Chapter 16

Bounds on Roots of Polynomials and Analytic Functions

In this chapter we make use of the Basic Family to derive an infinite family of lower bounds on the gap between two distinct zeros of a given analytic function $f(z)$. We then use the bounds to compute lower bounds on the distance from an arbitrary complex point to the nearest root of $f(z)$. In particular, when $f(z)$ is a polynomial, for each $m \geq 2$ we give explicit upper and lower bounds, U_m and L_m on the modulus of zeros. These bounds are efficiently computable and have many theoretical and practical applications, for instance in Weyl's classical quad-tree algorithm for computing all roots of a complex polynomial. McNamee and Olhovsky (2005) computational comparison shows even U_4 is superior to more than 45 existing bounds in the literature. Even for $m = 2$ our estimate of lower bound is more than twice as good as Smale's bound, Smale (1986), or its improved version given in . Blum *et al.* (1998). A significant property of these bounds, as proved by Jin (2006), is their asymptotic convergence to the radii of tightest annulus containing the zeros. Jin has also given an efficient, $O(mn)$-time algorithm, for the computation of the first m bounds for a polynomial of degree n.

16.1 Introduction

Computing apriori bound on the zeros of polynomials is an interesting and important problem with many theoretical and practical applications. There is a vast literature on this topic, as see the recent book McNamee (2007). As a consequence of the main results in this chapter we will show that if $f(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial with $a_n a_0 \neq 0$, for each $m \geq 2$ we can state upper (and lower) bounds on its zeros the first few of which are given below.

Let $r_m \in [1/2, 1)$ be the unique positive root of the polynomial $q(t) =$

$t^{m-1}+t-1$. Assume ξ is any root of $f(z)$. Then For $m=2$, $r_2=0.5$ and we have

$$|\xi| \le \frac{1}{r_2}\max\left\{\left|\frac{a_{n-k+1}}{a_n}\right|^{\frac{1}{k-1}} : k=2,\ldots,n+1\right\}.$$

For $m=3$, $r_3=0.618034$ and we have

$$|\xi| \le \frac{1}{r_3}\max\left\{\left|\frac{1}{{a_n}^2}\det\begin{pmatrix} a_{n-1} & a_{n-k+1}\\ a_n & a_{n-k+2}\end{pmatrix}\right|^{\frac{1}{k-1}} : k=3,\ldots,n+2\right\},$$

where $a_{-1}=0$.

For $m=4$, $r_4=0.682328$ and we have

$$|\xi| \le \frac{1}{r_4}\max\left\{\left|\frac{1}{{a_n}^3}\det\begin{pmatrix} a_{n-1} & a_{n-2} & a_{n-k+1}\\ a_n & a_{n-1} & a_{n-k+2}\\ 0 & a_n & a_{n-k+3}\end{pmatrix}\right|^{\frac{1}{k-1}} : k=4,\ldots,n+3\right\},$$

where $a_{-1}=a_{-2}=0$.

For $m=5$, $r_5=0.724492$ and we have

$$|\xi| \le \frac{1}{r_5}\max\left\{\left|\frac{1}{{a_n}^4}\det\begin{pmatrix} a_{n-1} & a_{n-2} & a_{n-3} & a_{n-k+1}\\ a_n & a_{n-1} & a_{n-2} & a_{n-k+2}\\ 0 & a_n & a_{n-1} & a_{n-k+3}\\ 0 & 0 & a_n & a_{n-k+4}\end{pmatrix}\right|^{\frac{1}{k-1}} : k=5,\ldots,n+4\right\},$$

where $a_{-1}=a_{-2}=a_{-3}=0$.

16.2 Estimate to Zeros of Analytic Functions

Let $f(z)$ be a complex-valued function analytic everywhere on the complex plane. Consider Newton's iteration function

$$N(z) = z - \frac{f(z)}{f'(z)}. \tag{16.1}$$

Define

$$\gamma(z) = \sup\left\{\left|\frac{f^{(k)}(z)}{f'(z)k!}\right|^{1/(k-1)}, k \ge 2\right\}. \tag{16.2}$$

From Smale's analysis of the one-point theory for Newton's method the following theorem is deducible:

Theorem 16.1 (Smale (1986)). *If ξ, ξ' are distinct zeros of f, ξ a simple zero, then they are separated by a distance according to*

$$|\xi - \xi'| \geq \frac{3 - \sqrt{7}}{2\gamma(\xi)} \approx \frac{.177}{\gamma(\xi)}. \tag{16.3}$$

The following stronger lower bound is given in Blum *et al.* (1998) (Corollary 1, page 158):

$$|\xi - \xi'| \geq \frac{5 - \sqrt{17}}{4\gamma(\xi)} \approx \frac{.219}{\gamma(\xi)}. \tag{16.4}$$

Such theorems are referred as *separation theorems*. Dediue (1997) gives separation theorems for system of complex polynomials and in particular polynomials in one complex variable.

