Chapter 5

Equivalence

THE RANK OF A MATRIX. A non-zero matrix A is said to have **rank** r if at least one of its r-square minors is different from zero while every $(r+1)$-square minor, if any, is zero. A zero matrix is said to have rank 0.

Example 1. The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ is $r = 2$ since $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$ while $|A| = 0$.

See Problem 1.

An n-square matrix A is called **non-singular** if its rank $r = n$, that is, if $|A| \neq 0$. Otherwise, A is called **singular**. The matrix of Example 1 is singular.

From $|AB| = |A| \cdot |B|$ follows

I. The product of two or more non-singular n-square matrices is non-singular; the product of two or more n-square matrices is singular if at least one of the matrices is singular.

ELEMENTARY TRANSFORMATIONS. The following operations, called **elementary transformations**, on a matrix do not change either its order or its rank:

(1) The interchange of the ith and jth rows, denoted by H_{ij};
The interchange of the ith and jth columns, denoted by K_{ij}.

(2) The multiplication of every element of the ith row by a non-zero scalar k, denoted by $H_i(k)$;
The multiplication of every element of the ith column by a non-zero scalar k, denoted by $K_i(k)$.

(3) The addition to the elements of the ith row of k, a scalar, times the corresponding elements of the jth row, denoted by $H_{ij}(k)$;
The addition to the elements of the ith column of k, a scalar, times the corresponding elements of the jth column, denoted by $K_{ij}(k)$.

The transformations H are called **elementary row transformations**; the transformations K are called **elementary column transformations**.

The elementary transformations, being precisely those performed on the rows (columns) of a determinant, need no elaboration. It is clear that an elementary transformation cannot alter the order of a matrix. In Problem 2, it is shown that an elementary transformation does not alter its rank.

THE INVERSE OF AN ELEMENTARY TRANSFORMATION. The **inverse** of an elementary transformation is an operation which undoes the effect of the elementary transformation; that is, after A has been subjected to one of the elementary transformations and then the resulting matrix has been subjected to the inverse of that elementary transformation, the final result is the matrix A.

Example 2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

The effect of the elementary row transformation $H_{21}(-2)$ is to produce $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$.

The effect of the elementary row transformation $H_{21}(+2)$ on B is to produce A again. Thus, $H_{21}(-2)$ and $H_{21}(+2)$ are inverse elementary row transformations.

The inverse elementary transformations are:

(1') $\quad H_{ij}^{-1} = H_{ij} \qquad\qquad K_{ij}^{-1} = K_{ij}$

(2') $\quad H_i^{-1}(k) = H_i(1/k) \qquad\qquad K_i^{-1}(k) = K_i(1/k)$

(3') $\quad H_{ij}^{-1}(k) = H_{ij}(-k) \qquad\qquad K_{ij}^{-1}(k) = K_{ij}(-k)$

We have

II. The inverse of an elementary transformation is an elementary transformation of the same type.

EQUIVALENT MATRICES. Two matrices A and B are called **equivalent**, $A \sim B$, if one can be obtained from the other by a sequence of elementary transformations.

Equivalent matrices have the same order and the same rank.

Example 3. Applying in turn the elementary transformations $H_{21}(-2)$, $H_{31}(1)$, $H_{32}(-1)$,

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Since all 3-square minors of B are zero while $\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} \neq 0$, the rank of B is 2; hence, the rank of A is 2. This procedure of obtaining from A an equivalent matrix B from which the rank is evident by inspection is to be compared with that of computing the various minors of A.

See Problem 3.

ROW EQUIVALENCE. If a matrix A is reduced to B by the use of elementary row transformations alone, B is said to be **row equivalent** to A and conversely. The matrices A and B of Example 3 are row equivalent.

Any non-zero matrix A of rank r is row equivalent to a **canonical matrix** C in which

(*a*) one or more elements of each of the first r rows are non-zero while all other rows have only zero elements.

