> $d := 7$

$$d := 7 \tag{1}$$

> $n := 7$

$$n := 7 \tag{2}$$

> $n\_min := evalf\left(d^{\frac{1}{4}} \cdot 3 - 1\right)$

$$n\_min := 3.879729686 \tag{3}$$

>

>

> $beta\_plus := \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) + d - 3) + I \cdot \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) - d + 3)$

$$beta\_plus := \sqrt{2\sqrt{7}+4} + \mathrm{I}\sqrt{2\sqrt{7}-4} \tag{4}$$

> $beta\_minus := \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) + d - 3) - I \cdot \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) - d + 3)$

$$beta\_minus := \sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4} \tag{5}$$

> $eta\_plus := d + 3 + \mathrm{sqrt}\left(d^2 - 10 \cdot d + 9\right)$

$$eta\_plus := 10 + 2\,\mathrm{I}\sqrt{3} \tag{6}$$

> $eta\_minus := d + 3 - \mathrm{sqrt}\left(d^2 - 10 \cdot d + 9\right)$

$$eta\_minus := 10 - 2\,\mathrm{I}\sqrt{3} \tag{7}$$

Part 1: Let's investage the denominator.

> $E\_denom := \mathrm{Im}\left(eta\_minus \cdot beta\_minus \cdot \mathrm{BesselJ}\left(\frac{d}{2} - 1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}\right) \cdot \mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}\right)\right)$

$$E\_denom := \tag{8}$$

$$-\frac{1}{\pi}\Bigg(8\,\Im\Bigg(\Bigg(\Bigg(\Bigg(\mathrm{I}\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\sqrt{2\sqrt{7}-4}\,\lambda^2$$

$$+ 3\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda$$

$$- 4\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\lambda^2$$

$$- 3\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda$$

$$+ 6\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\Bigg)\Bigg)$$

$$\Bigg(\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4} + \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\left(2\sqrt{7}-4\right)^{3/2}\lambda^3$$

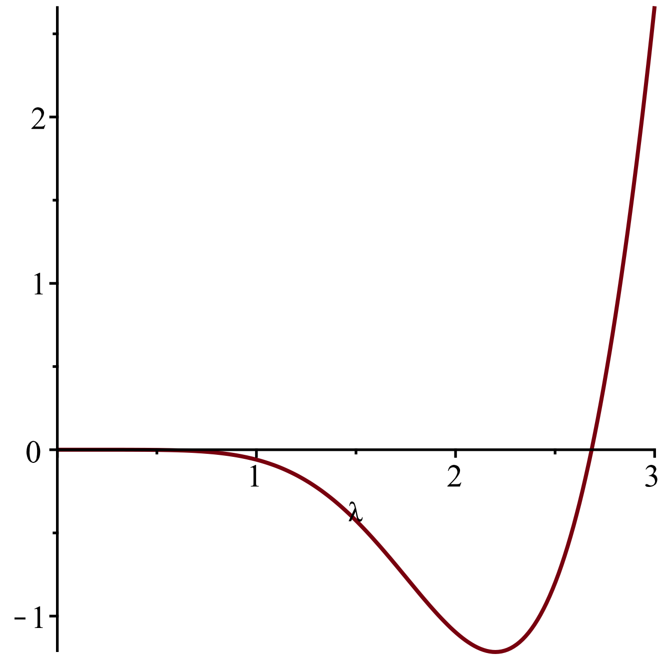
$$
\begin{aligned}
&- 6\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda^3\sqrt{7}\\
&- \cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\left(2\sqrt{7}+4\right)^{3/2}\lambda^3\\
&+ 6\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda^3\sqrt{7}\\
&- 12\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda^3\\
&+ 24\,\mathrm{I}\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\sqrt{2\sqrt{7}-4}\,\lambda^2\\
&- 12\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda^3\\
&+ 60\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda\\
&+ 60\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda\\
&+ 96\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\lambda^2\\
&- 120\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\Bigg)\left(10-2\,\mathrm{I}\sqrt{3}\right)\Bigg) \Bigg/\\
&\Bigg(\sqrt{\left(\sqrt{2\sqrt{7}+4}-\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}\,\lambda^5\sqrt{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}\,\Big(\\
&\sqrt{2\sqrt{7}+4}-\mathrm{I}\sqrt{2\sqrt{7}-4}\Big)\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)^3\Bigg)\Bigg)\Bigg)
\end{aligned}
$$

