CS 460/560
Introduction to Computational Robotics
Fall 2019, Rutgers University

Lecture 17
Multi-Robot Path Planning (2)

Instructor: Jingjin Yu

Outline

NP-Hardness of $\mathrm{MPP_r}$

- ⇨NP and NP-hardness
- ⇨Reduction from 3-SAT

Algorithms for multi-robot path and motion planning

- ⇨Graph search based algorithm for $\mathrm{MPP_p}$
- ⇨Integer linear programming models for $\mathrm{MPP_r}$

NP and NP-Hardness

Note that we are classifying **problems** here! In particular, we are NOT classifying **algorithms** (a common mistake).

A problem is in the class **non-deterministic polynomial** time **(NP)** if

- ⇨It can be solved by a **non-deterministic Turing machine** in polynomial time
- ⇨Equivalently, a given solution can be verified in polynomial time
- ⇨E.g., graph search
- ⇨E.g., given a graph G, find a Hamiltonian cycle
  - ⇨ To see that it is in NP, given a cycle, verifying it is part of G is doable in polynomial time

A problem P_1 is **NP-hard** if

- ⇨Solving it is harder than solving any other problems in NP
- ⇨I.e., any problem $P_2 \in$ NP can be solved in polynomial time via solving P_1
- ⇨Note that a problem is NP-hard does not require it is in NP
- ⇨A problem that is NP-hard and also in NP is NP-complete
- ⇨This implies that all NP-complete problems are in a sense “equal” in hardness

Some Classical NP-Complete Problems

Boolean satisfiability (SAT): first problem proven to be NP-hard

- ⇨ n **binary variables** $x_1, \dots, x_n$
- ⇨ A **literal** y of a variable x is x or $\neg x$, total $2n$ of these
- ⇨ m disjunctive clauses of literals, i.e. $c_j = y_{j_1} \vee \cdots \vee y_{j_k}$
- ⇨ Question: are there values for $x_i, \dots, x_n$ so that $c_1 \wedge \cdots \wedge c_m = 1$?
- ⇨ Shown to be NP-complete (Cook-Levin theorem)
  - ⇨ SAT is in NP because checking an answer is doable in polynomial time
  - ⇨ NP-hard via direct reduction from a generic nondeterministic Turing machine

3SAT: SAT with each clause containing up to 3 literals

- ⇨ NP-hard via the **reduction** from SAT
- ⇨ Reduction is how NP-hardness is proven in general

Vertex cover: $G = (V, E)$, is there a set of K vertices that covers V?

- ⇨ Reduction from **3SAT**

Numerous others: Hamiltonian cycle, Traveling Salesperson Problem (TSP), Set Cover, Knapsack, …

