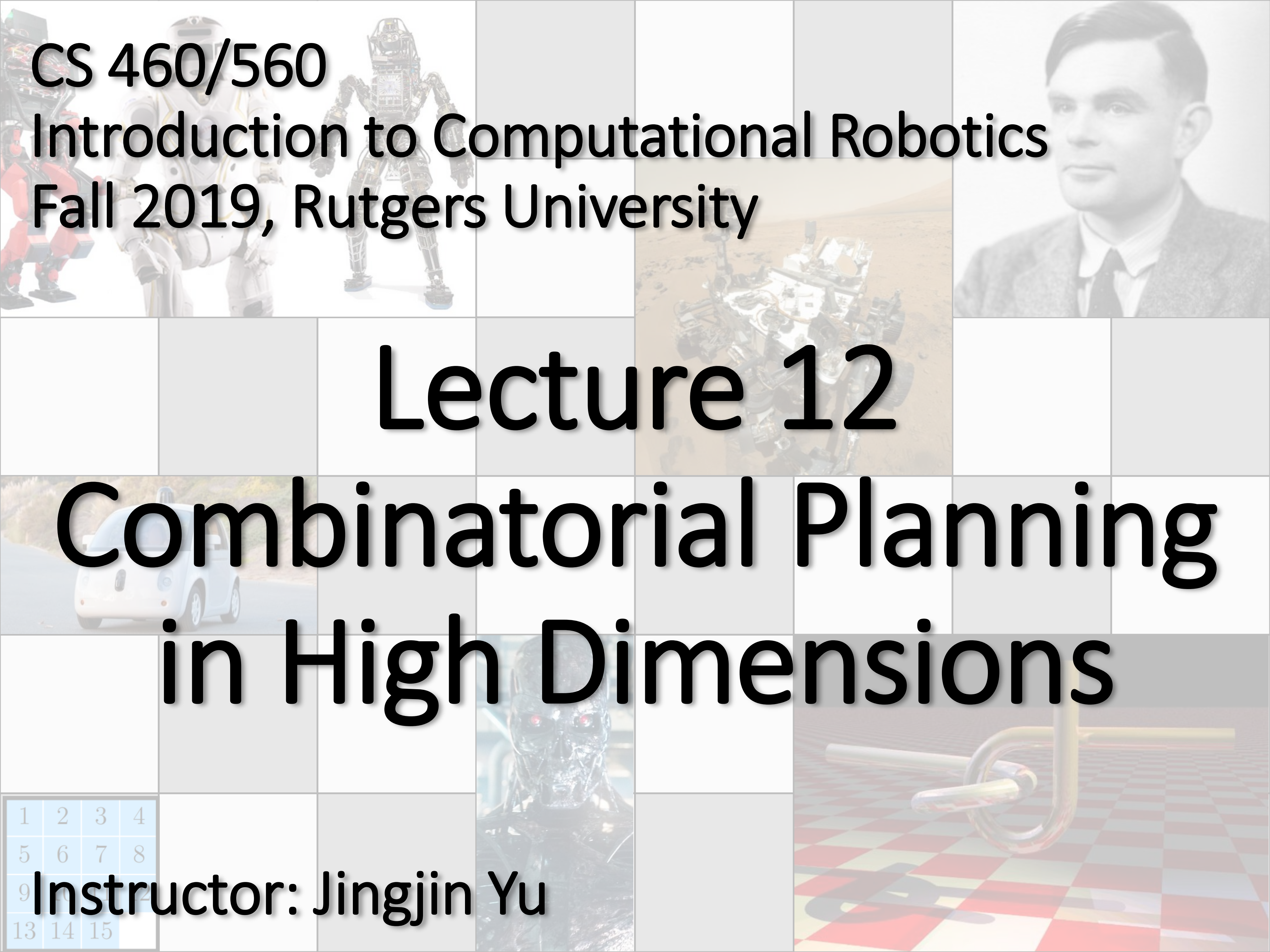


CS 460/560
Introduction to Computational Robotics
Fall 2019, Rutgers University

Lecture 12
Combinatorial Planning in High Dimensions

Instructor: Jingjin Yu

Outline

Review of combinatorial motion planning in the plane

Completeness and the halting problem

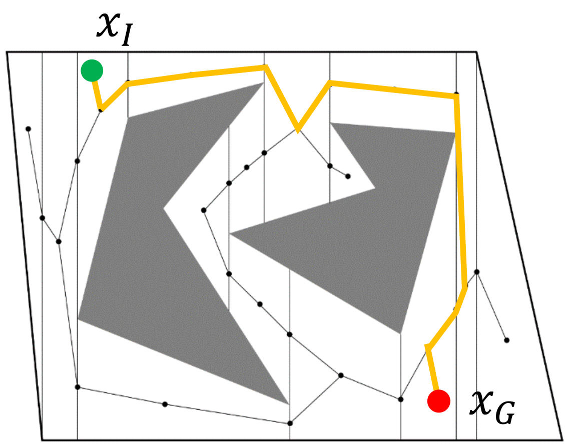
Cell decomposition in high dimensions

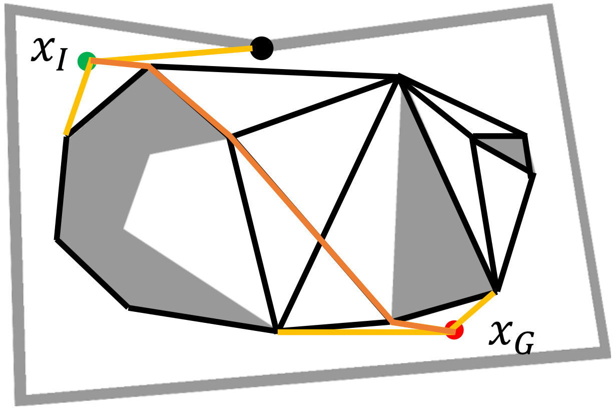
Complexity of motion planning

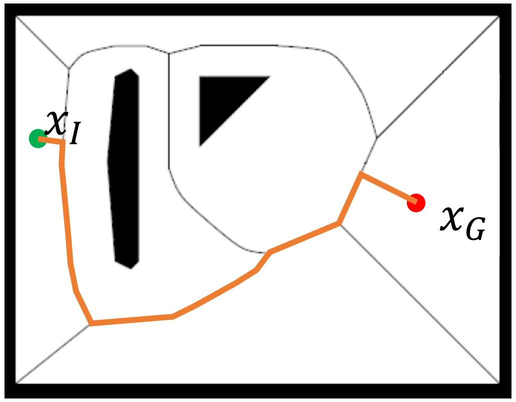
Combinatorial Motion Planning in the Plane

Last time, we covered several **combinatorial motion planning** algorithms in the plane

- ⇨Vertical cell decomposition
- ⇨Shortest-path roadmaps
- ⇨Maximum clearance roadmaps







What do these have in common?

- ⇨Each provides a (**combinatorial**) partitioning of the environment
- ⇨Which makes these algorithms **complete**

Completeness of Algorithms

An algorithm (in the sense of a **Turing machine**) is **complete** if it always returns in **finite time** a positive or negative answer

Behavior of a complete algorithm

- ⇨If the problem instance is solvable, it returns the solution
- ⇨If the problem instance is not solvable, it must return “no solution”

Are all algorithms complete?

Note that this is **different** from the question of <u>whether all problems are solvable with algorithms</u>

- ⇨How did we prove that not all problems can be solved with an algorithm?

	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$	$\boldsymbol{M_3}$	$\boldsymbol{M_4}$	$\boldsymbol{M_5}$	$\boldsymbol{M_6}$	…
$\boldsymbol{P_1}$	$\boxed{Yes}$	No	Yes	Yes	No	No	
$\boldsymbol{P_2}$	No	$\boxed{No}$	Yes	No	Yes	No	
$\boldsymbol{P_3}$	No	Yes	$\boxed{Yes}$	Yes	No	Yes	
$\boldsymbol{P_4}$	Yes	No	No	$\boxed{Yes}$	No	Yes	
…							
$\boldsymbol{P_{unsovlable}}$	No	Yes	No	No	…	…	…

The Halting Problem

The **halting problem**: is it possible to determine whether a program will stop?

⇨This problem is undecidable (using another algorithm)!

