v \in V : F(v) = 0\}.$$

Theorem 6.3: Let $F: V \to U$ be a linear mapping. Then (i) the image of F is a subspace of U and (ii) the kernel of F is a subspace of V.

Proof: (i) Since $F(0) = 0$, so $0 \in \operatorname{Im} F$.

Suppose $u, u' \in \operatorname{Im} F$ and $a, b \in K$.

Since u, u' belong to the image of F, so there exist vectors v, v' belong to V such that $F(v) = u$ and $F(v') = u'$.

Then $F(av + bv') = aF(v) + bF(v') = au + bu' \in \operatorname{Im} F$.

Hence the image of F is a subspace of U.

(ii) Since $F(0) = 0$ so $0 \in \ker F$.

Suppose $v, w \in \ker F$ and $a, b \in K$. Since $v, w \in \ker F$,

$$F(v) = 0 \text{ and } F(w) = 0.$$

Then $F(av + bw) = aF(v) + bF(w) = a.0 + b.0 = 0$

and so $av + bw \in \ker F$.

Thus the kernel of F is a subspace of V.

Problem: Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by $F(x, y, s, t) = (x - y + s + t,\ x + 2s - t,\ x + y + 3s - 3t)$.

Find a basis and the dimension of the (i) image U of F, (ii) Kernel W of F

Sol^n: (i) The images of the following generators of $\mathbb{R}^4$ generate the image U of F:

$$F(1,0,0,0) = (1,1,1)$$
$$F(0,1,0,0) = (-1,0,1)$$
$$F(0,0,1,0) = (1,2,3)$$
$$F(0,0,0,1) = (1,-1,-3)$$

Form the matrix whose rows are the generators of U and row reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $\{(1,1,1), (0,1,2)\}$ is a basis of U and so $\dim U = 2$.

(ii) We seek the set of (x, y, s, t) such that

$$F(x, y, s, t) = (0,0,0),$$

i.e., $F(x, y, s, t) = (x - y + s + t,\ x + 2s - t,\ x + y + 3s - 3t) = (0,0,0)$

$$\begin{aligned} x - y + s + t &= 0 \\ x + 2s - t &= 0 \\ x + y + 3s - 3t &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y + s + t &= 0 \\ y + s - 2t &= 0 \end{aligned}$$

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Here the free variables are s and t.
Hence $\dim W = 2$.

set ① $s=-1$, $t=0$ to obtain the solution $(2,1,-1,0)$
② $s=0$, $t=1$ to obtain the solution $(1,2,0,1)$

Thus $\{(2,1,-1,0),(1,2,0,1)\}$ is a basis of W.

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$\dim U + \dim W = 2+2 = 4 = \dim \mathbb{R}^4$ [$\mathbb{R}^4 \to$ domain of F]

Theorem 6.4: Let V be of finite dimension and let $F: V \longrightarrow U$ be a linear mapping. Then

$$\dim V = \dim(\operatorname{Ker} F) + \dim(\operatorname{Im} F).$$

Problem: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$. Find a basis and the ~~domai~~ dimension of the

ⓘ image U of T ⓘⓘ kernel W of T.

* Suppose that the vectors $v_1, \dots, v_n$ generate and that $F: V \to U$ is linear. We show that the vectors $F(v_1), \dots, F(v_n) \in U$ generate $\operatorname{Im} F$.

Proof: Suppose $u \in \operatorname{Im} F$, then $F(v) = u$ for some vector $v \in V$. Since $v_1, \dots, v_n$ generate V and since $v \in V$, so there exist scalars $a_1, \dots, a_n$ for which

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n .$$

So
$$u = F(v) = F(a_1 v_1 + a_2 v_2 + \dots + a_n v_n)$$
$$= a_1 F(v_1) + a_2 F(v_2) + \dots + a_n F(v_n)$$

and hence the vectors $F(v_1), \dots, F(v_n)$ generate $\operatorname{Im} F$.

