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3007

Vector Spaces and Subspaces

Field: $(F; +, \cdot)$

$F \to$ non-empty set and $+, \cdot$ are binary operations.

Axioms for addition:

(A1) $a, b \in F \Rightarrow a+b \in F$ (Closer law)

(A2) For all $a, b, c \in F$

$(a+b)+c = (b+c)+a$ (associative law)

(A3) There exists $0 \in F$ such that $a+0 = 0+a = a$ for all $a \in F$.

[$0 \to$ additive identity]

(A4) For every $a \in F$, there exists an element $-a \in F$ such that $a+(-a) = (-a)+a = 0$

[$-a \to$ ~~inverse~~ inverse of a]

(A5) Addition is commutative: $a+b = b+a$, for all $a, b \in F$

Axioms for multiplication:

(M1) If $a, b \in F$, then $ab \in F$

(M2) $(ab) \cdot c = a \cdot (bc)$ for all $a, b, c \in F$ [Multiplication is associative]

(M3) There exists 1 in F such that $a \cdot 1 = 1 \cdot a = a$ $\forall$ $a \in F$.

[$1 \to$ multiplicative identity].

(M4) For every $a \in F$, there exists an element $\frac{1}{a} \in F$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

[$\frac{1}{a} \to$ inverse of a for multiplication]

(M5) Multiplication is commutative: $a \cdot b = b \cdot a$ for all $a, b \in F$.

Distributive law:

(i) $a(b+c) = ab + ac$ $\forall$ $a, b, c \in F$

(ii) $(a+b)c = ac + bc$ $\forall$ $a, b, c \in F$

Examples:— (i) real number ($\mathbb{R}$), (ii) complex number ($\mathbb{C}$)

2 ~~vector spaces~~:

Vector space: Let K be a field and let V be a non-empty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$, $u+v \in V$ and to any $u \in V, k \in K$, $ku \in V$. Then V is called a vector space over K if the following axioms hold:

[The elements of V are called vectors]

Axioms for addition:

(A1) $u, v, w \in V$, $(u+v)+w = u+(v+w)$

(A2) There is a vector in V, denoted by 0 for which
$u+0 = 0+u = u \quad \forall\, u \in V$ [$0 \to$ zero vector]

(A3) For each vector $u \in V$ there is a vector in V, denoted by $-u$, for which $u+(-u) = 0$. [$-u \to$ inverse of u]

(A4) $u+v = v+u \quad \forall\, u, v \in V$

Axioms for multiplication:

(M1) For any $k \in K$ and any $u, v \in V$, $k(u+v) = ku+kv$.

(M2) For any scalars $a, b \in K$ and $u \in V$, $(a+b)u = au+bu$.

(M3) For any $a, b \in K$ and any $u \in V$, $(ab)u = a(bu)$

(M4) For the unit scalar $1 \in K$, $1 \cdot u = u \quad \forall\, u \in V$.

Example: (1) Let K be an arbitrary field. The set of all n-tuples of elements of K with vector addition and scalar multiplication defined by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$$

and $k(a_1, a_2, \dots, a_n) = (ka_1, ka_2, \dots, ka_n)$ where $a_i, b_i, k \in K$, is a vector space over K.

This space is denoted by K^n.

2 $\mathbb{R} \to$

$\mathbb{R}^2$, $\mathbb{R}^3$

ASM. 4.77 $\to$ P$\to$85 4.55 4.12 P$\to$83

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Theorems: 4.8, 4.9

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Subspace: Let W be a subset of a vector space V over a field K. W is called a subspace of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication on V.

Theorem 4.2: W is a subspace of V iff

(i) W is non-empty,

(ii) $v, w \in W$ implies $v+w \in W$,

(iii) $v \in W$ implies $kv \in W$ for every $k \in K$.

Corollary: W is a subspace of V iff

(i) $0 \in W$ $(W \neq \phi)$ and $v, w \in W \Rightarrow av+bw \in W$ for every $a, b \in K$.

Proof: 4.8 [P$\to$72]

Problem 4.9: Let $V = \mathbb{R}^3$. Show that W is a subspace of V where:

(i) $W = \{(a, b, 0) : a, b \in \mathbb{R}\}$.

(ii) $W = \{(a, b, c) : a+b+c = 0\}$.

