CS 460/560
Introduction to Computational Robotics
Fall 2019, Rutgers University

Lecture 03, Continued
Math. Foundations II

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Outline

Topological spaces

Manifolds

Path and connectivity

Homotopic paths

Connectedness of space

Fixed point theorems

Hairy ball theorem

Metric spaces

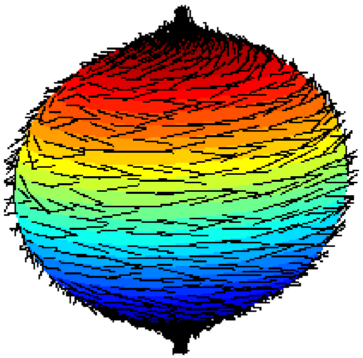
Geometric representations

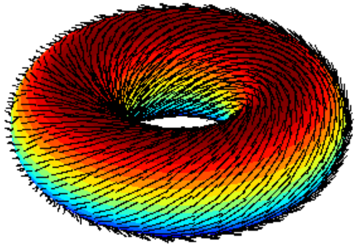
Sampling from a distribution

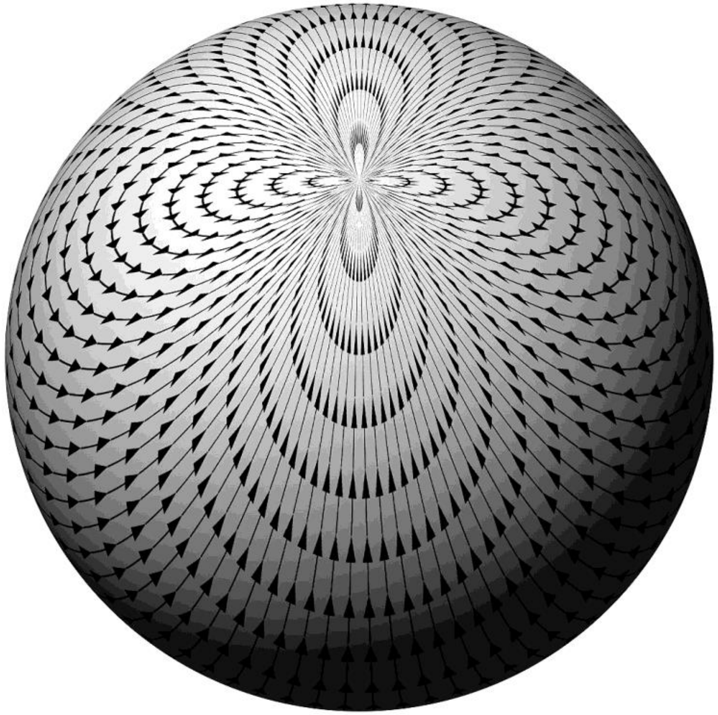
Hairy Ball Theorem

Hairy ball theorem. Given a 2-sphere S and let $f(p)$ assigns a vector to $p \in S$ such that $f(p)$ is tangential to p on S. Then there exists a $p \in S$ such that $f(p) = 0$.

- ⇨I.e., you cannot comb a hairy ball to make it "smooth" everywhere
- ⇨It turns out that one can construct a vector field with exactly one $f(p) = 0$
- ⇨Hairy donut can be combed!

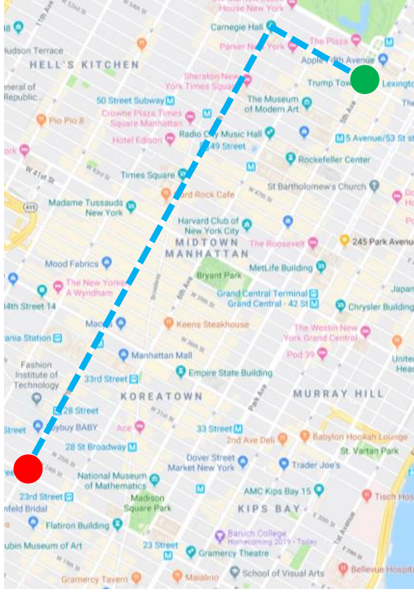






Images from wikipedia

Metric Spaces



We have all seen and used metric spaces. For example

⇨ $v_1 = (x_1, y_1), v_2 = (x_2, y_2) \in \mathbb{R}^2$,

⇨ $d(v_1, v_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

A more formal definition. A metric space is a pair (M, d) in which M is a set and d: $M \times M \to \mathbb{R}$ defines a **metric** s.t. for all $m_1, m_2, m_3 \in M$,

⇨ $d(m_1, m_2) = 0 \Leftrightarrow m_1 = m_2$ (definiteness)

⇨ $d(m_1, m_2) = d(m_2, m_1)$ (symmetry)

⇨ $d(m_1, m_2) \leq d(m_1, m_3) + d(m_3, m_2)$ (triangle inequality)