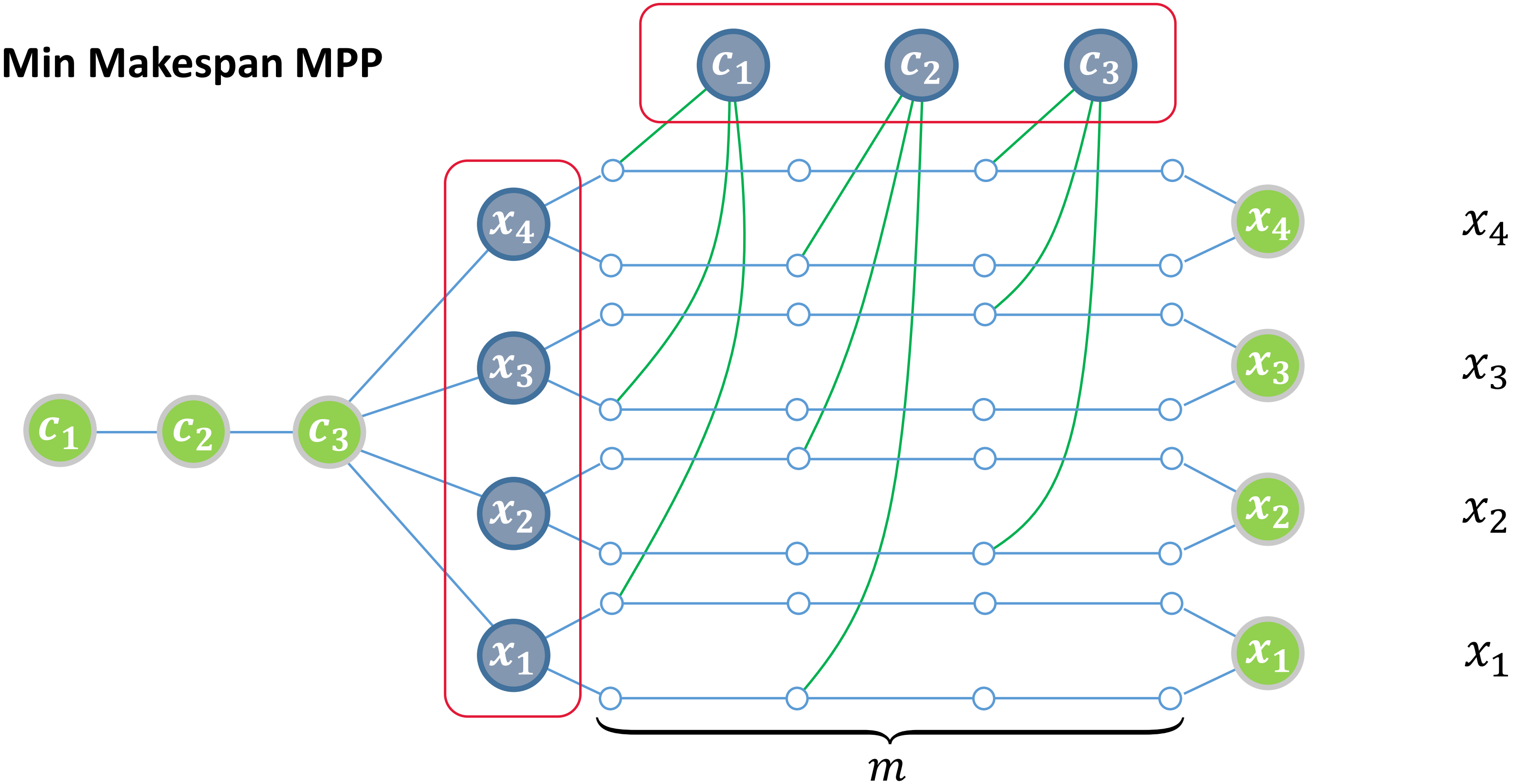
NP-Hardness of Makespan Optimal $\mathrm{MPP_r}$

Min Makespan $\mathrm{MPP_r}$ is NP-hard

3SAT $\underbrace{(x_1 \vee \neg x_3 \vee x_4)}_{c_1} \wedge \underbrace{(\neg x_1 \vee x_2 \vee \neg x_4)}_{c_2} \wedge \underbrace{(\neg x_2 \vee x_3 \vee x_4)}_{c_3}$

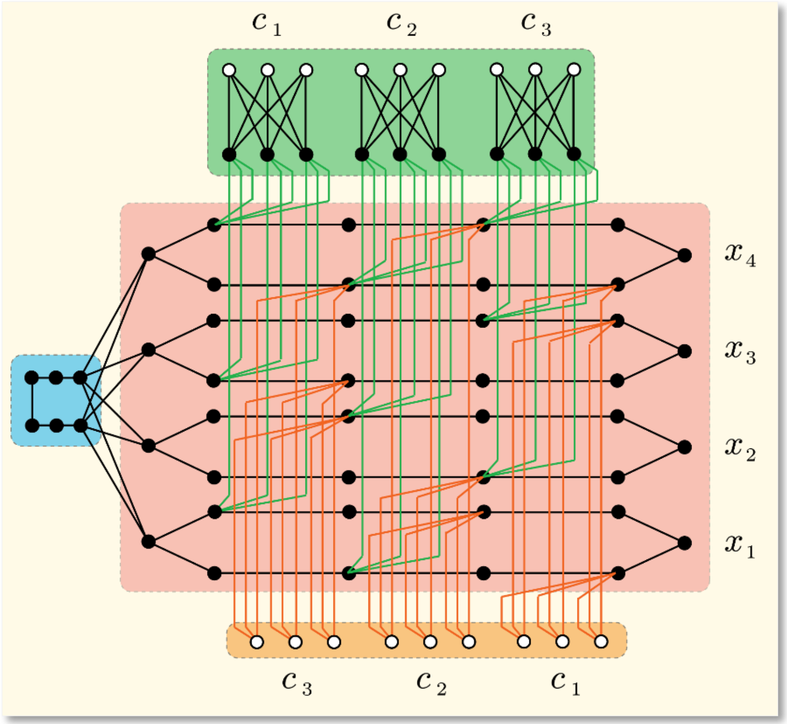
$n = 4$ variables
$m = 3$ clauses

Min Makespan MPP



NP-Hardness of Distance Optimal $\mathbf{MPP_r}$

NP-hardness of distance optimal MPP is slightly more tricky…



Theorem. MPP is NP-hard when optimizing min makespan, min total time, min max distance, and min total distance.

