> $d := 6$

$$d := 6 \tag{1}$$

> $n := 6$

$$n := 6 \tag{2}$$

> *alias*(*beta_plus* = sqrt(2 * sqrt(*d*^2 - 10 * *d* + 9) + 2 * *d* - 6), *beta_minus* = sqrt(-2 * sqrt(*d*^2 - 10 * *d* + 9) + 2 * *d* - 6))

$$beta\_plus,\ beta\_minus \tag{3}$$

> $alias\left(eta\_plus = d + 3 + \mathrm{sqrt}\left(d^2 - 10 \cdot d + 9\right), eta\_minus = d + 3 - \mathrm{sqrt}\left(d^2 - 10 \cdot d + 9\right)\right)$

$$beta\_plus,\ beta\_minus,\ eta\_plus,\ eta\_minus \tag{4}$$

> $J\_appx := (v, x, n) \rightarrow sum\left(\frac{\left(\frac{x}{2}\right)^v \cdot (-1)^k}{k! \cdot \mathrm{GAMMA}(k + v + 1)} \cdot \left(\frac{x^2}{4}\right)^k, k = 0 .. n\right)$

$$J\_appx := (v, x, n) \mapsto \sum_{k=0}^{n} \frac{\left(\frac{x}{2}\right)^v \cdot (-1)^k \cdot \left(\frac{x^2}{4}\right)^k}{k! \cdot \Gamma(k + v + 1)} \tag{5}$$

> $error\_bar := (x, n, v) \rightarrow \frac{\left|\frac{x}{2}\right|^{2\,n + v}}{\mathrm{factorial}(n) \cdot \mathrm{GAMMA}(n + v + 1) \cdot \left(1 - \frac{|x|^2}{(2\,n + 2)^2}\right)}$

$$error\_bar := (x, n, v) \mapsto \frac{\left|\frac{x}{2}\right|^{2 \cdot n + v}}{n! \cdot \Gamma(n + v + 1) \cdot \left(1 - \frac{|x|^2}{(2 \cdot n + 2)^2}\right)} \tag{6}$$

> $n1 := \mathrm{Im}\left(\frac{eta\_minus \cdot beta\_minus}{\mathrm{GAMMA}\left(\frac{d}{2}\right)} \cdot \left(\frac{\mathrm{lambda} \cdot beta\_minus}{4}\right)^{\frac{d}{2} - 1} \cdot \mathrm{BesselJ}\left(\frac{d}{2}, \frac{\mathrm{lambda} \cdot beta\_plus}{2}\right)\right)$

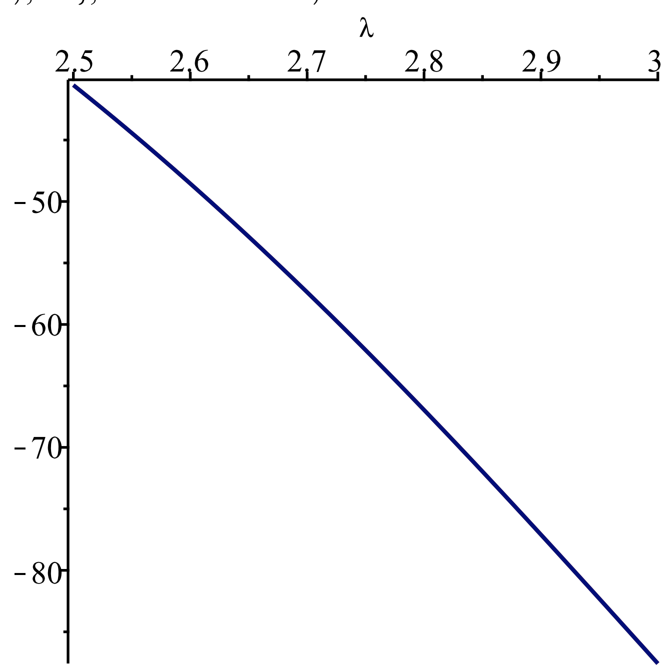
$$n1 := \frac{\Im\left(eta\_minus\, \left(-2\,\mathrm{I}\sqrt{15} + 6\right)^{3/2} \lambda^2\, \mathrm{BesselJ}\left(3, \frac{\lambda\, beta\_plus}{2}\right)\right)}{32} \tag{7}$$

