$B_1(z) = z - P(z)$ Taylor method

$P(\theta) = 0$

$B_1(\theta) = \theta$

$z_{k+1} = B_1(z_k)$, $k \geq 0$

$z_0, z_1, z_2 \cdots$

$B_1'(\theta)$

$|B_1'(\theta)|$ $\begin{cases} <1. & \text{attractive} \quad \text{converge} \\ >1. & \text{repulsive} \quad \text{No converge} \\ =1. & \text{indifferent} \quad \text{things happens} \end{cases}$

$z_k \to \theta$. $|z_{k+1} - \theta| \approx |z_k - \theta|^2$

Newton method

$$B_2(z) = z - \frac{P(z)}{P'(z)}$$

Halley method

$$B_3(z) = z - P(z) \frac{P'(z)}{P'(z)^2 - P(z)\frac{P''(z)}{2}}$$

$$B_m(z) = z - P(z) \frac{D_{m-2}(z)}{D_{m-1}(z)}$$

$$D_m(z) = \begin{vmatrix} P'(z) & \frac{P''(z)}{2} & & \\ P(z) & \ddots & \ddots & \frac{P''(z)}{2} \\ & \ddots & \ddots & \\ 0 & & P(z) & P'(z) \end{vmatrix}$$

$D_0(z) = 1$

$D_1(z) = P'(z)$

$D_2(z) = \begin{vmatrix} P' & P''/2 \\ P & P' \end{vmatrix}$

fixed point theorem

$g(z)$

θ is a fixed point. if $g(\theta) = \theta$.

$$|g'(\theta)| < 1$$

$$g(z) = g(\theta) + g'(\theta)(z-\theta) + g''(\theta)(z-\theta)^2 + \cdots$$

$$g(z) - \theta = g'(\theta)(z-\theta) + g''(\theta)(z-\theta)^2 + \cdots$$

$$= (z-\theta)(g'(\theta) + \tfrac{1}{2}g''(\theta)(z-\theta) + \cdots)$$

$$\frac{g(z_k) - \theta}{z_k - \theta} = g'(\theta) + \cdots$$

$$\lim_{k=\infty} g(z_k) = g'(\theta)$$

$$N(z) = z - \frac{P(z)}{P'(z)}$$

$$P(z) = (z-\theta)^m q(z), \quad m \geq 1$$

$m > 1$ $N'(\theta) =$

$$N'(z) = 1 - \frac{P'^2(z) - P(z)P''(z)}{P'^2(z)} = 1 - 1 + \frac{P(z)P''(z)}{P'^2(z)}$$

$$N'(\theta) = 0$$

Boards on zero

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

$t^{m-1} + t - 1$, let r_m be the real root of

$m \geq 2$

$m = 2,\ r_2 = \frac{1}{2}$, $m = 3,\ r_3 = 0.618$

Let θ be any root of $P(z)$

$$|\theta| \leq \frac{1}{r_2} \max\left\{ \left|\frac{a_{n-k+1}}{a_n}\right|^{\frac{1}{k-1}},\ k = 2, \cdots n+1 \right\}$$

$$|\theta| \le \frac{1}{r_3} \max\left\{ \left| \frac{1}{a_n^2} \begin{vmatrix} a_{n-1} & a_{n-k+1} \\ a_n & a_{n-k+2} \end{vmatrix} \right|^{\frac{1}{k-1}}, \ k=3, \dots, n+2 \right\}$$

$$z^2 + 3z - 4 \quad \begin{cases} z_1 = -1 \\ z_2 = 4 \end{cases}$$

$$|\theta| \le \frac{1}{\frac{1}{2}} \max\left\{ \left|\frac{3}{1}\right|^{\frac{1}{1}}, \left|\frac{-4}{1}\right|^{\frac{1}{2}} \right\}$$

$$\le 2 \max\{3, 2\}$$

$$\le 2 \times 3 = 6.$$

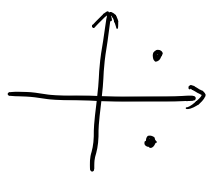
$$z_{j+1} = z_j - \frac{P(z_j)}{P'(z_j)}, \quad j \ge 0$$

$$p(z) = a_n z^n + \cdots + a_0$$

$$z = x + iy, \quad |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Conjugate

$$\bar{z} = x - iy$$



$$F(z) = |p(z)|^2 = p(z)\overline{p(z)}$$

$$F(\theta) = 0 \iff F(\theta) = 0$$

$$F(z_j) \quad ? \quad F(z_{j+1})$$

let $z_0 \in \mathbb{C}$ be a seed. Assume $P(z_0) \neq 0$

$$k = \min\left\{ j \in \{1, 2, \cdots, n\} \text{ such that } p^{(j)}(z_0) \neq 0 \right\}$$

$$u_k = \frac{1}{k!} P(z_0) \frac{1}{p^{(k)}(z_0)}$$

$$\gamma = 2\,\underline{\mathrm{Re}}\,(u_k^{k-1})$$

take real part of a complex

$$s = -2\,\mathrm{Im}(u_k^{k-1})$$

$$C_k = \max\{|\gamma|, |s|\}.$$

$$\theta = \begin{cases} 0, & \text{if } C_k = |\gamma|,\ \gamma < 0 \\ \pi/k, & \text{if } C_k = |\gamma|,\ \gamma > 0 \\ \pi/2k, & \text{if } C_k = |s|,\ s < 0 \\ 3\pi/2k, & \text{if } C_k = |s|,\ s > 0 \end{cases}$$