⇨That is, a computer (Turing machine) cannot really tell

Sketch of proof (via contradiction)

⇨Suppose an algorithm H_{TM} can decide whether an algorithm X will halt

⇨Construct a new algorithm A_{TM}

```
$A_{TM}$: $on\ input\ algorithm\ X$
  $\boldsymbol{if}\ H_{TM}(X) == true\ \boldsymbol{then}\ loop\ forever;$
  $\boldsymbol{else}\ return\ true;$
```

What happens if we run A_{TM} on input A_{TM}?

⇨If A_{TM} halts, then $H_{TM}(A_{TM}) == true$, A_{TM} should loop forever

⇨If A_{TM} does not halt, then $H_{TM}(A_{TM}) == false$, A_{TM} should halt

⇨Both are contradictions → H_{TM} cannot exist!

Implications of the Halting Problem

So, are all algorithms complete?

- ⇨No!
- ⇨Proof sketch
  - ⇨ There exist algorithms which we **cannot tell whether they will stop**
  - ⇨ Such algorithms **may run forever** and there is nothing we can do
  - ⇨ Such algorithms/programs are not complete
- ⇨In practice, this can be bad
  - ⇨ E.g., real time systems
  - ⇨ Solution: do not use full Turing machine

Combinatorial algorithms **are** complete

- ⇨This is because every single point in C_{free} is covered
- ⇨This is a big deal theoretically – a piece of mind
- ⇨Motivates the development of combinatorial methods for higher dimensions

The Piano Mover’s Problem

A classical “grand challenge” in robotics was the “Piano Mover’s Problem”

- ⇨Using autonomous robots, how to move a piano around without hitting other objects?

It was not immediately clear whether such problems are solvable

- ⇨Addressed by Schwartz and Sharir in a series of classical papers (‘83)
- ⇨The problem is shown to be PSPACE-hard (Reif ‘79)
- ⇨But solvable with Collins’ cylindrical algebraic decomposition (‘75)
- ⇨Canny provided a “faster” algorithm a few years later (’87)
- ⇨We will look at these algorithms briefly

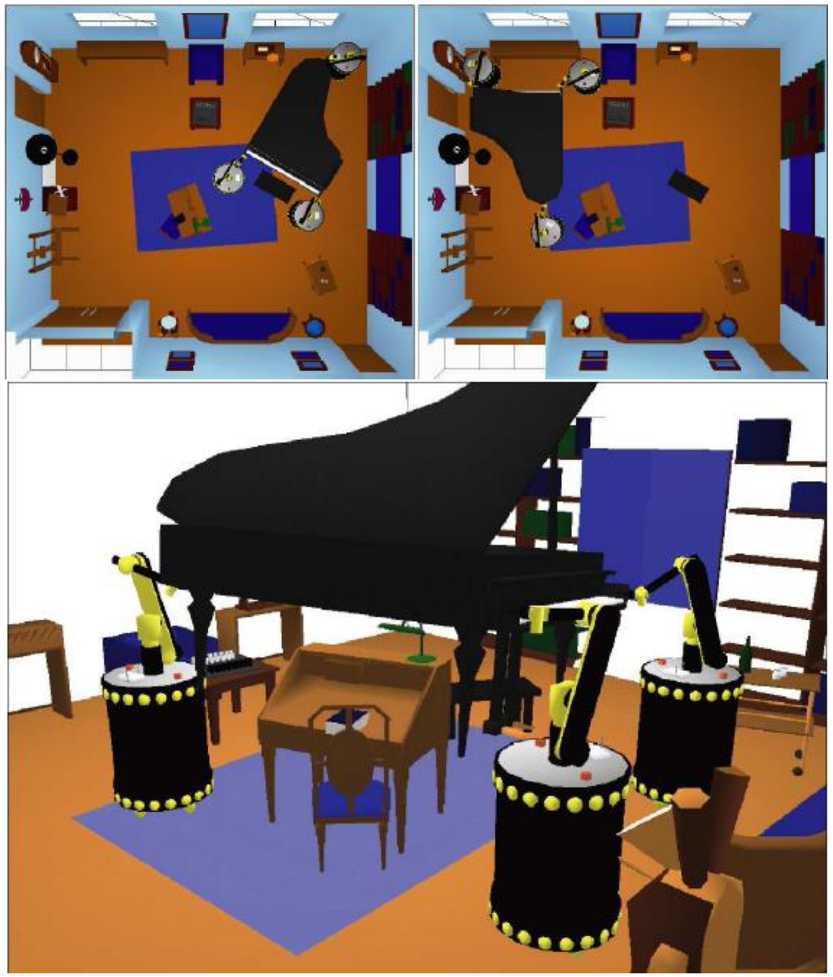
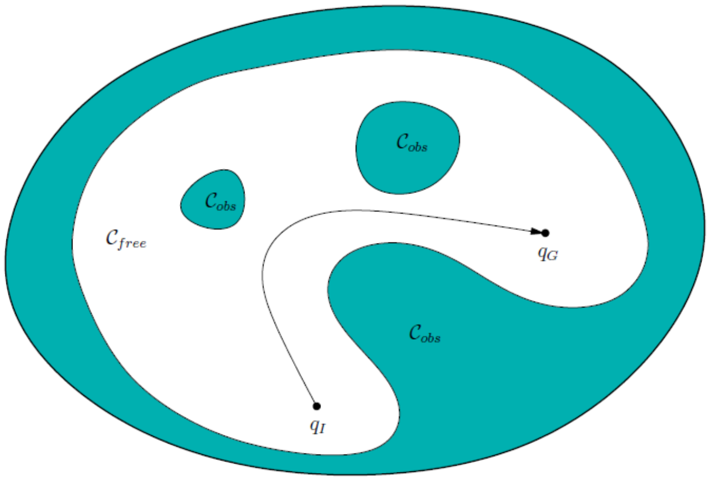


