CS 596 Homework #2 Xuenan Wang (xw336)

problem 1:

(a) $A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0.5 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$

$$\therefore \lambda_1 = \lambda_2 = 1$$

when $\lambda_1 = \lambda_2 = 1$, $\begin{pmatrix} 1-1 & 0.5 \\ 0 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore, matrix A is not diagonalizable.

(b) $A^n = \begin{pmatrix} 1 & \frac{n}{2} \\ 0 & 1 \end{pmatrix}$

suppose ~~that~~ the formula above is ~~true~~ for $n=k$.

$$A^{k+1} = A^k A = \begin{pmatrix} 1 & \frac{k}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}+\frac{k}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{k+1}{2} \\ 0 & 1 \end{pmatrix}$$

Therefore, $A^n = \begin{pmatrix} 1 & \frac{n}{2} \\ 0 & 1 \end{pmatrix}$.

(c) $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 1 & \frac{n}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} & \sum_{n=0}^{\infty} \frac{n}{2n!} \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} \end{pmatrix}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e \qquad \sum_{n=0}^{\infty} \frac{n}{2n!} = \frac{1}{2}e$$

$$\therefore e^A = \begin{pmatrix} e & \frac{1}{2}e \\ 0 & e \end{pmatrix}$$

Problem 2:

(a) suppose x, y, z are variables in $\mathcal{V}$ and α, β are real numbers.

1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

2) $x \oplus y = y \oplus x$

3) $x \oplus 0 = x$

4) $(-x) \oplus x = 0$

5) $\alpha \odot x \in \mathcal{V}$

6) $\alpha \odot (\beta \odot x) = (\alpha \cdot \beta) \odot x$

7) $1 \odot x = x$

8) $\alpha \odot (x \oplus y) = (\alpha \odot x) \oplus (\alpha \odot y)$

9) $(\alpha + \beta) \odot x = (\alpha \odot x) \oplus (\beta \odot x)$

Since x, y, z are all scalars and all above properties remain true for scarlars, we can say that they are satisfied as well here. And this proves that $\mathcal{V}$ is a vector space.

(b) Assume x, y are two variables.

$\therefore$ $x, y \in \mathcal{V}$

inner product is mapping from vector to $\mathbb{R}$.

$\langle x, y \rangle \in \mathbb{R}$

expecetation of the product $E[xy] \in \mathbb{R}$

we can say that expectation is also a mapping from vector x, y to $\mathbb{R}$. which shows that it's also a inner product.

(c) Assume $z_0 = 1$. then we have $\Omega = \{z_0, z_1, \cdots, z_K\}$.

To find a best proximation of subspace in a vector space, we can project x onto subspace Ω and find the minimal value using orthogonality principle:

$$E\{(x-\hat{x})\cdot z_i\} = 0, i \in [0, k] \text{ for each } i+1 \text{ equations.}$$

(d) The physical meaning is that $\hat{x}$ is the projection of x in Ω.

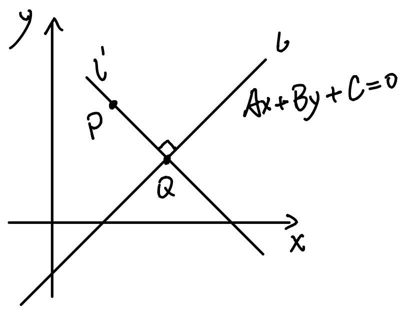
Problem 3.

See attachments.

Problem 4.

(a) Since the linear manifold is defined by $Ax+By+C=0$.

We assume that there's another linear manifold be perpendicular to the original one. we can know that the gradient of this new line is B/A.



$$\begin{cases} y - y_0 = \frac{B}{A}(x - x_0) \\ Ax + By + C = 0 \end{cases}$$

we can solve that $Q\left(\frac{B^2x_0 - ABy_0 - AC}{A^2+B^2}, \frac{A^2y_0 - ABx_0 - BC}{A^2+B^2}\right)$

$$|PQ|^2 = \left(\frac{B^2x_0 - ABy_0 - AC}{A^2+B^2} - x_0\right)^2 + \left(\frac{A^2y_0 - ABx_0 - BC}{A^2+B^2} - y_0\right)^2$$

$$= \frac{A^2(Ax_0+By_0+C)^2 + B^2(Ax_0+By_0+C)^2}{(A^2+B^2)^2}$$

$$= \frac{(Ax_0+By_0+C)^2}{A^2+B^2}$$

$$\therefore |PQ| = \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$$

In this case, $|PQ|$ is the smallest distance of a point (x_0, y_0).