(*b*) in the ith row, $(i = 1, 2, \ldots, r)$, the first non-zero element is 1; let the column in which this element stands be numbered j_i.

(*c*) $j_1 < j_2 < \cdots < j_r$.

(*d*) the only non-zero element in the column numbered j_i, $(i = 1, 2, \ldots, r)$, is the element 1 of the ith row.

To reduce A to C, suppose j_1 is the number of the first non-zero column of A.

($\mathbf{i}_1$) If $a_{1j_1} \neq 0$, use $H_1(1/a_{1j_1})$ to reduce it to 1, when necessary.

($\mathbf{i}_2$) If $a_{ij_1} = 0$ but $a_{pj_1} \neq 0$, use H_{1p} and proceed as in ($\mathbf{i}_1$).

(**ii**) Use row transformations of type (**3**) with appropriate multiples of the first row to obtain zeros elsewhere in the j_1st column.

If non-zero elements of the resulting matrix B occur only in the first row, $B = C$. Otherwise, suppose j_2 is the number of the first column in which this does not occur. If $b_{2j_2} \neq 0$, use $H_2(1/b_{2j_2})$ as in ($\mathbf{i}_1$); if $b_{2j_2} = 0$ but $b_{qj_2} \neq 0$, use H_{2q} and proceed as in ($\mathbf{i}_1$). Then, as in (**ii**), clear the j_2nd column of all other non-zero elements.

If non-zero elements of the resulting matrix occur only in the first two rows, we have C. Otherwise, the procedure is repeated until C is reached.

Example 4. The sequence of row transformations $H_{21}(-2)$, $H_{31}(1)$; $H_2(1/5)$; $H_{12}(1)$, $H_{32}(-5)$ applied to A of Example 3 yields

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 17/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= C$$

having the properties (a)-(d). See Problem 4.

THE NORMAL FORM OF A MATRIX. By means of elementary transformations any matrix A of rank $r > 0$ can be reduced to one of the forms

$$(5.1) \qquad I_r, \quad \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad [I_r \;\; 0], \quad \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

called its **normal form**. A zero matrix is its own normal form.

Since both row and column transformations may be used here, the element 1 of the first row obtained in the section above can be moved into the first column. Then both the first row **and** first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column, and so on.

For example, the sequence $H_{21}(-2)$, $H_{31}(1)$, $K_{21}(-2)$, $K_{31}(1)$, $K_{41}(-4)$, K_{23}, $K_2(1/5)$, $H_{32}(-1)$, $K_{42}(3)$ applied to A of Example 3 yields $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, the normal form.

See Problem 5.

ELEMENTARY MATRICES. The matrix which results when an elementary row (column) transformation is applied to the identity matrix I_n is called an **elementary row (column) matrix**. Here, an elementary matrix will be denoted by the symbol introduced to denote the elementary transformation which produces the matrix.

Example 5. Examples of elementary matrices obtained from $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are:

$$H_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = K_{12}, \qquad H_3(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix} = K_3(k), \qquad H_{23}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} = K_{32}(k)$$

Every elementary matrix is non-singular. (Why?)

The effect of applying an elementary transformation to an $m \times n$ matrix A can be produced by multiplying A by an elementary matrix.

To effect a given elementary row transformation on A of order $m \times n$, apply the transformation to I_m to form the corresponding elementary matrix H and multiply A **on the left** by H.

To effect a given elementary column transformation on A, apply the transformation to I_n to form the corresponding elementary matrix K and multiply A **on the right** by K.

Example 6. When $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $H_{13} \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ interchanges the first and third rows of A; $AK_{13}(2) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 3 \\ 16 & 5 & 6 \\ 25 & 8 & 9 \end{bmatrix}$ adds to the first column of A two times the third column.