⇨ Some metric spaces

⇨ L_2 metric on $\mathbb{R}^2$: the one at the beginning of this page

⇨ Manhattan (L_1) metric on $\mathbb{R}^2$: $d(v_1, v_2) = |x_1 - x_2| + |y_1 - y_2|$

⇨ L_∞ metric on $\mathbb{R}^2$: $d(v_1, v_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

⇨ Generalizes to $\mathbb{R}^n$

Geometric Representations

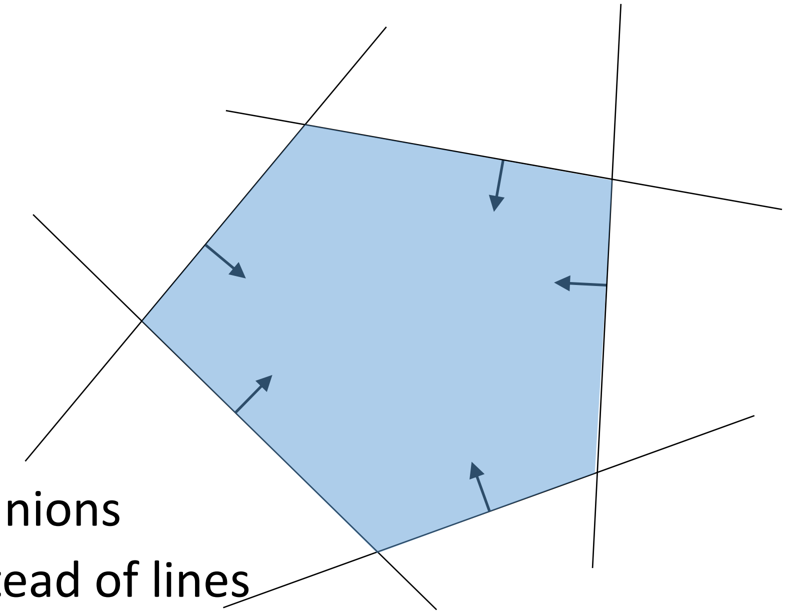
We will compute over geometric shapes – how do we represent them in computer?

We will look at two “simple” models

- ⇨Polygonal and polyhedral models
- ⇨Semi-algebraic models

Polygons and Polyhedrals

- ⇨A convex polygon
- ⇨Arbitrary polygons can be represented as unions
- ⇨Same trick for polyhedral, using planes instead of lines
- ⇨Can also use oriented edges



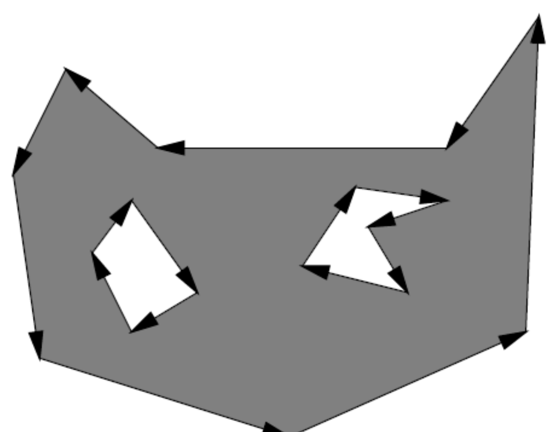
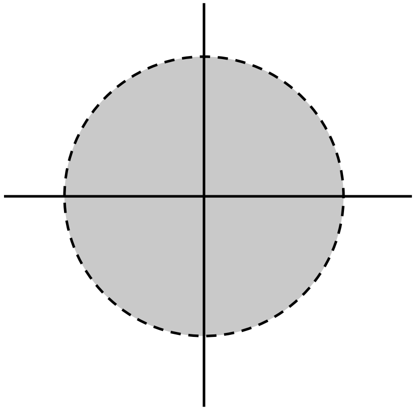


