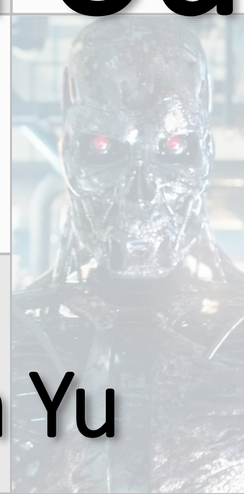
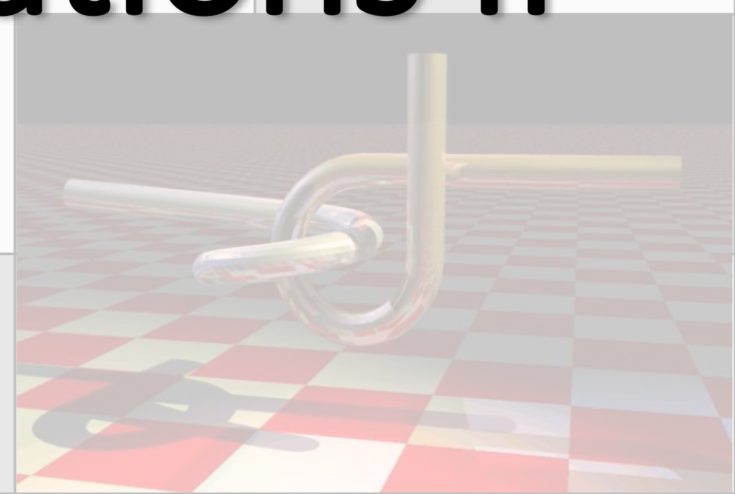
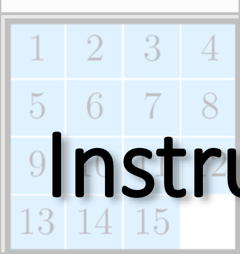
CS 460/560
Introduction to Computational Robotics
Fall 2019, Rutgers University

Lecture 03
Math. Foundations II







Instructor: Jingjin Yu

Outline

Topological spaces

Manifolds

Path and connectivity

Homotopic paths

Connectedness of space

Fixed point theorems

Topological Space

A set X and a collection Γ of subsets (called open sets) of X form a topological space if

⇨ $\emptyset \in \Gamma$ and $X \in \Gamma$

⇨ Arbitrary union ($\cup$) of elements of Γ is again in Γ

⇨ Finite intersection ($\cap$) of elements of Γ is again in Γ

Note: here, "open sets" are defined differently from earlier for Euclidean space; the definition here is more general

⇨ E.g., point set topologies with $X = \{1, 2, 3\}$ (from Wikipedia)

$\Gamma = \{\emptyset, \{1,2,3\}\}$

$\Gamma = \{\emptyset, \{1\}, \{1,2,3\}\}$

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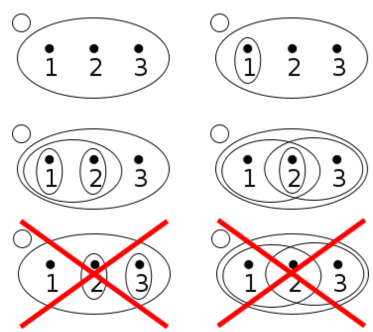
$\Gamma = \{\emptyset, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$



$\Gamma = \{\emptyset, \{2\}, \{3\}, \{1,2,3\}\}$,
$\{2\} \cup \{3\} = \{2,3\} \notin \Gamma$

$\Gamma = \{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}$,
$\{1,2\} \cap \{2,3\} = \{2\} \notin \Gamma$

A set A is closed if $(X - A)$ is open

Topological Spaces on $\mathbb{R}$

The "standard topology" on $\mathbb{R}$ is the one with basic open sets being (a, b) for all $a \leq b$, plus $\mathbb{R}$. This is the same as to what have defined before with Euclidean spaces

- ⇨Is $[0, 1]$ open or closed?
- ⇨Closed, because $(-\infty, 0) \cup (1, \infty)$ is open
- ⇨What about $\cup_{i=1}^{\infty} \left(i, i + \frac{1}{i}\right)$?

Alternatively, we can have $\Gamma = \{\emptyset, \mathbb{R}\}$

- ⇨This is the **trivial topology** on $\mathbb{R}$

Or, we can have $\Gamma = \{\, (-n, n) \mid n \in \mathbb{R}\,\} \cup \{\emptyset, \mathbb{R}\}$

So, many different topologies are possible!