Problem: Let $V = \mathbb{R}^3$. Show that W is not a subspace of V where:

(i) $W = \{(a, b, c) : a \geq 0\}$

(ii) $W = \{(a, b, c) : a^2+b^2+c^2 \leq 1\}$.

soln:— (i) $v = (1, 2, 3)$ and $k = -5 \in \mathbb{R}$. But $-5(1, 2, 3) = (-5, -10, -15) \notin W$, since $-5 < 0$. Hence W is not a subspace of V.

(ii) ~~ve~~ $v = (1, 0, 0) \in W$ and $w = (0, 1, 0) \in W$. But $v+w = (1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin W$, since $1^2+1^2+0^2 = 2 > 1$. Hence W is not a subspace of V.

Problem: Determine whether or not W is a subspace of $\mathbb{R}^3$ if W consists of those vectors $(a, b, c) \in \mathbb{R}^3$ for which: (i) $a = 2b$; (ii) $a \leq b \leq c$; (iii) $ab = 0$; (iv) $a = b = c$; (v) $a = b^2$.

4.3:- W is a subspace of V iff (i) $0 \in W$ and (ii) $v, w \in W$ implies $av + bw \in W$ for all $a, b \in K$.

Problem 4.9:- Let $V = \mathbb{R}^3$. Show that W is a subspace of V where: (i) $W = \{(a, b, 0) : a, b \in \mathbb{R}\}$, i.e., W is the xy plane consisting of those vectors whose third component is 0.

(ii) $W = \{(a, b, c) : a + b + c = 0\}$, [i.e. sum of its component 0]

Sol^n:- $0 = (0,0,0) \in W$, since the third component of ~~(0,0 is 00~~ $(0,0,0)$ is 0. For any vectors $v = (a, b, 0)$, $w = (c, d, 0)$ in W and any scalars k, k' (real numbers),

$$kv + k'w = k(a, b, 0) + k'(c, d, 0) = (ka, kb, 0) + (k'c, k'd, 0)$$
$$= (ka + k'c,\ kb + k'd,\ 0).$$

Thus $kv + k'w \in W$ and so W is a subspace of V.

(ii) $0 = (0, 0, 0) \in W$, since $0 + 0 + 0 = 0$.

Suppose $v = (a, b, c)$, $w = (a', b', c')$ belong to W, i.e., $a + b + c = 0$ and $a' + b' + c' = 0$.

Then for any scalars k and k',

$$kv + k'w = k(a, b, c) + k'(a', b', c')$$
$$= (ka, kb, kc) + (k'a', k'b', k'c')$$
$$= (ka + k'a',\ kb + k'b',\ kc + k'c')$$

By given condition

$$(ka + k'a') + (kb + k'b') + (kc + k'c') = k(a + b + c) + k'(a' + b' + c')$$
$$= k0 + k'0 = 0$$

Thus $kv + k'w \in W$ and so W is a subspace of V.

4

1 * If U and W be subspaces of a vector space V, then $U \cap W$ is also a subspace of V.

Proof :– $0 \in U$ and $0 \in W$, since U and W are subspaces. So, $0 \in U \cap W$. Suppose $u, v \in U \cap W$. Then $u, v \in U$ and $u, v \in W$. Since U and W are subspaces so $au + bv \in U$ and $au + bv \in W$ for any $a, b \in K$.
Hence $au + bv \in U \cap W$ and so $U \cap W$ is a subspace of V.

Theorem: The intersection of any number of subspaces of a vector space V is a subspace of V.

Linear combination :– Let V be a vector space over a field K and let $v_1, v_2, \dots, v_m \in V$. Any vector in V of the form $a_1 v_1 + a_2 v_2 + \dots + a_m v_m$ where $a_i \in K$, is called a linear combination of $v_1, v_2, \dots, v_m$.

Problem :– Write the vector $(1, 7, -4)$ as a linear combination of the vectors $u = (1, -3, 2)$ and $v = (2, -1, 1)$

Sol^n :– $(1, 7, -4) = x(1, -3, 2) + y(2, -1, 1)$ [$x, y \to$ unknown scalars]

$$= (x, -3x, 2x) + (2y, -y, y)$$
$$= (x + 2y, -3x - y, 2x + y)$$

$x = -3$ and $y = 2$

[P → 75, problem 4.17 → করে দেওয়া আছে]

Problem :– For which value of k will the vector $u = (1, -2, k)$ in $\mathbb{R}^3$ be a linear combination of the vectors $v = (3, 0, -2)$ and $w = (2, -1, -5)$?