> *true_deno* := *evalf*(*eval*(*E_denom*, lambda = *lambda0*))

$$true\_deno := 2.666108263 \tag{9}$$

>

> *plot*(*E_denom*, lambda = 0 ..3)



>

> $J\_appx := (v, x, n) \rightarrow sum\left(\frac{\left(\frac{x}{2}\right)^{v} \cdot (-1)^{k}}{k! \cdot \text{GAMMA}(k+v+1)} \cdot \left(\frac{x^2}{4}\right)^{k}, k = 0..n\right)$

$$J\_appx := (v, x, n) \mapsto \sum_{k=0}^{n} \frac{\left(\frac{x}{2}\right)^{v} \cdot (-1)^{k} \cdot \left(\frac{x^2}{4}\right)^{k}}{k! \cdot \Gamma(k+v+1)} \tag{10}$$

> $error\_bar := (x, n, v) \rightarrow \frac{\left|\frac{x}{2}\right|^{2n+v}}{(\text{factorial}(n))^2 \cdot n^{v} \cdot \left(1 - \frac{|x|^2}{(2n+2)^2}\right)} \cdot charfcn[0, 1, 2, 3, 4](v) + \frac{2}{\text{sqrt}(\text{Pi})} \cdot \frac{\left|\frac{x}{2}\right|^{2n+v}}{(\text{factorial}(n))^2 \cdot n^{v-0.5} \cdot \left(1 - \frac{|x|^2}{(2n+2)^2}\right)} charfcn[0.5, 1.5, 2.5, 3.5](v)$

$$error\_bar := (x, n, v) \mapsto \frac{\left|\frac{x}{2}\right|^{2 \cdot n+v} \cdot charfcn_{0, 1, 2, 3, 4}(v)}{n!^2 \cdot n^{v} \cdot \left(1 - \frac{|x|^2}{(2 \cdot n+2)^2}\right)} + \frac{2 \cdot \left|\frac{x}{2}\right|^{2 \cdot n+v} \cdot charfcn_{0.5, 1.5, 2.5, 3.5}(v)}{\sqrt{\pi} \cdot n!^2 \cdot n^{v-0.5} \cdot \left(1 - \frac{|x|^2}{(2 \cdot n+2)^2}\right)} \tag{11}$$

> $deno\_appx\_func :=$ **proc**$(d, n, lambda\_star)$ **local** $J1, beta\_minus, beta\_plus, eta\_minus,$

$eta\_plus, J1\_appx, J2, J2\_appx, err1, err2, err, deno\_appx, acum\_err, E\_denom, true\_deno;$
$beta\_plus := \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) + d - 3) + I \cdot \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) - d + 3); beta\_minus := \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) + d - 3) - I \cdot \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) - d + 3); eta\_plus := d + 3 + \mathrm{sqrt}(d^2 - 10 \cdot d + 9);$
$eta\_minus := d + 3 - \mathrm{sqrt}(d^2 - 10 \cdot d + 9); J1 := evalf\left(eval\left(\mathrm{BesselJ}\left(\frac{d}{2} - 1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}\right), \mathrm{lambda} = lambda\_star\right)\right); J1\_appx := evalf\left(eval\left(J\_appx\left(\frac{d}{2} - 1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}, n\right), \mathrm{lambda} = lambda\_star\right)\right); J2 := evalf\left(eval\left(\mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}\right), \mathrm{lambda} = lambda\_star\right)\right); J2\_appx := evalf\left(eval\left(J\_appx\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}, n\right), \mathrm{lambda} = lambda\_star\right)\right); err1 := evalf\left(eval\left(error\_bar\left(\frac{|beta\_plus| \cdot \mathrm{lambda}}{2}, n, \frac{d}{2} - 1\right), \mathrm{lambda} = 3\right)\right); err2 := evalf\left(eval\left(error\_bar\left(\frac{|beta\_plus| \cdot \mathrm{lambda}}{2}, n, \frac{d}{2}\right), \mathrm{lambda} = 3\right)\right); err := \max(err1, err2);$
printf("|J1-J1_appx|= %f; err %f", $|J1 - J1\_appx|, err$);
printf("|J2-J2_appx|= %f; err %f;", $|J2 - J2\_appx|, err$); $deno\_appx :=$
$evalf(eval(\mathrm{Im}(eta\_minus \cdot beta\_minus \cdot J1\_appx \cdot J2\_appx), \mathrm{lambda} = lambda\_star));$
$acum\_err := evalf\left(|eta\_minus \cdot beta\_minus| \cdot err^2 + |J2\_appx| \cdot |eta\_minus \cdot beta\_minus| \cdot err + |eta\_minus \cdot beta\_minus| \cdot |J1\_appx| \cdot err\right); E\_denom := \mathrm{Im}\left(eta\_minus \cdot beta\_minus \cdot \mathrm{BesselJ}\left(\frac{d}{2} - 1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}\right) \cdot \mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}\right)\right);$
$true\_deno := evalf\left(eval\left(E\_denom, \mathrm{lambda} = lambda\_star\right)\right);$
printf("the bound for denominator for lambda = %f: [%f, %f] (true value %f) \n", $lambda\_star, deno\_appx - acum\_err, deno\_appx + acum\_err, true\_deno$); **end proc**