Image sources: http://planning.cs.uiuc.edu/ch1.pdf

A Formal Problem Definition



Formulation 4.1 (The Piano Mover's Problem)

1. A *world* $\mathcal{W}$ in which either $\mathcal{W} = \mathbb{R}^2$ or $\mathcal{W} = \mathbb{R}^3$.

2. A semi-algebraic *obstacle region* $\mathcal{O} \subset \mathcal{W}$ in the world.

3. A semi-algebraic *robot* is defined in $\mathcal{W}$. It may be a rigid robot $\mathcal{A}$ or a collection of m links, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$.

4. The *configuration space* $\mathcal{C}$ determined by specifying the set of all possible transformations that may be applied to the robot. From this, $\mathcal{C}_{obs}$ and $\mathcal{C}_{free}$ are derived.

5. A configuration, $q_I \in \mathcal{C}_{free}$ designated as the *initial configuration*.

6. A configuration $q_G \in \mathcal{C}_{free}$ designated as the *goal configuration*. The initial and goal configurations together are often called a *query pair* (or *query*) and designated as (q_I, q_G).

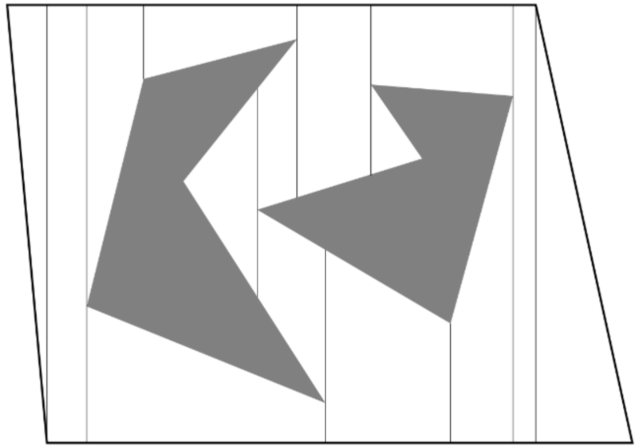
7. A complete algorithm must compute a (continuous) *path*, $\tau : [0, 1] \to \mathcal{C}_{free}$, such that $\tau(0) = q_I$ and $\tau(1) = q_G$, or correctly report that such a path does not exist.