Implications of MPP Intractability

A problem being NP-hard means likely no polynomial time algorithm exists for solving it exactly

- ⇨More precisely, unless P = NP (a million dollar question)

Implications:

- ⇨Practitioners should not try to find polynomial time algorithm
- ⇨Aiming for solving the problem approximately
- ⇨Or aiming for solving easier cases of the problem quickly
- ⇨This is what we will try with MPP

Algorithmic solutions for MPP fall into two flavors

- ⇨Discrete search based algorithms
- ⇨Integer programming solvers

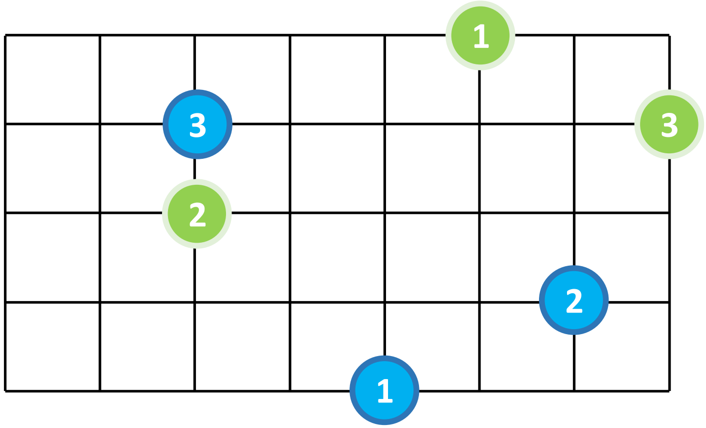
Discrete Search Algorithms

General methods

- ⇨**Coupled** search – treat all robots as a "single" robot
- ⇨**Decoupled** search – treat robots as individual ones as much as possible
- ⇨Recall the basic structure of search algorithms (still applies!)

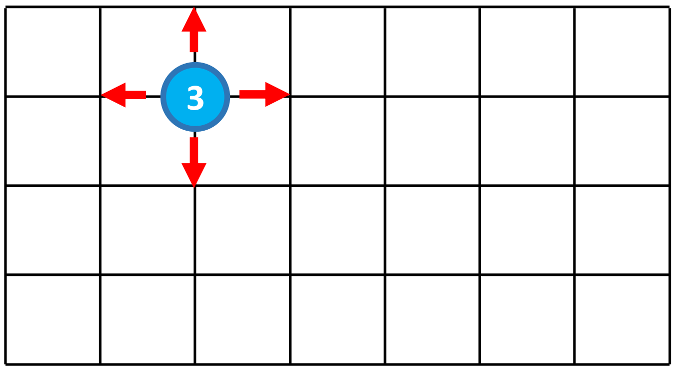
```
input: G = (V,E), x_I, x_G
AddToQueue(x_I, Queue);  // Add x_I to a queue of nodes to be expanded
while(!IsEmpty(Queue))
    x ← Front(Queue);               // Retrieve the front of the queue
    if(x.expanded == true) continue;   // Do not expand a node twice
    x.expanded = true;              // Mark x as expanded
    if(x == x_G) return solution;   // Return if goal is reached
    for each neighbor n_i of x // Add all neighbors of to the queue
        if(n_i.expanded == false) AddToQueue(n_i, Queue)
return failure;
```

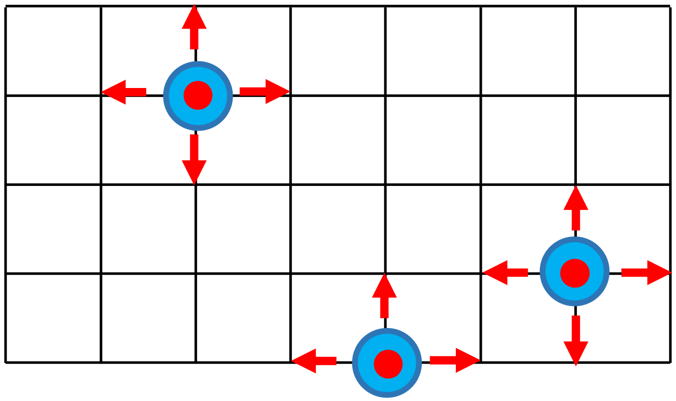
Coupled Search



Key: treat all robots as a whole

- ⇨ For a single robot, # neighbors?
  - ⇨ Up to 4 neighbors
- ⇨ What about multi-robot case?
  - ⇨ Up to 5 neighbors per robot, including staying put
- ⇨ The search works in a straightforward way
  - ⇨ For three robots, a neighboring node may be $(east, north, stay)$
- ⇨ But, huge branching factor!
  - ⇨ For n robots, 5^n "neighbors" in the graph
  - ⇨ For 3 robots, 125 neighbors per step
  - ⇨ Optimal, complete, but impractical (why exactly?)

