Lecture 7

CS 510

Triangle Algorithm Review

Input S, p, ϵ $\quad S = \{v_1, \dots, v_n\}$

step 0. Let $p' = v = \operatorname{argmin}\{\|p - v_i\|\}$

i.e. the closest v_i to p.

step 1. If $\|p - p'\| \leq \epsilon \|p - v\|$, output p' as an approximate solution, stop.

Step 2. Replace v with a pivot v_j, compute p''.

Replace p' with p'' and Goto step 1.

Solving $Ax = b$

Assume A is $n \times n$, invertible.

Thm. Suppose $x_* = A^{-1}b \geq 0$

There exist $\alpha > 0$, $x \geq 0$

such that

$$Ax = \alpha b$$

$$\sum x_i + \alpha = 1 \quad , \quad x_i \geq 0, \alpha \geq 0$$

In other words

$0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is in $\mathrm{Conv}(A, -b)$.

We can solve $Ax = b$

with the assumption that $x_* = A^{-1}b \geq 0$

via the triangle algorithm:

In each iteration we have a pair (x, α) such that

$$p' = Ax - \alpha b, \quad \sum x_i + \alpha = 1$$

$$x_i, \alpha \geq 0$$

we check if

$$\left\| A \frac{x}{\alpha} - b \right\| \leq \epsilon \max\{\|a_1\|, \dots, \|a_n\|, \|b\|\}.$$

a_i = i-th column of A

If so we stop.
Otherwise, we iterate.

What if $A^{-1}b \ngeq 0$?

If $Ax = b$

then $Ax + w = b + w$

suppose we choose $w = tAe$

where $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, t a scalar.

Then $A(x + te) = b + tAe$

$\exists\, t_*$ such that $x + t_* e \geq 0$

and t_* is the smallest such t such that $x + te \geq 0$.

If we knew such t_*, we add $t_* Ae$ to both sides and do as before.

Since we don't know t_x we try to pick a t & incrementally change it if necessary.

Incremental Triangle Alg.

Assume for a given $t_0 \geq 0$ we have tried to use triangle algorithm to find x_0 s.t.

$$\|A(x_0 - t_0 e) - b\| \leq \epsilon \max\{\|a_1\|, \dots, \|a_n\|, \|b\|\}$$

where a_i = i-th column of A

If this is possible we are done.

Otherwise, the simplest strategy is to increase t_0 to t_1 & repeat.

There are better ideas. (Later)

Eigenvalue - Eigenvector Remarks

Suppose A is $n \times n$ Hermitian

i.e. $A = A^*$ conjugate transpose.

we claim all eigenvalues are real (not eigenvectors).

How to prove this?

Theorem (Schur Normal Form)

Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. There exists Unitary matrix U, i.e. $U^* = U^{-1}$ so that

$$A = U T U^*$$

with T triangular

Proof is by induction on n.

The following is a important consequence of the thm.

Theorem (Principal Axes Thm)

Let A be Hermitian $(A = A^*)_{n\times n}$

Then all its eigenvalues are real, say $\lambda_1, \ldots, \lambda_n$, not necessarily distinct.

Additionally, there exist corresponding eigenvectors $u_1, \ldots, u_n$ such that they are orthonormal, i.e

$$u_i^* u_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

If A is real $u_1, \dots, u_n$ can be taken to be real vectors.

Finally,

$$A = U \Lambda U^*, \text{ where}$$

$$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

$$A = \begin{bmatrix} u_1 \,\Big|\, u_2 \,\Big|\, \cdots \,\Big|\, u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^* \\ \hline u_2^* \\ \hline \vdots \\ \hline u_n^* \end{bmatrix}.$$

Pf. From Schur Normal Form

$$A = UTU^*, \quad UU^* = I, \quad T \text{ Triangular}$$

Then

$$A^* = UT^*U^*$$

Since $A = A^*$,

it follows that $T = T^*$.

But $T = T^*$ $\Rightarrow$ T is diagonal & $t_{ii} = t_{ii}^*$ $\Rightarrow$ t_{ii} is real. Let $\lambda_i = t_{ii}$.

so $AU = UT$

$U = [u_1, \ldots, u_n]$

so $Au_i = \lambda_i u_i$.

