Periodic Matrix: A square matrix A is called periodic, if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then A is said to be of period k.

Example:
$$A = \begin{bmatrix} 4 & 5 & 3 \\ 1 & 6 & 4 \\ 3 & 9 & 5 \end{bmatrix}$$ is a periodic matrix,

and the period of the matrix is 2.

Because $A^3 = A$, $\therefore A^{2+1} = A$.

Idempotent Matrix: A square matrix A is called idempotent provided it satisfied the relation $A^2 = A$.

Example:
$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$ is an idempotent matrix.

Because $A^2 = A$.

Involutory matrix: A square matrix A is called involutory provided it satisfies the relation $A^2 = I$, where I is the identity matrix.

For example, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is involutory matrix. Since $A^2 = I$.

■ **Nilpotent Matrix:** A square matrix A is called Nilpotent matrix of order m, provided it satisfies the relation $A^m = 0$ and $A^{m-1} \neq 0$ where m is a positive integer and 0 is the null matrix.

For example, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a nilpotent matrix since $A \neq 0$ but $A^2 = 0$.

■ **Transposed Matrix:** The matrix of order $n \times m$ obtained by interchanging the rows and columns of a matrix A of order $m \times n$ is called the transposed matrix of A or transpose of the matrix A and is denoted by A' or A^t.

For example: If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ then

$$A' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

☐ **Conjugate of a Matrix:** The matrix obtained from any given matrix A of order $m \times n$ with complex element a_{ij} by replacing it's elements by the corresponding conjugate complex numbers is called the complex conjugate or conjugate of A denoted by $\bar{A}$ and is read as 'A conjugate'.

For example: If $A = \begin{bmatrix} 1+i & 2+3i \\ 2 & 3i \end{bmatrix}$,

then $\bar{A} = \begin{bmatrix} 1-i & 2-3i \\ 2 & -3i \end{bmatrix}$.

☐ **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is called symmetric provided $a_{ij} = a_{ji}$, for all values of i and j.

For example, $A = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 0 & 7 \\ 5 & 7 & 3 \end{bmatrix}$

A Skew-Symmetric Matrix: A square matrix $A = [a_{ij}]$ is called skew-symmetric provided $a_{ij} = -a_{ji}$, for all values of i and j.

For example, $A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

[H] Hermitian Matrix: A square matrix A such that $\bar{A}' = A$ is called Hermitian i.e. the matrix $[a_{ij}]$ is Hermitian provided $a_{ij} = \bar{a}_{ji}$, for all values of i and j.

For example: $$A = \begin{bmatrix} l & \alpha+i\beta & \gamma+i\delta \\ \alpha-i\beta & m & x+iy \\ \gamma-i\delta & x-iy & m \end{bmatrix}$$

[A] Skew-Hermitian Matrix: A square matrix A such that $\bar{A}' = -A$ is called skew-Hermitian i.e. the matrix $[a_{ij}]$ is ~~harem~~ skew-Hermitian provided $a_{ij} = -\bar{a}_{ij}$ for all values of i and j.

For example: $$A = \begin{bmatrix} 2i & -\alpha-i\beta & -3+i \\ \alpha-i\beta & -i & -\gamma+i\delta \\ 3+i & \gamma+i\delta & 0 \end{bmatrix}$$

1(a): Find the rank of the matrix $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$

Solⁿ: $A \sim \begin{bmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 6 & 3 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$

$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, replacing C_2 by $C_2 - 2C_3$

$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1, R_3 - 3R_1$.

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, ~~replacing~~ interchanging C_1 and C_3

$\sim \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$

Hence the rank of A is 1.

2: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$

Sol:

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 2 & -3 & -4 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -6 \\ 0 & -3 & -4 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 4R_1$, $R_3 - 2R_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -3 & -4 \end{bmatrix}$, replacing R_2 by $R_2 - R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, replacing C_2, C_3 by $-\frac{1}{3}C_2$, $-\frac{1}{2}C_3$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, interchanging C_2 and C_3.

$\sim [I_3]$

Hence the rank of A is 3.

Q: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ after reducing it to the normal form.

Sol: $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_2$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ replacing C_1 by $C_1 - C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, integrating C_1 and C_2

$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence the rank of A is 2.

6(a): Reduce $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ to normal form.

Sol: $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 6 & -9 & 6 \\ 2 & 4 & -6 & 2 \end{bmatrix}$, replacing C_2, C_3, C_4 by C_2+C_1, C_3-2C_1, C_4+C_1 respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 2 & 2 & 3 & 2 \end{bmatrix}$ replacing C_2, C_3 by C_2-C_3, $C_3-\frac{3}{2}C_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 2 & 2 & 0 & 0 \end{bmatrix}$, replacing C_3, C_4 by $C_3-\frac{3}{2}C_2$, C_4-C_2 respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, replacing R_2 and R_3 by $\frac{1}{2}R_2$ and $\frac{1}{2}R_3$.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ replacing R_2 and R_3 by R_2-2R_1 and R_3-R_1.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_3 by $\frac{1}{3}C_3$.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, interchanging C_2, C_4

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, interchanging C_3, C_4

$\sim [I_3 \; 0]$ $\therefore$ Rank of A is 3.