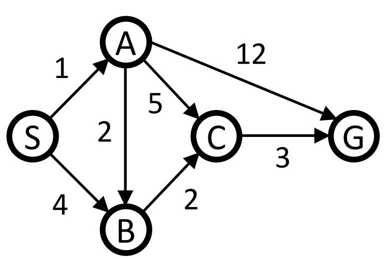


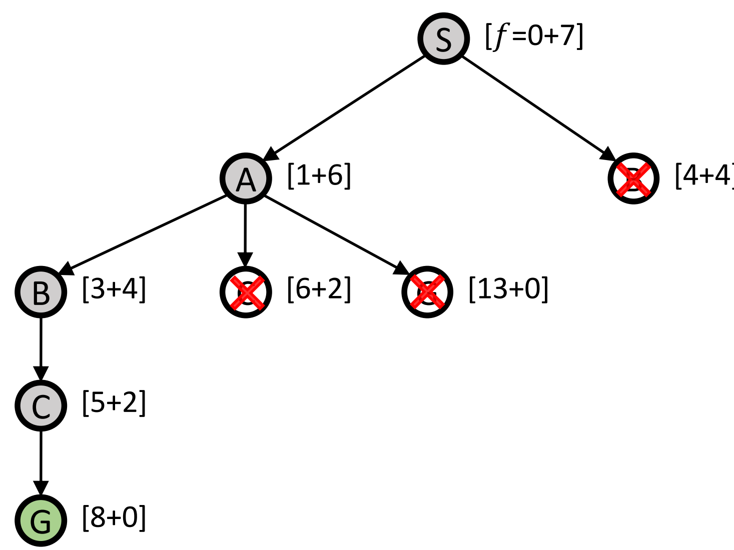


Coupled Search – Why Impractical?

How large can a priority queue be?

- ⇨ For a single robot, no more than $|V|$
- ⇨ For n robots, $|V|^n$
  - ⇨ Well, not exactly, a bit smaller, why?
  - ⇨ But close enough
- ⇨ Suppose $|V| = 10^3, n = 10$
- ⇨ $|V|^n = 10^{30}$!
- ⇨ We cannot hope to even store the queue on hard disk
- ⇨ So search will be extremely slow!

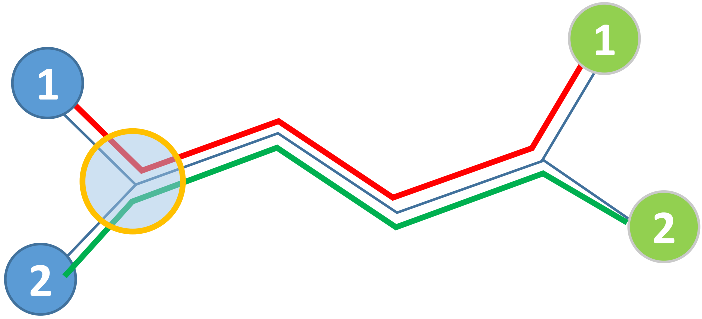




Decoupled Search

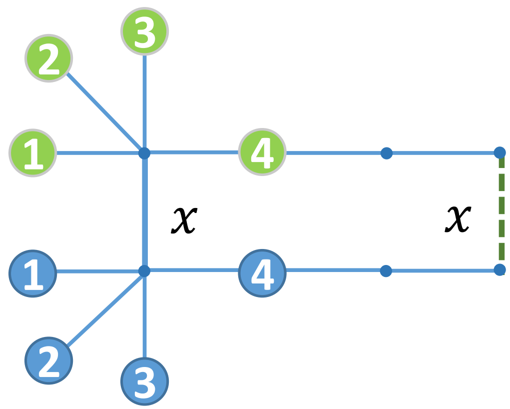
Key idea: treat robots as individual ones as much as possible

- ⇨ To start, plan optimal paths for individual robots
  - ⇨ This reduces branching factor: $5^n \Rightarrow 4n$
- ⇨ Then, simulate the "execution" of the paths
- ⇨ When there are conflicts, push all choices onto the priority queue
- ⇨ Then continue the "execution"
- ⇨ Initial paths may get updated/changed
- ⇨ In the example
  - ⇨ One queue node corresponding to robot 1 takes the junction first
  - ⇨ One queue node corresponding to robot 2 takes the junction first
  - ⇨ Both will add an additional (makespan) cost of 1
- ⇨ A rough sketch: practical implementations require lots of care-taking