Image sources: http://planning.cs.uiuc.edu/ch4.pdf

Vertical Cell Decomposition, Higher Dimensions

Extending VCD to higher dimensions (say n dimensions)

- ⇨Pick a dimension x_i
- ⇨Sweep the configuration space using an $(n-1)$-dimensional plane orthogonal to x_i (this is unique)
  - ⇨ Similar to the 2D case, stop only when “critical connectivity events” happen
  - ⇨ That is, with the appearance/disappearance/change of C_{obs}
  - ⇨ Every time a critical change happens, a new $(n-1)$-dimensional slice is produced
- ⇨Repeat the process to each of the $(n-1)$-dimensional slices
- ⇨The sweeping yields convex n-**cells**, $(n-1)$-**cells**, and so on…
  - ⇨ An n-**cell** is a n-dimensional polytope sandwiched between $(n-1)$-dimensional slices
  - ⇨ E.g., in 2D, 2-cells and 1-cells



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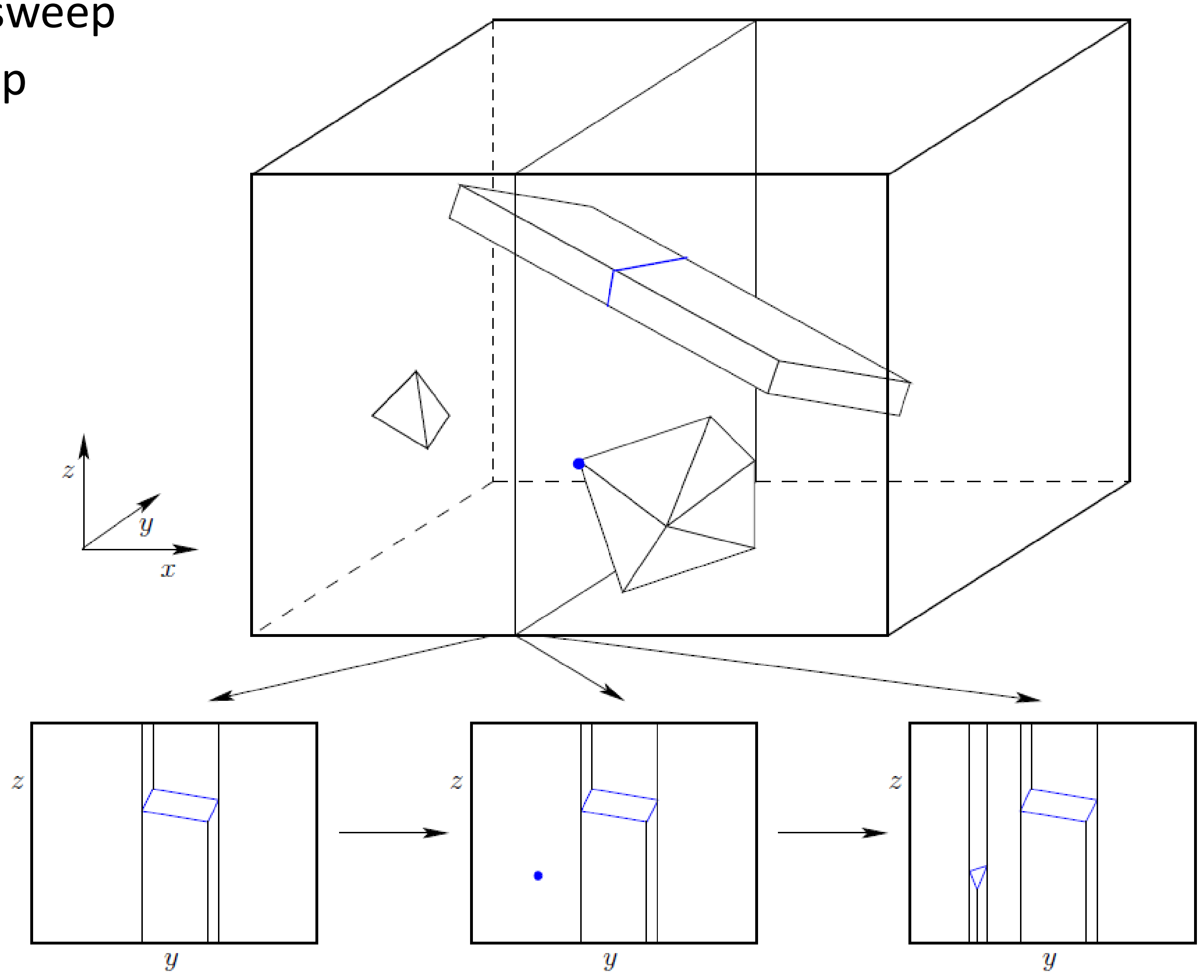
Vertical Cell Decomposition in 3D