If A is real then we can take U to be real.

This can be seen as follows:

For each λ_i we solve $Ax = \lambda_i x$

But this gives u_i that is real.

Suppose $\operatorname{rank}(A - \lambda_i I) = n-1$.

Then if $\lambda_i \neq \lambda_j$

u_i & u_j are orthogonal $\to$

This is because on the
on hand we have

$$A v_i = \lambda_i v_i$$

$$A v_j = \lambda_j v_j$$

But $v_j^T A v_i = \lambda_i v_i^T v_j$

$$v_i^T A v_j = \lambda_j v_i^T v_j$$

But $v_j^T A v_i = v_i^T A v_j$

So if $\lambda_i \neq \lambda_j$, $v_i^T v_j = 0$.

Ex. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$

$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,

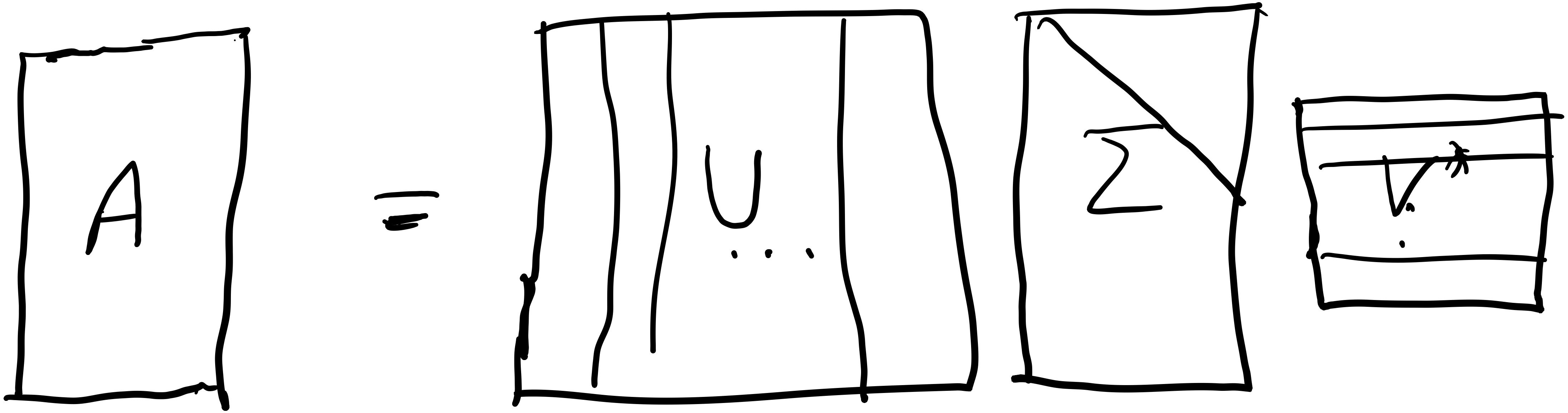
$u_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

