CS512 LECTURE NOTES - LECTURE 19

1 Correctness of the Ford and Fulkeron Algorithm

Notation:
We will use the following notation to simplify the summations:

$$\sum_{A \to B} w = \sum_{u \in A, v \in B, (u,v) \in E} w(u,v)$$

if one of the sets contains a single element $A = \{a\}$, then we will write it as:

$$\sum_{a \to B} w = \sum_{v \in B, (a,v) \in E} w(a,v)$$

similar notation in the case of $B = \{b\}$

Generalized Flow conservation
The flow conservation property described before can be easily generalized to a subset $A \in V - \{s,t\}$ (flow into A equals flow out of A):

$$\sum_{(V-A) \to A} f = \sum_{A \to (V-A)} f$$

Definition
A *cut* (S,T) is a partition of V such that $s \in S$ and $t \in T$.

Definition
The *capacity of a cut* $cap(S,T)$ is equal to the sum of the capacities of the edges that go from S to T:

$$cap(S,T) = \sum_{S \to T} c$$

Definition
The *flow through a cut* denoted by $flow(S,T)$ is defined as the total flow from S to T minus the flow from T to S:

$$flow(S,T) = \sum_{S \to T} f - \sum_{T \to S} f$$

Definition
The *value of the flow* $v(f)$ is equal to the total flow out of the source s:

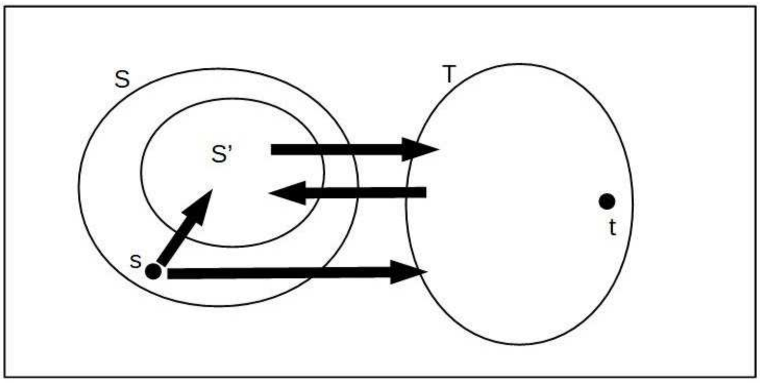
$$v(f) = \sum_{s \to V} f$$

Theorem 1
For every cut (S, T)

$$v(f) = flow(S, T)$$

Proof

Let (S, T) be a cut, and $S' = S - \{s\}$. We can illustrate all flows into and out of S' in the following figure:



Since s, t are not in S' we can use the *generalized flow conservation* property given above, so we have that the summation of all the flows into S' is equal to the summation of all the flows out of S':
Notice that:

Flow into S':

$$\sum_{s \to S'} f + \sum_{T \to S'} f$$

Flow out of S':

$$\sum_{S' \to T} f$$

and we get that

$$\begin{aligned}
\sum_{s\to S'} f + \sum_{T\to S'} f &= \sum_{S'\to T} f \\
\Rightarrow \quad \sum_{s\to S'} f &= \sum_{S'\to T} f - \sum_{T\to S'} f \quad \textbf{(1)}
\end{aligned}$$

Let us now compute the $flow(S,T)$. From the figure we have:

$$flow(S,T) = \sum_{s\to T} f + \sum_{S'\to T} f - \sum_{T\to S'} f$$

Substituting equation **(1)** into the equation above we have

$$\begin{aligned}
flow(S,T) &= \sum_{s\to S'} f + \sum_{s\to T} f \\
&= \sum_{s\to V} f \\
&= v(f) \qquad \square
\end{aligned}$$

In order to prove our next result recall that:

1. $f(u, v) \leq c(u, v)\ \forall(u, v) \in E$
2. From the execution of the algorithm for each $(u, v) \in E$:

$$c^f(u, v) = c(u, v) - f(u, v)$$

$$c^f(v, u) = f(u, v)$$

Theorem 2
The value of the flow $v(f)$ is bounded from above by the capacity of any cut $cap(S, T)$. $(v(f) \leq cap(S, T)\ \ \forall(S, T))$
Proof

$$\begin{aligned} v(f) &= flow(S, T) \quad \text{(from theorem 1)} \\ &= \sum_{S \to T} f - \sum_{T \to S} f \\ &\leq \sum_{S \to T} f \\ &\leq \sum_{S \to T} c \quad \text{(from property 1 (above))} \\ &= cap(S, T) \qquad \square \end{aligned}$$

Theorem 3

If f^* is the flow returned by Ford and Fulkerson's Algorithm then $v(f^*)$ is a maximum flow ($\forall$ flow f, $v(f) \leq v(f^*)$).

Proof

Let f^* be the flow returned by Ford and Fulkerson's algorithm, this implies that there is no augmenting path on the residual network computed from that flow. Let S^* be the set of vertices reachable from s through an augmenting path. We know that t is not in S^*. So we have

$$S^* = \{v|v \text{ is reachable from } s \text{ in } G^f\}, \quad T^* = V - S^*$$

Notice that (S^*, T^*) is a cut.

Let $(u, v) \in E$ be an edge that crosses the cut, i.e. $u \in S^*, v \in T^*$

Vertex u is reachable from s but v is not, therefore the residual capacity $c^f(u, v) = 0$. This implies that there are two cases (according to the Residual method described above):

1. $(u, v) \in E$ then $c^f(u, v) = c(u, v) - f^*(u, v) = 0 \Rightarrow f^*(u, v) = c(u, v)$.
2. $(v, u) \in E$ then $c^f(u, v) = f^*(v, u) = 0$

Since all the flows from S^* to T^* are at capacity (case 1 above) and all the flows from T^* to S^* are 0 (case 2 above) we have that

$$\begin{aligned} v(f^*) &= flow(S^*, T^*) \text{ from theorem 1} \\ &= \sum_{S^* \to T^*} f - \sum_{T^* \to S^*} f \\ &= \sum_{S^* \to T^*} c \\ &= cap(S^*, T^*) \end{aligned}$$

Let f be a flow, since (S^*, T^*) is a cut, we know from theorem 2 that

$$v(f) \leq cap(S^*, T^*)$$

and from the previous property, we know that

$$cap(S^*, T^*) = v(f^*)$$

Therefore

$$\forall \text{ flow } f, \ \ v(f) \leq v(f^*)$$

Hence $v(f^*)$ is a maximum flow.