Logistics

First two recitations (this week and the next week)

- ⇨Python basics (installation, basic syntax, basic programming), optional
- ⇨Making models for 3D printing w/ Blender, optional
- ⇨Will announce details through Sakai per recitation
- ⇨PTL: Han, Shuai (shuai.han@rutgers.edu)
- ⇨For recitation related questions, direct them to Han, Shuai

Regarding Python level necessary for the course

- ⇨First of all, Python is SIMPLE!
- ⇨We will only use basic Python features and basic/math/plotting libraries
- ⇨We will not do heavy GUI programming

TAs and office hours (office hours held at Hill 275):

- ⇨Feng, Siwei (siwei.feng@rutgers.edu), office hour: Thu 1-3pm
- ⇨Huang, Baichuan (baichuan.huang@Rutgers.edu), office hour: Tue 9-11am

CS 460/560
Introduction to Computational Robotics
Fall 2019, Rutgers University

Lecture 02
Math. Foundations I

Instructor: Jingjin Yu

Outline

Sets, functions, and cardinality of Sets

Open sets and continuous functions

Group theory concepts

Topological space concepts

Some notes:

⇨We will discuss how some of the concepts are useful in the context of computational robotics: you will see most of these if you study further or work with robotics computations!

⇨But, as an introductory course, we will not extensively use these advanced concepts

⇨In terms of course work, these will mainly appear in HWs; midterm will cover only some basic definitions

Set, Set Operations, and Venn Diagram

A **set** is a collection of elements. Examples:

- ⇨ $\{1, a, cup, \pi, \quad \}$ – elements do need not be of the same type
- ⇨ Natural numbers (an infinite set), $\mathbb{N} = \{0, 1, 2, \dots\}$
- ⇨ n-dimensional Euclidean spaces, $\mathbb{R}^n$ (e.g., $\mathbb{R}^3$ is the 3-dimensional space)

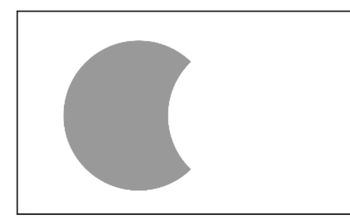
What is an element here?

Set operations

such that

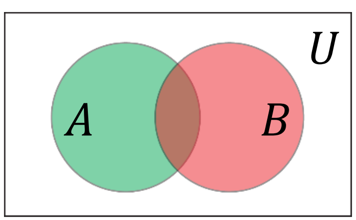
- ⇨ Union: $A \cup B = \{x \mid x \in A \lor x \in B\}$
- ⇨ Intersection: $A \cap B = \{x \mid x \in A \land x \in B\}$
- ⇨ Complement: $\overline{A} = \{x \mid x \in U \land x \notin A\}$
- ⇨ Difference: $A - B = \{x \mid x \in A \land x \notin B\}$ (or $A \backslash \mathrm{B}$)
- ⇨ Symmetric difference: $\mathrm{A} \ominus B = A \cup B - A \cap B$

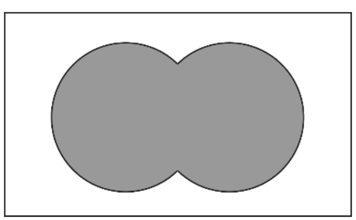
E.g., $U = \{1, \dots, 10\}$,
$A = \{2, 3, 5, 6, 7\}$,
$B = \{3,4,7,8, 9\}$