LET A AND B BE EQUIVALENT MATRICES. Let the elementary row and column matrices corresponding to the elementary row and column transformations which reduce A to B be designated as $H_1, H_2, \ldots, H_s$; $K_1, K_2, \ldots, K_t$ where H_1 is the first row transformation, H_2 is the second, ...; K_1 is the first column transformation, K_2 is the second, ... Then

$$H_s \ldots H_2 \cdot H_1 \cdot A \cdot K_1 \cdot K_2 \ldots K_t = PAQ = B \tag{5.2}$$

where

$$P = H_s \ldots H_2 \cdot H_1 \quad \text{and} \quad Q = K_1 \cdot K_2 \ldots K_t \tag{5.3}$$

We have

III. Two matrices A and B are equivalent if and only if there exist non-singular matrices P and Q defined in (5.3) such that $PAQ = B$.

Example 7. When $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$, $H_{31}(-1) \cdot H_{21}(-2) \cdot A \cdot K_{21}(-2) \cdot K_{31}(1) \cdot K_{41}(-2) \cdot K_{42}(1) \cdot K_3(\frac{1}{2})$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & -2 & \frac{1}{2} & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = PAQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = B$$

Since any matrix is equivalent to its normal form, we have

IV. If A is an n-square non-singular matrix, there exist non-singular matrices P and Q as defined in (5.3) such that $PAQ = I_n$.

See Problem 6.

INVERSE OF A PRODUCT OF ELEMENTARY MATRICES. Let

$$P = H_s \ldots H_2 \cdot H_1 \quad \text{and} \quad Q = K_1 \cdot K_2 \ldots K_t$$

as in (5.3). Since each H and K has an inverse and since the inverse of a product is the product in reverse order of the inverses of the factors

$$P^{-1} = H_1^{-1} \cdot H_2^{-1} \ldots H_s^{-1} \quad \text{and} \quad Q^{-1} = K_t^{-1} \ldots K_2^{-1} \cdot K_1^{-1} \tag{5.4}$$

Let A be an n-square non-singular matrix and let P and Q defined above be such that $PAQ = I_n$. Then

$$A = P^{-1}(PAQ)Q^{-1} = P^{-1} \cdot I_n \cdot Q^{-1} = P^{-1} \cdot Q^{-1} \tag{5.5}$$

We have proved

V. Every non-singular matrix can be expressed as a product of elementary matrices.

See Problem 7.

From this follow

VI. If A is non-singular, the rank of AB (also of BA) is that of B.

VII. If P and Q are non-singular, the rank of PAQ is that of A.

CANONICAL SETS UNDER EQUIVALENCE. In Problem 8, we prove

VIII. Two $m \times n$ matrices A and B are equivalent if and only if they have the same rank.

A set of $m \times n$ matrices is called a **canonical set** under equivalence if every $m \times n$ matrix is equivalent to one and only one matrix of the set. Such a canonical set is given by (5.1) as r ranges over the values $1, 2, \ldots, m$ or $1, 2, \ldots, n$ whichever is the smaller.

See Problem 9.

RANK OF A PRODUCT. Let A be an $m \times p$ matrix of rank r. By Theorem **III** there exist non-singular matrices P and Q such that

$$PAQ = N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Then $A = P^{-1}NQ^{-1}$. Let B be a $p \times n$ matrix and consider the rank of

$$AB = P^{-1}NQ^{-1}B \tag{5.6}$$

By Theorem **VI**, the rank of AB is that of $NQ^{-1}B$. Now the rows of $NQ^{-1}B$ consist of the first r rows of $Q^{-1}B$ and $m-r$ rows of zeros. Hence, the rank of AB cannot exceed r, the rank of A. Similarly, the rank of AB cannot exceed that of B. We have proved

IX. The rank of the product of two matrices cannot exceed the rank of either factor.

Suppose $AB = 0$; then from (5.6), $NQ^{-1}B = 0$. This requires that the first r rows of $Q^{-1}B$ be zeros while the remaining rows may be arbitrary. Thus, the rank of $Q^{-1}B$ and, hence, the rank of B cannot exceed $p-r$. We have proved

X. If the $m \times p$ matrix A is of rank r and if the $p \times n$ matrix B is such that $AB = 0$, the rank of B cannot exceed $p-r$.

