CS512 LECTURE NOTES - LECTURE 2

Example:

Show that $3n^2 - 2n + 7 \in \Theta(n^2)$.

1. Using the definition (we will show $3n^2 - 2n + 7 \in O(n^2)$, the proof that $3n^2 - 2n + 7 \in \Omega(n^2)$ is similar).

   We have to show that $\exists c > 0$ such that

   $$3n^2 - 2n + 7 \leq c \quad \Rightarrow 3 - \frac{2}{n} + \frac{7}{n^2} \leq c$$

   Notice that

   $$3 - \frac{2}{n} + \frac{7}{n^2} \leq 3 + \frac{7}{n^2} \leq 3 + 7 = 10$$

   Since the largest possible value of $7/n^2$ when n is a positive integer is obtained when $n = 1$.

   Therefore $c = 10$

2. Using limits:

   $$\lim_{n\to\infty} \frac{3n^2 - 2n + 7}{n^2} = \lim_{n\to\infty} (3 - \frac{2}{n} + \frac{7}{n^2}) = 3$$

   Since the limit is equal to a constant, then $3n^2 - 2n + 7 \in O(n^2)$.

1 Common functions

function name	order
constant	$\Theta(1)$
logarithmic	$\Theta(\lg n)$
polylogarithmic	$\Theta(\lg^k(n)) \;\; k \geq 0$
linear	$\Theta(n)$
quadratic	$\Theta(n^2)$
polynomial	$\Theta(n^k) \;\; k \geq 0$
exponential	$\Theta(b^n) \;\; b > 1$
factorial	$\Theta(n!)$

This functions are written in ascending order (remember that o, ω, O, Ω, and Θ induce a poset).

We can show that the function in each row is in little o of the function in the next row.

2 Order of growth of common functions

We can show that the function in each row is in little o of the function in the next row, so we have that:

1. $1 \in o(\lg n)$
2. $\lg n \in o(\lg^k n); \quad k > 1$
3. $\lg^k n \in o(n); \quad k \geq 0$
4. $n \in o(n^2)$
5. $n^2 \in o(n^k); \quad k > 2$
6. $n^k \in o(b^n); \quad k \geq 0, \quad b > 1$
7. $b^n \in o(n!); \quad b > 1$

We now use limits to show each one.

1. $1 \in o(\lg n)$

$$\lim_{n\to\infty} \frac{1}{\lg n} = 0$$

2. $\lg n \in o(\lg^k n)$; $k > 1$

$$\lim_{n\to\infty} \frac{\lg n}{\lg^k n} = \lim_{n\to\infty} \frac{1}{\lg^{k-1} n} = 0$$

because $k - 1 > 0$

3. $\lg^k n \in o(n)$; $k \geq 0$

$$\lim_{n\to\infty} \frac{\lg^k n}{n}$$

L'Hopital 1 time:

$$= \lim_{n\to\infty} k \lg^{k-1} n \left(\frac{1}{n \ln 2} \right) = \frac{k}{\ln 2} \lim_{n\to\infty} \frac{\lg^{k-1} n}{n}$$

L'Hopital 2 times:

$$= \frac{k(k-1)}{\ln^2 2} \lim_{n\to\infty} \frac{\lg^{k-2} n}{n}$$

L'Hopital 3 times:

$$= \frac{k(k-1)(k-2)}{\ln^3 2} \lim_{n\to\infty} \frac{\lg^{k-3} n}{n}$$

L'Hopital $\lceil k \rceil$ times:

$$= \frac{k(k-1)(k-2)\ldots(k - \lceil k \rceil + 1)}{\ln^{\lceil k \rceil} 2} \lim_{n\to\infty} \frac{\lg^{k-\lceil k \rceil} n}{n}$$

Let

$$c = \frac{k(k-1)(k-2)\ldots(k - \lceil k \rceil + 1)}{\ln^{\lceil k \rceil} 2}$$

CASE 1: k is integer, then $k = \lceil k \rceil$, $k - \lceil k \rceil = 0$, and $k - \lceil k \rceil + 1 = 1$. Therefore $c > 0$ and

$$\lim_{n\to\infty} \frac{\lg^k n}{n} = c \lim_{n\to\infty} \frac{1}{n} = 0$$

CASE 2: k is not an integer, then $k - \lceil k \rceil < 0$, $\lceil k \rceil - k > 0$, and $\lceil k \rceil - k + 1 > 0$. Clearly $c > 0$ and

$$\lim_{n\to\infty} \frac{\lg^k n}{n} = c \lim_{n\to\infty} \frac{1}{n \lg^{\lceil k \rceil - k} n} = 0$$