Image from Planning Algorithms

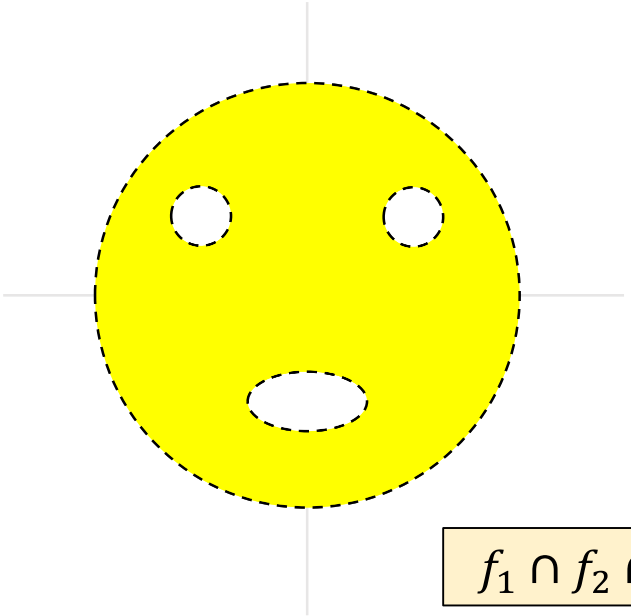
Geometric Representations, Continued

Semi-algebraic models



- ⇨Define sets using algebraic inequalities, e.g. $x^2 + y^2 < 1$
- ⇨Then do set operations over them
- ⇨These sets are known as **semi-algebraic** sets
- ⇨Example
  - ⇨ $f_1 = x^2 + y^2 < 1$
  - ⇨ $f_2 = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 > 0.01$
  - ⇨ $f_3 = \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 > 0.01$
  - ⇨ $f_4 = \frac{x^2}{2} + \left(y + \frac{1}{2}\right)^2 > 0.01$

Semi: with inequalities > and <



$f_1 \cap f_2 \cap f_3 \cap f_4$

Probability Essentials

Probability space (Ω, F, P)

- ⇨ Ω: **sample space** or space of unique outcomes from an experiment
- ⇨ F: **event space** – each event $e \in F$ is a subset of Ω, i.e., $e \subset \Omega$
- ⇨ P: a **probability measure** that assigns probabilities to events
  - ⇨ $P(\emptyset) = 0, P(\Omega) = 1$

Example: a single toss of a fair dice

- ⇨ $\Omega = \{f_1, f_2, f_3, f_4, f_5, f_6\}$
- ⇨ $F = \{\emptyset, \{f_1\}, \dots, \{f_6\}, \{f_1, f_2\}, \dots, \{f_5, f_6\}, \dots, \Omega = \{f_1, \dots, f_6\}\}$
- ⇨ $P(\mathrm{e}) = \frac{1}{6}|\mathrm{e}|, \mathrm{e.g.},$
  - ⇨ $e = \emptyset, P(e) = 0$
  - ⇨ $e = \{f_1\}, P(e) = \frac{1}{6}$
  - ⇨ $e = \{f_1, f_3, f_5\}, P(e) = \frac{1}{2}$

A **random variable** (**RV**) is a function $X: \Omega \to D$ with D often being $\mathbb{R}$

- ⇨ E.g., for the dice toss, we may let $X: f_i \mapsto i$

Probability Essentials – Expectation

Expectation: the expected value of a random variable

⇨In the discrete case, for an RV X with n values $x_1, \dots, x_n$

$$E[X] = \sum_{1 \le i \le n} x_i P(X = x_i) = x_1 P(X = x_1) + \cdots + x_n P(X = x_n)$$

⇨This is also commonly known as the "mean" or "weighted average"

⇨E.g., the average score of this class

⇨E.g., single dice toss

- ⇨ If we let $X: f_i \mapsto i$, that is, giving each face a number 1-6,
- ⇨ Then $E[X] = 1 * \frac{1}{6} + \cdots + 6 * \frac{1}{6} = 3.5$

Linearity of Expectation

Linearity of Expectation: the expectation of an RV is the sum of the expectation of the component RVs

- Very handy in practice!
- Q: tossing a coin, how many tosses to get a first head, on average?
  - The RV: # of tosses to get a first head
  - Decompose
    - Get a head in first toss: probability $\frac{1}{2}$
    - Get a first head in second toss: $\frac{1}{4}$
    - …
    - Get a first head in n-th toss: $\frac{1}{2^n}$
    - …
    - Each of the above is an event disjoint from the others
  - Apply linearity: $T = 1 * \frac{1}{2} + 2 * \frac{1}{4} + \cdots + n * \frac{1}{2^n} + \cdots = 2$
  - You can verify things like this easily using a python program

Linearity of Expectation, Continued

Q: tossing a coin, how many tosses to get both sides, on average?

- The RV: # of tosses to get both sides
- Decompose:
  - # of tosses to get a first side (doesn’t matter head or tail)
    - What is this #?
    - Yes, 1, because the first toss must produce a side
  - # of tosses to get a different side
    - What is this #?
    - This is the same as asking for a specific side, like a head
    - So the # is 2 from the previous calculation
- To total # of tosses to get both sides, in expectation, is $1 + 2 = 3$
- Do this experiment yourself to verify (after class please…)

Probability Essentials, Continued

Sampling from a distribution is very useful

- ⇨ Used very often in computation applications including robotics
  - ⇨ E.g., Monte-Carlo methods (used in Alpha-Go)
  - ⇨ You will see this in intro AI quite a bit as well
- ⇨ Involves drawing “samples” from a probability distribution
- ⇨ To do this, we work w/ the CDF – cumulative distribution function
- ⇨ Steps (for continuous distributions)
  - ⇨ Generate a random number y in (0,1)
  - ⇨ Locate y on the Y axis
  - ⇨ Find the corresponding x
  - ⇨ x is the sample