CS512 LECTURE NOTES - LECTURE 4

1 Proof of the Master Theorem

- **CASE 1)** If $f(n) \in O(n^{\log_b a-\epsilon}$ then $T(n) \in \Theta(n^{\log_b a})$
- **CASE 2)** If $f(n) \in \Theta(n^{\log_b a})$ then $T(n) \in \Theta(n^{\log_b a} \lg n)$
- **CASE 3)** If $f(n) \in \Omega(n^{\log_b a+\epsilon})$ and $af(n/b) \leq f(n)$, $0 \leq c < 1$ then $T(n) \in \Theta(f(n))$

The condition in case 1 and case 3 including ϵ requires $f(n)$ to grow exponentially faster (slower) than $n^{\log_b a}$. The condition $af(n/b) \leq f(n)$, $0 \leq c < 1$ in case 3 is called the *Regularity Condition* and all it is saying is that $f(n)$ must increase with the length of the input.

Proof

The first part of the proof is common for the three cases, and we use the iterative method to solve the recurrence relation:

$$
\begin{array}{rrcl}
 & T(n) & = & aT(n/b) + f(n) \\
\text{iteration 1:} & T(n) & = & a(aT(n/b^2) + f(n/b)) + f(n) \\
 & & = & a^2T(n/b^2) + af(n/b) + f(n) \\
\text{iteration 2:} & T(n) & = & a^2(aT(n/b^3) + f(n/b^2)) + af(n/b) + f(n) \\
 & & = & a^3T(n/b^3) + a^2f(n/b^2) + af(n/b) + f(n) \\
 & \vdots & \vdots & \vdots \\
\text{iteration k-1:} & T(n) & = & a^kT(n/b^k) + \sum_{i=0}^{k-1} a^i f(n/b^i)
\end{array}
$$

Repeat until $\frac{n}{b^k} = 1$ (in order to avoid unnecessary complication we will assume that n is a power of b). Therefore $k = \log_b n$

Therefore we have that:

$$T(n) = a^{\log_b n} + \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)$$

Notice also that

$$a^{\log_b n} = a^{\log_a n / \log_a b} = a^{\log_a n^{\frac{1}{\log_a b}}} = n^{\log_b a}$$

Therefore

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)$$

CASE 1 $f(n) \in O(n^{\log_b a-\epsilon})$

$$\begin{aligned}
\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(\frac{a}{b^{\log_b a-\epsilon}}\right)^i \\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(\frac{a}{b^{-\epsilon} b^{\log_b a}}\right)^i \\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} \left(\frac{a}{ab^{-\epsilon}}\right)^i \\
&= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i \\
&= cn^{\log_b a-\epsilon} \left(\frac{(b^\epsilon)^{\log_b n} - 1}{b^\epsilon - 1}\right) \\
&= cn^{\log_b a-\epsilon} \left(\frac{n^\epsilon - 1}{b^\epsilon - 1}\right) \\
&\in O(n^{\log_b a})
\end{aligned}$$

So we have that

$$T(n) = \Theta(n^{log_b a}) + O(n^{log_b a})$$

We can see that $T(n)$ cannot grow slower than $n^{log_b a}$
So we have that $T(n) \in \Theta(n^{\log_b a})$

CASE 2 $f(n) \in \Theta(n^{\log_b a})$

$$\Rightarrow \; c_1 n^{\log_b a} \leq f(n) \leq c_2 n^{\log_b a}$$

where $c1 > 0, c_2 > 0$. Substituting in the summation we have

$$\begin{array}{rcccl}
c_1 \sum_{i=0}^{\log_b n-1} a^i (n/b^i)^{\log_b a} & \leq & \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) & \leq & c_2 \sum_{i=0}^{\log_b n-1} a^i (n/b^i)^{\log_b a} \\
c_1 \sum_{i=0}^{\log_b n-1} n^{\log_b a} \left(\frac{a}{b^{\log_b a}}\right)^i & \leq & \vdots & \leq & c_2 \sum_{i=0}^{\log_b n-1} n^{\log_b a} \left(\frac{a}{b^{\log_b a}}\right)^i \\
c_1 n^{\log_b a} \sum_{i=0}^{\log_b n-1} (1)^i & \leq & \vdots & \leq & c_2 n^{\log_b a} \sum_{i=0}^{\log_b n-1} (1)^i \\
c_1 n^{\log_b a} \log_b n & \leq & \vdots & \leq & c_2 n^{\log_b a} \log_b n \\
\frac{c_1}{\lg b} n^{\log_b a} \lg n & \leq & \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) & \leq & \frac{c_2}{\lg b} n^{\log_b a} \lg n
\end{array}$$

Therefore,

$$T(n) \in \Theta(n^{\log_b a} \lg n)$$

CASE 3 $f(n) \in \Omega(n^{\log_b a+\epsilon}$ and $af(n/b) \le f(n)$, $0 \le c < 1$

Notice that $f(n/b) \le \frac{c}{a} f(n)$ then

$$f(n/b^i) \le \frac{c^i}{a^i} f(n) \quad \Rightarrow a^i f(n/b^i) \le c^i f(n)$$

So we have that

$$\begin{aligned} \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) &\le \sum_{i=0}^{\log_b n-1} c^i f(n) \\ &= f(n) \sum_{i=0}^{\log_b n-1} c^i \end{aligned}$$

Since $c < 1$ the series

$$\sum_{i=0}^{\infty} c^i = \frac{1}{1-c} \text{ which is a constant}$$

We have that

$$\sum_{i=0}^{\log_b n-1} a^i f(n/b^i) \le \frac{f(n)}{1-c} \in \Theta(f(n))$$

Therefore,

$$T(n) \in \Theta(f(n))$$