4. $n \in o(n^2)$

$$\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$$

5. $n^2 \in o(n^k);\quad k > 2$

$$\lim_{n\to\infty} \frac{n^2}{n^k} = \lim_{n\to\infty} \frac{1}{n^{k-2}} = 0$$

because $k - 2 > 0$

6. $n^k \in o(b^n);\quad k \geq 0,\quad b > 1$

$$\lim_{n\to\infty} \frac{n^k}{b^n}$$

L'Hopital 1 time:

$$= \frac{k}{\ln b} \lim_{n\to\infty} \frac{n^{k-1}}{b^n}$$

L'Hopital 2 times:

$$= \frac{k(k-1)}{\ln^2 b} \lim_{n\to\infty} \frac{n^{k-2}}{b^n}$$

L'Hopital 3 times:

$$= \frac{k(k-1)(k-2)}{\ln^3 b} \lim_{n\to\infty} \frac{n^{k-3}}{b^n}$$

L'Hopital $\lceil k \rceil$ times:

$$= \frac{k(k-1)(k-2)\ldots(k-\lceil k \rceil + 1)}{\ln^{\lceil k \rceil} b} \lim_{n\to\infty} \frac{n^{k-\lceil k \rceil}}{b^n}$$

Let

$$c = \frac{k(k-1)(k-2)\dots(k-\lceil k\rceil+1)}{\ln^{\lceil k\rceil} b}$$

CASE 1: k is integer, then $k = \lceil k\rceil$, $k - \lceil k\rceil = 0$, and $k - \lceil k\rceil + 1 = 1$. Therefore $c > 0$ and

$$\lim_{n\to\infty} \frac{n^k}{b^n} = c \lim_{n\to\infty} \frac{1}{b^n} = 0$$

since $b > 1$

CASE 2: k is not an integer, then $k - \lceil k\rceil < 0$, $\lceil k\rceil - k > 0$, and $\lceil k\rceil - k + 1 > 0$. Clearly $c > 0$ and

$$\lim_{n\to\infty} \frac{n^k}{b^n} = c \lim_{n\to\infty} \frac{1}{b^n n^{\lceil k\rceil - k}} = 0$$

7. $b^n \in o(n!)$; $b > 1$
   It is possible to show in general that this case is true, but it requires the use of the Γ function to approximate the factorial, which leads to the Stirling approximation (see your textbook for the formula). We will show a weaker case:

8. $2^n \in O(n!)$
   To show this we need to show that $\exists c > 0$ such that

   $$2^n \le cn!$$

   Let $c = 2$, and notice that:

   $$cn! = (2)(1)(2)(3)(4)\dots(n) = (2)(2)(3)(4)\dots(n)$$

   Since each factor is ≥ 2 and there are n factors,

   $$2^n \le cn!$$

2.1 Generalization of the factorial function

It is possible to generalize the factorial function to all real numbers by using the following integral:

$$\mathrm{fact}(n) = \int_0^\infty x^n e^{-x} \mathrm{dx}$$

We can easily show that this function is equal to $n!$ where $n \in \mathbb{N}$. This can be done by induction if we prove the following two conditions:

- $\mathrm{fact}(n) = n\ \mathrm{fact}(n-1)$
- $\mathrm{fact}(0) = 1$

To show $\mathrm{fact}(n) = n\ \mathrm{fact}(n-1)$ we can try to solve the integral by parts: $u = x^n$, $du = nx^{n-1}\mathrm{dx}$, $dv = e^{-x}$, $v = -e^{-x}$. So we have:

$$\begin{aligned} \mathrm{fact}(n) &= -x^n e^{-x}|_0^\infty + \int_0^\infty e^{-x} n x^{n-1} \mathrm{dx} \\ &= n \int_0^\infty x^{n-1} e^{-x} \mathrm{dx} \end{aligned}$$

Since $x^n e^{-x} = \frac{x^n}{e^x} = 0$ when $x \to \infty$ because e^x grows faster than x^n (you can prove it using limits too!). And since

$$\int_0^\infty x^{n-1} e^{-x} dx = \mathrm{fact}(n-1)$$

We get the desired result: $\mathrm{fact}(n) = n\mathrm{fact}(n-1)$
Now we only have to show that $\mathrm{fact}(0) = 1$. Using the definition of fact we have that:

$$\begin{aligned} \mathrm{fact}(0) &= \int_0^\infty x^0 e^x \mathrm{dx} \\ &= -e^x|_0^\infty \\ &= 0 + 1 = 1 \end{aligned}$$

The function that we discussed is called the Gamma Function: $\Gamma(z) = \mathrm{fact}(z-1)$. With a variable change $x = ny$ we can obtain an integral for which a closed-form approximation can be given by using Laplace's method, and we get:

$$n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

This formula is called the *Stirling Approximation* which can be used to show the growth rate of $n!$ with respect to n^n