*/Ex. 6.19 4×3 matrix A and $A: K^3 \to K^4$

Theorem 6.4: Let V be of finite dimension and let $F: V \to U$ be a linear mapping. Then

$$\dim V = \dim(\operatorname{Ker} F) + \dim(\operatorname{Im} F).$$

** Let $F: V \to U$ be a linear mapping. Then the rank of F is defined to be the dimension of its image and the nullity of F is defined to be the dimension of its kernel: $\operatorname{rank}(F) = \dim(\operatorname{Im} F)$ and $\operatorname{nullity}(F) = \dim(\operatorname{Ker} F)$

$$\dim V = \operatorname{nullity}(F) + \operatorname{rank}(F).$$

Singular and nonsingular mappings:

A linear mapping $F: V \to U$ is said to be singular if the image of some nonzero vector under F is 0, i.e. if there exists $v \in V$ for which $v \neq 0$ but $F(v) = 0$.

A linear mapping $F: V \to U$ is nonsingular if only $0 \in V$ maps into $0 \in U$ or equivalently, if $\operatorname{Ker} F = \{0\}$.

generate the that

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Theorem 6.5: A linear mapping $F: V \to U$ is an isomorphism if and only if it is non-singular.

Proof: If the linear mapping $F: V \to U$ is an isomorphism, then only $0 \in V$ can map into $0 \in U$, i.e. $\ker F = \{0\}$ and so F is nonsingular.

Conversely, suppose F is non singular and $F(v) = F(w)$; then $F(v-w) = F(v) - F(w) = 0$ and hence $v - w = 0$ or $v = w$.

Thus $F(v) = F(w)$ implies $v = w$ and so F is one-to-one. Hence F is an isomorphism.

[A one-to-one linear mapping is called an isomorphism.]

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Operations with linear mappings

Suppose $F: V \to U$ and $G: V \to U$ are linear mappings of vector spaces over a field K.

We define the sum $F+G$ to be the mapping from V into U which assigns $F(v) + G(v)$ to $v \in V$:

$$(F+G)(v) = F(v) + G(v).$$

Also, for any scalar $k \in K$, we define the product kF to be the mapping from V into U which assigns $kF(v)$ to $v \in V$:

$$(kF)(v) = kF(v).$$

Now we show that if F and G are linear, then $F+G$ and kF are also linear.

For any vectors $v, w \in V$ and any scalars $a, b \in K$,

$$(F+G)(av+bw) = F(av+bw) + G(av+bw)$$
$$= aF(v) + bF(w) + aG(v) + bG(w)$$
$$= a\big(F(v)+G(v)\big) + b\big(F(w)+G(w)\big)$$
$$= a(F+G)(v) + b(F+G)(w)$$

and
$$(kF)(av+bw) = kF(av+bw) = k\big(aF(v)+bF(w)\big)$$
$$= akF(v) + bkF(w)$$
$$= a(kF)(v) + b(kF)(w).$$

Thus $F+G$ and kF are linear.

Theorem 6.6: Let V and U be vector spaces over a field K. Then the collection of all linear mappings from V into U with above operations of addition and scalar multiplication form a vector space over K.

Definition: Let V and U be vector spaces over a field K. Then the collection of all linear mappings from V into U with the operations of addition $[(F+G)(v)=F(v)+G(v)]$ and scalar multiplication $[(kF)(v)=kF(v)]$ form a vector space over K. This space is usually denoted by

$$\mathrm{Hom}(V, U)$$

Theorem 6.7: Suppose $\dim V = m$ and $\dim U = n$. Then $\dim H(V, U) = mn$.

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Algebra of linear operators

Let V be a vector space over a field K. We consider the special case of linear mappings $T: V \to V$. They are called linear operators on V. We will write $A(V)$, instead of $Hom(V, V)$ for the space of all such mappings.

$A(V)$ is a vector space over K; it is of dimension n^2 if V is of dimension n. If $S, T \in A(V)$, then the composition $S \circ T$ exists and is also a linear mapping from V into itself, i.e. $S \circ T \in A(V)$

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Invertible operators

A linear operator $T: V \to V$ is said to be invertible if it has an inverse, i.e. if there exists $T^{-1} \in A(V)$ such that $TT^{-1} = T^{-1}T = I$.

$$T: V \to V$$