SOLVED PROBLEMS

1. (a) The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 5 \end{bmatrix}$ is 2 since $\begin{vmatrix} 1 & 2 \\ -4 & 0 \end{vmatrix} \neq 0$ and there are no minors of order three.

(b) The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 8 \end{bmatrix}$ is 2 since $|A| = 0$ and $\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} \neq 0$.

(c) The rank of $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$ is 1 since $|A| = 0$, each of the nine 2-square minors is 0, but

every element is 0.

2. Show that the elementary transformations do not alter the rank of a matrix.

We shall consider only row transformations here and leave consideration of the column transformatio as an exercise. Let the rank of the $m \times n$ matrix A be r so that every $(r+1)$-square minor of A, if any, is ze Let B be the matrix obtained from A by a row transformation. Denote by $|R|$ any $(r+1)$-square minor of A a by $|S|$ the $(r+i)$-square minor of B having the same position as $|R|$.

Let the row transformation be H_{ij}. Its effect on $|R|$ is either (i) to leave it unchanged, (ii) to intercha two of its rows, or (iii) to interchange one of its rows with a row not of $|R|$. In the case (i), $|S| = |R| =$ in the case (ii), $|S| = -|R| = 0$; in the case (iii), $|S|$ is, except possibly for sign, another $(r+1)$-square min of $|A|$ and, hence, is 0.

Let the row transformation be $H_i(k)$. Its effect on $|R|$ is either (i) to leave it unchanged or (ii) to mul ply one of its rows by k. Then, respectively, $|S| = |R| = 0$ or $|S| = k|R| = 0$.

Let the row transformation be $H_{ij}(k)$. Its effect on $|R|$ is either (i) to leave it unchanged, (ii) to increa one of its rows by k times another of its rows, or (iii) to increase one of its rows by k times a row not of $|R$ In the cases (i) and (ii), $|S| = |R| = 0$; in the case (iii), $|S| = |R| \pm k$ (another $(r+1)$-square minor of A) $0 \pm k \cdot 0 = 0$.

Thus, an elementary row transformation cannot raise the rank of a matrix. On the other hand, it cann lower the rank for, if it did, the inverse transformation would have to raise it. Hence, an elementary ro transformation does not alter the rank of a matrix.

3. For each of the matrices A obtain an equivalent matrix B and from it, by inspection, determine th rank of A.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = B$

The transformations used were $H_{21}(-2)$, $H_{31}(-3)$; $H_2(-1/3)$, $H_3(-1/4)$; $H_{32}(-1)$. The rank is 3.

(b) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$. The rank is 3.

(c) $A = \begin{bmatrix} 1 & 1+i & -i \\ 0 & i & 1+2i \\ 1 & 1+2i & 1+i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 1+2i \\ 1 & i & 1+2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 1+2i \\ 0 & 0 & 0 \end{bmatrix} = B$. The rank is 2.

Note. The equivalent matrices B obtained here are not unique. In particular, since in (a) and (b) only row transformations were used, the reader may obtain others by using only column transformations. When the elements are rational numbers, there generally is no gain in mixing row and column transformations.

4. Obtain the canonical matrix C row equivalent to each of the given matrices A.

(a) $$A = \begin{bmatrix} 0 & 0 & 1 & 3 & -2 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 2 & 3 & 9 & 2 \\ 0 & 1 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 2 & 3 & 9 & 2 \\ 0 & 0 & 1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C$$

(b) $$A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 & 7 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = C$$

5. Reduce each of the following to normal form.

(a) $$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 2 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 11 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$= [I_3 \;\; 0]$$

The elementary transformations are:

$H_{21}(-3),\ H_{31}(2);\ K_{21}(-2),\ K_{41}(1);\ K_{23};\ H_{32}(-2);\ K_{32}(2),\ K_{42}(-5);\ K_3(1/11),\ K_{43}(7)$

(b) $$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 2 & 8 & 13 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

The elementary transformations are:

$H_{12};\ K_1(\tfrac{1}{2});\ H_{31}(-2);\ K_{21}(-3),\ K_{31}(-5),\ K_{41}(-4);\ K_2(\tfrac{1}{2});\ K_{32}(-3),\ K_{42}(-4);\ H_{32}(-1)$

6. Reduce $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ to normal form N and compute the matrices P_1 and Q_1 such that $P_1 A Q_1 = N$.