Handling Different Objectives

Different objectives cause the queue to be sorted differently



- ⇨Total distance
  - ⇨ Initially all choose left path
  - ⇨ Then 1-4 have conflict at $t = 1$, generating 4 new nodes (robot i goes first)
  - ⇨ Then at $t = 2$, suppose we pick the node letting 1 go first, three new nodes are created
  - ⇨ These three new nodes can be inserted into the front of the queue using a secondary heuristic
  - ⇨ After one more iteration, 2 new nodes are generated
  - ⇨ Then one last iteration resolves all conflicts
  - ⇨ The total distance remains the same for all nodes, which is $4x + 8$
- ⇨Total time
  - ⇨ Initial node cost is $4x + 8$
  - ⇨ Here, at $t = 1$, 4 new nodes, cost is now $4x + 11$ for all
  - ⇨ At $t = 2$, if 4 goes through the right, cost is $4x + 13$, otherwise, $4x + 14$

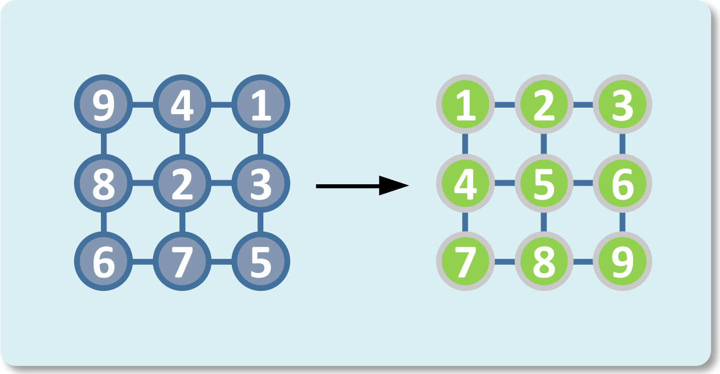
Strengths and Weakness of Discrete Search

Strengths of discrete search solutions

- ⇨ When it works, the algorithm generally runs rather fast
  - ⇨ Because the overall algorithm is relatively simple due to its discrete nature
- ⇨ Capable of solving large (sparse) problems
- ⇨ Generally straightforward to implement and tweak

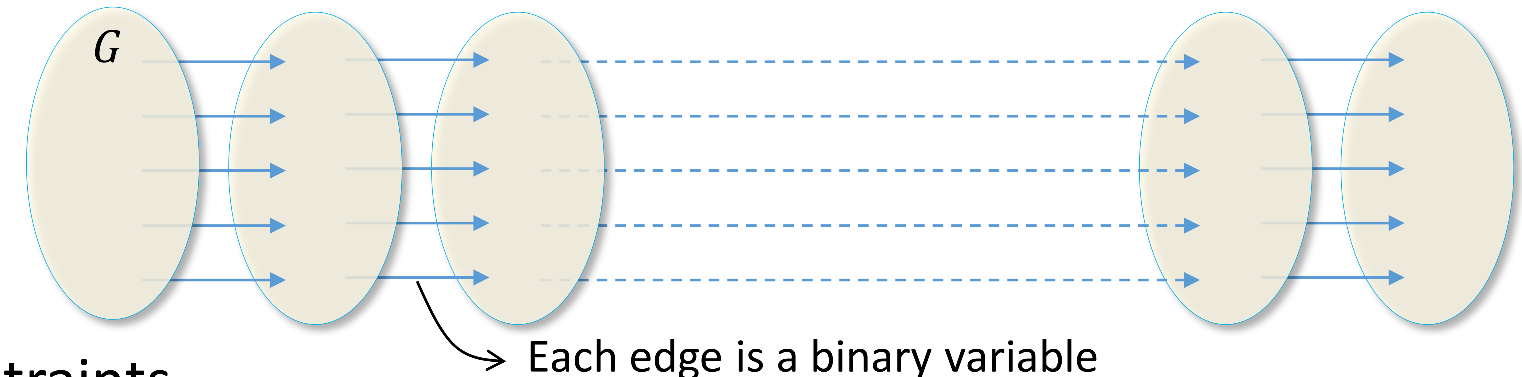
Weaknesses

- ⇨ As the interactions among the robots grow, performance degrades quickly
- ⇨ As such, not suitable for solving very dense problems
- ⇨ Not suitable for handling $\mathrm{MPP}_{\mathrm{r}}$ as the number of possible rotations can be very large; huge branching factor
  - ⇨ For 16-puzzle, >1000 possible cycles
  - ⇨ Each cycle has two directions
  - ⇨ Enumerating becomes impossible