> $n1\_appx := mtaylor\left(\frac{eta\_minus \cdot beta\_minus}{\mathrm{GAMMA}\left(\frac{d}{2}\right)} \cdot \left(\frac{\mathrm{lambda} \cdot beta\_minus}{4}\right)^{\frac{d}{2} - 1} \cdot J\_appx\left(\frac{d}{2},\right.\right.$

$$\frac{\text{lambda} \cdot beta\_plus}{2}, n\Big), \text{lambda} = 2.5, 2\, n + d\Bigg)$$

$$n1\_appx := 13.24973762 - 40.53352375\,\mathrm{I} + (-12.66609651 + 16.90055418\,\mathrm{I})\,(\lambda - 2.5)^4 + (0.5564596431 - 0.4398899575\,\mathrm{I})\,(\lambda - 2.5)^7 - (0.009970834531 + 0.1690475484\,\mathrm{I})\,(\lambda - 2.5)^8 + (1.329651653\ 10^{-8} - 3.623876131\ 10^{-8}\,\mathrm{I})\,(\lambda - 2.5)^{16} - (8.891742712 + 75.75692475\,\mathrm{I})\,(\lambda - 2.5) + (2.526543991 + 0.8121442328\,\mathrm{I})\,(\lambda - 2.5)^6 + (2.460497215 + 7.627093444\,\mathrm{I})\,(\lambda - 2.5)^5 + (0.00005180283738 - 0.00003525936112\,\mathrm{I})\,(\lambda - 2.5)^{13} + (3.128592125\ 10^{-10} - 8.526767366\ 10^{-10}\,\mathrm{I})\,(\lambda - 2.5)^{17} + (3.084791837\ 10^{-7} - 6.569201091\ 10^{-7}\,\mathrm{I})\,(\lambda - 2.5)^{15} + (0.0008405589112 + 0.0006159642216\,\mathrm{I})\,(\lambda - 2.5)^{11} + (0.0003223613334 - 0.00004966463347\,\mathrm{I})\,(\lambda - 2.5)^{12} + (-43.78612888 + 4.448440869\,\mathrm{I})\,(\lambda - 2.5)^3 - (49.21269794 + 44.09838406\,\mathrm{I})\,(\lambda - 2.5)^2 + (4.919711119\ 10^{-6} - 6.515123434\ 10^{-6}\,\mathrm{I})\,(\lambda - 2.5)^{14} - (0.02419661591 + 0.01591905192\,\mathrm{I})\,(\lambda - 2.5)^9 + (-0.002550059876 + 0.002690630066\,\mathrm{I})\,(\lambda - 2.5)^{10} \quad \textbf{(8)}$$

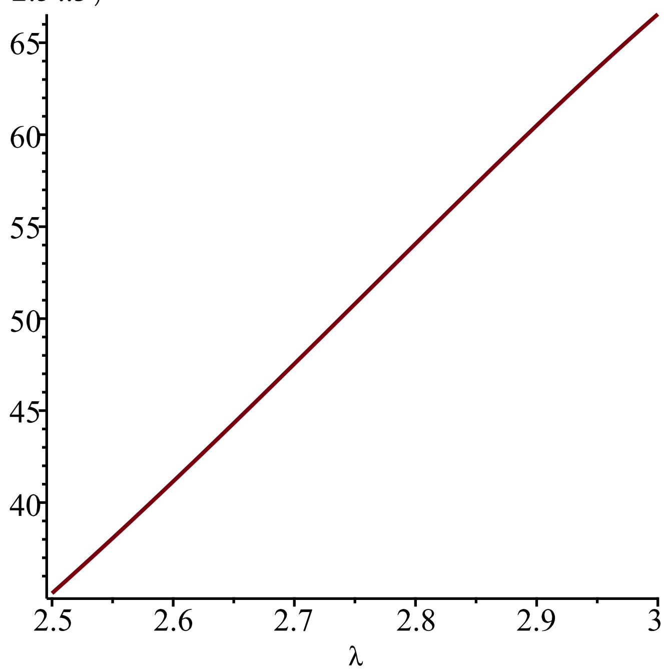
> $plot(\{\mathrm{Im}(n1\_appx), n1\}, \text{lambda} = 2.5..3)$



> $n2 := \mathrm{Im}\left(\frac{beta\_plus \cdot beta\_minus \cdot eta\_plus \cdot eta\_minus}{8 \cdot d \cdot \mathrm{GAMMA}\left(\frac{d}{2}\right)} \cdot \left(\frac{\mathrm{lambda} \cdot beta\_plus}{4}\right)^{\frac{d}{2}-1} \cdot \mathrm{BesselJ}\left(\frac{d}{2}-1, \frac{\mathrm{lambda} \cdot beta\_minus}{2}\right)\right)$