$$A = \max_{j \geq 0} \left\{ \left| \frac{p^{(j)}(z_0)}{j!} \right| \right\}$$

Robust Newton Iteration

$$\hat{N}_p(z_0) = z_0 + \frac{c_k}{3} \frac{u_k}{|u_k|} e^{i\theta}$$

$$c_k = \frac{C_k |u_k|^{2-k}}{6A^2}$$

If $k=1$, $C_1 = 2$, $\theta = \pi$, $e^{i\theta} = e^{i\pi} = -1$, $c_1 = \frac{|u_1|}{3A^2}$

$$\hat{N}_p(z_0) = z_0 - \frac{p(z_0)\overline{p(z_0)}}{9A}$$

$$\hat{N}_p(z) = z_0 - \frac{|p'(z_0)|^2}{9A^2} \left(\frac{p(z_0)}{p'(z_0)} \right)$$

Generic Robust Newton Method

p, k, $z_0 \in C$

$t \leftarrow 0$

while $|p(z_t)p'(z_t)| \neq 0$ do

  $z_{t+1} \leftarrow \hat{N}_p(z_t)$, $t \leftarrow t+1$

end while.

pick ε in $(0,1)$

iterate until $|p(z_\varepsilon)| \leq \varepsilon$

But the algorithm may give a pt z_ε

when $|p(z_t)p'(z_t)| \le \varepsilon$

It can be shown that as long as $|p(z_t)p'(z_t)| \ge \varepsilon$, RNM. decrease $F(z)$ by at least $\frac{\varepsilon^2}{9A^2}$

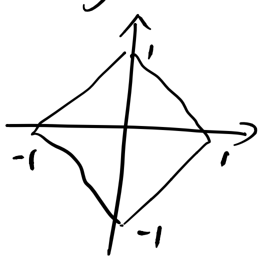
If $|p(z_t)| > \varepsilon$. but $|p(z_t)p'(z_t)| < \varepsilon$

the decrease in $F(z)$ can be small, To avoid this, we treat z_t as if it is a critical pt, if $p'(z_t) = 0$ and redefine its index as the smallest k such that $\left|\frac{p(z)\, p^{(k)}(z_t)}{k!}\right| \ge \varepsilon$

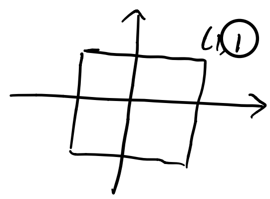
Convex sets

$x, y \in C$, $\alpha x + (1-\alpha) y \in C$. $\forall \alpha \in [0,1]$

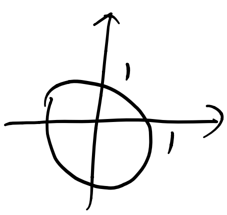
$\{x : \|x\|_1 < 1\}$



$\{x : \|x\|_\infty < 1\}$



$\{x : \|x\|_2 < 1\}$



Unit ball under any norm is a convex set.

$x, y \quad \alpha \in [0,1]$

$N(\alpha x + (1-\alpha) y)$

$\le N(\alpha x) + N((1-\alpha) y)$

$= \alpha N(x) + (1-\alpha) N(y)$

$\le \alpha + (1-\alpha) = 1$

$x^0, x^1, x^2, \cdots, \in \mathbb{R}^n$

$\lim_{k\to\infty} x^{(k)} = x_*$

$\| x^{(k)} - x_* \| \to 0$

$\|A\|_1$ = maximum of 1-norm of its colomns.

$\|A\|_\infty$ = maximum of 1-norm of its rows.

Let $r(A)$ be the max $|\lambda|$ where λ is a eigenvalue of A.

Theorm. For any norm $\|A\|$, $r(A) \leq \|A\|$.

Frobenius norm

A is a $n\times n$ matrix

$F(A) = \left(\sum_{1\leq i, j\leq n} (a_{ij})^2 \right)^{\frac{1}{2}}$ $\qquad F(AB) \leq F(A)\, F(B)$

Gaussian Elimination

$Ax = b, \quad x = A^{-1}b$

cramer's rule

$x_i = \frac{|A_i|}{|A|}$ $\qquad A_i = \begin{array}{c} \text{ith} \\ \begin{bmatrix} & b & \end{bmatrix} \end{array}$

Hadamard inequality

$A = [a_1, a_2, \cdots, a_n]$

$\det(A) \leq \prod_{i=1}^{n} \|a_i\|$

$a_{11} \neq 0$

$$A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & & & \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} \to \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & & & \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \to \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn}^{(n)} \end{bmatrix}$$

$m_{i1} = \frac{a_{i1}}{a_{11}}, \quad i = 2 \cdots n$

multiply row 1 by $-m_i$ and add to row i.

$i = 2, 3, \cdots, n$

\# of i times.

$(n-1)^2$

suppose $a_{22}^{(2)} \neq 0$

$m_{i2} = \frac{a_{i2}^{(2)}}{a_{22}^{(2)}}, \quad i = 3 \cdots n$

$(n-2)^2$

$$L = \begin{bmatrix} 1 & & & \\ m_{21} & 1 & & 0 \\ \vdots & & \ddots & \\ m_{n1} & & & 1 \end{bmatrix}$$

Thm. $A = LU$

$Ax = b \qquad LUx = b$

let $y = Ux$

$Ly = b$

$Ux = y$

solve backward substitution

$1 + 2 + \cdots + (n-1) \doteq \frac{n^2}{2}$

total: $\frac{1}{3}n^3 + n^2$

$AA^{-1} = I = [e_1, e_2, \cdots, e_n]$

$A^{-1} = [x^{(1)} \cdots x^{(n)}]$

$Ax^{(i)} = e_i$

$\frac{1}{3}n^3 + n \cdot n^2 = \frac{4}{3}n^3$

In practice we use pivot row is the one that has maximum entry in absolute value among the column we want to make zero.

maximum

$$A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & & & \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

$LU = PA$ → purmultation matrix.