$deno\_appx\_func :=$ **proc**$(d, n, lambda\_star)$ **(12)**

```
local J1, beta_minus, beta_plus, eta_minus, eta_plus, J1_appx, J2, J2_appx, err1, err2, err,
deno_appx, acum_err, E_denom, true_deno;
beta_plus := sqrt(2 * sqrt(d) + d - 3) + I * sqrt(2 * sqrt(d) - d + 3);
beta_minus := sqrt(2 * sqrt(d) + d - 3) - I * sqrt(2 * sqrt(d) - d + 3);
eta_plus := d + 3 + sqrt(d^2 - 10 * d + 9);
eta_minus := d + 3 - sqrt(d^2 - 10 * d + 9);
J1 := evalf(eval(BesselJ(1/2 * d - 1, 1/2 * beta_minus * lambda), lambda
 = lambda_star));
J1_appx := evalf(eval(J_appx(1/2 * d - 1, 1/2 * beta_minus * lambda, n), lambda
 = lambda_star));
J2 := evalf(eval(BesselJ(1/2 * d, 1/2 * beta_plus * lambda), lambda = lambda_star));
J2_appx := evalf(eval(J_appx(1/2 * d, 1/2 * beta_plus * lambda, n), lambda
 = lambda_star));
err1 := evalf(eval(error_bar(1/2 * abs(beta_plus) * lambda, n, 1/2 * d - 1), lambda
 = 3));
err2 := evalf(eval(error_bar(1/2 * abs(beta_plus) * lambda, n, 1/2 * d), lambda = 3));
```

```
    err := max(err1, err2);
    printf("|J1-J1_appx|= %f; err %f", abs(J1 - J1_appx), err);
    printf("|J2-J2_appx|= %f; err %f;", abs(J2 - J2_appx), err);
    deno_appx := evalf(eval(Im(eta_minus * beta_minus * J1_appx * J2_appx), lambda
     = lambda_star));
    acum_err := evalf(abs(eta_minus * beta_minus) * err^2 + abs(J2_appx) * abs(eta_minus
     * beta_minus) * err + abs(eta_minus * beta_minus) * abs(J1_appx) * err);
    E_denom := Im(eta_minus * beta_minus * BesselJ(1/2 * d - 1, 1/2 * beta_minus
     * lambda) * BesselJ(1/2 * d, 1/2 * beta_plus * lambda));
    true_deno := evalf(eval(E_denom, lambda = lambda_star));
    printf("the bound for denominator for lambda = %f: [%f, %f] (true value %f) \n",
    lambda_star, deno_appx - acum_err, deno_appx + acum_err, true_deno)
end proc
```

```
> deno_appx_func(d, n, 3)
|J1-J1_appx|= 0.000029; err 0.002462|J2-J2_appx|= 0.000006; err
0.002462;the bound for denominator for lambda = 3.000000:
[2.530460, 2.802594] (true value 2.666108)
```

```
> deno_appx_func(d, n, 2.5)
|J1-J1_appx|= 0.000001; err 0.002462|J2-J2_appx|= 0.000000; err
0.002462;the bound for denominator for lambda = 2.500000:
[-0.907522, -0.693811] (true value -0.800674)
```