(b) cost function $= f = \sum_{i=1}^{n} \frac{(Ax_i + By_i + C)^2}{A^2 + B^2}$

$$\frac{\partial f}{\partial C} = \sum_{i=0}^{n} \frac{2(Ax_i + By_i + C)}{A^2 + B^2} = \frac{2}{A^2+B^2} \sum_{i=0}^{n} (Ax_i + Bx_i + C)$$

Assume $\frac{\partial f}{\partial C} = 0$

$$\frac{2}{A^2+B^2} \sum_{i=0}^{n} (Ax_i + By_i + C) = 0$$

$$C = -\sum_{i=0}^{n} (Ax_i + By_i)/n$$

$\therefore$ cost function : $f = \sum_{i=0}^{n} \frac{(Ax_i + By_i - \sum_{i=0}^{n} (Ax_i + By_i)/n)^2}{A^2 + B^2}$

assume $X = \sum_{i=0}^{n} x_i \quad Y = \sum_{i=0}^{n} y_i$

$$f = \frac{(AX + BY - \frac{A}{n}X - \frac{B}{n}Y)^2}{A^2 + B^2}$$

$$= \frac{[A(X - \frac{1}{n}X) + B(Y - \frac{1}{n}Y)]^2}{A^2 + B^2}$$

$$= \frac{(A B) \begin{pmatrix} (X - \frac{1}{n}X)^2 & (X - \frac{1}{n}X)(Y - \frac{1}{n}Y) \\ (X - \frac{1}{n}X)(Y - \frac{1}{n}Y) & (Y - \frac{1}{n}Y)^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}}{(A B) \begin{pmatrix} A \\ B \end{pmatrix}}$$

$$= \frac{X^T \Omega X}{X^T X} \qquad X = \begin{pmatrix} A \\ B \end{pmatrix} \quad \Omega = \begin{pmatrix} (X - \frac{1}{n}X)^2 & (X - \frac{1}{n}X)(Y - \frac{1}{n}Y) \\ (X - \frac{1}{n}X)(Y - \frac{1}{n}Y) & (Y - \frac{1}{n}Y)^2 \end{pmatrix}$$

(c) distance from point to circle :

$$\begin{cases} Ax + Bx + C = 0 \\ (X - A)^2 + (Y - B)^2 = C^2 \end{cases} \Rightarrow d = \left| \sqrt{(x-A)^2 + (y-B)^2} - C \right|$$

Similar to what we did above :

cost function : $f = \sum_{i=0}^{n} \left(\sqrt{(x_i - A)^2 + (y_i - B)^2} - C \right)^2$

$$\frac{\partial f}{\partial C} = \sum_{i=0}^{n} 2\left(\sqrt{(x_i - A)^2 (Y_i - B)^2} - C \right)$$

assume $\frac{\partial f}{\partial C} = 0$

we have $\sum_{i=0}^{n} 2\left(\sqrt{(x_i - A)^2 (Y_i - B)^2} \right) = 0$

$$C = \sum_{i=0}^{n} \sqrt{(x_i - A)^2 (Y_i - B)^2}$$

$\therefore$ cost function: $f = \sum_{i=0}^{n} \left(\sqrt{(x_i - A)^2 + (Y_i - B^2)} - \sum_{i=0}^{n} \sqrt{(x_i - A)^2 (Y_i - B)^2} \right)$

(4) Step 0 : Given x^0, set $k=0$

Step 1 : $d_k = -\nabla f(x_k)$. If $d_k = 0$, then stop.

$$f(x_k) = \sqrt{(x_k - A)^2 + (Y_k - B)^2} - \sqrt{(x_k - A)^2 (Y_k - B)^2}$$

Step 2 : solve $\min_\alpha f(x_k + \alpha d_k)$ for the stepsize α_k, perhaps chosen by an exact or inexact linesearch.

Step 3 : Set $x_{k+1} \leftarrow x_k + \alpha_k d_k$, $k \leftarrow k+1$. Go to step 1.