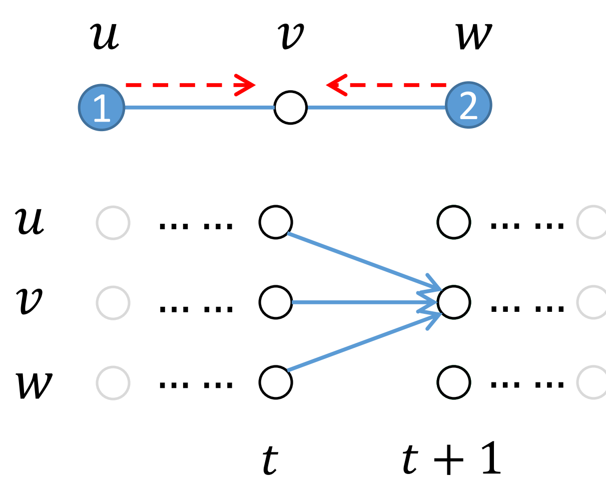


An Integer Programming Based Solver for $\mathbf{MPP_r}$

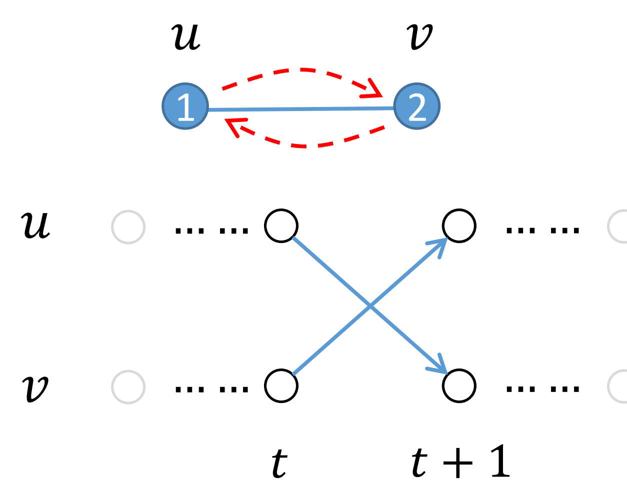
Transform MPP into multiflow over time steps



Constraints

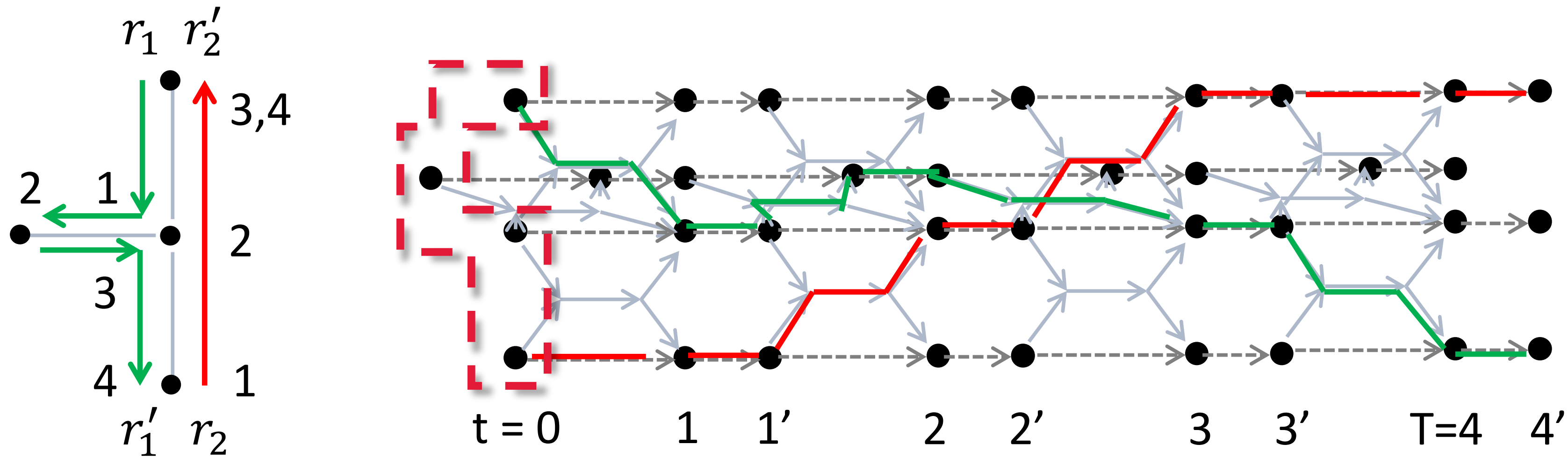


$$\sum_{1 \le i \le n} (x^i_{uv,t,t+1} + x^i_{vv,t,t+1} + x^i_{wv,t,t+1}) \le 1$$