Similar topologies can be defined for $\mathbb{R}^n$

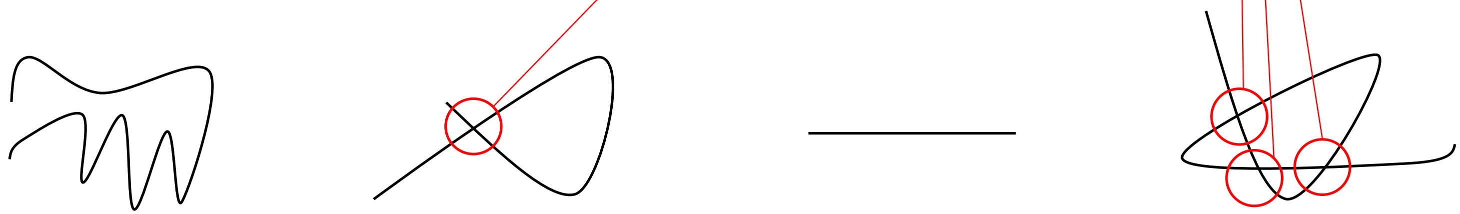
Homeomorphism (I)

These parts are slightly “two-dimensional”

Why study topology?

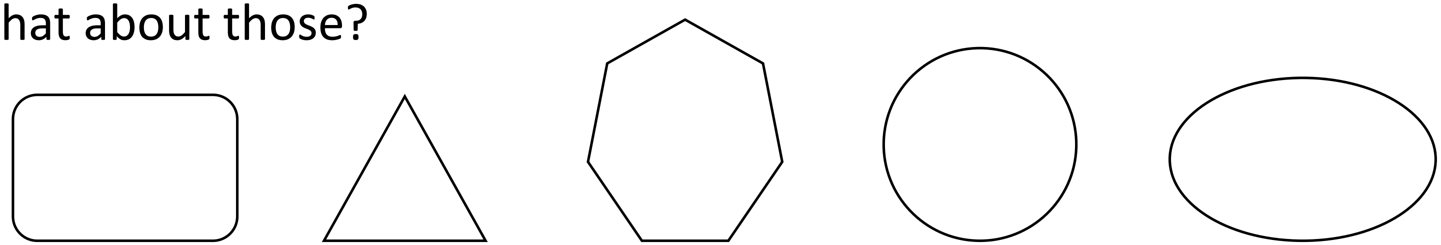
⇨One of the use is that it helps us to classify spaces

⇨Intuitively, which two of the following spaces are similar?



⇨The first and the third are both “one dimensional”

⇨What about those?



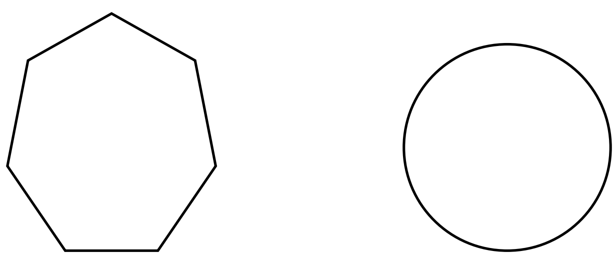
⇨All “similar” to a circle

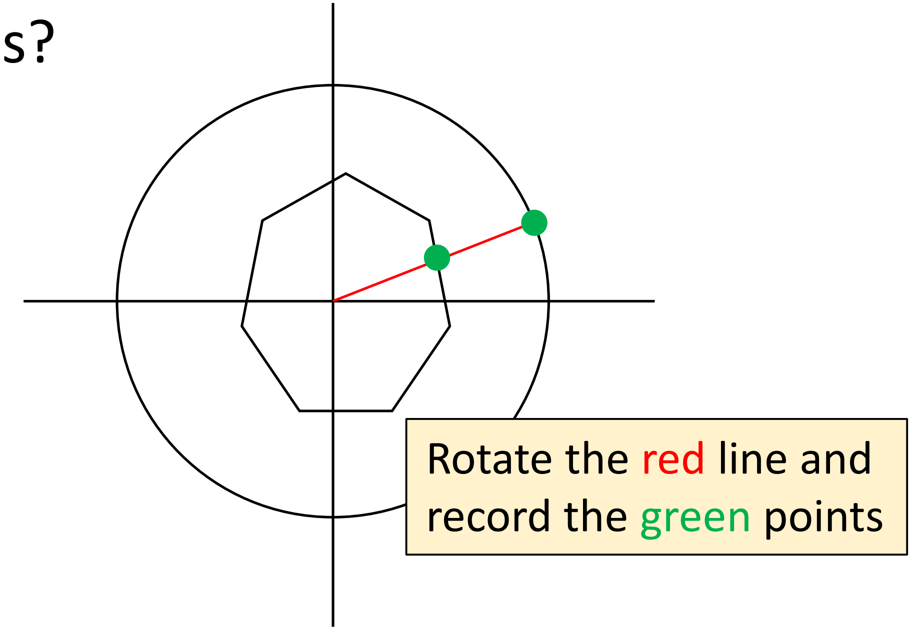
Homeomorphism: two spaces X and Y are **homeomorphic** if there is a continuous bijective function $F: X \to Y$ (that is, every member of X is mapped to a unique member of Y and vice versa)

Homeomorphism (II)

We can build a **bijection** $f: X \to Y$ by “sliding” from the green end to the red end on both lines

What about the following two shapes?