Sol^n :– $u = xv + yw$

$(1, -2, k) = x(3, 0, -2) + y(2, -1, -5) = (3x + 2y, -y, -2x - 5y)$

$3x + 2y = 1 \to (1)$, $-y = -2 \to (2)$, $-2x - 5y = k \to (3)$

From (1) and (2) $x = -1$, $y = 2$

$k = -8$

~~Theorem 4.5 :–~~ Problem: For which value of k will the vector $(1, k, 5)$ in $\mathbb{R}^3$ be a linear combination of the vectors $u = (1, -3, 2)$ and $v = (2, -1, 1)$?

Problems: 4.19, 4.20, 4.21, 4.23, 4.24, 4.28

Theorem 4.5: Let S be a nonempty subset of V. Then £ $L(S)$, the set of all linear combinations of vectors in S, is a subspace of V containing S. Furthermore, if W is any other subspace of V containing S, then $L(S) \subset W$.

Sum and Direct Sum

Sum: Let U and W be subspaces of a vector space V. The sum of U and W, written $U+W$, consists of all sums $u+w$ where $u \in U$ and $w \in W$:

$$U+W = \{u+w : u \in U,\ w \in W\}.$$

** Ex.

Direct Sum: The vector space V is said to be the direct sum of its subspaces U and W, denoted by $V = U \oplus W$ if every vector $v \in V$ can be written in one and only one way as $v = u+w$ where $u \in U$ and $w \in W$.

Ex:- $\mathbb{R}^3 \to$ vector space. Let U and W be the subspaces of $\mathbb{R}^3$ defined by $U = \{(a,b,0) : a,b \in \mathbb{R}\}$ and $W = \{(0,0,c) : c \in \mathbb{R}\}$.
Show that $\mathbb{R}^3 = U \oplus W$.

Any vector $(a,b,c) \in \mathbb{R}^3$ can be written as the sum of a vector in U and a vector in W in one and only one way:

$$(a,b,c) = (a,b,0) + (0,0,c).$$

$$\therefore \mathbb{R}^3 = U \oplus W.$$

** Ex:- $V \to$ vector space of 2×2 matrices over $\mathbb{R}$.

$U = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a,b \in \mathbb{R}\right\}$, $W = \left\{\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} : a,c \in \mathbb{R}\right\}$

U and W are subspaces of V and $U+W = \left\{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} : a,b,c \in \mathbb{R}\right\}$ /

Examples:- (i) real number ($\mathbb{R}$), (ii) complex number ($\mathbb{C}$)

Prove that.

Theorem: the sum ~~of the~~ $U+W$ of the subspaces U and W of V is also a subspace of V.

Proof:— $0=0+0\in U+W$, since $0\in U$ and $0\in W$.

Suppose $u+w$ and $u'+w'$ belong to $U+W$ with $u,u'\in U$ and $w,w'\in W$.
Then $(u+w)+(u'+w')=(u+u')+(w+w')\in U+W$ and for any scalar k,

$k(u+w)=ku+kw\in U+W$.

Theorem 4.9:— The vector space V is the direct sum of its subspaces U and W iff (i) $V=U+W$ and (ii) $U\cap W=\{0\}$.

Proof: Suppose $V=U\oplus W$. Then any vector $v\in V$ can be uniquely written in the form $v=u+w$ where $u\in U$ and $w\in W$.
Thus in particular, $V=U+W$.
Suppose $v\in U\cap W$. Then ① $v=v+0$ where $v\in U$, $0\in W$
② $v=0+v$ where $0\in U$, $v\in W$.

[$v\in U\cap W \Rightarrow v\in U$ and $v\in W$.
Also $v\in V$, since U and W are subspaces of V]

Since such a sum for v must be unique so $v=0$.
Hence $U\cap W=\{0\}$.

Conversely, suppose $V=U+W$ and $U\cap W=\{0\}$.
Let $v\in V$. Since $V=U+W$, there exist $u\in U$ and $w\in W$ such that $v=u+w$.
Also, suppose that $v=u'+w'$ where $u'\in U$ and $w'\in W$.