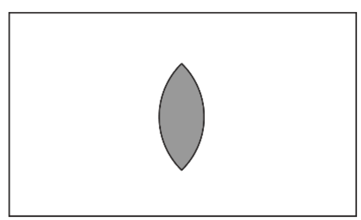
$A - B$

Venn diagram

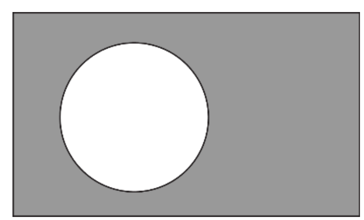




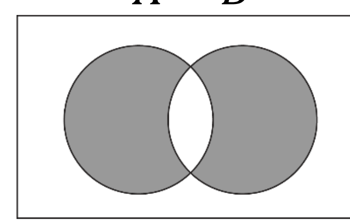
$A \cup B$



$A \cap B$



$\overline{A}$



$A \ominus B$

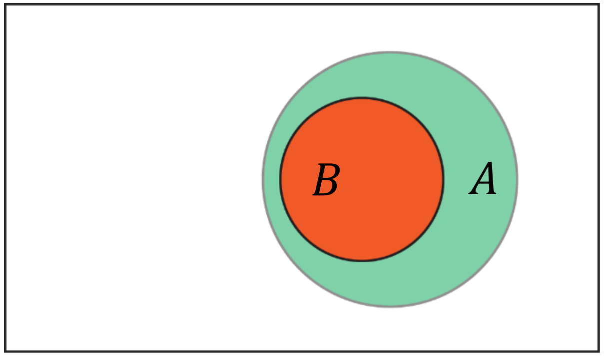
More Sets and Functions

Subset ($\subset$): $B \subset A \Leftrightarrow \forall x \in B, x \in A$

Superset ($\supset$): $A \supset B \Leftrightarrow B \subset A$

Powerset: $\mathcal{P}(S) = \{A \mid A \subset S\}$, example:

- ⇨ $S = \{1,2\}$
- ⇨ $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$



$B \subset A$

Functions

- ⇨ To fully specify a function, we write $f: X \to Y, x \mapsto f(x)$
  - ⇨ $x \in X, f(x) \in Y$
- ⇨ For f to be a valid function, $\forall x \in X, f(x) \in Y$ must be uniquely defined
- ⇨ Ex: for $f_1(x) = x^2$, we may write $f_1: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$
- ⇨ A function is **surjective** if $f(X) = Y$. Ex: $f_2: \mathbb{R} \to \mathbb{R}^+ \cup \{0\}, x \mapsto x^2$
- ⇨ A function is **injective** if $\forall x_1 \neq x_2, f(x_1) \neq f(x_2)$
- ⇨ Ex: $f_3: \mathbb{N} \to \mathbb{N}, x \mapsto x^2$
- ⇨ A function is **bijective** if it is both surjective and injective
- ⇨ Ex: $f_4: \mathbb{R} \to \mathbb{R}, x \mapsto x^3$

Q: Is f_1 surjective?

Q: Is f_1 injective? What about f_2? Is f_3 surjective?

Q: Are f_1-f_3 bijective? How can you make $x \mapsto x^2$ bijective?

Cardinality of Sets

Cardinality: essentially the "size" of a set

- $|\emptyset| = 0$

  Q: What about $|\{\emptyset\}|$?
- $|\{1, 2\}| = 2$
- $|\mathcal{P}(\{1,2\})| = |\{\emptyset, \{1\}, \{2\}, \{1,2\}\}| = 4$
- In general, $|\mathcal{P}(S)| = 2^{|S|}$
- $|\mathbb{N}| = \aleph_0$ - the "smallest" infinite (cardinal) number, read "Aleph 0"
- $|\mathbb{R}| = \aleph_1$ - there are "more" real number than natural numbers

Measuring the relative cardinality of sets

- $|A| \leq |B|$ if there exists an injective function $f: A \to B$
- If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$
  - This means there is a **bijective function** between A and B
- Cardinality of rational numbers, $\mathbb{Q}$?
  - Same as the set of natural numbers $\mathbb{N}$

    Q: How to prove this?
  - We call this "size" as "countably infinite"
- Cardinality of real numbers?
  - "Uncountably infinite" (Cantor's diagonalization argument)

    Q: How does this work?

A (Related) Fun Result

Can all problems be solved by algorithms?

Algorithm (roughly): a (Python) program that takes in a (finite) input

- ⇨E.g., a program deciding whether a number n is prime

Problem: a parametrized question that also has a (finite) input

- ⇨E.g., is n a prime number?

Not all problems can be solved by algorithms

- ⇨The number of programs is countable (i.e., as many as the natural numbers)
- ⇨But there are "more" problems!
- ⇨The number of parametrized questions is uncountable
  - ⇨ This can be shown via contradiction using Cantor's diagonalization argument
- ⇨One algorithm can solve one problem
- ⇨Therefore, some (actually, most) questions do not have algorithmic solutions!

> Geeky friend: I can solve all problems with computer algorithms!
> You: yeah, like that's mathematically possible…

Why Sets?