$Ax = p$

$x_1 - x_2 + 2x_3 = 2$

$-x_1 + 2x_2 + x_3 = 2$

$2x_1 - 4x_2 + x_3 = -1$

$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & -4 & 1 \end{bmatrix} \quad p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -4 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad p = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$m_{21} = -\frac{1}{2} \quad m_{31} = \frac{1}{2}$

$\begin{bmatrix} 2 & -4 & 1 \\ 0 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$

$\begin{bmatrix} 2 & -4 & 1 \\ m_{31} & 1 & \frac{3}{2} \\ m_{21} & 0 & \frac{3}{2} \end{bmatrix} \quad p = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$

$LU = \begin{bmatrix} 2 & -4 & 1 \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

$Ly = Pb$

$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$y_1 = -1$

$\frac{1}{2}y_1 + y_2 = 2$

$y_2 = \frac{5}{2}$

$-\frac{1}{2}y_1 + y_3 = 2$

$y_3 = \frac{3}{2}$

$Ux = y$

$$\begin{bmatrix} 2 & -4 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{5}{2} \\ \frac{3}{2} \end{bmatrix}$$

$x_3 = 1$

$x_2 + \frac{3}{2}x_3 = \frac{5}{2}$

$x_2 = 1$

$2x_1 - 4x_2 + x_3 = -1$

$x_1 = 1$

Iterative method for solving $Ax=b$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = D - L - U$$

$$\Downarrow$$

$$\overset{D}{\begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & \ddots & a_{nn} \end{bmatrix}} - \overset{L}{\begin{bmatrix} 0 & & 0 \\ -a_{21} & \ddots & \\ \vdots & & \\ -a_{n1} & & 0 \end{bmatrix}} - \overset{U}{\begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ & \ddots & & \\ 0 & & \ddots & 0 \end{bmatrix}}$$

$Ax = b$ Jacobi method

$Dx = (L+U)x + b$ suppose $a_{ii} \neq 0, \forall i$

$x = D^{-1}(L+U)x + D^{-1}b$

$x^{(k)} = D^{-1}(L+U)x^{(k-1)} + D^{-1}d$

$x^{(0)}$ is a seed.

$T = D^{-1}(L+U)$

$C = D^{-1}b$

$$D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{a_{nn}} \end{bmatrix}$$

① $x^{(k)} = Tx^{(k-1)} + C$

let x_t for solution $Ax=b$

② $X_K = TX_K + C$

②$-$①

$$X_K - X^{(K)} = T(X_K - X^{(K-1)})$$

$$e^{(K)} = Te^{(K-1)} = T^2 e^{(K-2)} \cdots = T^{(k)} e^{(0)}$$

suppose $\|T\| < 1$. $\|e^{(K)}\| \leq \|T^k\| \, \|e^{(0)}\|$

$$\leq \|T\|^k e^{(0)}$$

Def. we say A is diagonally dominant if $|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$

$(D-L)x = Ux+b$. suppose $(D-L)^{-1}$ exist.

$$x = \underbrace{(D-L)^{-1}U}_{T_g}x + \underbrace{(D-L)^{-1}b}_{C_g}$$

$$X = T_g x + C_g$$

$$x^{(k)} = T_g x^{(k-1)} + C_g$$

Gaussian seidel.

$$l^{(k)} = T_g \, l^{(k-1)}$$

$$= T_g^k \, l^{(k-1)}$$

want $\|T_g\| < 1$

what is cost of each iteration?

$$T_g x^{(k-1)} = y$$

$$(D-L)^{-1} U x^{(k-1)} = y$$

$$Ux^{(k-1)} = (D-L)y$$

Successive over-relaxition (SOR)

$$Lx = (-D+U)x + b$$

let w be any scalar

$$wLx = w(-D+U)x + wb$$

add $Dx = Dx$

$$(D-wL)x^{(k)} = [(1-w)D + wU]x^{(k-1)} + wb$$

$$T_w = [D-wL]^{-1}[(1-w)D+wU], \quad C_w = w(D-wL)^{-1}b$$

$$x^{(k)} = T_w x^{(k-1)} + C_w$$

Thm (Kahom)

If $a_{ii} \neq 0$, $\forall i$. then $\rho(T_w) \geq |w-1|$ so that if SOR converge, then $w \in (0,2)$.

Thm (Ostrouski-Reich)

If A is Positive definite and $w \in (0,2)$, then SOR converge for any $x^{(0)}$.

a symmatric matrix A is PSD, if $x^T A x \geq 0$, $\forall x$ and posive definite if $x^T A x > 0$, $\forall x \neq 0$

Thm. if A is positive definite and tridiagonal, then $\rho(T_g) = \rho(T_j)^2 < 1$ and the optimal w is $w_* = \dfrac{2}{1+\sqrt{1-\rho(T_j)^2}}$. $\rho(T_{w_*}) = w_* - 1$.