Homeomorphism (III) – Deformation

One can build a series of homeomorphisms to **deform** between two objects, that is,

- ⇨ $F_t, t \in [0, 1]$, is a bijective continuous function with domain X
- ⇨ $F_0(X) = X, F_1(X) = Y$
- ⇨ There are many intermediate $F_{0.x}$, e.g., $F_{0.1}, F_{0.2}, F_{0.3}$, ...
  - ⇨ Every $F_{0.x}(X)$ represents a **deformed** X
- ⇨ This is known as a **deformation**
- ⇨ Classic example: coffee mug and donut

$X =$ cup
$F_0(X) =$ cup, $F_1(X) =$ donut

Image from wikipedia

Topological Manifolds

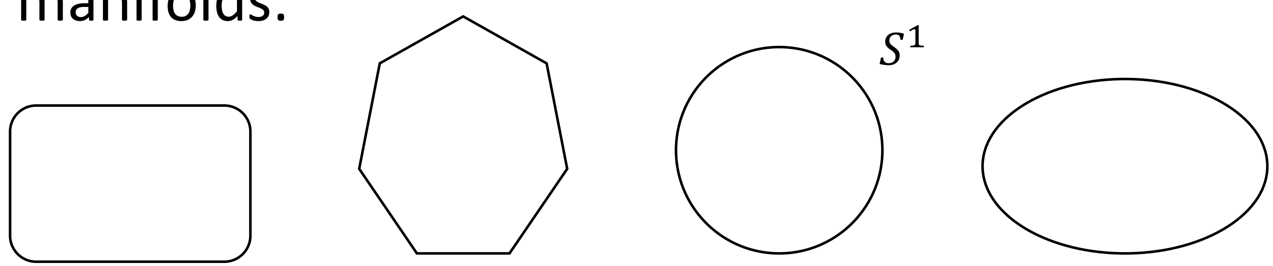
Roughly speaking, an ***n*-dimensional topological manifold *M*** is a space such that for all $x \in M$, there exists a neighborhood U of x **homeomorphic** to $\mathbb{R}^n$

Alternative intuition: take a piece, and smash it flat… it should look like $\mathbb{R}^n$

0-dimensional manifolds: discrete spaces

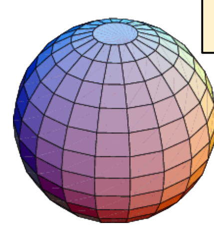
1-dimensional manifolds:

(a, b), $\mathbb{R}$



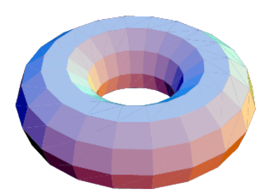
2-dimensional manifolds: $\mathbb{R}^2, S^2, T^2, \ldots$

$\mathbb{R}^2$ - 2D plane



Think Earth!

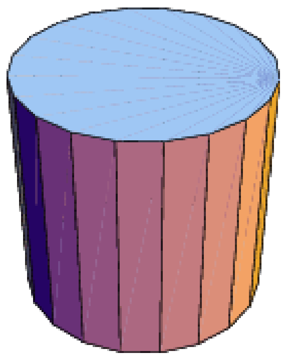
S^2 - 2 sphere

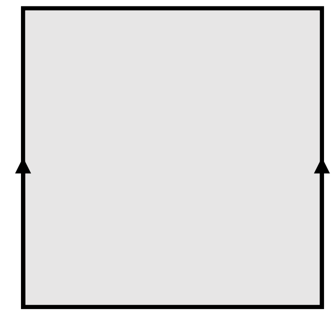


T^2 - torus

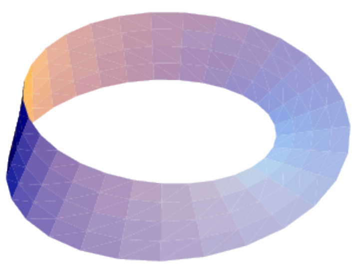
"Flat Representation" of 2D Manifolds

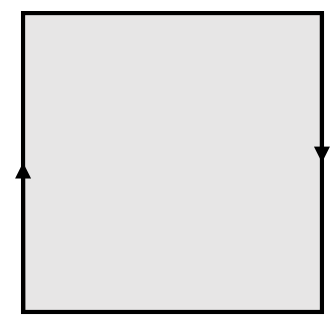
2D manifolds can be represented as **unit squares w/ sides identified**.





