Inner product space: Let V be a (real or complex) vector space over K. Suppose to each pair of vectors $u, v \in V$ there is assigned a scalar $\langle u, v\rangle \in K$. This mapping is called an inner product in V if it satisfies the following axioms:

(i) $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$.

(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$

(iii) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$.

$F : V \to K$

$F(u, v) = \langle u, v\rangle$

The vector space V with an inner product is called an inner product space.

Note: ① $\langle u, v\rangle$ is always real [by (iii)].

Also $\|u\| = \sqrt{\langle u, u\rangle}$ and this nonnegative real number $\|u\|$ is called the norm or length of u.

Note: A real inner product space is called a Euclidean space and a complex inner product space is called a Unitary space.

Orthogonal complement: Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal if $\langle u, v\rangle = 0$. If u is orthogonal to v, then $\langle v, u\rangle = \overline{\langle u, v\rangle} = \bar{0} = 0$ and so v is orthogonal to u.

Suppose W is any subset of V. Then the orthogonal complement of W, denoted by $W^{\perp}$, consists of those vectors in V which are orthogonal to every $w \in W$;

$$W^{\perp} = \{v \in V : \langle v, w\rangle = 0 \text{ for every } w \in W\}.$$

Note: If $\|v\| = 1$; i.e. if $\langle v, v\rangle = 1$, then v is called a unit vector or is said to be normalized. Every nonzero vector $u \in V$ can be normalized by setting $v = u/\|u\|$.

* Show that $W^{\perp}$ is a subspace of V.

Proof: Clearly $0 \in W^{\perp}$.

Suppose $u, v \in W^{\perp}$. Then for any $a, b \in K$ and any $w \in W$,

$$\langle au+bv, w\rangle = a\langle u,w\rangle + b\langle v,w\rangle$$
$$= a\cdot 0 + b\cdot 0 = 0.$$

Hence $au+bv \in W^{\perp}$ and so $W^{\perp}$ is a subspace of V.

Theorem 13.2: Let W is a subspace of V.

Then V is the direct sum of W and $W^{\perp}$,

i.e. $$V = W \oplus W^{\perp}$$

— x —

Orthonormal sets: A set $\{u_i\}$ of vectors in V is said to be orthogonal if its distinct elements are orthogonal, i.e. if $\langle u_i, u_j\rangle = 0$ for $i \neq j$

Then the set $\{u_i\}$ is said to be orthonormal if it is orthogonal and if each u_i has length 1.

That is, if $$\langle u_i, u_j\rangle = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

Note: An orthonormal set can always be obtained from an orthogonal set of nonzero vectors by normalizing each vector.

Theorem: Let $\{v_1, v_2, \ldots, v_n\}$ be an arbitrary basis of an inner product space V. Then there exists an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of V such that the transition matrix from $\{v_i\}$ to ~~$\{u_i\}$~~ $\{u_i\}$ is triangular; that is, for $i = 1, 2, \ldots, n$.

$$u_i = a_{i1} v_1 + a_{i2} v_2 + \cdots + a_{in} v_i$$

Ex. 13.12 Consider the following basis of Euclidian space $\mathbb{R}^3$:

$$\{v_1 = (1,1,1),\ v_2 = (0,1,1),\ v_3 = (0,0,1)\}.$$

We use the Gram-Schmidt orthogonalization process to transform $\{v_i\}$ into an orthonormal basis $\{u_i\}$.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1 = \qquad = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Then we normalize w_2

$$u_2 = \frac{w_2}{\|w_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$$

$$= \qquad = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

and then we normalize w_3:

$$u_3 = \frac{w_3}{\|w_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Therefore the required orthonormal basis of $\mathbb{R}^3$ is

$\{ u_1 = \quad , u_2 = \quad , u_3 = \quad \}$

1. Define basis and dimension of a vector space with an example. Find the dimension and a basis of the subspace $W = \{(a,b,c,d) : a+b=0,\ c=2d\}$ of $\mathbb{R}^4$.

2. Let V be a finite dimensional vector space. Then prove that every basis of V has the same number of vectors.

3. Define a linear mapping. Also define image and kernel of a linear mapping. Let $F: V \longrightarrow U$ be a linear mapping. Prove that kernel of F is a subspace of V.

4. ~~Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear mapping defined by~~ Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the linear mapping for which

$T(1,1) = 3$ and $T(0,1) = -2$.

Find $T(a,b)$