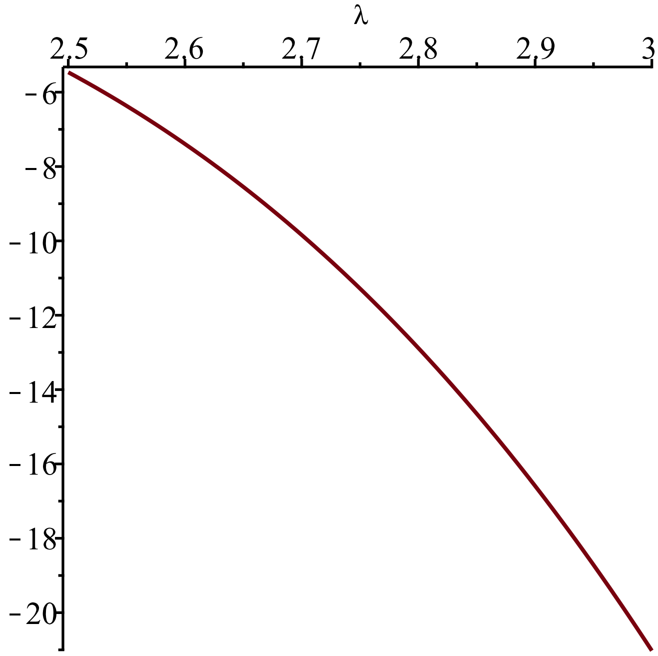
$$n2 := \frac{\Im\left(\left(2\,\mathrm{I}\sqrt{15}+6\right)^{3/2} beta\_minus\ eta\_plus\ eta\_minus\ \lambda^2\ \mathrm{BesselJ}\left(2, \frac{\lambda\, beta\_minus}{2}\right)\right)}{1536} \quad \textbf{(9)}$$

> $plot(\ n2, \mathrm{lambda} = 2.5\,..3)$



>

> $plot(\ n1 + n2, \mathrm{lambda} = 2.5\,..3)$



$>$

$$> numerator\_appx := \frac{eta\_minus \cdot beta\_minus}{\mathrm{GAMMA}\left(\frac{d}{2}\right)} \cdot \left(\frac{\mathrm{lambda} \cdot beta\_minus}{4}\right)^{\frac{d}{2}-1} \cdot J\_appx\left(\frac{d}{2}, \frac{\mathrm{lambda} \cdot beta\_plus}{2}, n\right) + \frac{beta\_plus \cdot beta\_minus \cdot eta\_plus \cdot eta\_minus}{8 \cdot d \cdot \mathrm{GAMMA}\left(\frac{d}{2}\right)} \cdot \left(\frac{\mathrm{lambda} \cdot beta\_plus}{4}\right)^{\frac{d}{2}-1} \cdot J\_appx\left(\frac{d}{2}-1, \frac{\mathrm{lambda} \cdot beta\_minus}{2}, n\right)$$