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CourseWork / CS596 Foundations of Computer and Data Science / Homework #2 / hw2_Q3.m

hello-roderickwang finish 596 hk2

148d133 1 hour ago

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```
% @Date    : 2019-10-12 17:15:20
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% Generation of a random matrix of rank equal to 3
A = rand(20, 5);
B = matrix_filling(A, 3);
disp('My random matrix of rank equal to 3 is:')
disp(B)
disp('Rank of this matrix is:');
disp(rank(B));

% Question (a)
B2 = put_random_zeros(B, 2);
B3 = put_random_zeros(B, 3);
B4 = put_random_zeros(B, 4);
B5 = put_random_zeros(B, 5);

[U2, S2, V2] = svd(B2);
[U3, S3, V3] = svd(B3);
[U4, S4, V4] = svd(B4);
[U5, S5, V5] = svd(B5);

disp('Rank of B2 matrix is:');
disp(rank(B2));
disp('Rank of B3 matrix is:');
disp(rank(B3));
disp('Rank of B4 matrix is:');
disp(rank(B4));
disp('Rank of B5 matrix is:');
disp(rank(B5));

% Question (b)
B2f = matrix_filling(B2, 3)
B3f = matrix_filling(B3, 3)
B4f = matrix_filling(B4, 3)
B5f = matrix_filling(B5, 3)

% Question (c)
var2 = var(var(B2f-B))
var3 = var(var(B3f-B))
var4 = var(var(B4f-B))
var5 = var(var(B5f-B))

function B = matrix_filling(A, target_rank)
    [U, S, V] = svd(A);
    B = U(:, 1:target_rank)*S(1:target_rank, 1:target_rank)*V(:,1:target_rank)';
end

function A = put_random_zeros(A, zero_num)
```

```
    for i = 1:zero_num
        A(randi(20), randi(5)) = 0;
    end
end
```

```
>> hw2_Q3
My random matrix of rank equal to 3 is:
    0.7595    0.5430    0.6031    0.5085    0.6016
    0.5534    0.6484   -0.0604    0.5779    0.7058
    0.5506    0.5050    0.9045    0.4355    0.3462
    0.6921    1.0408    0.3689    0.8792    0.8543
    0.6613    0.7026    0.3235    0.6222    0.6906
    0.2914    0.9128    0.9982    0.6959    0.3235
    0.9525    0.4616    0.4291    0.4849    0.7523
    0.9382    0.1784    0.2838    0.2754    0.6558
    0.7007    0.2988    0.1034    0.3337    0.5988
    0.2895    0.3436    0.5721    0.2841    0.1888
    0.5072    0.4303    0.2802    0.3947    0.4700
    0.0280    0.8879    0.1218    0.6660    0.3919
    0.4890    0.4214    0.5840    0.3742    0.3613
    0.6508    0.4714    0.2721    0.4491    0.5923
    0.7691    0.7575    0.3191    0.6810    0.7930
    0.3968    0.2945    0.7203    0.2593    0.1968
    0.8059    0.7215    0.8254    0.6419    0.6492
    0.8566    0.9692    0.3624    0.8524    0.9389
    0.2138    0.8953    0.9001    0.6738    0.2906
    0.2221    0.3286    0.9162    0.2496    0.0305