Then $u+w=u'+w'$
or $u-u'=w'-w$.

But $u-u'\in U$ and $w'-w\in W$; hence by $U\cap W=\{0\}$
$u-u'=0$, $w'-w=0$ and so $u=u'$ and $w=w'$
Therefore such a sum for $v\in V$ is unique and $V=U\oplus W$.

Problem: Let U and W be the subspaces of $\mathbb{R}^3$ defined by
$U=\{(a,b,c): a=b=c\}$ and $W=\{(0,b,c)\}$.
Show that $\mathbb{R}^3=U\oplus W$.

Basis and Dimension

Linearly dependence: Let V be a vector space over a field K. The vectors $v_1, v_2, \ldots, v_m \in V$ are said to be linearly dependent over K if there exist scalars $a_1, a_2, \ldots, a_m \in K$, not all of them 0, such that $a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$.

That is, if $a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$ when at least one of the a's is not 0, then the vectors are linearly dependent.

* Show that the vectors $u = (1,-1,0)$, $v = (1,3,-1)$ and $w = (5,3,-2)$ are linearly dependent.

Sol^n:

$xu + yv + zw = 0$ where x, y, z unknown scalars

$x(1,-1,0) + y(1,3,-1) + z(5,3,-2) = (0,0,0)$

$$\Rightarrow \begin{array}{l} x + y + 5z = 0 \\ -x + 3y + 3z = 0 \\ -y - 2z = 0 \end{array} \Rightarrow \begin{array}{l} x + y + 5z = 0 \\ 4y + 8z = 0 \\ y + 2z = 0 \end{array} \Rightarrow \begin{array}{l} x + y + 5z = 0 \\ y + 2z = 0 \end{array}$$

Here z is a free variable.

Set $z = 1$, ~~we~~ obtain, $y = -2$ and $x = -3$

Problem: Determine whether or not the following vectors in $\mathbb{R}^3$ are linearly dependent:

(i) $(1,-2,1)$, $(2,1,-1)$, $(7,-4,1)$

(ii) $(1,-3,7)$, $(2,0,-6)$, $(3,-1,-1)$, $(2,4,-5)$

(iii) $(1,2,-3)$, $(1,-3,2)$, $(2,-1,5)$

(iv) $(2,-3,7)$, $(0,0,0)$, $(3,-1,-4)$.

** Sol^n

Note: ① Two vectors u and v are dependent iff one is a multiple of the other.

② If one of the vectors $v_1, \ldots, v_m$ say $v_1 = 0$, then the vectors must be dependent.

③ If two of the vectors $v_1, \ldots, v_m$ are equal, say $v_1 = v_2$, then the vectors are dependent.

(i) $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 7 & -4 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 10 & -6 \end{pmatrix}$ to $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

Since the echelon matrix has a zero row, the vectors are dependent

(ii) ~~Yes~~, Since any four (or more) vectors in $\mathbb{R}^3$ are dependent

(iii) $\begin{pmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 2 & -1 & 5 \end{pmatrix}$ to $\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 5 & 11 \end{pmatrix}$ to $\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

Since the echelon matrix has no zero rows, so the vectors are independent.

(iv) Since $0 = (0, 0, 0)$ is one of the vectors are dependent.

Linearly independent: Let V be a vector space over a field K. The vectors $v_1, \cdots, v_m \in V$ are said to be linearly independent over K, if there exist scalars $a_1, \cdots, a_m \in K$, all of them 0 such that $a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$.
That is,

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \text{ only if } a_1 = 0, \cdots, a_m = 0$$

then the vectors are linearly independent.

* Show that the vectors $u = (6, 2, 3, 4)$, $v = (0, 5, -3, 1)$ and $w = (0, 0, 7, -2)$ are linearly independent.

Solⁿ: Suppose $xu + yv + zw = 0$ where x, y, z are unknown scalars.

$$x(6, 2, 3, 4) + y(0, 5, -3, 1) + z(0, 0, 7, -2) = (0, 0, 0, 0)$$

$$\begin{aligned} 6x &= 0 \\ 2x + 5y &= 0 \\ 3x - 3y + 7z &= 0 \\ 4x + y - 2z &= 0 \end{aligned} \implies x = 0,\ y = 0 \text{ and } z = 0$$

$xu + yv + zw = 0$ implies $x = 0, y = 0$ and $z = 0$

Hence u, v and w are independent.