Ex

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \qquad x^T A x = (x_1, x_2, x_3) A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 4x_1^2 + 4x_2^2 + 4x_3^2 + 3x_1x_2 + 3x_1x_2 - x_2x_3 - x_2x_3$$

$$= 3(x_1+x_2)^2 + x_1^2 + 3x_3^2 + (x_2 - x_3)^2$$

$$T_j = D^{-1}(L+U)$$

$$= \begin{bmatrix} \frac{1}{4} & & 0 \\ & \frac{1}{4} & \\ 0 & & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & -\frac{3}{4} & 0 \\ -\frac{3}{4} & -\lambda & \frac{1}{4} \\ 0 & \frac{1}{4} & -\lambda \end{bmatrix}$$

$$\det(T_j - \lambda I)$$

$$= -\lambda(\lambda^2 - \tfrac{1}{16})$$

$$\lambda = \tfrac{1}{4} = \rho(T_j)$$

from this $\omega_* = \dfrac{2}{1+\sqrt{1-\frac{1}{16}}} \approx 1.24$

$$\rho(T_{\omega_*}) \doteq 0.24$$

$$\|e^{(k)}\| \le \|T\|^k \|e^{(0)}\| \le \epsilon$$

$$\|T\| = \rho$$

$$\rho^k e \le \epsilon$$

$$\frac{e}{\epsilon} \le \left(\frac{1}{\rho}\right)^k$$

$$\ln\left(\frac{e}{\epsilon}\right) \le k \ln\left(\frac{1}{\rho}\right) \qquad k = \frac{\ln(\frac{e}{\epsilon})}{\ln(\frac{1}{\rho})} \qquad O(\ln \tfrac{1}{\epsilon}) \quad \epsilon = 2^{-10}$$

Accelerated SOR

$$x^{(k)} = T_{\sigma,w} x^{(k-1)} + T_{\sigma,w}$$

$$T_{\sigma,w} = (D - \sigma L)^{-1}[(1-w)D + (w-\sigma)L + wU]$$

$$C_{\sigma,w} = w(D-\sigma L)^{-1} b$$

σ, w constraints: if $\sigma = 0$, $w=1$. Jacobi

if $\sigma = w = 1$, Gaussian-seidel.

if $\sigma = w$, SOR

$Ax = b$ ← want to solve this

$A - m \times n$

if x is a symmetric

$X \succeq$ ← x is semi-definate

Triangular Alg.

suppose that there is a solution $x \geq 0$

$$Ax - \alpha b = 0$$

$$\Sigma x_i + \alpha = 1$$

$$x \geq 0, \alpha \geq 0$$

$$x_1 - x_2 = 0$$

$$2x_1 + x_2 = 3$$

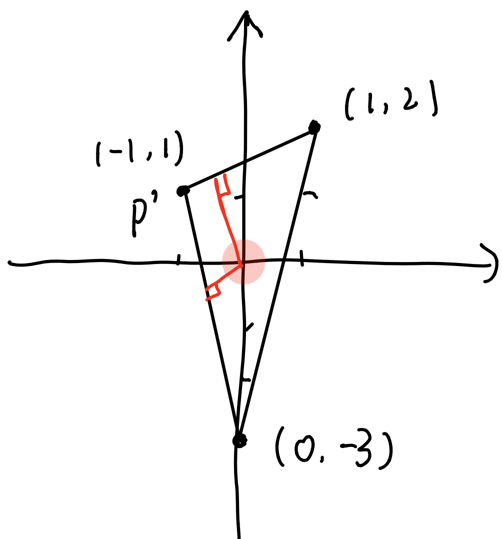
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathrm{Conv}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, -\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$$

$$p' = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\|p' - v\| \geq \|p - v\|$$

$$(p - p')^T v \geq \frac{1}{2}(\|p\|^2 - \|p'\|^2)$$

$$v_1 - v_2 \geq -\frac{1}{2} 2$$



$v_1 - v_2 \geq -1$

$p'' = Ax' - \alpha' b$

check if $\|A\frac{x'}{\alpha'} - b\| \leq \epsilon$

singular value decomposition

A $n \times n$, $A = A^T$ symmetric

$A = A^*$ conjugate transpose, Hermitian

eigenvalue are real

$Ax = \lambda x$

$x^T A x = \lambda x^T x$

Hermitian

$x^* A x = \lambda x^* x$

take conjugate of both side:

$x^* A^* x = \lambda^* x^* x$

$\lambda = \lambda^* \Rightarrow \lambda$ is real.

Thm. A is $n \times n$ over $\mathbb{C}$ (schur Normal Form) Then there exists unitary matrix U such that $A = UTU^*$

T triangular ($UU^* = U^*U = I$)

Thm. principal axis Thm.

suppose $A = A^*$

All eigenvalue are real and eigenvector are orthonormal.

$A = UTU^*$

$A^* = (U^*)^* T^* U^* = UT^*U^*$ $A = A^*$. $T = T^*$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \text{get eigenvalue.}$$

$$(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 4)$$

$$A \cdot u_1 = \lambda_1 \cdot u_1 \quad A u_1 = u_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \|u_1\| = \sqrt{3}$$

$$u_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \leftarrow \text{orthonorgonal}$$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A = U \Sigma V^*$$

$$M = A^* A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -2 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

a trick: suppose A is A^* here.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$M = A^* A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\det(M - \lambda I) = 0 \Rightarrow \begin{cases} \lambda_1 = 6 \\ \lambda_2 = 1 \end{cases}$$