Rank of this matrix is:
     3

Rank of B2 matrix is:
     5

Rank of B3 matrix is:
     5

Rank of B4 matrix is:
     5

Rank of B5 matrix is:
     5


B2f =

    0.7754    0.5497    0.5945    0.5153    0.5765
    0.5543    0.6504   -0.0608    0.5756    0.7050
    0.4395    0.4608    0.9648    0.3840    0.1754
    0.7009    1.0458    0.3643    0.8809    0.8409
    0.6703    0.7072    0.3187    0.6248    0.6767
    0.3094    0.9195    0.9883    0.7050    0.2948
    0.9670    0.4682    0.4212    0.4904    0.7296
    0.9510    0.1842    0.2769    0.2803    0.6358
    0.7075    0.3025    0.0998    0.3353    0.5885
    0.3016    0.3481    0.5655    0.2903    0.1695
    0.5152    0.4341    0.2759    0.3975    0.4576
    0.0270    0.8886    0.1224    0.6638    0.3939
    0.5028    0.4268    0.5765    0.3807    0.3395
    0.6597    0.4758    0.2673    0.4519    0.5785
    0.7787    0.7626    0.3140    0.6835    0.7782
    0.4129    0.3002    0.7115    0.2677    0.1711
    0.7156    0.6751    0.0482    0.6164    0.7879
    0.8670    0.9750    0.3569    0.8547    0.9231
```

0.2294	0.9011	0.8917	0.6815	0.2659
0.2403	0.3347	0.9062	0.2599	0.0012

B3f =

0.5793	0.4299	0.7202	0.3940	0.3374
0.5455	0.6438	-0.0552	0.5722	0.7207
0.5659	0.5116	0.8942	0.4501	0.3163
0.6801	1.0305	0.3764	0.8761	0.8755
0.6620	0.7022	0.3229	0.6241	0.6888
0.2806	0.8984	1.0044	0.7014	0.3407
0.9769	0.4790	0.4135	0.4969	0.7076
0.9717	0.2038	0.2624	0.2897	0.5947
0.7167	0.3116	0.0933	0.3394	0.5701
0.2959	0.3452	0.5677	0.2921	0.1759
0.5131	0.4338	0.2764	0.3987	0.4589
-0.0074	0.8602	0.1442	0.6524	0.4559
0.1026	0.5137	0.5202	0.3941	0.1803
0.6603	0.4781	0.2660	0.4540	0.5748
0.7715	0.7587	0.3175	0.6832	0.7882
0.4121	0.3020	0.7102	0.2723	0.1673
0.8203	0.7284	0.8158	0.6545	0.6214
0.8540	0.9664	0.3640	0.8527	0.9433
0.0253	0.9165	0.8842	0.6813	0.2453
0.2325	0.3308	0.9089	0.2633	0.0091

B4f =

0.7768	0.3332	0.6135	0.2908	0.5376
0.5583	0.6440	-0.0617	0.5874	0.6977
0.5419	0.5289	0.9056	0.3973	0.3633
0.6983	1.0414	0.3668	0.8830	0.8452
0.6613	0.7093	0.3230	0.6134	0.6918
0.2950	0.9237	0.9961	0.6842	0.3203
0.9411	0.4833	0.4313	0.4477	0.7730
0.9225	0.2027	0.2872	0.2316	0.6831
0.6938	0.3102	0.1049	0.3134	0.6111
0.2856	0.3569	0.5724	0.2636	0.1970
0.5243	0.2605	0.2875	0.2377	0.4136
0.0453	0.8709	0.1172	0.7016	0.3635
0.4832	0.4376	0.5848	0.3484	0.3728
0.7104	0.2076	0.2357	0.2189	0.5434
0.7318	0.7402	0.0140	0.6760	0.8442
0.3883	0.3151	0.7216	0.2256	0.2130
0.7983	0.7440	0.8262	0.6064	0.6645
0.8586	0.9751	0.3613	0.8461	0.9372
0.2195	0.9024	0.8977	0.6686	0.2836
0.2152	0.3496	0.9169	0.2167	0.0444

B5f =

0.6736	0.5447	0.5805	0.6075	0.6177
0.6635	0.6463	0.0259	0.4603	0.6788
0.4637	0.5067	0.8812	0.5365	0.3618
0.8496	-0.0031	0.4107	0.6969	0.8252
0.7072	0.7017	0.3362	0.5675	0.6832

```
    0.4248    0.9102    1.0337    0.5410    0.2992
    0.8059    0.4644    0.3904    0.6538    0.7798
    0.7036    0.0046    0.2214    0.5473    0.6988
    0.6080    0.3006    0.0791    0.4402    0.6166
    0.2606    0.3442    0.5643    0.3178    0.1940
    0.4947    0.4305    0.2772    0.4084    0.4729
    0.3481   -0.0062    0.2068    0.2952    0.3331
    0.4334    0.4225    0.5692    0.4386    0.3715
    0.6156    0.4721    0.2631    0.4888    0.5994
    0.8072    0.7567    0.3299    0.6350    0.7872
    0.2966    0.2964    0.6935    0.3761    0.2147
    0.7450    0.7226    0.8094    0.7119    0.6607
    0.9478    0.9674    0.3875    0.7444    0.9238
    0.3703   -0.0030    0.9411    0.4943    0.2606
    0.1554    0.3299    0.8981    0.3279    0.0419


var2 =

   1.7120e-04


var3 =

   1.7426e-05


var4 =

   8.5757e-06


var5 =

    0.0025

>>
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