Observe that the vectors in the above problem form a matrix in echelon form:

$$\begin{pmatrix} 6 & 2 & 3 & 4 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 7 & -2 \end{pmatrix}.$$

Thus we have shown that the nonzero rows of the above echelon matrix are independent.

Note: ① Any nonzero vector v is, by itself, independent; for $kv = 0$, $v \neq 0$ implies $k = 0$.

② The set $\{v_1, \dots, v_m\}$ is called a dependent or independent set according as the vectors $v_1, \dots, v_m$ are dependent or independent. Also the empty set ϕ to be independent.

③ A set which contains a dependent subset is itself dependent.

Problem:- Let V be the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$. Determine whether $u, v, w \in V$ are independent or dependent where:

$$u = t^3 - 3t^2 + 5t + 1, \quad v = t^3 - t^2 + 8t + 2, \quad w = 2t^3 - 4t^2 + 9t + 5.$$

Solⁿ: Set a linear combination of the polynomials u, v and w equal to the zero polynomial; that is

$$xu + yv + zw = 0 \text{ where } x, y, z \text{ are unknown scalars}$$

$$\therefore x(t^3 - 3t^2 + 5t + 1) + y(t^3 - t^2 + 8t + 2) + z(2t^3 - 4t^2 + 9t + 5) = 0$$

or $$(x + y + 2z)t^3 + (-3x - y - 4z)t^2 + (5x + 8y + 9z)t + (x + 2y + 5z) = 0$$

The coefficients of the powers of t must each be 0:

$$\begin{aligned} x + y + 2z &= 0 \\ -3x - y - 4z &= 0 \\ 5x + 8y + 9z &= 0 \\ x + 2y + 5z &= 0 \end{aligned}$$

Solving the above homogeneous system, we obtain only the zero solution: $x = 0, y = 0, z = 0$.

Hence u, v and w are independent.

* Let u, v and w be independent vectors. Show that $u+v$, $u-v$ and $u-2v+w$ are also independent.

$$x(u+v)+y(u-v)+z(u-2v+w)=0 \quad \text{where } x,y,z \text{ scalars}$$

or $(x+y+z)u+(x-y-2z)v+zw=0$.

But u, v and w are linearly independent; hence the coefficients in the above relation are each 0:

$$x+y+z=0$$
$$x-y-2z=0$$
$$z=0$$

$\therefore x=0,\ y=0,\ z=0$. Thus $u+v$, $u-v$ and $u-2v+w$ are independent.

* Let $v_1, v_2, \dots, v_m$ be independent vectors and suppose u is a linear combination of the v_i, say $u=a_1v_1+a_2v_2+\dots+a_mv_m$ where the a_i are scalars. Show that the above representation of u is unique.

Proof: Suppose $u=b_1v_1+b_2v_2+\dots+b_mv_m$ where the b_i are scalars

$$0=u-u=(a_1-b_1)v_1+(a_2-b_2)v_2+\dots+(a_m-b_m)v_m.$$

But v_i are independent, so

$$a_1-b_1=0,\ a_2-b_2=0,\ \dots,\ a_m-b_m=0$$

or $a_1=b_1,\ a_2=b_2,\ \dots,\ a_m=b_m$

* Suppose $\{v_1, v_2, \dots, v_m\}$ is independent, but $\{v_1, v_2, \dots, v_m, w\}$ is dependent. Show that w is a linear combination of the v_i.

Proof: suppose $a_1v_1+a_2v_2+\dots+a_mv_m+bw=0$.

If $b=0$, then one of the a_i is not zero and $a_1v_1+\dots+a_mv_m=0$

But $\{v_1,\dots,v_m\}$ is independent, so $a_1=0,\dots,a_m=0$. Hence $b\neq 0$ and

~~so $w = b^{-1}(-a_1 v_1$~~

so $w = -b^{-1}a_1 v_1 - \cdots - b^{-1}a_m v_m$

That is, w is a linear combination of the v_i.

