Exercise 2.1

10011011×10111010

$$xy = 2^n x_1 y_1 + 2^{n/2}(x_1 y_2 + x_2 y_1) + x_2 y_2$$
$$= 2^n x_1 y_1 + 2^{n/2}[(x_1+x_2)(y_1+y_2) - x_1 y_1 - x_2 y_2] + x_2 y_2$$

round 1: $n=8$

$$2^n(1001 \times 1011) + 2^{n/2}[(1001+1011)(1011+1010) - (1001\times 1011) - (1011 \times 1010)] + (1011 \times 1010)$$

Therefore: 10011011×10111010 was devided in three subquestions:

1001×1011 $\quad$ 1011×1010 $\quad$ $(1001+1011)(1011+1010) = 10100 \times 10101$

round 2:

$$1001 \times 1011 = 2^{n-1}(10\times 10) + 2^{n-1/2}[(10+01)(10+11) - (10\times 10) - (01\times 11)] + (01 \times 11)$$

three subquestions are: 10×10 $\quad$ 01×11 $\quad$ $(10+01)(10+11) = 11 \times 101$

$$1011 \times 1010 = 2^{n-1}(10\times 10) + 2^{n-1/2}[(10+11)(10+10) - (10\times 10) - (11\times 10)] + (11\times 10)$$

three subquestions are: 10×10 $\quad$ 11×10 $\quad$ $(10+11)(10+10) = 101 \times 100$

$$10100 \times 10101 = 2^n(0001 \times 0001) + 2^{n/2}[(0001+0100)(0001+0101) - (0001\times 0001) - (0100 \times 0101)] + (0100 \times 0101)$$

three subquestions are: $0001 \times 0001 = 01 \times 01$ $\quad$ 0100×0101
$(0001+0100)(0001+0101) = 0101 \times 0110$

there are actually seven subquestions now:

10×10 $\quad$ 01×11 $\quad$ 11×10 $\quad$ 11×101 $\quad$ 101×100 $\quad$ 01×01 $\quad$ 101×110

round 3:

$$10 \times 10 = 2^{n-2}(1\times 1) + 2^{n-2/2}[(1+0)(1+0) - (1\times 1) - (0 \times 0)] + (0\times 0)$$

three subquestions are: 1×1 $\quad$ 0×0 $\quad$ $(1+0)(1+0) = 1\times 1$

$$01 \times 11 = 2^{n-2}(0\times 1) + 2^{n-2/2}[(0+1)(1+1) - (0\times 1) - (1\times 1)] + (1\times 1)$$

three subquestions are: 0×1 $\quad$ 1×1 $\quad$ $(0+1)(1+1) = 1\times 10$

$$11 \times 10 = 2^{n-2}(1\times 1) + 2^{n-2/2}[(1+1)(1+0) - (1\times 1) - (1\times 0)] + (1\times 0)$$

there subquestions are: 1×1 $\quad$ 1×0 $\quad$ $(1+1)(1+0) = 10 \times 1$

$$01 \times 01 = 2^{n-2}(0\times 0) + 2^{n-2/2}[(0+1)(0+1) - (0\times 0) - (1\times 1)] + (1\times 1)$$

three subquestions are: 0×0 $\quad$ 1×1 $\quad$ $(0+1)(0+1) = 1\times 1$

$11 \times 101 = 2^{n-1}(00 \times 01) + 2^{n-1/2}[(00+11)(01+01) - (00 \times 01) - (11 \times 01)] + (11 \times 01)$

three subquestions are: 00×01 $\quad 11 \times 01$ $\quad (00+11)(01+01) = 11 \times 10$

$101 \times 100 = 2^{n-1}(01 \times 01) + 2^{n-1/2}[(01+01)(01+00) - (01 \times 01) - (01 \times 00)] + (01 \times 00)$

three subquestions are: 01×01 $\quad 01 \times 00$ $\quad (01+01)(01+00) = 10 \times 01$

$101 \times 110 = 2^{n-1}(01 \times 01) + 2^{n-1/2}[(01+01)(01+10) - (01 \times 01) - (01 \times 10)] + (01 \times 10)$

three subquestions are: 01×01 $\quad 01 \times 10$ $\quad (01+01)(01+10) = 10 \times 11$

there are actually four subquestions

$1 \times 1 = 1$ $\quad 0 \times 0 = 0$ $\quad 1 \times 0 = 0$ $\quad 01 \times 10$

round 4:

$01 \times 10 = 2^{n-2}(0 \times 1) + 2^{n-3/2}[(0+1)(1+0) - (0 \times 1) - (1 \times 0)] + (1 \times 0)$

three subquestions are: 0×1 $\quad 1 \times 0$ $\quad (0+1)(1+0) = 1 \times 1$

Now, combine these:

return 4:

$01 \times 10 = 4 \times 0 + 2 \times (1 - 0 - 0) + 0 = 10$

return 3:

$10 \times 10 = 4 \times 1 + 2 \times (1 - 1 - 0) + 0 = 100$

$01 \times 11 = 4 \times 0 + 2 \times (2 - 0 - 1) + 1 = 11$

$11 \times 10 = 4 \times 1 + 2 \times (2 - 1 - 0) + 0 = 110$

$01 \times 01 = 1$

$11 \times 101 = 16 \times 0 + 4 \times (6 - 0 - 3) + 3 = 1111$

$101 \times 100 = 16 \times 1 + 4 \times (2 - 1 - 0) + 0 = 10100$

$101 \times 110 = 16 \times 1 + 4 \times (6 - 1 - 2) + 2 = 11110$

return 2:

$1001 \times 1011 = 16 \times 4 + 4 \times (15 - 4 - 3) + 3 = 1100011$

$1011 \times 1010 = 16 \times 4 + 4 \times (20 - 4 - 6) + 6 = 1101110$

$10100 \times 10101 = 256 \times 1 + 16 \times (30 - 1 - 20) + 20 = 110100100$

return 1:

$1001011 \times 1011010 = 256 \times 99 + 16 \times (420 - 99 - 110) + 110 = 111000010011110$

Exercise 2.2.

if $n > b$;

  if $n > b^2$;

    if $n > b^3$;

      ⋮

    else $n \leq b^3$, $nb \geq b^3$.

      b^3 is in the range of $[n, bn]$

  else $n \leq b^2$, $nb \geq b^2$

    b^2 is in the range of $[n, bn]$

else $n \leq b$, $nb > b$

  b is in the range of $[n, bn]$

no matter how big n goes, there are always a b^k where k is a positive integer that satisfy $n \leq b^k \leq bn$.

Therefore, for any integer n and base b, there must some power of b lying in the range $[n, bn]$.

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