$M\varphi_1 = 6\varphi_1$ $\quad \begin{bmatrix}5 & 2\\ 2 & 2\end{bmatrix}\varphi_1 = 6\varphi_1 \quad \begin{bmatrix}-1 & 2\\ 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad \begin{matrix}x_1 = 2\\ x_2 = 1\end{matrix}$

$$\varphi_1 = \begin{bmatrix}2\\ 1\end{bmatrix}\frac{1}{\sqrt{5}}$$

$\begin{bmatrix}5 & 2\\ 2 & 2\end{bmatrix}\varphi_2 = 1\varphi_2, \quad \begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad \begin{matrix}x_1 = 1\\ x_2 = -2\end{matrix}$

$$\varphi_2 = \begin{bmatrix}1\\ -2\end{bmatrix}\frac{1}{\sqrt{5}}$$

$r=2 \quad \lambda_1 = 6 \quad \lambda_2 = 1$

$w_1 = A\varphi_1 = \begin{bmatrix}2 & 1\\ 0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}\frac{1}{\sqrt{5}} = \begin{bmatrix}5\\ 1\\ -2\end{bmatrix}\frac{1}{\sqrt{5}}$

$w_2 = A\varphi_2 = \begin{bmatrix}2 & 1\\ 0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}\frac{1}{\sqrt{5}} = \begin{bmatrix}0\\ -2\\ -1\end{bmatrix}\frac{1}{\sqrt{5}}$

$\begin{cases} u_1 = \begin{bmatrix}5\\ 1\\ -2\end{bmatrix}\frac{1}{\sqrt{30}} \\ u_2 = \begin{bmatrix}0\\ -2\\ -1\end{bmatrix}\frac{1}{\sqrt{5}} \end{cases} \quad \begin{matrix}5x_1 + x_2 - 2x_3 = 0\\ 0 - 2x_2 - x_3 = 0\end{matrix} \Rightarrow \begin{matrix}x_2 = 1\\ x_3 = -2\\ x_1 = -1\end{matrix} \quad u_3 = \begin{bmatrix}-1\\ 1\\ -2\end{bmatrix}\frac{1}{\sqrt{6}}$

$\frac{1}{\sqrt{\lambda}}$

$$\begin{bmatrix}u_1, u_2, u_3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}$$

$$U\Sigma V^*$$

A . $n \times n$ matrix

Can we find bound on norm of eigenvalue?

Thm. Gerschgorin Thm.

$A = (a_{ij})$, $n \times n$

Let $r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Let $Z_i = \{z \in \mathbb{C} : |z - a_{ij}| \leq r_i\}$

If λ is an eigenvalue of A.

Then $\lambda \in Z_i$ for some i.

$$\begin{bmatrix} 1 & -2 & 5 \\ -3 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

$Z_1 = \{z : |z-1| \leq 7\}$

$Z_2 = \{z : |z-2| \leq 4\}$

$Z_3 = \{z : |z-4| \leq 2\}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\|X\| = \sum |x_i|$

$\|X\|_2 = \sum x^2$

$\|X\|_\infty = \max |x_i|$

root finding   newton's method / robust newton's method

linear system   how to solve system using factorization

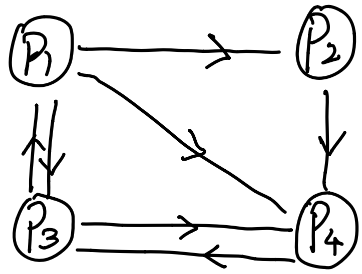
. . . .   jacobi . norm of matrix. SVD. Triangular.

MIDTERM

Page rank matrix

Webpages can be considered as a direct graph edge = link, nodes are pages.

Example.



How to assign ranking to them?

In general there are n pages, we define a matrix A, $n \times n$

$A = (a_{ij})$

$$a_{ij} = \begin{cases} \frac{1}{n_j}, & \text{if there is an edge from } j \text{ to } i. \\ 0, & \text{otherwise} \end{cases}$$

n_j = num of outgoing edge at j.

For the example.

$$\begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array}$$
$$\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

A is column stochastic (sum of column entries of each column is 1)

$e = (1, 1, \cdots, 1)^T \qquad e^T A = e^T$ or $A^T e = e$

So $\lambda = 1$ is an eigenvalue with eigenvector e.

A & A^T have same eigenvalue.

$\therefore Ax = A$ for some $x \neq 0$.

Thm. If A is column stochastic, then $\lambda = 1$ is an eigenvalue and all other eigenvalue λ_i satisfies $|\lambda_i| \leq 1$.

proof. suppose $Ax = \lambda_i x$, $x \neq 0$

let $\|x\| = 1$

So $|\lambda_i x| = |\lambda_i| \cdot |x| = |\lambda_i|$

$= \|Ax\| \leq \|A\| \cdot \|x\| \leq \|A\| \leq 1$

Thm. A column stochastic if $Ax = x$, then $x \geq 0$.
$(x \neq 0)$

So we can say $Ax = x$, $e^T x = \sum x_i = 1$. $x \geq 0$ is feasible.

Let $M = A - I$. $Mx = 0$. $\sum x_i = 1$, $x \geq 0$

Thm. to use Gordan's Thm, for any $m \times n$ matrix H either

$Hx = 0$

$\sum x_i = 1$ either $H^T y < 0$ for some y.

$x \geq 0$

Thm. (Perron - Frobenius)

If M is column stochastic with positive entries, then $\exists !$ (there is a unique) $x \geq 0$ such that $Mx = x$.
ie. $\lambda = 1$ is the largest eigenvalue and it's unique.