Infinite Cylinder



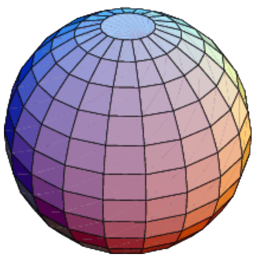


(infinite)
Mobius band

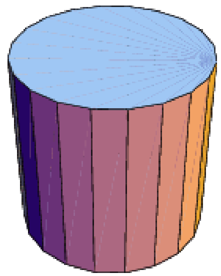
Such a "flat" representation is called the **fundamental polygon**

Image from wolfram mathworld

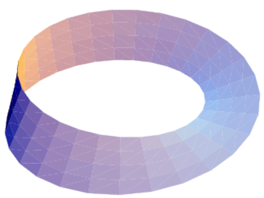
Some Common 2-Dimensional Manifolds



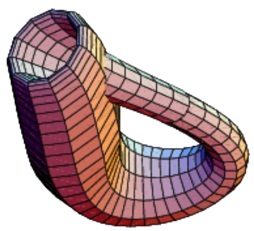
Sphere



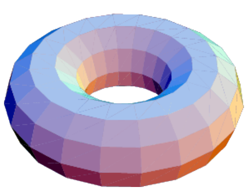
Cylinder



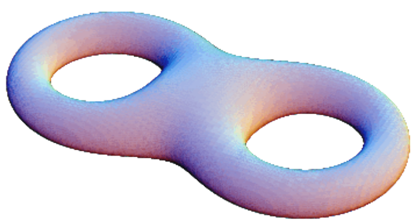
Mobius band



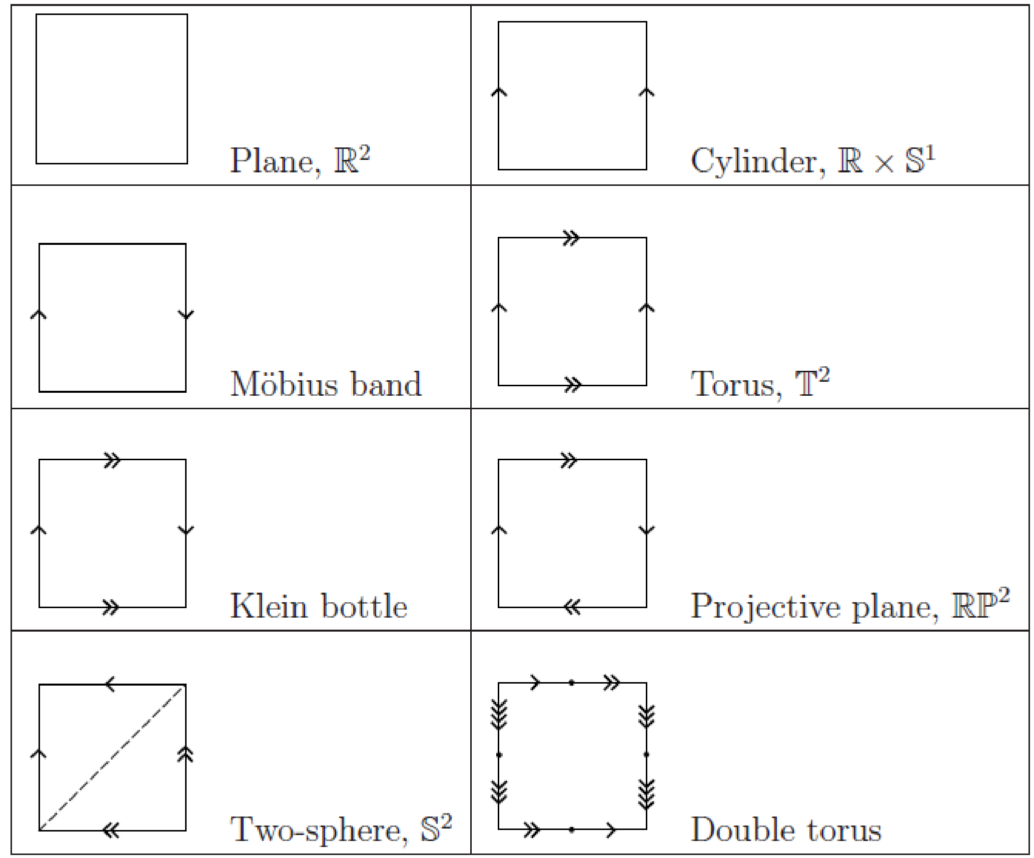
Klein bottle



Torus



Double torus



There are infinitely many different types of 2-dimensional manifolds

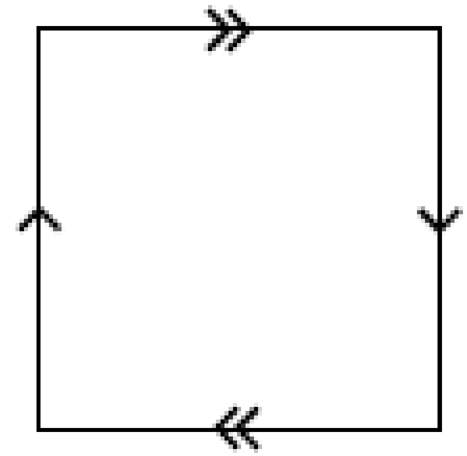
Image from wolfram mathworld, Planning Algorithms