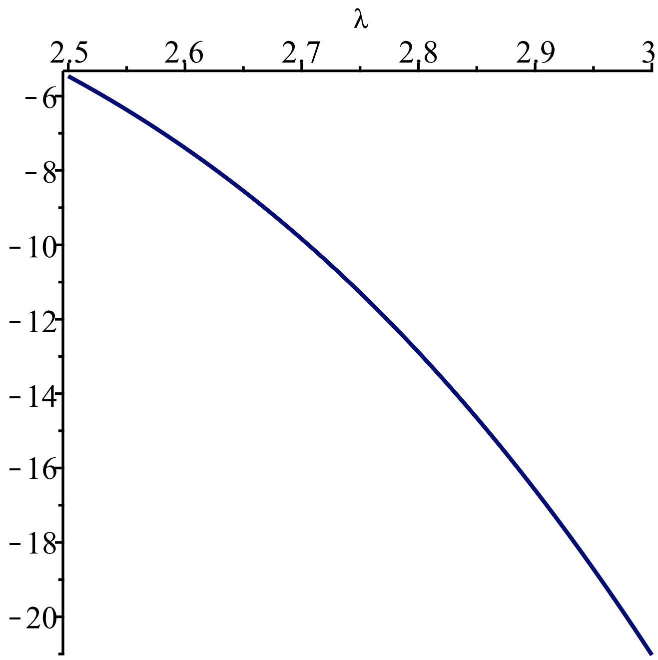
$$numerator\_appx := \frac{1}{32}\Bigg(eta\_minus\,(-2\,\mathrm{I}\sqrt{15}+6)^{3/2}\lambda^2\Bigg(\frac{\lambda^3(2\,\mathrm{I}\sqrt{15}+6)^{3/2}}{384} - \frac{\lambda^5(2\,\mathrm{I}\sqrt{15}+6)^{5/2}}{24576} + \frac{\lambda^7(2\,\mathrm{I}\sqrt{15}+6)^{7/2}}{3932160} - \frac{\lambda^9(2\,\mathrm{I}\sqrt{15}+6)^{9/2}}{1132462080} + \frac{\lambda^{11}(2\,\mathrm{I}\sqrt{15}+6)^{11/2}}{507343011840} - \frac{\lambda^{13}(2\,\mathrm{I}\sqrt{15}+6)^{13/2}}{324699527577600} + \frac{\lambda^{15}(2\,\mathrm{I}\sqrt{15}+6)^{15/2}}{280540391827046400}\Bigg)\Bigg) + \frac{1}{1536}\Bigg((2\,\mathrm{I}\sqrt{15}+6)^{3/2}\,beta\_minus\;eta\_plus\;eta\_minus\,\lambda^2\Bigg(\frac{\lambda^2(-2\,\mathrm{I}\sqrt{15}+6)}{32} - \frac{\lambda^4(-2\,\mathrm{I}\sqrt{15}+6)^2}{1536} + \frac{\lambda^6(-2\,\mathrm{I}\sqrt{15}+6)^3}{196608} - \frac{\lambda^8(-2\,\mathrm{I}\sqrt{15}+6)^4}{47185920} + \frac{\lambda^{10}(-2\,\mathrm{I}\sqrt{15}+6)^5}{18119393280} \tag{10}$$

$$- \frac{\lambda^{12} \left(-2\,\mathrm{I}\sqrt{15}+6\right)^{6}}{10146860236800} + \frac{\lambda^{14} \left(-2\,\mathrm{I}\sqrt{15}+6\right)^{7}}{7792788661862400} \Bigg)\Bigg)$$

> $numerator\_appx := mtaylor\left(numerator\_appx, \text{lambda} = 2.5, 2\,n + \frac{d}{2}\right)$

$$numerator\_appx := 26.74444362 - 5.46501652\,\mathrm{I} - (29.85380220 + 16.95221322\,\mathrm{I})\,(\lambda - 2.5) - (105.8469600 + 22.05262545\,\mathrm{I})\,(\lambda - 2.5)^{2} - (78.46108943 + 12.90829849\,\mathrm{I})\,(\lambda - 2.5)^{3} - (15.29096734 + 1.03030609\,\mathrm{I})\,(\lambda - 2.5)^{4} + (7.669301172 + 2.708466714\,\mathrm{I})\,(\lambda - 2.5)^{5} + (4.686542930 + 1.311352591\,\mathrm{I})\,(\lambda - 2.5)^{6} + (0.7756967809 + 0.1209550592\,\mathrm{I})\,(\lambda - 2.5)^{7} - (0.07146181030 + 0.07050962424\,\mathrm{I})\,(\lambda - 2.5)^{8} - (0.03952415581 + 0.02181151659\,\mathrm{I})\,(\lambda - 2.5)^{9} - (0.000163219665 + 0.001614204590\,\mathrm{I})\,(\lambda - 2.5)^{10} + (0.002468506182 + 0.0001773091618\,\mathrm{I})\,(\lambda - 2.5)^{11} + (0.0006679711699 + 0.00001534905061\,\mathrm{I})\,(\lambda - 2.5)^{12} + (0.00009475632155 - 0.00001094461081\,\mathrm{I})\,(\lambda - 2.5)^{13} + (8.441254417\;10^{-6} - 3.397019242\;10^{-6}\,\mathrm{I})\,(\lambda - 2.5)^{14} \tag{11}$$

> $plot(\{\mathrm{Im}(numerator\_appx), n1 + n2\}, \text{lambda} = 2.5\,..3)$



> $poly\_upperbound := evalf(subs(\text{lambda} = 2.5, \mathrm{Im}(numerator\_appx)))$

$$poly\_upperbound := -5.46501652 \tag{12}$$

> $error\_appx := evalf\left(\left|\frac{eta\_minus \cdot beta\_minus}{\mathrm{GAMMA}\left(\frac{d}{2}\right)} \cdot \left(\frac{3 \cdot beta\_minus}{4}\right)^{\frac{d}{2}-1}\right|\right.$