$\lambda_1 = 1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$

If f is column stochastic but not all entries ≥ 0, then place ϵ for zero entries. $M_\epsilon x \approx x$

$M_\epsilon x_\epsilon = \lambda_\epsilon x_\epsilon$. $\|x_\epsilon\| = 1$

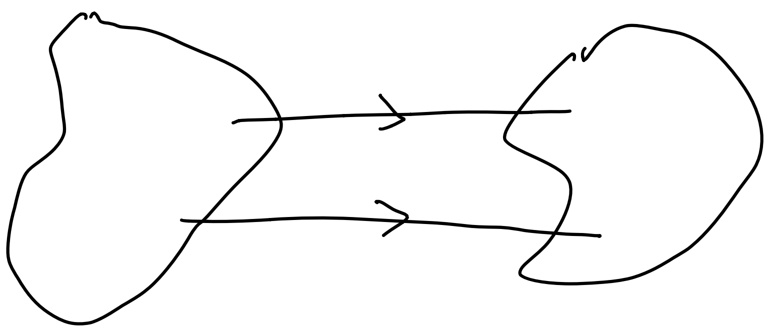
in the limit is $\epsilon \to 0$.

In general place $\begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$ when we have a dangling node.

$\bar{A} = (\bar{a}_{ij})$

$$\bar{a}_{ij} = \begin{cases} \frac{1}{n_j}, & \text{if there is edge from } j \text{ to } i. \ (n_j = \# \text{ of outgoing edge at } j) \\ \frac{1}{n}, & \text{for dangling node.} \\ 0, & \text{otherwise} \end{cases}$$

Another pattology is reducible webgraphs



To avoid this we define Google page rank matrix $M = d\bar{A} + \frac{(1-d)}{n} ee^T$ when $d \in (0,1)$

$d = 0.85$

$$ee^T = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [1 \ \cdots \ 1] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & & 1 \end{bmatrix}$$

$$e^T M = d e^T \bar{A} + \frac{(1-d)}{n} e^T e e^T$$
$$= de^T + (1-a)e^T = e^T$$

so m is column stochastic and $M = (m_{ij})$, $m_{ij} > 0$.

Projects.

#1 Robust Newton Method

#2 Iterative method to solve linear system (SOR, Jacobi, Gauss-Seidel)
+ triangular Alg.

#3 page rank (power method & triangular method)

A $n \times n$

$e^T A = e^T$

$Ax = x$, $x \neq 0$, $x \geq 0$

Thm. If A is column stochastic. $\exists x \geq 0$, $x \neq 0$. such that $Ax = x$.

proof. Let $H = A - I$

$$Hx = 0$$

$$\sum x_i = 1, x \geq 0 \quad \text{or} \quad H^T y < 0$$

$$H^T = A^T - I$$

$$[A^T - I] y < 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^T y < y$$

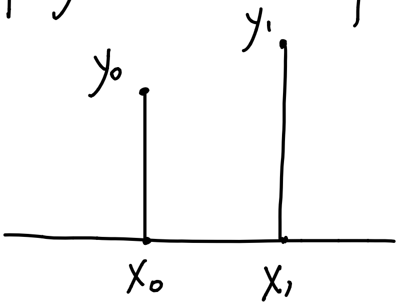
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} < \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

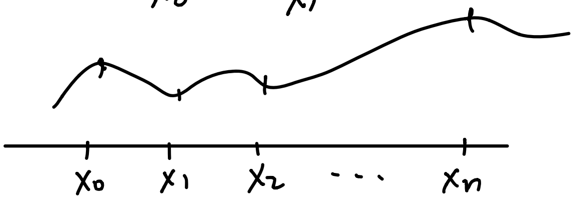
$$\leq a_{11} y_1 + a_{21} y_2 + a_{31} y_3 < y_1$$

$$a_{11} y_1 + a_{21} y_1 + a_{31} y_1 < y_1$$

$$y_1 < y_1$$

Polynomial Interpolation





suppose we are given $(n+1)$ distant points.

$x_0, x_1, \cdots, x_n$

and we are given

$y_0, y_1 \cdots y_n$

want to find a polynomial of least degree $P(x)$ so that

$P(x_i) = y_i \,.\, i=0,1\cdots n$

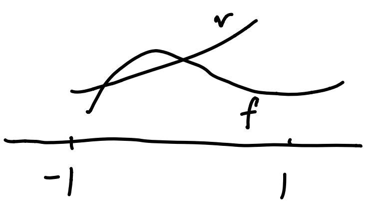
Also degree of $p(x) \leq n$

Thm. It is unique if it exists. If not we have at least two polynomials $P(x)$ and $q(x)$ such that

$f \in C[a,b]$

$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$

$\|f-r\|_\infty = \max_{x\in[a,b]} |f(x)-r(x)|$



$a_0 + a_1 x$

least square approximation in function space

$g \in C[a,b]$

$\|g\|_2 = [\int_a^b (g(x))^2 \, dx]$

suppose $f \in C[a,b]$

For $n \geq 0$. Let $M_n(f) = \inf_{\deg(r)\leq n} \|f-r\|_2$

If inf is attained at some polynomial r_n^* of degree $\le n$

$M_n(f) = \| f - r_n^* \|$

Ex. $f(x) = e^x$, $[-1, 1]$

$n = 1$, $r_1(x) = a_0 + a_1 x$

$\| f - r_1 \|_2^2 = \int_{-1}^{1} (e^x - (b_0 + b_1 x))^2 dx$

minimize $F(b_0, b_1)$ $\quad F(b_0, b_1)$

$\frac{\partial F}{\partial b_0} = b_0 = \frac{\partial F}{\partial b_1}$

$b_0 = \frac{1}{2} \int_{-1}^{1} e^x dx$, $\quad b_1 = \frac{3}{2} \int_{-1}^{1} x e^x dx$

General least-square problem

Suppose we have $w(x) \geq 0$ on $[a, b]$ is satisfied.