\>

\>

\>

Part2: Let's study the numerator at lambda within the interval [2.5, 3]

\> $lambda0 := 3$

$$\lambda 0 := 3 \qquad (13)$$

\> $num\_c1 := evalf\left(eval\left(\mathrm{Im}\left(eta\_minus \cdot beta\_minus \cdot \mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}\right)\right), \mathrm{lambda} = lambda0\right)\right)$

$$num\_c1 := -12.08181828 \qquad (14)$$

\> $num\_c2 := evalf\left(eval\left(\mathrm{Im}\left(\mathrm{BesselJ}\left(\frac{d}{2} - 1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}\right)\right), \mathrm{lambda} = lambda0\right)\right)$

$$num\_c2 := 0.5071037176 \qquad (15)$$

\>

\> $num\_c1\_err := evalf(err \cdot |eta\_minus \cdot beta\_minus|)$

$$num\_c1\_err := 0.08477081569 \qquad (16)$$

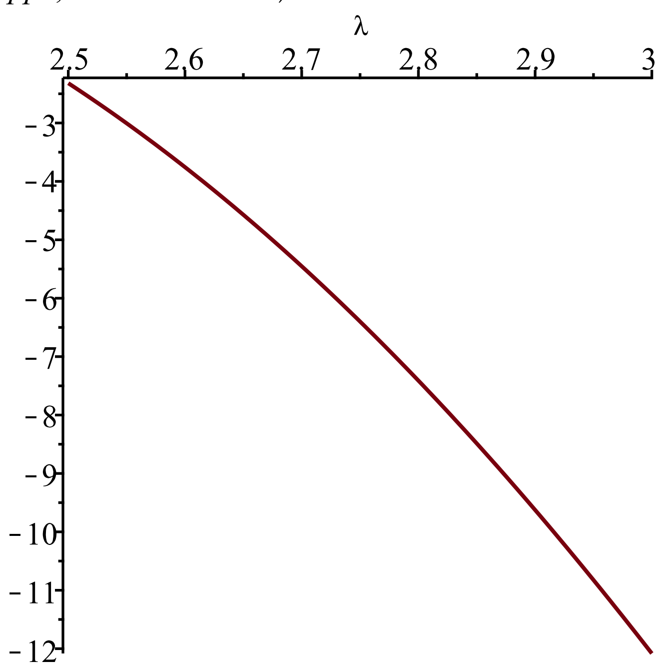
\>

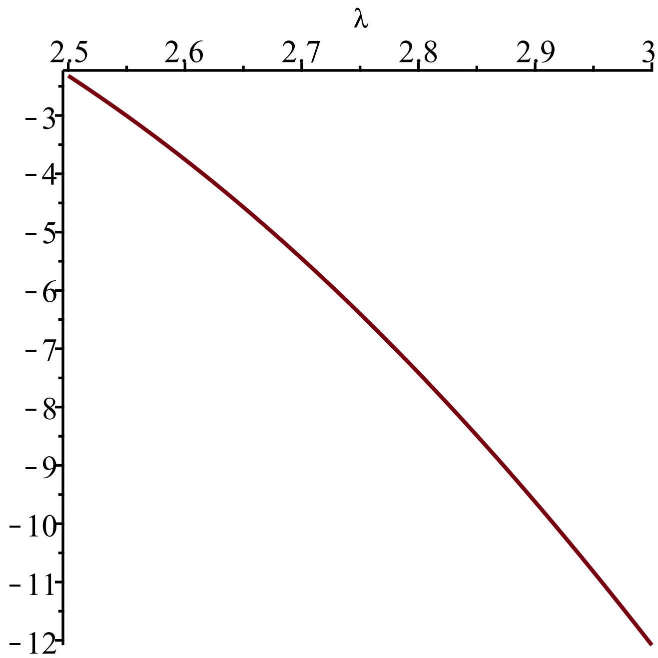
\> $num\_c1\_appx := evalf\left(\mathrm{Im}\left(eta\_minus \cdot beta\_minus \cdot J\_appx\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}, n\right)\right)\right)$ assuming lambda ≥ 0