$$\sum_{1 \le i \le n} (x^i_{uv,t,t+1} + x^i_{vu,t,t+1}) \le 1$$

An Example



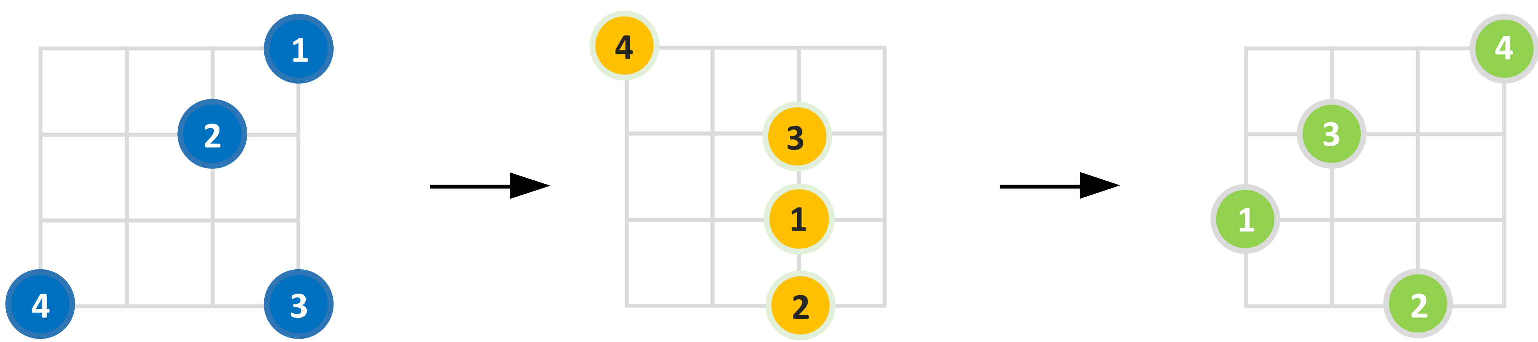
Additional Splitting Heuristic

The ILP-base algorithm can require big models

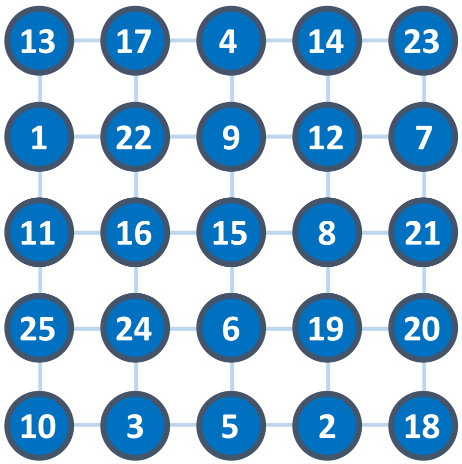
⇨ 20×15 grid, 20 steps, 20 robots $\rightarrow \sim 1$ million variables

⇨ We can use a divide-and-conquer like heuristic through **splitting over time**

⇨ The algorithm is no longer complete and may yield sub-optimal solutions

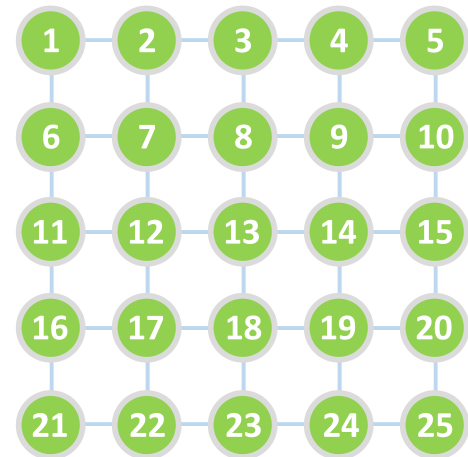


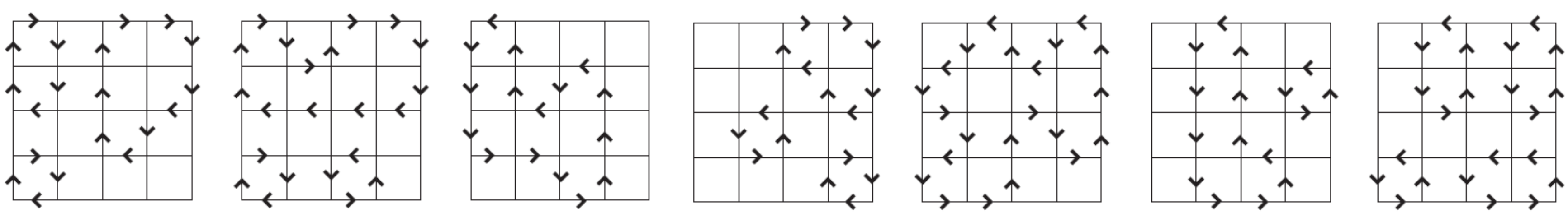
The Approach Can Solves Some Tough Problem