Since A is 3×4, we shall work with the array $\begin{matrix} I_4 & \\ A & I_3 \end{matrix}$. Each row transformation is performed on a row of seven elements and each column transformation is performed on a column of seven elements.

$$\begin{array}{rrrrrrr} 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 1 & 2 & 3 & -2 & 1 & 0 & 0 \\ 2 & -2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 0 & 4 & 1 & 0 & 0 & 1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 1 & 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & -6 & -5 & 7 & -2 & 1 & 0 \\ 0 & -6 & -5 & 7 & -3 & 0 & 1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1 & -2 & -3 & 2 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -6 & -5 & 7 & -2 & 1 & 0 \\ 0 & -6 & -5 & 7 & -3 & 0 & 1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1 & -2 & -3 & 2 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -6 & -5 & 7 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{array}$$

$$\rightarrow \begin{array}{rrrrrrr} 1 & 1/3 & -3 & 2 & & & \\ 0 & -1/6 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -5 & 7 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1 & 1/3 & -4/3 & -1/3 & & & \\ 0 & -1/6 & -5/6 & 7/6 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{array} \quad \text{or} \quad \begin{array}{cc} Q_1 & \\ N & P_1 \end{array}$$

Thus. $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$, $Q_1 = \begin{bmatrix} 1 & 1/3 & -4/3 & -1/3 \\ 0 & -1/6 & -5/6 & 7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $P_1AQ_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = N$.

7. Express $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ as a product of elementary matrices.

The elementary transformations $H_{21}(-1)$, $H_{31}(-1)$; $K_{21}(-3)$, $K_{31}(-3)$ reduce A to I_3, that is, [see (5

$$I \;=\; H_2\cdot H_1\cdot A\cdot K_1\cdot K_2 \;=\; H_{31}(-1)\cdot H_{21}(-1)\cdot A\cdot K_{21}(-3)\cdot K_{31}(-3)$$

From (5.5), $$A = H_1^{-1}\cdot H_2^{-1}\cdot K_2^{-1}\cdot K_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Prove: Two $m\times n$ matrices A and B are equivalent if and only if they have the same rank.

If A and B have the same rank, both are equivalent to the same matrix (5.1) and are equivalent to e other. Conversely, if A and B are equivalent, there exist non-singular matrices P and Q such that $B = P$ By Theorem VII, A and B have the same rank.

9. A canonical set for non-zero matrices of order 3 is

$$I_3,\quad \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A canonical set for non-zero 3×4 matrices is

$$[I_3\;\; 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

10. If from a square matrix A of order n and rank r_A, a submatrix B consisting of s rows (columns) of A is selected, the rank r_B of B is equal to or greater than $r_A + s - n$.

The normal form of A has $n - r_A$ rows whose elements are zeros and the normal form of B has $s - r_B$ rows whose elements are zeros. Clearly

$$n - r_A \;\geq\; s - r_B$$

from which follows $r_B \geq r_A + s - n$ as required.

SUPPLEMENTARY PROBLEMS

11. Find the rank of (*a*) $\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}$, (*b*) $\begin{bmatrix}1&2&1&2\\1&3&2&2\\2&4&3&4\\3&7&4&6\end{bmatrix}$, (*c*) $\begin{bmatrix}1&2&-2&3\\2&5&-4&6\\-1&-3&2&-2\\2&4&-1&6\end{bmatrix}$, (*d*) $\begin{bmatrix}3&4&5&6&7\\4&5&6&7&8\\5&6&7&8&9\\10&11&12&13&14\\15&16&17&18&19\end{bmatrix}$.