For three dimensions

- ⇨First do plane sweep
- ⇨Then line sweep

Example



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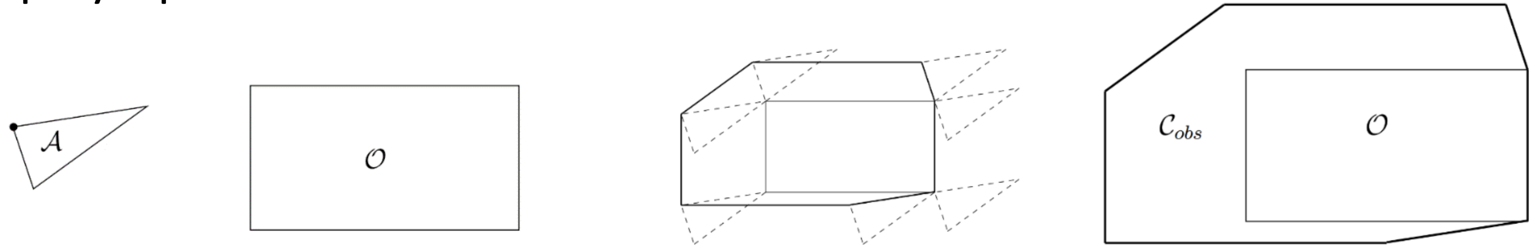
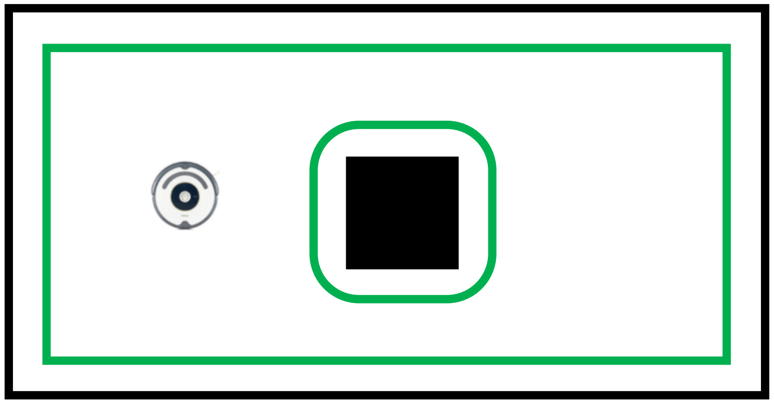
Issues with Vertical Cell Decomposition

VCD requires C_{obs} be polytopes (high dimensional "polygons")

⇨This is the case for translating rigid bodies among obstacles that are polytopes



⇨However, configuration spaces are more like this

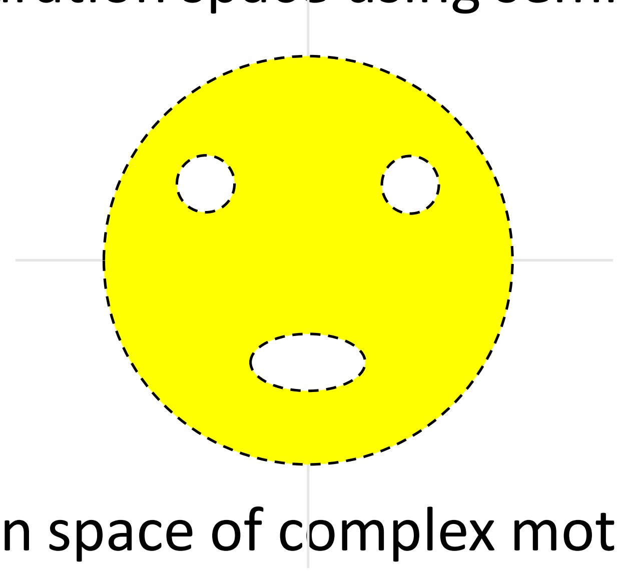


Need more powerful methods!