10^{25} states

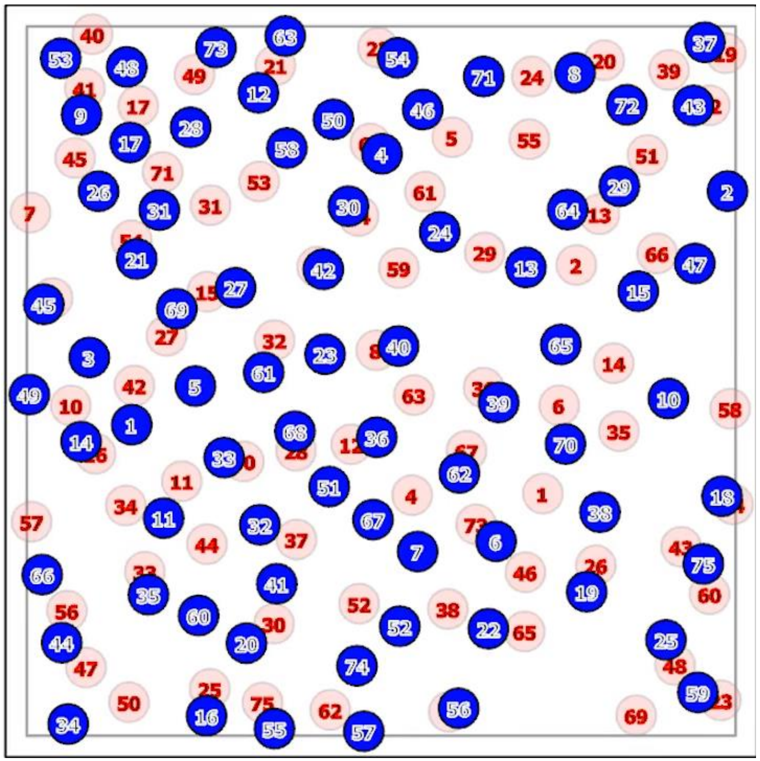
$> 10^4$ branching factor



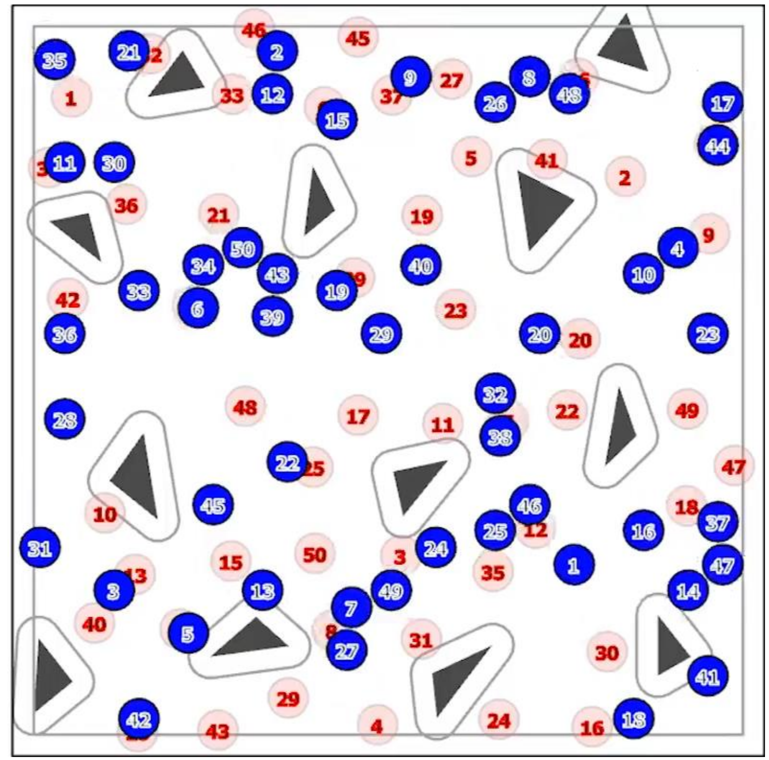


A 7-step min makespan plan

Some Examples in 2D Continuous Domain



No static obstacles, 75 robots
1.7 seconds to compute, 1.6-optimal



Random obstacles, 50 robots
4.0 seconds to compute, 1.9-optimal