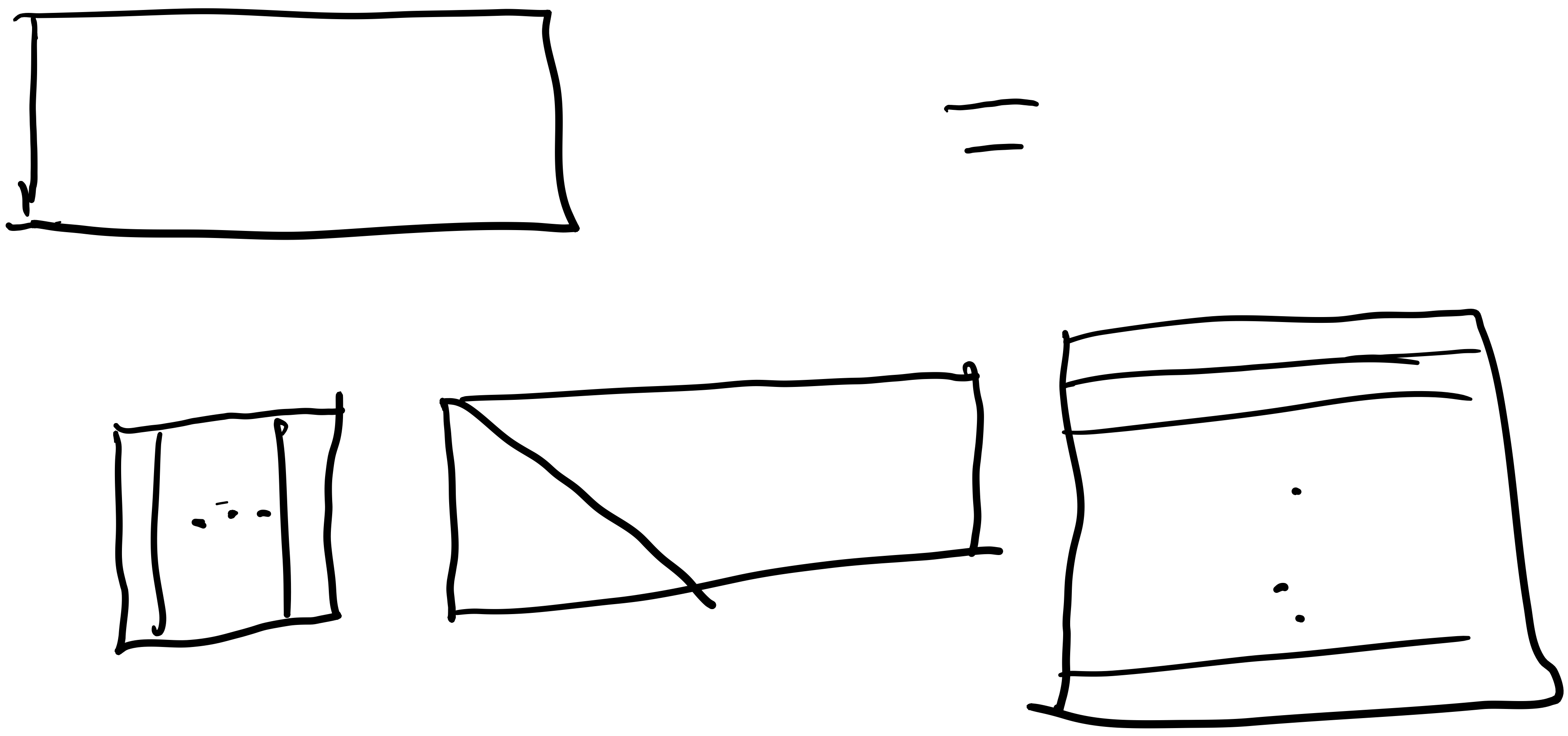
Singular Value Decomposition

Thm (SVD) Let A be $m \times n$ complex matrix.

Then there exist unitary matrices U, $m \times m$, V $n \times n$, & $m \times n$ matrix Σ with at most $\min\{m, n\}$ positive diagonal entries & all others zero

$$\underset{m \times n}{A} = \underset{m \times m}{U} \, \underset{m \times n}{\Sigma} \, \underset{n \times n}{V^*}$$





Σ diagonal entries ≥ 0 and can be written as

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$$

$$\begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \\ & & \\ & O & \end{bmatrix}$$

The μ_i's are called singular values of A

If Σ is like

$$\begin{bmatrix} \mu_1 & & & \\ & \ddots & & 0 \\ 0 & & \mu_m & \end{bmatrix}$$

$$\mu_1 \geq \mu_2 \cdots \geq \mu_m$$

Pf. Let $M = A^* A$ $\therefore$ M is $n \times n$

Then M is Hermitian.

By Principal axis thm

$$M = V \Lambda V^*, \quad VV^* = V^* V = I$$

so

$$V^* M V = \Lambda, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

$$V^* A^* A V = \Lambda$$

$\lambda_i \geq 0$ since M is positive semidefinite.

Let

$$W = AV = [w_1, w_2, \dots, w_n]$$

$$W^* = \begin{bmatrix} w_1^* \\ w_2^* \\ \vdots \\ w_n^* \end{bmatrix}$$

suppose $r \le \min\{m, n\}$ of the λ_i's are positive.

We can assume U is such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > 0$.