$$\cdot error\_bar\left(\frac{3\cdot beta\_plus}{2}, n, \frac{d}{2}\right) + \left| \frac{beta\_plus\cdot beta\_minus\cdot eta\_plus\cdot eta\_minus}{8\cdot d\cdot \mathrm{GAMMA}\left(\frac{d}{2}\right)} \cdot\left(\frac{3\cdot beta\_plus}{4}\right)^{\frac{d}{2}-1} \right| \cdot error\_bar\left(\frac{3\cdot beta\_plus}{2}, n, \frac{d}{2}-1\right)\Bigg)$$

$$error\_appx := 0.4558764407 \qquad \textbf{(13)}$$

\>

\> $with(plots)$

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] **(14)**

\>

\> *numerator_appx_func* := **proc**(*d*, *n*)
local *numerator, numerator_appx, eta_minus, beta_minus, beta_plus, eta_plus, J_appx, error_bar, error_appx, poly_upperbound, A*;

$beta\_plus := \mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)+d-3) + I\cdot\mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)-d+3);$

$beta\_minus := \mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)+d-3) - I\cdot\mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)-d+3);$

$eta\_plus := d+3+\mathrm{sqrt}(d^2-10\cdot d+9);$

$eta\_minus := d+3-\mathrm{sqrt}(d^2-10\cdot d+9);$

$$J\_appx := (v, x, n) \rightarrow sum\left(\frac{\left(\frac{x}{2}\right)^{v}\cdot(-1)^{k}}{k!\cdot\mathrm{GAMMA}(k+v+1)}\cdot\left(\frac{x^2}{4}\right)^{k}, k=0..n\right);$$

$$error\_bar := (x, n, v) \rightarrow \frac{\left|\frac{x}{2}\right|^{2n+v}}{\mathrm{factorial}(n)\cdot\mathrm{GAMMA}(n+v+1)\cdot\left(1-\frac{|x|^2}{(2n+2)^2}\right)};$$

$$numerator := \frac{eta\_minus\cdot beta\_minus}{\mathrm{GAMMA}\left(\frac{d}{2}\right)}\cdot\left(\frac{\mathrm{lambda}\cdot beta\_minus}{4}\right)^{\frac{d}{2}-1}\cdot J\_appx\left(\frac{d}{2}, \frac{\mathrm{lambda}\cdot beta\_plus}{2}, n\right) + \frac{beta\_plus\cdot beta\_minus\cdot eta\_plus\cdot eta\_minus}{8\cdot d\cdot\mathrm{GAMMA}\left(\frac{d}{2}\right)}$$

$$\cdot\left(\frac{\text{lambda}\cdot beta\_plus}{4}\right)^{\frac{d}{2}-1}\cdot\text{BesselJ}\left(\frac{d}{2}-1,\ \frac{\text{lambda}\cdot beta\_minus}{2}\right);$$

$$numerator\_appx := \text{Im}\left(\frac{eta\_minus\ \cdot beta\_minus}{\text{GAMMA}\left(\frac{d}{2}\right)}\cdot\left(\frac{\text{lambda}\cdot beta\_minus}{4}\right)^{\frac{d}{2}-1}\cdot J\_appx\left(\frac{d}{2},\right.\right.$$

$$\left.\left.\frac{\text{lambda}\cdot beta\_plus}{2}, n\right)\right) + \text{Im}\left(\frac{beta\_plus\cdot beta\_minus\cdot eta\_plus\cdot eta\_minus}{8\cdot d\cdot\ \text{GAMMA}\left(\frac{d}{2}\right)}\right.$$

$$\left.\cdot\left(\frac{\text{lambda}\cdot beta\_plus}{4}\right)^{\frac{d}{2}-1}\cdot J\_appx\left(\frac{d}{2}-1,\ \frac{\text{lambda}\cdot beta\_minus}{2}, n\right)\right);$$

$$error\_appx :=\ evalf\left(\left|\frac{eta\_minus\ \cdot beta\_minus}{\text{GAMMA}\left(\frac{d}{2}\right)}\cdot\left(\frac{3\cdot beta\_minus}{4}\right)^{\frac{d}{2}-1}\right|\right.$$

$$\cdot error\_bar\left(\frac{3\cdot beta\_plus}{2}, n,\ \frac{d}{2}\right) + \left|\frac{beta\_plus\cdot beta\_minus\cdot eta\_plus\cdot eta\_minus}{8\cdot d\cdot\ \text{GAMMA}\left(\frac{d}{2}\right)}\right.$$

$$\left.\left.\cdot\left(\frac{3\cdot beta\_plus}{4}\right)^{\frac{d}{2}-1}\right|\cdot error\_bar\left(\frac{3\cdot beta\_plus}{2}, n,\ \frac{d}{2}-1\right)\right);$$