Computational Algebraic Geometry

Recall that we have defined the configuration space using semi-algebraic sets

- ⇨ $f_1 = x^2 + y^2 < 1$
- ⇨ $f_2 = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 > 0.01$
- ⇨ $f_3 = \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 > 0.01$
- ⇨ $f_4 = \frac{x^2}{2} + \left(y + \frac{1}{2}\right)^2 > 0.01$



Such sets can describe the configuration space of complex motion planning problems

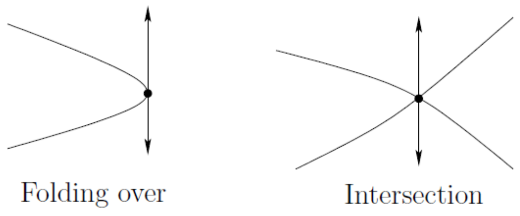
Usually this is done over **rational** polynomials $\mathbb{Q}[x_1, \dots, x_n]$

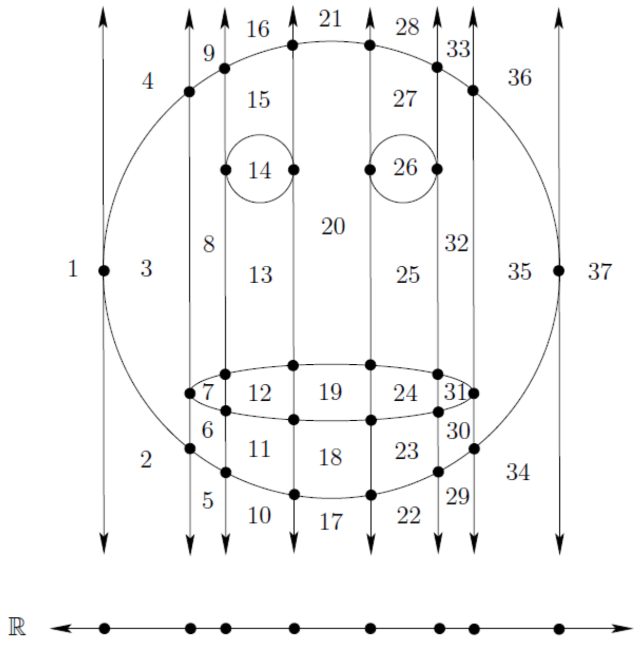
- ⇨ Computing roots for real polynomials is tricky (precision wise)
- ⇨ Note that this doesn't mean no irrational numbers will appear!
  - ⇨ E.g., $x^2 - 2 \geq 0$. But all these can be represented compactly
- ⇨ Polynomials with rational coefficient are nice: the projection of a semi-algebraic set from dimension n to dimension $n - 1$ is a semi-algebraic set!

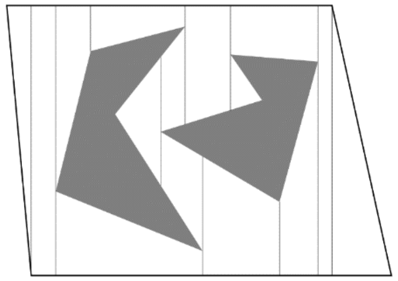
Cylindrical Algebraic Decomposition

Cylindrical algebraic decomposition is similar to VCD in spirit

- ⇨Start with a set of n dimensional polynomials $\mathcal{F}_n$
- ⇨Projects from n to $n-1$ dimensions
  - ⇨ Figure out where the polynomials fold or intersect
- ⇨The information is contained in the projection $\mathcal{F}_{n-1}$
- ⇨Once we get to $\mathcal{F}_1$, this is a one dimensional polynomial
- ⇨Partitions $\mathbb{R}$ into 0-cells and 1-cells
- ⇨Then reverse the process
  - ⇨ From $\mathbb{R}^1 \to \mathbb{R}^2$, 0-cells expand to 0-cells and 1-cells
  - ⇨ 1-cells expand to 2-cells
- ⇨Going from $\mathbb{R}^{n-1} \to \mathbb{R}^n$, the full decomposition
- ⇨2D example:







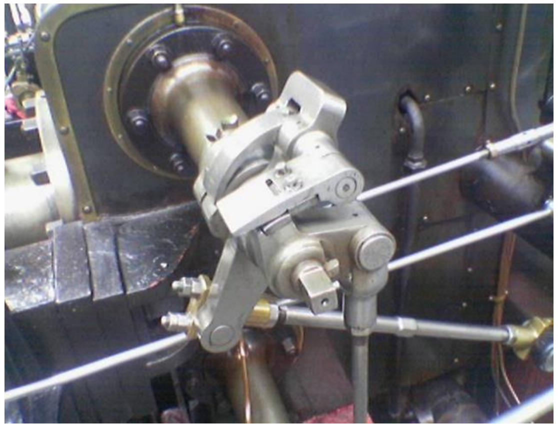
Canny's Algorithm

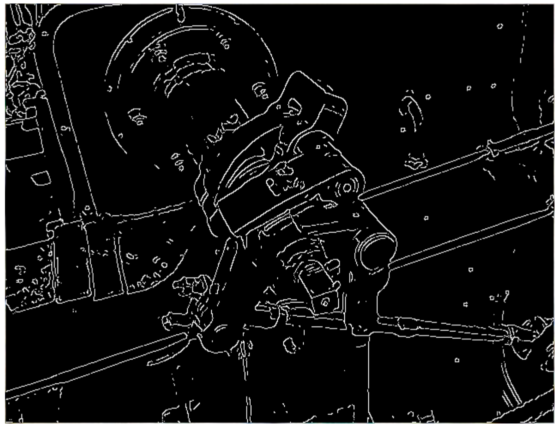
Cylindrical algebraic decomposition is slow!

- ⇨It is doubly exponential with respect to the number of dimensions
- ⇨That is, the running time contains a term a^{b^n} for $a, b > 1$
- ⇨John Canny invented a method to "speed" it up
- ⇨The basic idea is to project directly from $\mathbb{R}^n$ to $\mathbb{R}^2$ many times
- ⇨The technique drops the computation time to singly exponential (a^n)
- ⇨Still not very practical but theoretically much better

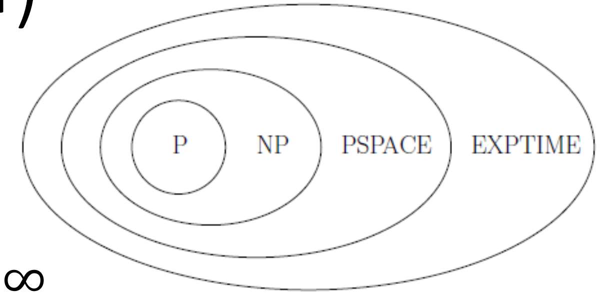
This was John Canny's Ph.D. Thesis that won an ACM best thesis award

But he may be more famous for his MS thesis: Canny's edge detector





Complexity of Motion Planning (I)



Complexity classes: P, NP, PSPACE, EXPTIME

- ⇨For a problem with an input size n
- ⇨P: the problem is solvable in time n^k for some $k < \infty$
- ⇨NP: the solution to the problem can be verified in time n^k for some $k < \infty$
- ⇨PSPACE: the problem can be solved by a Turing machine using only n^k space for some $k < \infty$
- ⇨EXPTIME: the problem can be solved in time 2^{n^k} for some $k < \infty$
- ⇨We know P ⊂ NP ⊂ PSPACE ⊂ EXPTIME
- ⇨And P ≠ EXPTIME

NP-hard and PSPACE-hard

- ⇨A (decision) problem is NP-hard if it is “harder” to solve than any other problem in NP
- ⇨If it is also in NP, then it is NP-complete
- ⇨Similarly, we can define PSPACE-hardness and PSPACE-completeness

Complexity of Motion Planning (II)

Complexity of Motion Planning

- ⇨The Piano Movers Problem was shown to be PSPACE-HARD (Reif, '79)
- ⇨Note that this does not mean the problem is in PSPACE!
  - ⇨ Many problem outside of PSPACE are also PSAPCE-hard
- ⇨This is a "lower bound" on the difficulty of the problem
- ⇨Canny's algorithm shows that the problem in in PSPACE
- ⇨This is an "upper bound" on the difficulty of the problem
- ⇨Together the problem is then **PSPACE-complete**

Implications

- ⇨PSPACE-hard is believed to be (much) harder than NP-hard
- ⇨Need alternative solution methods
- ⇨Perhaps we must sacrifice completeness
- ⇨This leads to sampling-based algorithms, which has a weaker notion of completeness – probabilistic completeness