Sets (and related concepts) are the cornerstones of mathematics

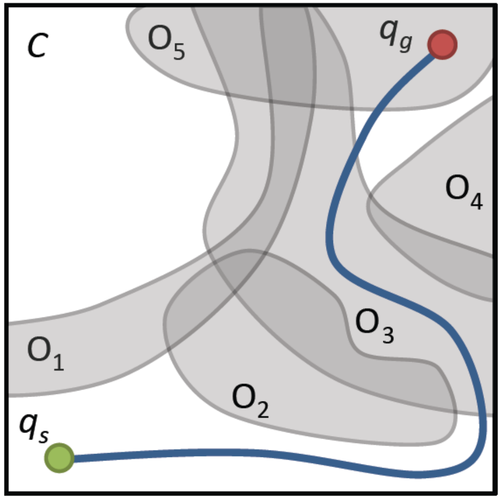
Formalize mathematical study, i.e., making things more precise

An example in the context of computer science: **set cover**

- ⇨ U: the finite universe with n elements
  e.g., $U = \{1, 2, \dots, 20\}$
- ⇨ $S = \{s_1, s_2, \dots, s_m\}$, each $s_i \subset U$
  e.g., $s_1 = \{1, 3, 14, 18\}$, $s_2 = \{3, 6, 11\}$, …
- ⇨ Question: find the smallest collection from $s_1, s_2, \dots, s_m$ whose union is U
- ⇨ Turns out this problem is very hard for computers – NP-hard

Connection to robotics: Minimum Constraint Removal

- ⇨ A robot wants to go from q_s to q_g
- ⇨ There are multiple obstacles blocking the way
- ⇨ Q: remove the least number of obstacles to go to q_g
- ⇨ Turns out this is as hard as set cover
- ⇨ So, yeah, robotics problems can be hard…

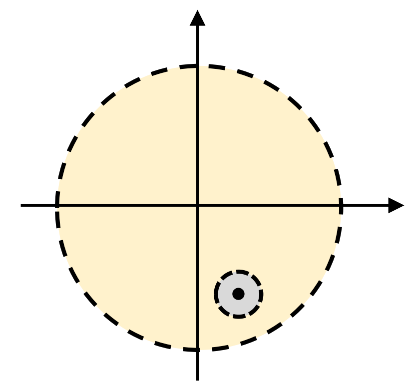


Kris Hauser, The minimum constraint removal problem with three robotics applications, WAFR, 2012

Open Sets, Closed Set, Boundary on $\mathbb{R}^n$

In **Euclidean spaces**, by convention, a set X is **open** if for all $x \in X$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X$.

- ⇨Ex: $\mathbb{R}$ is open, $\forall a, b \in R, a < b, (a, b)$ is open
- ⇨Ex: The set $\{ (x, y) \mid x^2 + y^2 < 1 \} \subset \mathbb{R}^2$ is open
- ⇨The union of any number of open set is open



A set is **closed** if its complement is open

- ⇨A set may be both open and closed, e.g., $\mathbb{R}$ Why?
- ⇨A set may be neither open nor closed, e.g., $(a, b]$

The **closure** of a set is the set plus all its **limit points** not in the set

- ⇨The closure of a set is closed, e.g. $Cl(S)$ is always closed
  - ⇨Also, $Cl(S) = Cl(Cl(S))$

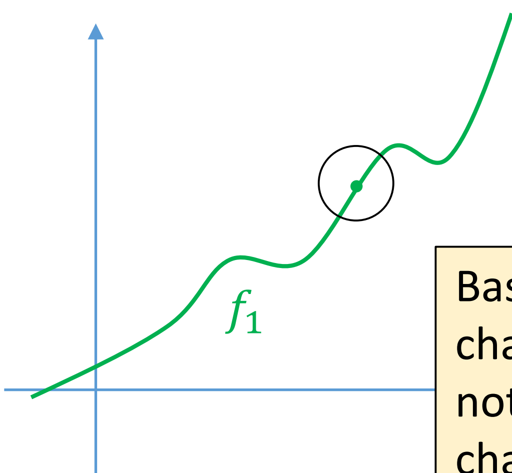
The **interior** of a set S, denoted S^o, is the union of all open sets in S

The **boundary** of a set S, denoted ∂S, is $Cl(S) - S^o$.