$num\_c1\_appx := \Im\big((26.54521225 - 21.92369252\,\mathrm{I})\,\big(0.0006716542660\,\big((3.048196618$ **(17)**

$$+ 1.136442969\,I)\,\lambda)^{7/2} - (0.00007462825178$$
$$+ 0.00006462996187\,I)\,((3.048196618 + 1.136442969\,I)\,\lambda)^{7/2}\,\lambda^{2}$$
$$+ (8.480483165\;10^{-7} + 5.875451083\;10^{-6}\,I)\,((3.048196618$$
$$+ 1.136442969\,I)\,\lambda)^{7/2}\,\lambda^{4} + (1.087241430\;10^{-7}$$
$$- 1.694841658\;10^{-7}\,I)\,((3.048196618 + 1.136442969\,I)\,\lambda)^{7/2}\,\lambda^{6}$$
$$+ (-4.258362270\;10^{-9} + 1.255438266\;10^{-9}\,I)\,((3.048196618$$
$$+ 1.136442969\,I)\,\lambda)^{7/2}\,\lambda^{8} + (6.288945529\;10^{-11}$$
$$+ 2.861660748\;10^{-11}\,I)\,((3.048196618 + 1.136442969\,I)\,\lambda)^{7/2}\,\lambda^{10}$$
$$- (3.342697039\;10^{-13} + 7.287760823\;10^{-13}\,I)\,((3.048196618$$
$$+ 1.136442969\,I)\,\lambda)^{7/2}\,\lambda^{12} + (-2.019516306\;10^{-15}$$
$$+ 6.926953313\;10^{-15}\,I)\,((3.048196618 + 1.136442969\,I)\,\lambda)^{7/2}\,\lambda^{14}))$$

> $plot\left(\mathrm{Im}\left(eta\_minus \cdot beta\_minus \cdot \mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}\right)\right), \mathrm{lambda} = 2.5\,..3\right);$
$plot(\ num\_c1\_appx, \mathrm{lambda} = 2.5\,..3\,)$





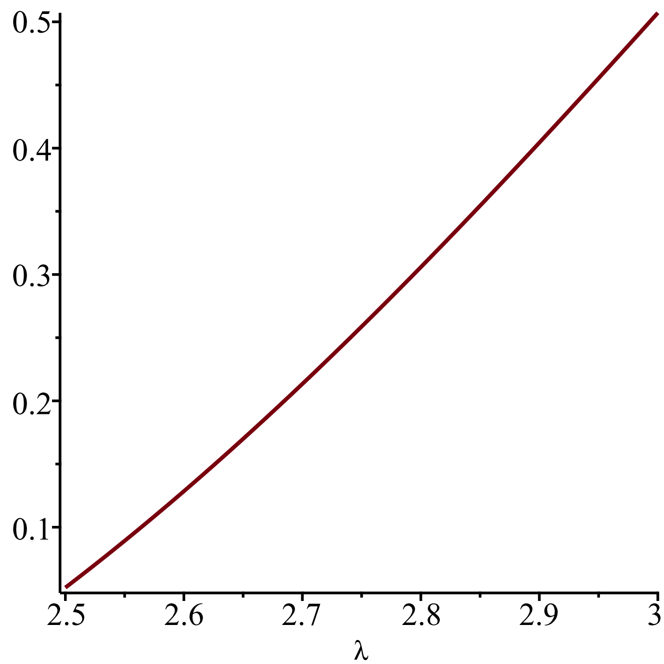
>

>

> $num\_c2\_appx := evalf\left(\mathrm{Im}\left(J\_appx\left(\frac{d}{2}-1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}, n\right)\right)\right)$ assuming lambda ≥ 0

