1. [Vertex cover and ILP] Given an undirected graph G(V, E), a set of vertices S is said to be a vertex cover in G if for every e=(u, v)∈E, at least one of u, v is in S. A vertex cover of minimum size in G is said to be a minimum vertex cover of G.

(a) Formulate the problem of finding the size of the minimum vertex cover in a graph as an integer linear program.

(b) Using the first part of this question and the fact that finding the size of the minimum vetted cover in a graph is NP-hard, prove that solving an integer linear program is NP-hard.

2.

(a) Prove that P⊆NP∩coNP.

(b) Show that if P=NP, then coNP=NP.

3. Given an undirected graph G(V, E), a set of vertices S is said to be a vertex cover in G if for every e=(u, v)∈E, at least one of u, v is in S.

A set of edges M⊆E is said to be a matching in G if for every vertex v∈V, v is incident to at most one edge in M. A matching is said to be maximal if for every matching M' in G such that M⊆M', M and M' are the same i.e M=M'.

(a) Show that if G has a maximal matching of size equal to k, then it has a vertex cover of size at most 2k.

(b) Show that if G has a matching of size k, then the size of any vertex cover in G has to be at least k.

4. Given an undirected and unweighted graph G=(V, E) consider the following transformation from G to a weighted supergraph G' of G.

G' has the same vertices as G and a pair (x, y) gets weight 1 if (x, y) is an edge in G; otherwise the pair (x, y) gets weight $1+\alpha$ for some non negative number α.

(a) [Hamiltonian cycle to TSP Reduction]

Prove that if G has a Hamiltonian Cycle then G' has a tour of total weight less than $n+\alpha$.

Prove that if G has no Hamiltonian Cycle then G' has no tour of weight $< n+\alpha$.

(b) [Approximability of a special case of the TSP]

Prove that if $\alpha=1$ then we can find efficiently in G' a tour of total weight not more than twice the Optimum Tour in G. In this part, you do need to provide an efficient algorithm.

(c) [Inapproximability of general TSP]

Show that if $\alpha>1$ then we can use any constant ratio approximation algorithm to the TSP on G', to solve the Hamiltonian path Problem in G.