In this chapter we will derive a family of lower bounds indexed by an integer $m \geq 2$ on the gap of Theorem 16.1 which in particular when $m = 2$ improves (16.3) as well as (16.4) by replacing their lower bounds with $1/(2\gamma(\xi))$ which is more than twice as good. Our results make use of the Basic Family, $\{B_m(z), m = 2, \dots\}$.

The chapter is organized as follows: In Section 2, we describe the Basic Family and its significant relevant properties for complex polynomials. We then extend these to the case of analytic functions. In Section 3, we make use of the Basic Family to derive lower bounds on the distance from a simple zero of f to its nearest distinct zero. In Section 4, we make use of the preceding lower bounds to derive lower bounds on the distance between an arbitrary point and the nearest root of f. In particular using the latter result we show that given a complex polynomial f, for each $m \geq 2$ we can compute an annulus containing the roots. In Section 5, we consider the application of the bounds on the modulus of roots within Weyl's algorithm. We conclude the chapter in Section 6.

16.3 The Basic Family for General Analytic Functions

Assume that $f(z)$ is a complex polynomial of degree n. Consider the Basic Family:

$$B_m(z) \equiv z - f(z) \ \frac{D_{m-2}(z)}{D_{m-1}(z)}, \tag{16.5}$$

where for each $m \geq 2$, $D_0(z) = 1$, and for each $m \geq 1$

$$D_m(z) = \det \begin{pmatrix} f'(z) & \frac{f''(z)}{2!} & \dots & \frac{f^{(m-1)}(x)}{(m-1)!} & \frac{f^{(m)}(z)}{(m)!} \\ f(z) & f'(z) & \ddots & \ddots & \frac{f^{(m-1)}(z)}{(m-1)!} \\ 0 & f(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{f''(z)}{2!} \\ 0 & 0 & \dots & f(z) & f'(z) \end{pmatrix}. \tag{16.6}$$

If ξ is a root of f, $D_m(\xi) = f'(\xi)^m$. Thus, whether or not ξ is a simple root of f it is a fixed-point of B_m since we have

$$B_m(\xi) = \xi - f(\xi)\frac{f'(\xi)^{m-2}}{f'(\xi)^{m-1}} = \xi - \frac{f(\xi)}{f'(\xi)} = \xi.$$

With

$$\widehat{D}_{m,k}(z) = \det \begin{pmatrix} \frac{f''(z)}{2!} & \frac{f'''(z)}{3!} & \dots & \frac{f^{(m)}(z)}{(m)!} & \frac{f^{(k)}(z)}{k!} \\ f'(z) & \frac{f''(z)}{2!} & \ddots & \frac{f^{(m-1)}(z)}{(m-1)!} & \frac{f^{(k-1)}(z)}{(k-1)!} \\ f(z) & f'(z) & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \frac{f''(z)}{2!} & \frac{f^{(k-m+2)}(z)}{(k-m+2)!} \\ 0 & 0 & \dots & f'(z) & \frac{f^{(k-m+1)}(z)}{(k-m+1)!} \end{pmatrix} \tag{16.7}$$

where $m \geq 1$, and $k \geq (m+1)$, the following theorem is already proved in Chapter 11 (Corollary 11.2, Chapter 11) is a consequence of the main determinantal theorem.

Theorem 16.2. *Assume that $f(z)$ is a complex polynomial of degree n. Let ξ be a root of $f(z)$. Then, except for finitely many values of $z \in C$, $B_m(z) \in C$, and*

$$B_m(z) = \xi + \sum_{k=m}^{m+n-2} (-1)^m \frac{\widehat{D}_{m-1,k}(z)}{D_{m-1}(z)} (\xi - z)^k. \tag{16.8}$$

We will now proceed by proving a more general version of Theorem 16.2

Theorem 16.3. *Let $f(z)$ be a complex-valued function analytic over the entire complex plane. For each $m \geq 2$, define $B_m(z)$ as in (16.5). Then $B_m(z)$ satisfies (16.8). Then,*

$$B_m(z) = \xi + \sum_{k=m}^{\infty} (-1)^m \frac{\widehat{D}_{m-1,k}(z)}{D_{m-1}(z)} (\xi - z)^k. \tag{16.9}$$