1. $\int_a^b |x|^n w(x) dx$ exists $< \infty$
2. If $g(x) \geq 0$ on $[a, b]$ then $\int_a^b g(x) w(x) = 0 \Rightarrow g(x) \equiv 0$

Ex.

1. $w(x) = 1$, $x \in [a, b]$
2. $w(x) = \frac{1}{\sqrt{1 - x^2}}$, $x \in [-1, 1]$
3. $w(x) = e^{-x}$, $x \in [0, \infty)$
4. $w(x) = e^{-x^2}$, $x \in (-\infty, \infty)$

given $f \in C[a, b]$

$$M_{n,w}(f) = \min_{\substack{r(x) \text{ a polynomial} \\ \text{of degree} \le n}} \int w(x) (f(x) - r(x))^2 dx$$

$r(x) = a_0 + a_1 x + \cdots + a_n x^n$

$F(a_0, a_1, \cdots, a_n) = \int_a^b w(x) (f(x) - (a_0 + \cdots + a_n x^n))^2 dx$

$\frac{\partial F}{\partial a_i} = 0, \quad i = 0, \cdots, n$

$w(x) \equiv 1, \quad [a, b] = [0, 1]$

this gives linear equations

$\sum \frac{a_j}{i+j+1} = \int_0^1 f(x) x^j dx$

$i = 0, 1, \cdots, n$

$w(x)$ as before

$f, g \in C[a, b]$

we can define inner product $(f, g) \equiv \langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx$

1. $\langle \alpha f, g \rangle = \langle f, \alpha g \rangle = \alpha \langle f, g \rangle$
2. $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$
3. $\langle f, g \rangle = \langle g, f \rangle$
4. $\langle f, f \rangle \geq 0$ is $\langle f, f \rangle = 0 \Leftrightarrow f \equiv 0$

Inner product induces a norm

$\|f\|_2 = [\int_a^b w(x) f^2(x) dx]^{1/2} = \langle f, f \rangle^{1/2}$

we can show

Cauchy - Schwatz inequality holds:

$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$

Thm (Gram - Schmit)

Given any $n \geq 0$, then exists polynomials $a_0(x), a_1(x), \cdots a_n(x)$ when $a_i(x)$ is of degree i.

so that $\langle a_i, a_j \rangle = 0, \ i \neq j$

$\langle a_i, a_i \rangle = \| a_i(x) \|_2^2$

we can scale a_i's so that $\langle a_i, a_i \rangle = 1$

we say $\{a_0, a_1, \cdots a_n\}$ form an orthogonal basis.

Particular case

case 1. $w(x) \equiv 1$, legendre polynomials

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x^2)^n]$$

$n \geq 1$

case 2. $w(x) = \frac{1}{\sqrt{1-x^2}}$, $[-1, 1]$

chebyshev polynomial:

$T_n(x) =$

$T_0(x) = 1, \ T_1(x) = x$

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$

$n \geq 1$

Case 3. $w(x) = e^{-x}$, $x \in [0, \infty)$

Laguwre polynomials.

$$L_n(x) = \frac{1}{n! \, e^{-x}} \cdot \frac{d^n}{dx^n} (x^n e^{-x}) \quad n \geq 0$$

orthogonal polynomials satisfy 3-term recursion.

$$L_{n+1}(x) = \frac{1}{n+1} [2n+1 - x] L_n(x) + \frac{n}{n+1} L_{n-1}(x)$$

Legendre

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

Numerical Integration

$f(x)$ on $C[a,b]$

$I(f) = \int_a^b f(x)\,dx$

Quadrature formular

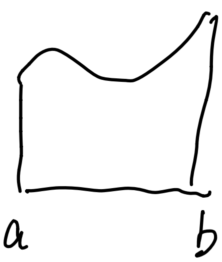
we may want to have a family $f_n(x)$

$I(f_n) = \int_a^b f_n(x)\,dx \equiv I_n(f)$

$\| f - f_n \|_\infty \to 0$ as $n \to \infty$

$E_n(f) = I(f) - I_n(f)$

$= \int_a^b (f(x) - f_n(x))\,dx$



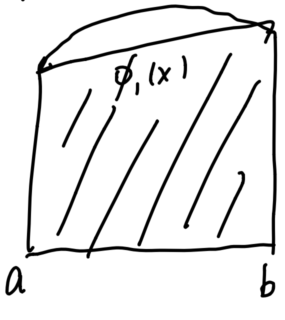
$E_n(f) \leq (b-a) \| f - f_n \|_\infty$

$I_n(f) = \sum_{j=1}^{n} w_{j,n} f(x_{j,n})$

$w_{j,n}$ are collece weight

x_j are collect nodes

Trapezoidel Rule



$I_1(f) = \int_a^b P_1(x)\,dx = \frac{b-a}{2}[f(a) + f(b)]$

$f(x) - P_1(x) = f(x) - \frac{(b-x)f(a) + (a-x)f b}{b-a}$

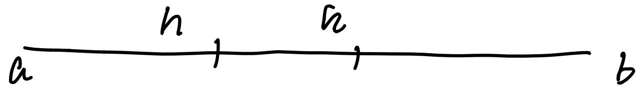
$= (x-a)(x-b) f[a,b,x]$

$E_1(x) = \int_a^b (f(x) - P_1(x)) = \int_a^b (x-a)(x-b) f[a,b,x]$

integral thm $= f[a,b,\varepsilon] \int_a^b (x-a)(x-b)\,dx$

$E_1(f) = \frac{1}{2} f''(\varepsilon)\left(-\frac{1}{6}(b-a)^3\right) \quad \varepsilon \in [a,b]$

