Midterm Exam

Foundations of Computer and Data Science CS-596

Problem 1: Let $\mathcal{V}$ denote the space of all polynomials $p(x)$ of order *up to* some fixed integer value n. a) Show that $\mathcal{V}$ is a vector space. Specify the addition and multiplication. b) Is $\mathcal{V}$ finite dimensional? If yes what is its dimension? c) Define a straightforward basis. d) Define at least three linear subspaces of $\mathcal{V}$. e) If $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$ there is a one-to-one correspondence between $p(x)$ and the vector $[p_0\ p_1 \cdots p_n]^\intercal$ of its coefficients. Using this correspondence define an inner product for $\mathcal{V}$ and then use it to define a norm for polynomials of order up to n.

Problem 2: If Q is a real *symmetric* matrix of dimensions $k \times k$, with eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_k$, which are real, then we recall that we have *already proved* that for any *real* vector X we have

$$\rho_1 \geq \frac{X^\intercal Q X}{X^\intercal X} \geq \rho_k.$$

a) Using the special eigen-decomposition of real symmetric matrices, extend the previous inequalities to *complex* vectors X as follows

$$\rho_1 \geq \frac{(X^*)^\intercal Q X}{(X^*)^\intercal X} \geq \rho_k,$$

where X^* denotes the conjugate of X. b) If A is a square matrix of dimensions $k \times k$ with real elements, denote with $\lambda_1, \ldots, \lambda_k$ its eigenvalues that may be complex numbers (and the corresponding eigenvectors complex vectors) and with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k$ its singular values which are real and nonnegative. Using question a) show that all eigenvalues λ_i satisfy

$$\sigma_1 \geq |\lambda_i| \geq \sigma_k.$$

Hint: The σ_i^2 are the eigenvalues of the symmetric matrix $A^\intercal A$.

Problem 3: A square matrix P is called a *projection* if $P^2 = P$. a) Show that the eigenvalues of P are either 0 or 1. b) Show that if P is a projection so is $I - P$ where I the identity matrix. c) If P is also symmetric $P^\intercal = P$ then P is called an *orthogonal projection*. Prove that for an orthogonal projection P and any vector X we have that $X - PX$ and PX are orthogonal. d) If the two matrices A, B have the same dimensions $m \times n$ then show that $P = A(B^\intercal A)^{-1} B^\intercal$ is a projection matrix. What is the condition on the dimensions m, n and on the product $B^\intercal A$ for this P to be well defined ? When is this matrix an orthogonal projection? e) If b is a fixed vector of length m and $\hat{b}$ some arbitrary vector, we are interested in minimizing the square distance $\min_{\hat{b}} \|b - \hat{b}\|^2$ where $\|\cdot\|$ is the Euclidean norm. To avoid the trivial solution we constrain $\hat{b}$ to satisfy $\hat{b} = AX$ where A is a matrix of dimensions $m \times n$ with $m > n$ and X an arbitrary vector of length n. Show that the optimum $\hat{b}$ is the orthogonal projection of b with some proper projection matrix which *you must identify.*

Problem 4: As discussed in the class, the space of all random variables constitutes a vector space. We can also define an inner product (also mentioned in class) between two random variables x, y using the expectation of the product

$$<x, y> = \mathbb{E}[xy].$$

Consider now the random variables x, z, w. We are interested in linear combinations of the form $\hat{x} = az + bw$ where a, b are real deterministic quantities. a) By using the orthogonality principle find the $\hat{x}_*$ (equivalently the optimum coefficients a_*, b_*) that is closest to x in the sense of the norm induced by the inner product. b) Compute the optimum (minimum) distance and its optimum approximation $\hat{x}_*$ in terms of $\mathbb{E}[xz], \mathbb{E}[xw], \mathbb{E}[z^2], \mathbb{E}[zw], \mathbb{E}[w^2]$.

You have 48 hours to complete the exam. Your reports, in *hard copy*, must be submitted to Mr. Stathopoulos our TA, on Wednesday, October 23, between 10:00-11:00AM, in CBIM. (NOT SOONER and NOT LATER!!!)

There will be no meeting in CoRE 101 and you are not allowed to ask me or the TA for any help.