Ex-3: [P-13]

If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ find AB and show that $AB \neq BA$.

Soln:

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 2 + (-2)\cdot 4 + 3\cdot 2 & 1\times 3 + (-2)\cdot 5 + 3\cdot 1 \\ -4\cdot 2 + 2\cdot 4 + 5\cdot 2 & -4\cdot 3 + 2\cdot 5 + 5\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 1 + 3\cdot(-4) & 2\cdot(-2) + 3\cdot 2 & 2\cdot 3 + 3\cdot 5 \\ 4\cdot 1 + 5\cdot(-4) & 4\times(-2) + 5\cdot 2 & 4\cdot 3 + 5\cdot 5 \\ 2\cdot 1 + 1\cdot(-4) & 2\times(-2) + 1\cdot 2 & 2\cdot 3 + 1\cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Hence $AB \neq BA$.

5.72|| $V = \{(a,b,c,d) : b - 2c + d = 0\}$

$W = \{(a,b,c,d) : a = d,\ b = 2c\}$

i) Basis and dimension of V:

We seek a basis of the set of solutions (a,b,c,d) of the equation

$$b - 2c + d = 0 \quad \text{or} \quad 0.a + b - 2c + d = 0$$

The free variable are a, c and d. Set

(1) $a = 1, c = 0, d = 0$ (2) $a = 0, c = 1, d = 0$

(3) $a = 0, c = 0, d = 1$

to obtain the respective solutions

$$v_1 = (1,0,0,0) \qquad v_2 = (0,2,1,0)$$

$$v_3 = (0,-1,0,1)$$

The set $\{v_1, v_2, v_3\}$ is a basis of V, and $\dim V = 3$.

II) Basis & dimension of W:

We seek a basis of the set of solutions (a,b,c,d) of the system,

$$a = d \qquad b = 2c$$

$$a - d = 0 \qquad b - 2c = 0$$

The free variables are d and c. Set

(1) $c = 1, d = 0$ (2) $c = 0, d = 1$

to obtain the respective solutions

$$v_1 = (0,2,1,0) \qquad v_2 = (1,0,0,1)$$

The set $\{v_1, v_2\}$ is a basis of W and $\dim W = 2$.

iii) $V \cap W$ consists of those vectors (a, b, c, d) which satisfied the conditions defining V and the conditions defining W. i.e. the three equation:

$$b - 2c + d = 0 \qquad \begin{aligned} a - d &= 0 \\ b - 2c + d &= 0 \\ b - 2c &= 0 \end{aligned} \quad \text{or}$$

$$a = d \qquad \text{or}$$

$$b = 2c$$

$$\begin{aligned} a - d &= 0 \\ b - 2c + d &= 0 \\ 2c - d - 2c &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} a - d &= 0 \\ b - 2c + d &= 0 \\ d &= 0 \end{aligned}$$

$\therefore$ The free variable is c. Set $c = 1$ to obtain the solution $v = (0, 2, 1, 0)$. Thus $\{v\}$ is a basis of $V \cap W$ and $\dim(V \cap W) = 1$.

5.78 ||

$U = \{t^3 + 4t^2 - t + 3,\ t^3 + 5t^2 + 5,\ 3t^3 + 10t^2 - 5t + 5\}$

$= \{(1,4,-1,3),\ (1,5,0,5),\ (3,10,-5,5)\}$

$W = \{t^3 + 4t^2 + 6,\ t^3 + 2t^2 - t + 5,\ 2t^3 + 2t^2 - 3t + 9\}$

$= \{(1,4,0,6),\ (1,2,-1,5),\ (2,2,-3,9)\}$

U+W is the space spanned by all six vectors. Hence form the matrix whose rows are the given six vectors. and then row reduce to echelon form:

$$\begin{pmatrix} 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 5 \\ 3 & 10 & -5 & 5 \\ 1 & 4 & 0 & 6 \\ 1 & 2 & -1 & 5 \\ 2 & 2 & -3 & 9 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & -6 & -1 & 3 \end{pmatrix} \text{ to }$$

$$\begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -7 & -9 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & -6 & -1 & 3 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ to }$$

$$\begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since to echelon matrix has three nonzero rows. $\dim(U+W) = 3$.

ii) First find the $\dim U$ and $\dim W$. form the two matrices whose rows are the generators of U and W respectively and then row reduce each echelon form!

$$\begin{pmatrix} 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 5 \\ 3 & 10 & -5 & 5 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 4 & 0 & 6 \\ 1 & 2 & -1 & 5 \\ 2 & 2 & -3 & 9 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & 0 & 6 \\ 0 & -2 & -1 & -1 \\ 0 & -6 & -3 & -3 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 4 & 0 & 6 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each the echelon matrices has two nonzero rows $\dim U = 2$ and $\dim W = 2$. Using the equation

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$\therefore 3 = 2 + 2 - \dim(U \cap W)$$

$$\therefore \dim(U \cap W) = 1.$$