Composite Trapezoidel rule



$h = \frac{b-a}{n}$, $x_j = a + hj$

$j = 0, 1 \dots n$

do TR on each substitus

$$I(f) = \int_a^b f(x)dx \equiv \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx$$

$$= \sum_{j=1}^{n} \frac{h}{2} [f(x_{j-1}) + f(x_j)] - \frac{h^3}{12} f''[\xi_j]$$

$$I_n(f) = h[\tfrac{1}{2}f_0 + f_1 + \dots + f_{n-1} + \tfrac{1}{2}f_n]$$

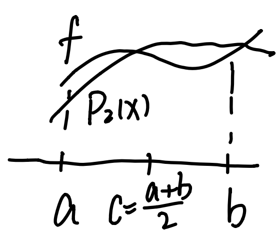
$$f_i = f(x_i)$$

$$E_n(f) = I(f) - I_n(f) = -\frac{h^3}{12}\sum f''(\xi_j)$$

$$E_n(f) = \frac{-(b-n)}{12} h^2 f''(\xi) = \frac{-h^3 n}{12} \sum \frac{f''(\xi_j)}{n} = \frac{-h^3 n}{12} f''(\xi)$$

$$P_2(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-a)(x-c)}{(b-a)(b-c)} f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c)$$

Next case.



$$I_2(f) = \int_a^b P_2(x)dx$$

$$I_2(f) = \frac{h}{3}[f(a) + 4f(\tfrac{a+b}{2}) + f(b)]$$

$$E_2(f) = \int_a^b (f(x) - P_2(x))dx$$

$$= \int_a^b (x-a)(x-b)(x-c) f[a,b,c,x]\, dx$$

let $w(x) = \int_a^x (t-a)(t-b)(t-c)\, dt$

$w(a) = 0$, $w(b) = 0$, $w(x) > 0$ on (a,b)

Integrating $E_2(f)$ by parts:

$$E_2(f) = \int_a^b w(x)\, f[a,b,c,x]\, dx$$

$$= \{w(x) f[a,b,c,x]\}_a^b - \int_a^b w(x) \frac{d}{dx} f[a,b,c,x]\, dx$$

$$E_2(f) = \frac{-f^{(4)}(\xi)}{24} \frac{4}{15} h^5$$

$$h = \frac{b-a}{2}$$

$$E_2(f) = -\frac{h^5}{90} f^{(4)}(\xi), \quad \xi \in [a,b]$$

Simpson's method

Composite Simpson's Rule

$[a,b]$, $h = \frac{b-a}{n}$, $x_j = a + jh$, $j = 0, 1, \cdots, n$

$$I_n(f) = \frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n]$$

$$E_n(f) = \frac{-h^4 (b-a)}{180} f^{(4)}(\xi)$$

In general: Newton–Cotes

$h = \frac{b-a}{n}$, $x_j = a + jh$, $j = 0 \cdots n$

$$I_n(f) = \int_a^b P_n(x)\, dx$$

$$= \int_a^b \sum_{j=0}^{n} l_{j,n}(x) f(x_j)\, dx$$

$$= \sum_{j=0}^{n} w_{j,n} f(x_j)$$

$$w_{j,n} = \int_a^b l_{j,n}(x)\, dx$$

Thm. n even

$$I(f) - I_n(f) = C_n h^{n+3} f^{(n+2)}(\xi)$$

$$C_n = \frac{1}{(n+2)!}\int_0^n \mu(\mu-1)\cdots(\mu-n)\,d\mu$$

n odd

$$I(f) - I_n(f) = C_n h^{n+2} f^{(n+1)}(\xi)$$

$$C_n = \frac{1}{(n+1)!}\int_0^n \mu(\mu-1)\cdots(\mu-n)\,d\mu$$

Gaussian Quadrature.

a b

Given n w_j weights, x_j nodes

$j = 1, \cdots, n$

$[a, b] = [-1, 1]$

$$\sum_{j=1}^{n} w_j f(x_j)$$

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=1}^{n} w_j f(x_j)$$

So that the formula is exist for polynomial of degree.

$$\int_{-1}^{1} f(x)\,dx \approx \underset{w_1}{2} f(\underset{x_1}{0})$$

For $n=2$

$$w_1 f(x_1) + w_2 f(x_2)$$

$$\begin{cases} w_1 + w_2 = 2 \\ w_1 x_1 + w_2 x_2 = 0 \\ w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3} \\ w_1 x_1^3 + w_2 x_2^3 = 0 \end{cases}$$

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m$$

$$E_n(f) = E_n(a_0 + a_1 x + \cdots + a_m x^m)$$

$$= a_0 E_n(1) + a_1 E_n(x) + \cdots + a_m E_n(x^m)$$

$$E_n(x^j) = \int_{-1}^{1} x^j \, dx$$

In case $n=1$

$$E_n(1)=0 \quad , \quad E_n(x)=0$$

$$\int_{-1}^{1} 1\,dx = w_1 = 0 \qquad \int_{-1}^{1} x\,dx - w_1 w_1 = 0$$

or $w_1 = 2$, $x_1 = 0$