Then
$$w_i^* w_j = \begin{cases} 0, & i \neq j, \\ \lambda_i, & i = j \end{cases}$$

For $i = 1, \ldots, r$ let $u_i = \frac{1}{\sqrt{\lambda_i}} w_i$

For $i = r+1, \ldots, n$, choose u_i so that $u_1, \ldots, u_n$ forms an orthonormal set.

Let $U = [u_1, \ldots, u_n]$.

This can be done by solving equation: For instance to compute u_{r+1} we solve r equation

$$u_i^T x = 0, \quad i = 1, \ldots, r$$

for a nontrivial solution.

So in this fashion we can extend $u_1, \ldots, u_r$ to an orthonormal basis of $\mathbb{R}^n$

To extend: we can do this one at a time: Let u_{r+1} be a solution to

$$u_i^T x = 0, \quad i = 1, \ldots, r$$

$\|u_{r+1}\| = 1$, etc. Then

$U^*U = I$

and if we let

$$\Sigma = \begin{bmatrix} \mu_1 & & 0 & \\ & \ddots & & \\ 0 & & \mu_r & \\ & & & 0 \end{bmatrix}, \quad \mu_i = \sqrt{\lambda_i}$$

then

$$U\Sigma = AV$$

or

$$U\Sigma V^* = A .$$

Remark. Suppose $n > m$.

We can then let $M = AA^*$.

Then we get

$A^* = U \Sigma V^*$ for appropriate U, Σ, V^*

Taking $*$ of both sides we get

$$A = V \Sigma U^*$$

.

Corollary:

If A is real then

$$A = U \Sigma V^T$$

where $VV^T = I$, $UU^T = I$,

U, V real.

EX. suppose

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad A \text{ is } 2 \times 3$$

Try $M = AA^T$

$$M = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}$$

eigenvalues:

$$(5-\lambda)(2-\lambda) - 4 =$$

$$\lambda^2 - 7\lambda + 10 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda = 1, \quad \lambda = 6$$

eigenvector

$$\begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -x_1 + 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{cases}$$

$$x_1 = 2x_2$$

$$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$

$$2x_1 = -x_2$$

$$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$V = [v_1, v_2] = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$V^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Next we need to find

$$U = [u_1, u_2, u_3]$$

$$A^T V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 0 \\ 1 & -2 \\ -2 & -1 \end{bmatrix} = [w_1, w_2]$$

Then

$$u_1 = \frac{w_1}{\|w_1\|}, \quad u_2 = \frac{w_2}{\|w_2\|}$$

$$u_1 = \frac{1}{\sqrt{30}}\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

We compute u_3 as a vector of norm one s.t. $u_3^T u_1 = 0$, $u_3^T u_2 = 0$

or we solve

$$5x_1 + x_2 - 2x_3 = 0$$
$$-x_2 - x_3 = 0$$

let $x_3 = 1$, $x_2 = -1$, $x_1 = \frac{3}{5}$

or $\frac{1}{\sqrt{59}}\begin{bmatrix} 3 \\ -5 \\ 5 \end{bmatrix}$

Note $\mu_1 = \sqrt{6}$, $\mu_2 = 1$

So we have

$$A^T = U \Sigma V^T,$$

$$\Sigma = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \\ 0 & 0 \end{bmatrix}$$

& $$A = V \Sigma^T U^T.$$

Thm. Gerschgorin Thm.

Let A be $n \times n$, $A = (a_{ij})$

Let $r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$, $i = 1, \dots, n$

Let $Z_i = \{ z \in C : |z - a_{ii}| \leq r_i \}$

$i = 1, \dots, n$.

If λ is an eigenvalue of A, then λ must belong to one of the circles Z_i, $i=1,\dots,n$.

Pf. We have $Ax = \lambda x$ for some $x \neq 0$ $\therefore$

Assume x is s.t.

$$\|x\|_\infty = 1$$

suppose $|x_k| = 1$, for some k.

we have

$$\sum_{j=1}^{n} a_{kj} x_j = \lambda x_k$$

so $x_k(\lambda - a_{kk}) = \sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{kj} x_j$.

or $|\lambda - a_{kk}| \leq \sum_{\substack{j=1 \\ j=1}}^{n} |a_{kj}| = r_k$.