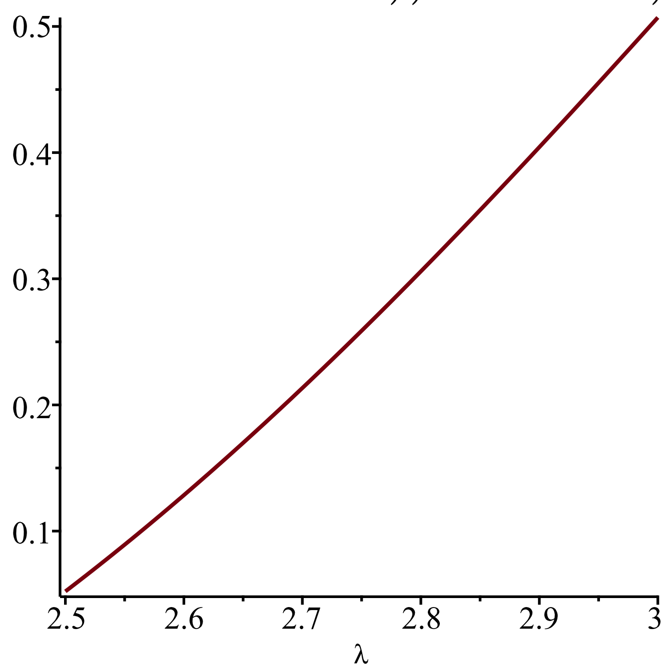
$$num\_c2\_appx := 1.806291924\ 10^{-20}\ \Im\Big(5.205780748\ 10^{17}\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2} + \big(-7.436829641\ 10^{16} + 6.440483391\ 10^{16}\ \mathrm{I}\big)\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2}\ \lambda^2 + \big(1.032893007\ 10^{15} - 7.156092662\ 10^{15}\ \mathrm{I}\big)\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2}\ \lambda^4 + \big(1.564989402\ 10^{14} + 2.439577042\ 10^{14}\ \mathrm{I}\big)\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2}\ \lambda^6 - \big(7.072548262\ 10^{12} + 2.085108585\ 10^{12}\ \mathrm{I}\big)\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2}\ \lambda^8 + \big(1.183774035\ 10^{11} - 5.386530500\ 10^{10}\ \mathrm{I}\big)\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2}\ \lambda^{10} + \big(-7.032223628\ 10^{8} + 1.533168075\ 10^{9}\ \mathrm{I}\big)\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2}\ \lambda^{12} - \big(4.695790515\ 10^{6} + 1.610659026\ 10^{7}\ \mathrm{I}\big)\ \big((3.048196618 - 1.136442969\ \mathrm{I})\ \lambda\big)^{5/2}\ \lambda^{14}\Big) \quad \mathbf{(18)}$$

> $plot(num\_c2\_appx, \mathrm{lambda} = 2.5\,..3)$



>

> $plot\left(\mathrm{Im}\left(\mathrm{BesselJ}\left(\frac{d}{2}-1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}\right)\right), \mathrm{lambda}=2.5..3\right)$



>

> $num\_c1\_appx := eval(num\_c1\_appx, \mathrm{lambda} = lambda0)$

$$num\_c1\_appx := -12.08180791 \tag{19}$$

> $num\_c2\_appx := eval(num\_c2\_appx, \mathrm{lambda} = lambda0)$

$$num\_c2\_appx := 0.5071118662 \tag{20}$$

> $err1 := evalf\left(eval\left(error\_bar\left(1/2 * \mathrm{abs}(beta\_plus) * \lambda, n, 1/2 * d - 1\right), \lambda = 3\right)\right);$
$err2 := evalf\left(eval\left(error\_bar\left(1/2 * \mathrm{abs}(beta\_plus) * \lambda, n, 1/2 * d\right), \lambda = 3\right)\right);$
$err := \max(err1, err2);$

$$err1 := 0.002462253962$$

$$err2 := 0.0008582238388$$

$$err := 0.002462253962 \quad \mathbf{(21)}$$

> $printf(\text{"the numerator = [\%f, \%f] (true value \%f)"}, num\_c1\_appx + num\_c2\_appx - (1 + |beta\_minus \cdot eta\_minus|) \cdot err, num\_c1\_appx + num\_c2\_appx + (1 + |beta\_minus \cdot eta\_minus|) \cdot err, num\_c1 + num\_c2)$

```
the numerator  = [-11.661929, -11.487463] (true value
-11.574715)
```

> $|num\_c1| - |num\_c2|$

$$11.57471456 \quad \mathbf{(22)}$$