$poly\_upperbound := evalf(subs(\text{lambda}=2.5,\ numerator\_appx));$ *printf* ("d = %g \n", d)
printf ("the error bounds of polynomail approximation= %f\n", *error_appx*);
printf ("the truncated polynomail of the numerator at lambda 2.5= %f", *poly_upperbound*);
$A := Array(1..2):$

$$A[1] :=\ plot\left(\text{Im}\left(\frac{beta\_plus\cdot beta\_minus\cdot eta\_plus\cdot eta\_minus}{8\cdot d\cdot\ \text{GAMMA}\left(\frac{d}{2}\right)}\cdot\left(\frac{\text{lambda}\cdot beta\_plus}{4}\right)^{\frac{d}{2}-1}\right.\right.$$

$$\left.\cdot J\_appx\left(\frac{d}{2}-1,\ \frac{\text{lambda}\cdot beta\_minus}{2}, n\right)\right), \text{lambda}=2.5..3, legend=[typeset(N$$

```
^{(2)})] ) :
A[2] := plot( [ Im(numerator), numerator_appx], lambda = 2.5 ..3,  legend = [typeset(N,
    "approx"), typeset(N) ]) :
display(A);
end proc
```

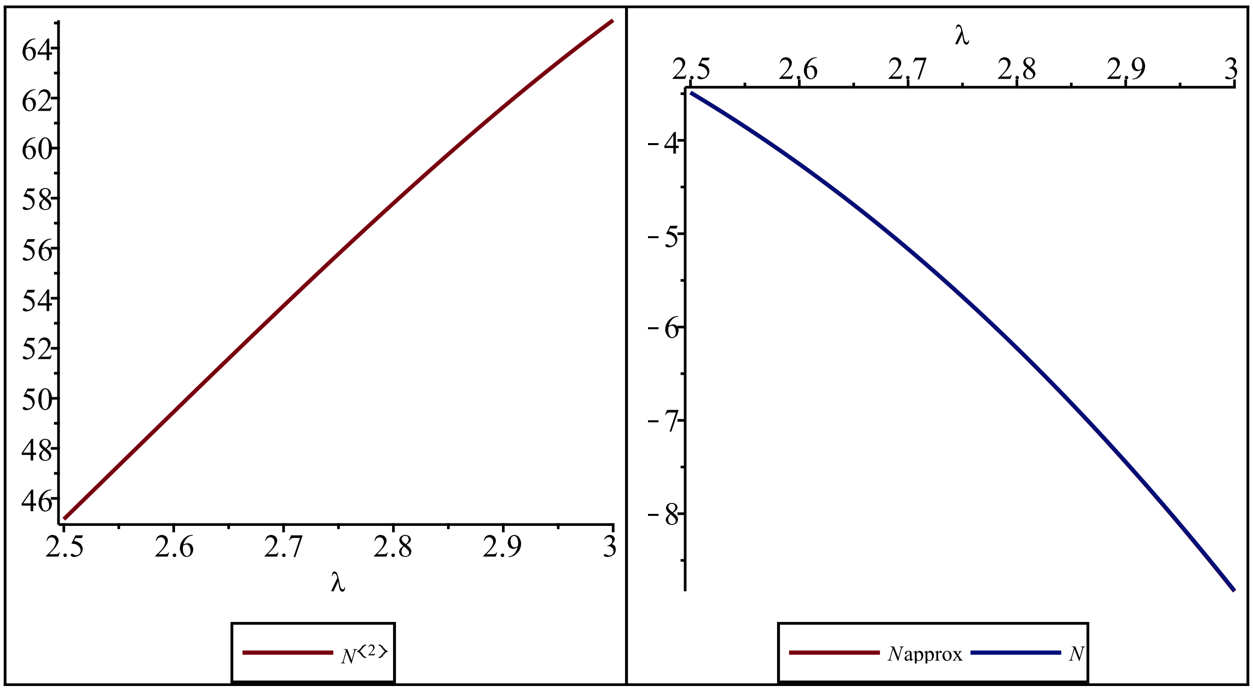
numerator_appx_func := proc(d, n) **(15)**

```
local numerator, numerator_appx, eta_minus, beta_minus, beta_plus, eta_plus, J_appx,
error_bar, error_appx, poly_upperbound, A;
beta_plus := sqrt(2 * sqrt(d) + d − 3) + I * sqrt(2 * sqrt(d) − d + 3);
beta_minus := sqrt(2 * sqrt(d) + d − 3) − I * sqrt(2 * sqrt(d) − d + 3);
eta_plus := d + 3 + sqrt(d^2 − 10 * d + 9);
eta_minus := d + 3 − sqrt(d^2 − 10 * d + 9);
J_appx := (v, x, n) → sum( (1/2 * x)^v * ( −1)^k * (1/4 * x^2)^k / (factorial(k)
 * GAMMA(k + v + 1)), k = 0 ..n);
error_bar := (x, n, v) → abs(1/2 * x)^(2 * n + v) / (factorial(n) * GAMMA(n + v + 1)
 * (1 − abs(x)^2 / (2 * n + 2)^2));
numerator := beta_minus * eta_minus * (1/4 * lambda * beta_minus)^(1/2 * d − 1)
 * J_appx(1/2 * d, 1/2 * lambda * beta_plus, n) / GAMMA(1/2 * d) + 1/8 * beta_plus
 * beta_minus * eta_plus * eta_minus * (1/4 * lambda * beta_plus)^(1/2 * d − 1)
 * BesselJ(1/2 * d − 1, 1/2 * lambda * beta_minus) / (d * GAMMA(1/2 * d));
numerator_appx := Im(beta_minus * eta_minus * (1/4 * lambda * beta_minus)^(1/2
 * d − 1) * J_appx(1/2 * d, 1/2 * lambda * beta_plus, n) / GAMMA(1/2 * d)) + Im(1/8
 * beta_plus * beta_minus * eta_plus * eta_minus * (1/4 * lambda * beta_plus)^(1/2
 * d − 1) * J_appx(1/2 * d − 1, 1/2 * lambda * beta_minus, n) / (d * GAMMA(1/2 * d)));
error_appx := evalf(abs(beta_minus * eta_minus * (3/4 * beta_minus)^(1/2 * d − 1)
/GAMMA(1/2 * d)) * error_bar(3/2 * beta_plus, n, 1/2 * d) + abs(1/8 * beta_plus
 * beta_minus * eta_plus * eta_minus * (3/4 * beta_plus)^(1/2 * d − 1) / (d * GAMMA(1
/2 * d))) * error_bar(3/2 * beta_plus, n, 1/2 * d − 1));
poly_upperbound := evalf(subs(lambda = 2.5, numerator_appx));
printf("d = %g \n", d) * printf("the error bounds of polynomail approximation= %f\n",
error_appx);
printf("the truncated polynomail of the numerator  at lambda 2.5= %f", poly_upperbound);
A := Array(1 ..2);
A[1] := plot(Im(1/8 * beta_plus * beta_minus * eta_plus * eta_minus * (1/4 * lambda
 * beta_plus)^(1/2 * d − 1) * J_appx(1/2 * d − 1, 1/2 * lambda * beta_minus, n) / (d
 * GAMMA(1/2 * d))), lambda = 2.5 ..3, legend = [typeset(N^{2})]);
A[2] := plot( [Im(numerator), numerator_appx], lambda = 2.5 ..3, legend = [typeset(N,
"approx"), typeset(N) ]);
plots:-display(A)
end proc
```