The Real Projective Plane

Real projective spaces are something of an oddity

$\mathbb{RP}^n$ or real projective space of dimension n is formed by making each line of $\mathbb{R}^{n+1}$ that goes through the origin into a point

$\mathbb{RP}^2$ can be thought of the top half of a sphere, plus half of the equator



Generally, we cannot easily "visualize" high dimensional manifolds, but can work through the math

Can be useful however – camera projections are related to $\mathbb{RP}^2$

Why Topology and Manifolds?

Sensing, planning, and control are all related to manifolds

Robotics examples

- ⇨A point robot in 2D take any position $x \in \mathbb{R}^2$
  - ⇨ This is also a group $E(2)$
  - ⇨ 2-dimensional Euclidean group
- ⇨A car in 2D has one more dimension
  - ⇨ This is called $SE(2) = \mathbb{R}^2 \times S^1$
  - ⇨ $SE(2)$ reads: Special Euclidean group of dimension 2
  - ⇨ Yes, each point in the space is also a group element, just like $\mathbb{R}$ and $\mathbb{R}^2$
  - ⇨ Using (x, y, θ), can describe all possible positions of the car

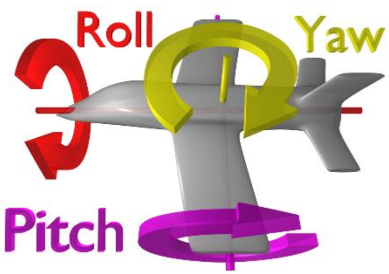
(x, y, θ)

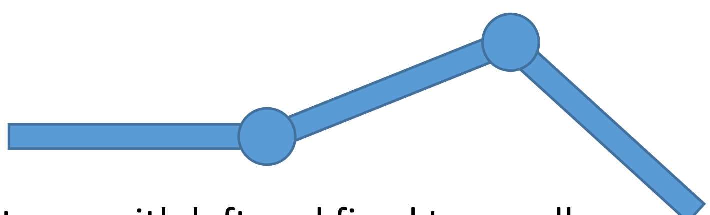
Why Topology and Manifolds? Continued

Robotics examples, continued

- ⇨ A quadcopter is in a six-dimensional manifold
  - ⇨ Three positions (x, y, z)
  - ⇨ Three rotations $(yaw, pitch, roll)$
  - ⇨ This is $SE(3) = \mathbb{R}^3 \times SO(3)$
  - ⇨ Special Euclidean group of three dimensions
- ⇨ A 2-link robot arm has a 2-dimensional manifold
  - ⇨ For rotations in the plane, this is T^2 (torus)
  - ⇨ Yes, a pose of such a robot arm corresponds to a point on a donut
- ⇨ These are the **configuration spaces** of the robots
- ⇨ More on this later

$(x, y, z, yaw, pitch, roll)$





A robot arm with left end fixed to a wall and remaining horizontal. The arm may rotate in the plane along the two joints

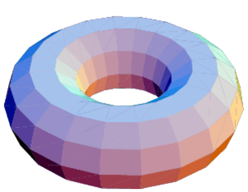


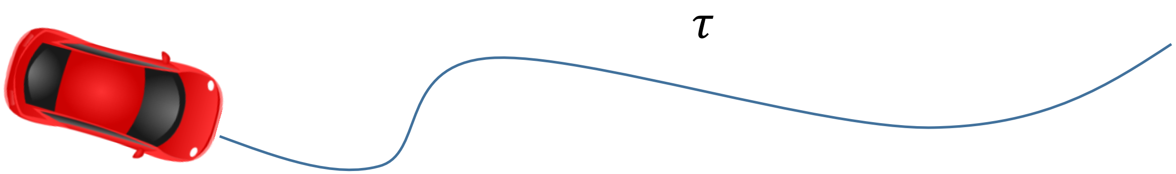
Image from wolfram wikipedia

Path and Notions of Connectivity

Path. A **path** in a manifold X is a continuous function $\tau: [0,1] \rightarrow X$

⇨Note that $[0,1]$ is the same as $[0, t]$ – simple scaling – time is relative

⇨E.g., for a car, $\tau: [0,1] \rightarrow \mathbb{R}^2 \times S^1$



⇨Important: it's not (just) a set of points!

⇨The points are chained together through time

⇨It is important to see how they are **connected**

A topological space X is **connected** if it cannot be partitioned into two disjoint, nonempty open sets.

A topological space X is **path connected** if for any $x, y \in X$, there exists a path $\tau: [0,1] \rightarrow X$ s.t. $\tau(0) = x$ and $\tau(1) = y$.