Ans. (*a*) 2, (*b*) 3, (*c*) 4, (*d*) 2

12. Show by considering minors that A, A', $\bar{A}$, and $\bar{A}'$ have the same rank.

13. Show that the canonical matrix C, row equivalent to a given matrix A, is uniquely determined by A.

14. Find the canonical matrix row equivalent to each of the following:

(*a*) $\begin{bmatrix}1&2&-3\\2&5&-4\end{bmatrix}\sim\begin{bmatrix}1&0&-7\\0&1&2\end{bmatrix}$ (*b*) $\begin{bmatrix}1&2&3&4\\3&4&1&2\\4&3&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&1/9\\0&1&0&1/9\\0&0&1&11/9\end{bmatrix}$ (*c*) $\begin{bmatrix}1&1&1&2\\2&1&-3&-6\\3&-3&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\end{bmatrix}$

(*d*) $\begin{bmatrix}1&2&1&0\\3&2&1&2\\2&-1&2&5\\5&6&3&2\\1&3&-1&-3\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ (*e*) $\begin{bmatrix}1&-1&1&1&1\\1&-1&2&3&1\\2&-2&1&0&2\\1&1&-1&-3&3\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1&2\\0&1&0&0&1\\0&0&1&2&0\\0&0&0&0&0\end{bmatrix}$

15. Write the normal form of each of the matrices of Problem 14.

Ans. (*a*) $[I_2\ 0]$, (*b*), (*c*) $[I_3\ 0]$ (*d*) $\begin{bmatrix}I_3&0\\0&0\end{bmatrix}$ (*e*) $\begin{bmatrix}I_3&0\\0&0\end{bmatrix}$

16. Let $A = \begin{bmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\end{bmatrix}$.

(*a*) From I_3 form H_{12}, $H_2(3)$, $H_{13}(-4)$ and check that each HA effects the corresponding row transformation.
(*b*) From I_4 form K_{24}, $K_3(-1)$, $K_{42}(3)$ and show that each AK effects the corresponding column transformation.
(*c*) Write the inverses H_{12}^{-1}, $H_2^{-1}(3)$, $H_{13}^{-1}(-4)$ of the elementary matrices of (*a*). Check that for each H, $H\cdot H^{-1} = I$.
(*d*) Write the inverses K_{24}^{-1}, $K_3^{-1}(-1)$, $K_{42}^{-1}(3)$ of the elementary matrices of (*b*). Check that for each K, $KK^{-1} = I$.
(*e*) Compute $B = H_{12}\cdot H_2(3)\cdot H_{13}(-4) = \begin{bmatrix}0&3&0\\1&0&-4\\0&0&1\end{bmatrix}$ and $C = H_{13}^{-1}(-4)\cdot H_2^{-1}(3)\cdot H_{12}^{-1} = \begin{bmatrix}0&1&4\\1/3&0&0\\0&0&1\end{bmatrix}$.
(*f*) Show that $BC = CB = I$.

17. (*a*) Show that $K'_{ij} = H_{ij}$, $K'_i(k) = H_i(k)$, and $K'_{ij}(k) = H_{ij}(k)$.
(*b*) Show that if R is a product of elementary column matrices, R' is the product in reverse order of the same elementary row matrices.

18. Prove: (*a*) AB and BA are non-singular if A and B are non-singular n-square matrices.
(*b*) AB and BA are singular if at least one of the n-square matrices A and B is singular.

19. If P and Q are non-singular, show that A, PA, AQ, and PAQ have the same rank.
Hint. Express P and Q as products of elementary matrices.

20. Reduce $B = \begin{bmatrix}1&3&6&-1\\1&4&5&1\\1&5&4&3\end{bmatrix}$ to normal form N and compute the matrices P_2 and Q_2 such that $P_2BQ_2 = N$.