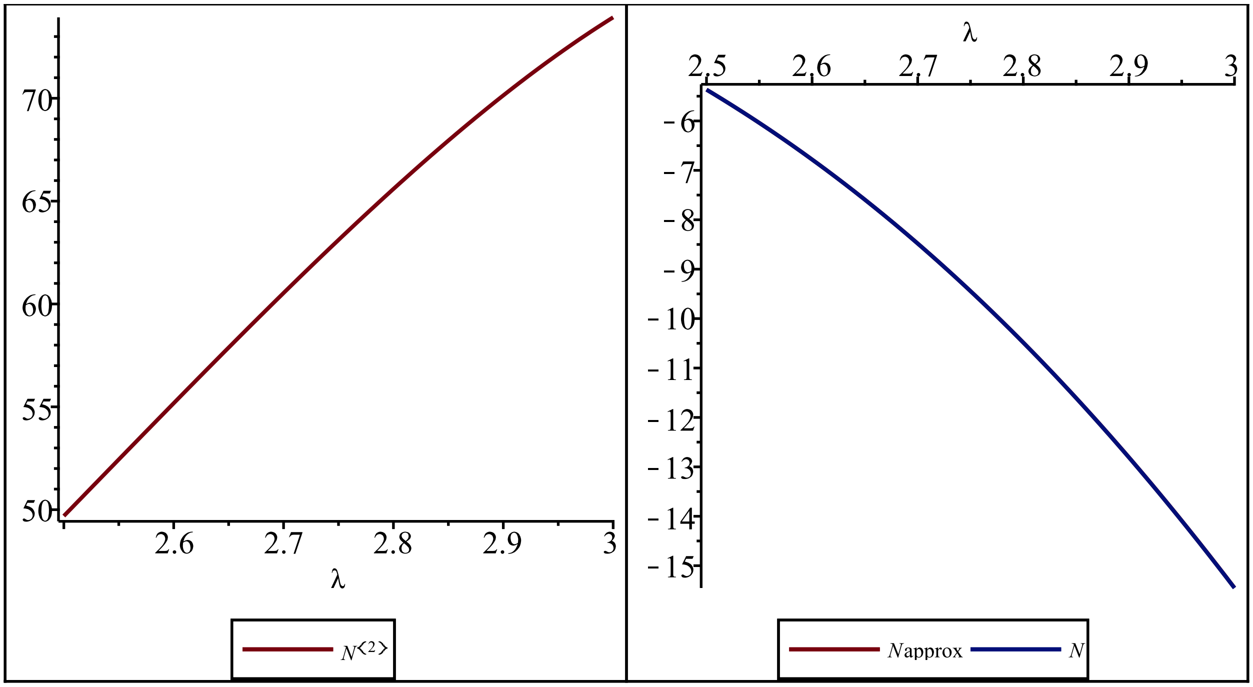
```
> n := 6;
  for d from 3 by 1 to 8 do
  numerator_appx_func(d, n)
  end do;
```