Topologist's Sine Curve

Topologist's sine curve: $X = \{y = \sin\frac{1}{x}, x > 0\} \cup \{x = 0\}$

- ⇨Connected: you cannot separate the two parts
- ⇨But not path connected: a point on $\{x = 0\}$ is infinitely far from a point on $\{y = \sin\frac{1}{x}, x > 0\}$

How to get rid of the problem?

- ⇨Require X be a manifold
- ⇨Recall: **roughly speaking**, an $\boldsymbol{n}$**-dimensional topological manifold** $\boldsymbol{M}$ is a space such that for $x \in M$, there exists a neighborhood U of x **homeomorphic** to $\mathbb{R}^n$
- ⇨X is homeomorphic to $\mathbb{R}$ at any $x > 0$
- ⇨X is not homeomorphic to $\mathbb{R}$ at $x = 0$

Generally want path connectivity

- ⇨A robot cannot follow topologist's sine curve!

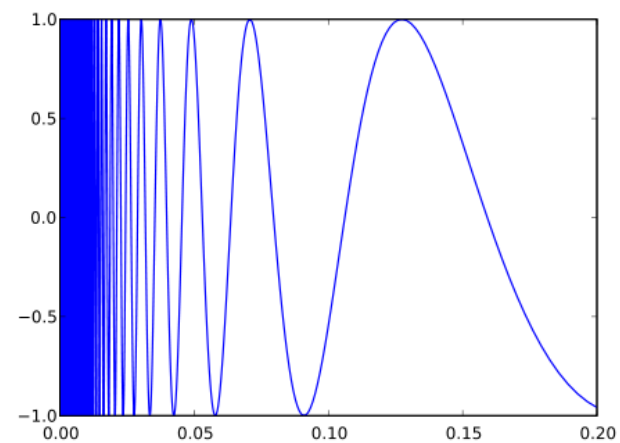


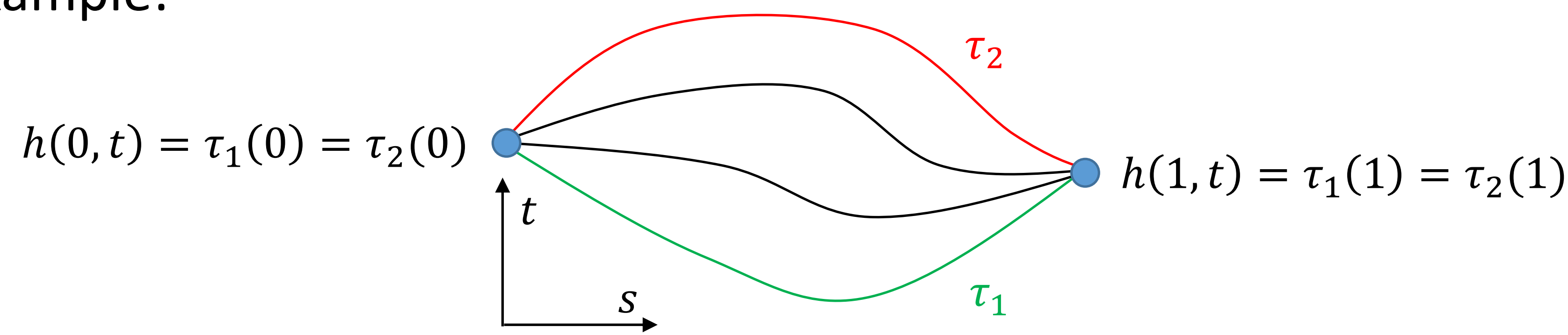
Image from scientific America

Homotopic Paths

Two paths τ_1 and τ_2 are **homotopic** to each other if there exists a map $h: [0,1] \times [0,1] \rightarrow X$ s.t.

- ⇨ $h(s,0) = \tau_1(s)$ for all $s \in [0,1]$
- ⇨ $h(s,1) = \tau_2(s)$ for all $s \in [0,1]$
- ⇨ $h(0,t) = h(0,0)$ for all $t \in [0,1]$
- ⇨ $h(1,t) = h(1,0)$ for all $t \in [0,1]$
- ⇨ The definition can be a bit confusing on a first look

Example:

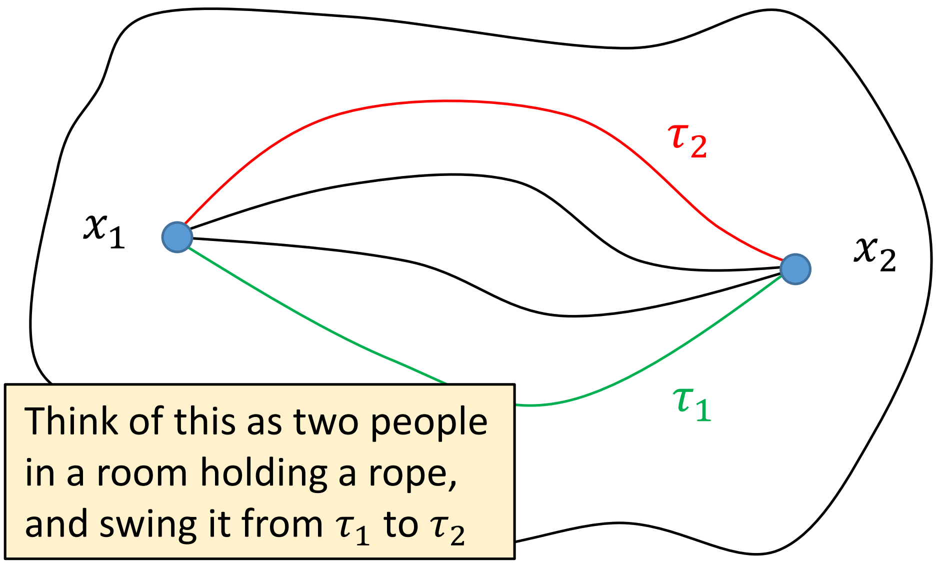


So, τ_1 and τ_2 are homotopic (roughly speaking) if they share the same end points and can be continuously morphed into each other

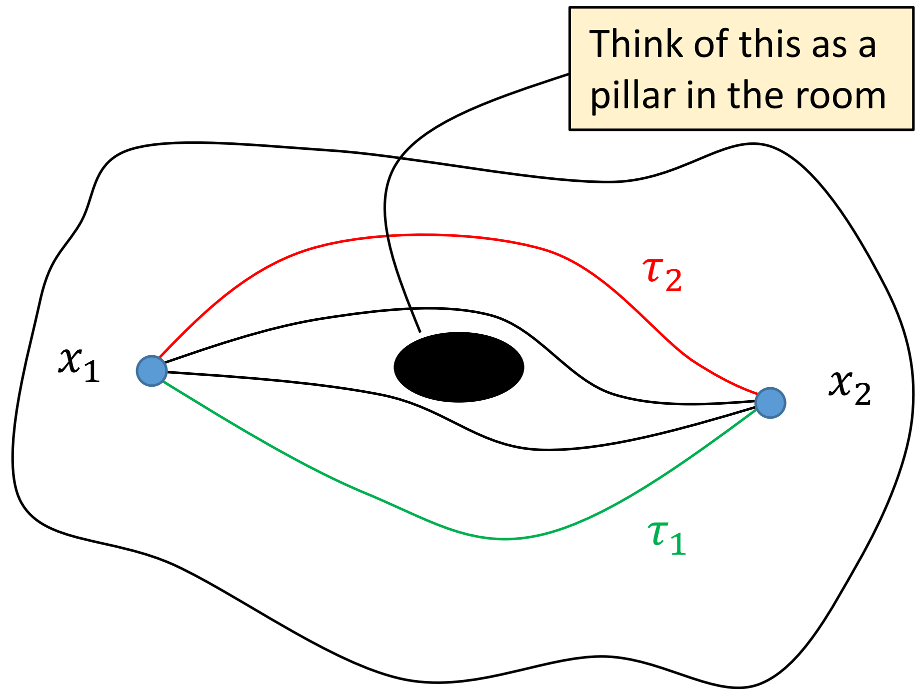
Simply Connected Space

A topological space X space is **simply connected** if $\forall x_1, x_2 \in X$ and any τ_1, τ_2 with $\tau_1(0) = \tau_2(0) = x_1$ and $\tau_1(1) = \tau_2(1) = x_2$, τ_1 and τ_2 are **homotopic**. Otherwise, X is **multiply connected**

An example:



Simply connected



Multiply connected

Importance in robotics: partition paths into different “classes”

Another Example

Let X be a 2-sphere with a circular area removed. Is X simply connected?

⇨Yes!

What about 2 holes?

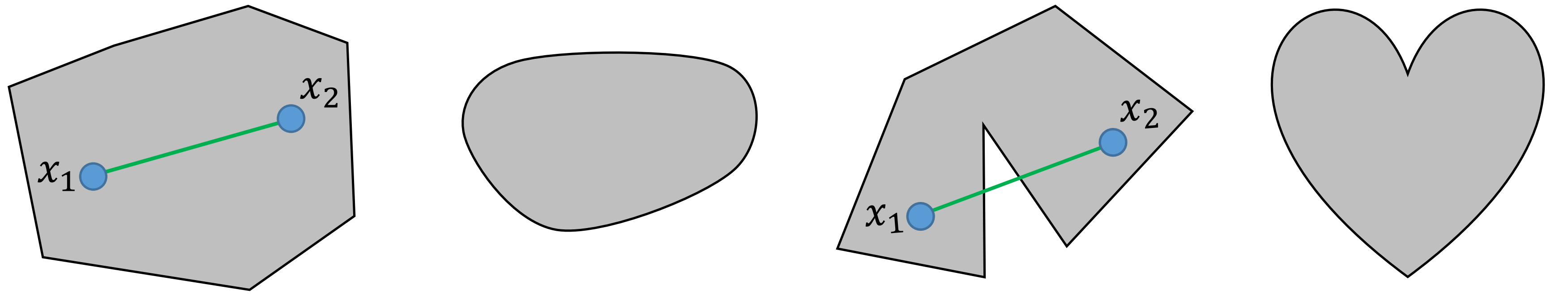
⇨Not anymore!