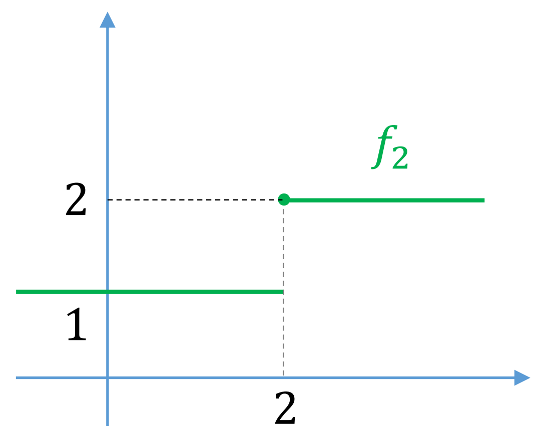
Continuous Functions

A function $f: \mathbb{R} \to \mathbb{R}$ is continuous around $x_0 \in R$ if $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $\forall x \in B(x_0, \delta)$, $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

⇨Readily generalize to $f: \mathbb{R}^n \to \mathbb{R}$



Basically, small change in x should not result in large change in f



$$f_2 = \begin{cases} 1, & x < 2 \\ 2, & x \geq 2 \end{cases}$$

f_2 is not continuous at $x = 2$: let $\varepsilon = 0.1$, $\forall \delta > 0$, $f_2\left(2 - \frac{\delta}{2}\right) = 1 \notin (2 - 0.1, 2 + 0.1)$

Why Open/Closed Sets and Continuous Functions?

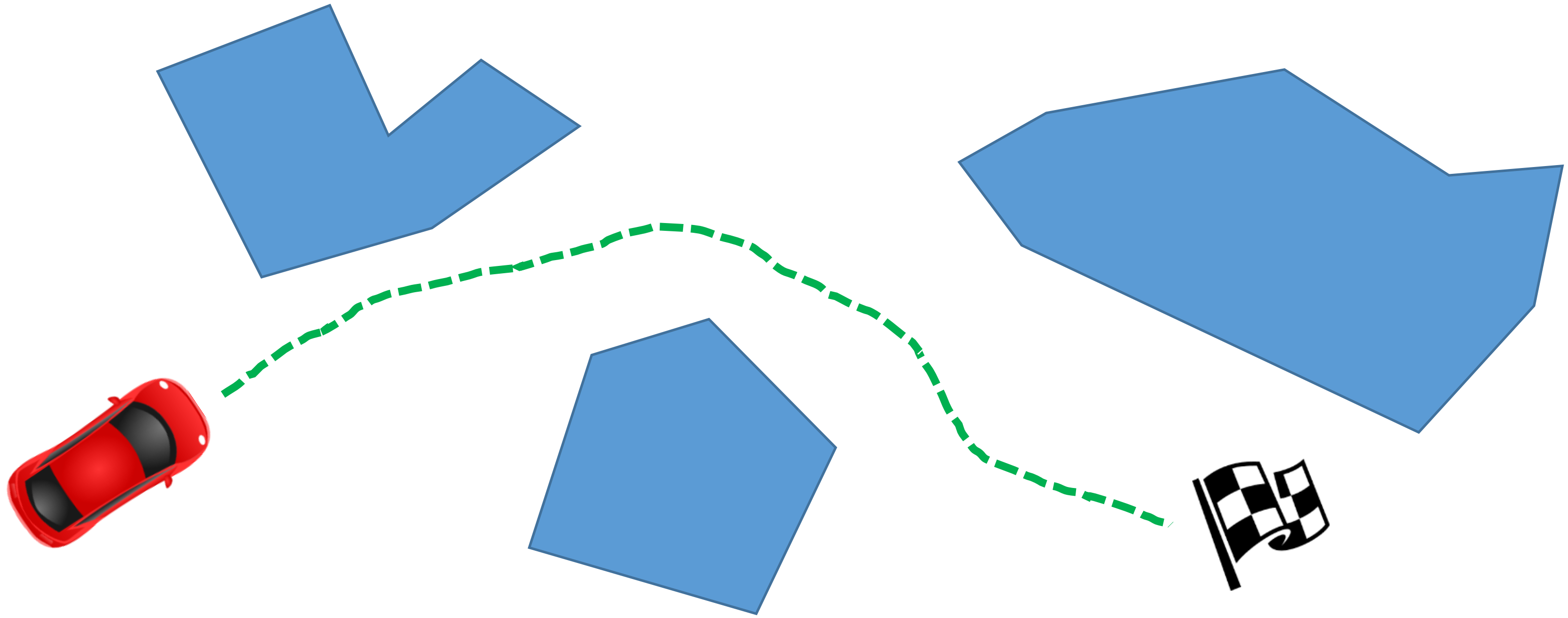
Open and closed sets: for precisely describing and solving a problem

- ⇨Obstacles are generally modeled as closed sets (not always)
- ⇨The free space (the space with obstacles removed) is then open
- ⇨If our robot stays in the free space, then no collisions