Lemma 5.2: The nonzero vectors $v_1, \ldots, v_m$ are linearly dependent iff one of them, say v_i is a linear combination of the preceding vectors: $v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1}$.

Proof:- Suppose $v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1}$.

Then $a_1 v_1 + \cdots + a_{i-1} v_{i-1} - v_i + 0 v_{i+1} + \cdots + 0 v_m = 0$ and the coefficient of v_i is not 0.

Hence the vectors $v_1, \ldots, v_m$ are linearly dependent.

Conversely, suppose the nonzero vectors $v_1, \ldots, v_m$ are linearly dependent. Then there exist scalars $a_1, \ldots, a_m$, not all of them 0 such that $a_1 v_1 + \cdots + a_m v_m = 0$.

Let k be the largest integer such that $a_k \neq 0$.

Then $a_1 v_1 + \cdots + a_k v_k + 0 v_{k+1} + \cdots + 0 v_m = 0$

or $a_1 v_1 + \cdots + a_k v_k = 0$.

Suppose $k = 1$, then $a_1 v_1 = 0$, $a_1 \neq 0$ and so $v_1 = 0$.

But $v_1, \ldots, v_m$ are nonzero vectors; hence $k > 1$ and

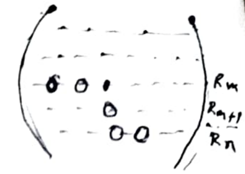
$$v_k = -a_k^{-1} a_1 v_1 - \cdots - a_k^{-1} a_{k-1} v_{k-1}.$$

That is, v_k is a linear combination of the preceding vectors.

Theorem 5.1: The nonzero rows $R_1, \ldots, R_n$ of a matrix in echelon form are linearly independent.

Proof:- Suppose $\{R_n, R_{n-1}, \ldots, R_1\}$ is dependent. Then one of of the rows, say R_m, is a linear combination of the preceding rows:

$$R_m = a_{m+1}R_{m+1} + a_{m+2}R_{m+2} + \cdots + a_nR_n \longrightarrow (1)$$



Suppose the kth component of R_m is its first nonzero entry. Then, since the matrix is in echelon form, the kth components of $R_{m+1}, \cdots, R_n$ are all 0 and so the kth component of (1) is $a_{m+1}\cdot 0 + a_{m+2}\cdot 0 + \cdots + a_n\cdot 0 = 0$. But this contradicts the assumption that the kth component of R_m is not 0.
Hence $R_1, \cdots, R_n$ are linearly independent.

Basis and Dimension:

A vector space V is said to be of finite dimension n or to be n-dimensional if there exist linearly independent vectors $e_1, e_2, \cdots, e_n$ which span V.

Then the sequence $\{e_1, e_2, \cdots, e_n\}$ is called a basis of V. $\dim V = n$.

Examples: (1) Let K be any field. Consider the vector space K^n which consists of n-tuples of elements of K.

The vectors
$$e_1 = (1,0,0,\cdots,0,0)$$
$$e_2 = (0,1,0,\cdots,0,0)$$
$$\cdots\cdots\cdots$$
$$e_n = (0,0,0,\cdots,0,1)$$
form a basis, called the usual basis of K^n and $\dim K^n = n$

(2) $\mathbb{R} \rightarrow$ field, $\mathbb{R}^3 \rightarrow$ vector space

The vectors
$$e_1 = (1,0,0)$$
$$e_2 = (0,1,0)$$
$$e_3 = (0,0,1)$$
form a basis, called the usual basis of $\mathbb{R}^3$ and $\dim \mathbb{R}^3 = 3$

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Problem: Determine whether or not the following form a basis for the vector space $\mathbb{R}^3$:

(i) $(1,1,1)$ and $(1,-1,5)$

(ii) $(1,2,3)$, $(1,0,-1)$, $(3,-1,0)$ and $(2,1,-2)$

(iii) $(1,1,1)$, $(1,2,3)$ and $(2,-1,1)$

Solⁿ: (i) and (ii) no; for a basis of $\mathbb{R}^3$ must contain exactly 3 elements, since $\mathbb{R}^3$ is of dimension 3.

(iii) The vectors form a basis if and only if they are independent. Thus form the matrix whose rows are the given vectors and row reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

The echelon matrix has no zero rows. Hence the three vectors are independent and so form a basis for $\mathbb{R}^3$.