Intuition: the outer surface is equivalent to a disc

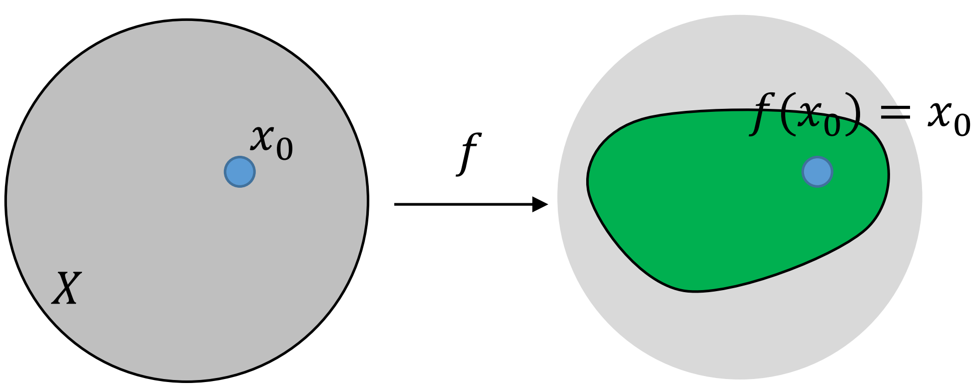
An empty ball with a “cap” removed

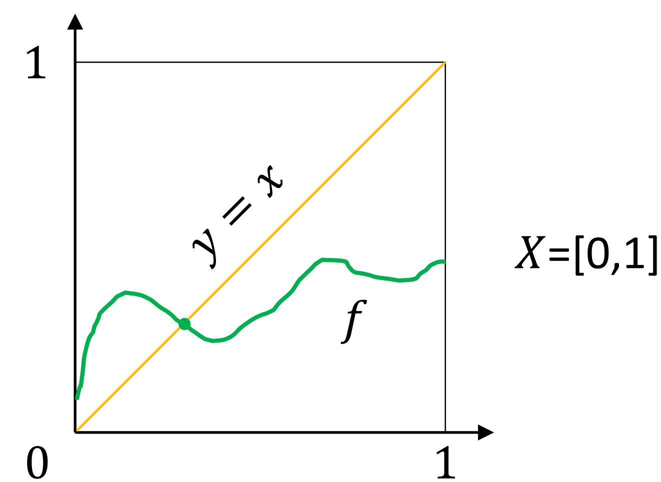
Fixed Point Theorems

Convexity. In a Euclidean space, a set X is **convex** if given any $x_1, x_2 \in X$, all points on the straight-line segment $x_1 x_2$ belong to X.



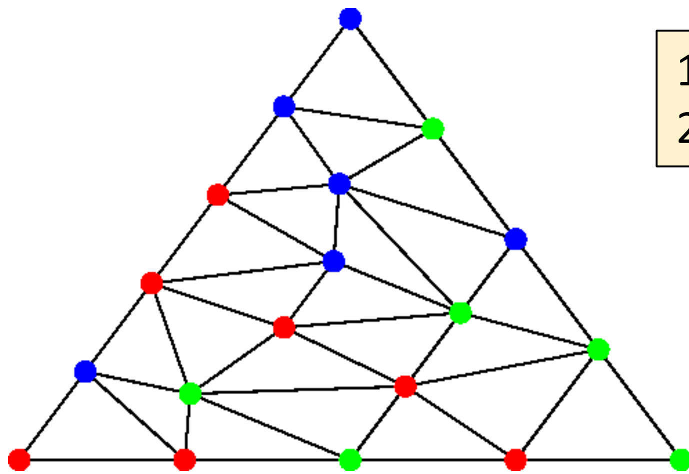
Brouwer's fixed point theorem. Let X be a bounded, closed, convex set. Let $f: X \rightarrow X$ be a continuous function. Then there exists a point $x_0 \in X$ s.t. $f(x_0) = x_0$



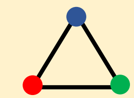


Fixed Point Theorems, Continued

Discrete case: Sperner’s lemma



1. Must be at least one
2. An odd number of



Many other ones: https://en.wikipedia.org/wiki/Fixed-point_theorem

Why interesting?

- ⇨Fundamental in the study of topological manifolds
- ⇨Used to prove the Jordan Curve Theorem