Robot trajectories are mostly continuous functions

- ⇨E.g., in 2D, $f: [0, t] \to \mathbb{R}^2$

If not, your robot may be teleporting…



Group Theory Concepts

A **set** G together with a **binary operation** $\cdot$ is a **group** if the following group axioms are satisfied

- ⇨**Closed**: $\forall a, b \in G, a \cdot b \in G$
- ⇨**Associative**: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ⇨**Identity**: $\exists e \in G, \forall a \in G, a \cdot e = e \cdot a = a$
- ⇨**Inverse**: $\forall a \in G, \exists b \in G$ s.t. $a \cdot b = b \cdot a = e$

Examples?

- ⇨The set of integers under addition
- ⇨The set of positive rational numbers under multiplication

Why Groups?

A mathematical field full of sad (but curious) stories!

- ⇨Niels Henrik Abel (Norwegian, 1802-1829)
  - ⇨ Invented group theory!
  - ⇨ Proved no explicit algebraic solutions for quintic polynomials
  - ⇨ And many other fundamental contributions…
  - ⇨ Very unlucky!
    - ⇨ Sent group theory paper to Gauss, Gauss tossed it into garbage…
    - ⇨ Sent another seminal paper to Cauchy, Cauchy "misplaced" it…
    - ⇨ Died at the age of 26…
    - ⇨ Then he got a letter appointing him Professor at the University of Berlin
  - ⇨ Abel Prize is basically the Nobel Prize in math
- ⇨Évariste Galois (French, 1811-1832)
  - ⇨ Also invented group theory (independently, no internet then)
  - ⇨ Galois theory (more general than Abel's work)
  - ⇨ Also many, many other important work…
  - ⇨ But this guy was very passionate
  - ⇨ Political activist, went to prison
  - ⇨ Then chose to dual with an army officer and died…
  - ⇨ 20 years old…

Niels Henrik Abel

Évariste Galois

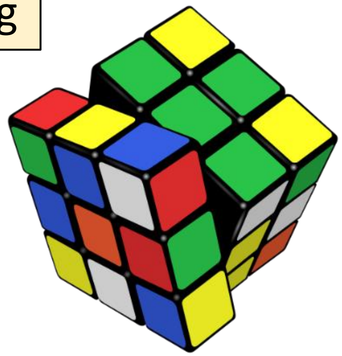
Images from wikipedia

Why Groups? Seriously…

Many types of discrete and continuous spaces are also groups!

⇨ Ex: The Rubik's cube

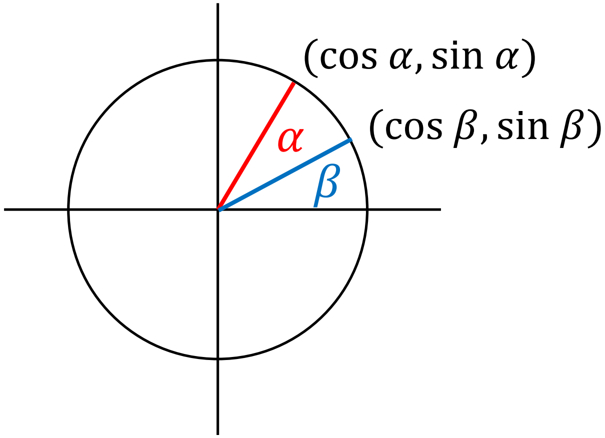
“God's number”: http://cube20.org

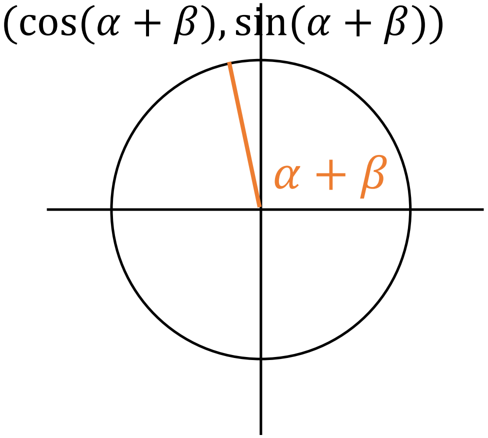


- ⇨ It's a planning problem! The “configurations” form a graph

⇨ Ex: $\mathbb{R}$ under addition

⇨ Ex: The unit circle under rotation





⇨ Can also do this using matrix multiplication

Matrices are also groups

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

⇨ More on this when we do coordinate transformations