$$n := 6$$

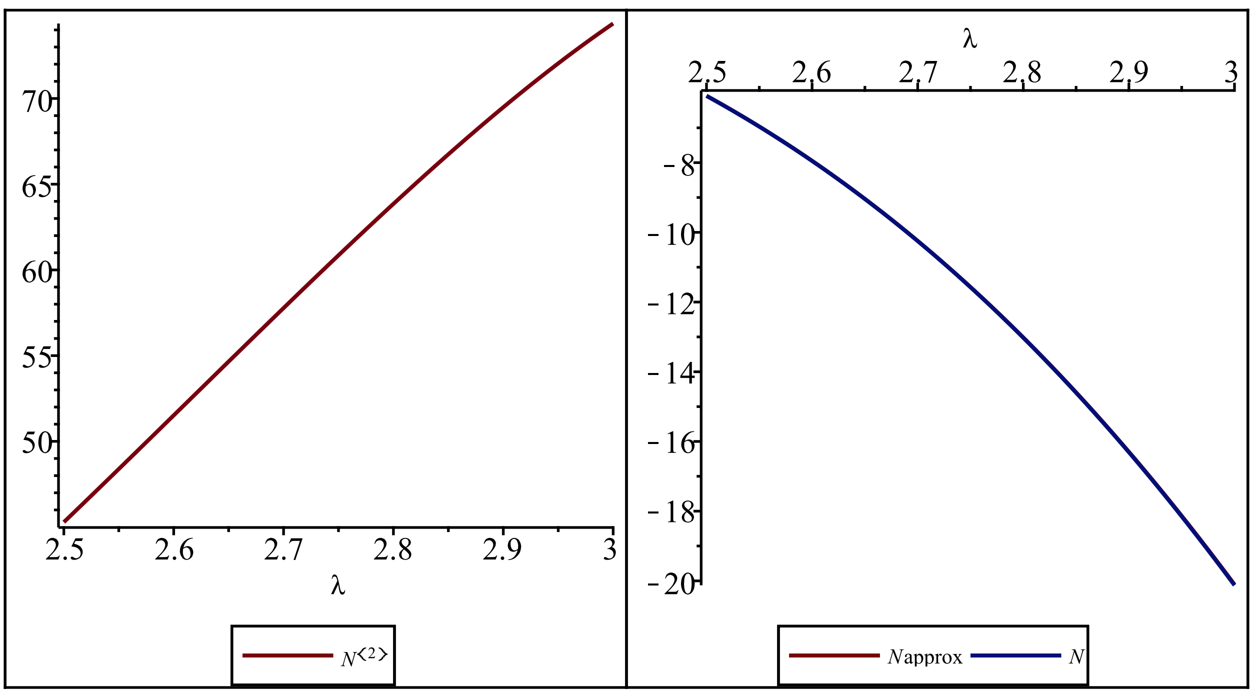
```
d = 3
the error bounds of polynomail approximation= 0.117392
the truncated polynomail of the numerator  at lambda 2.5=
-3.487949
```



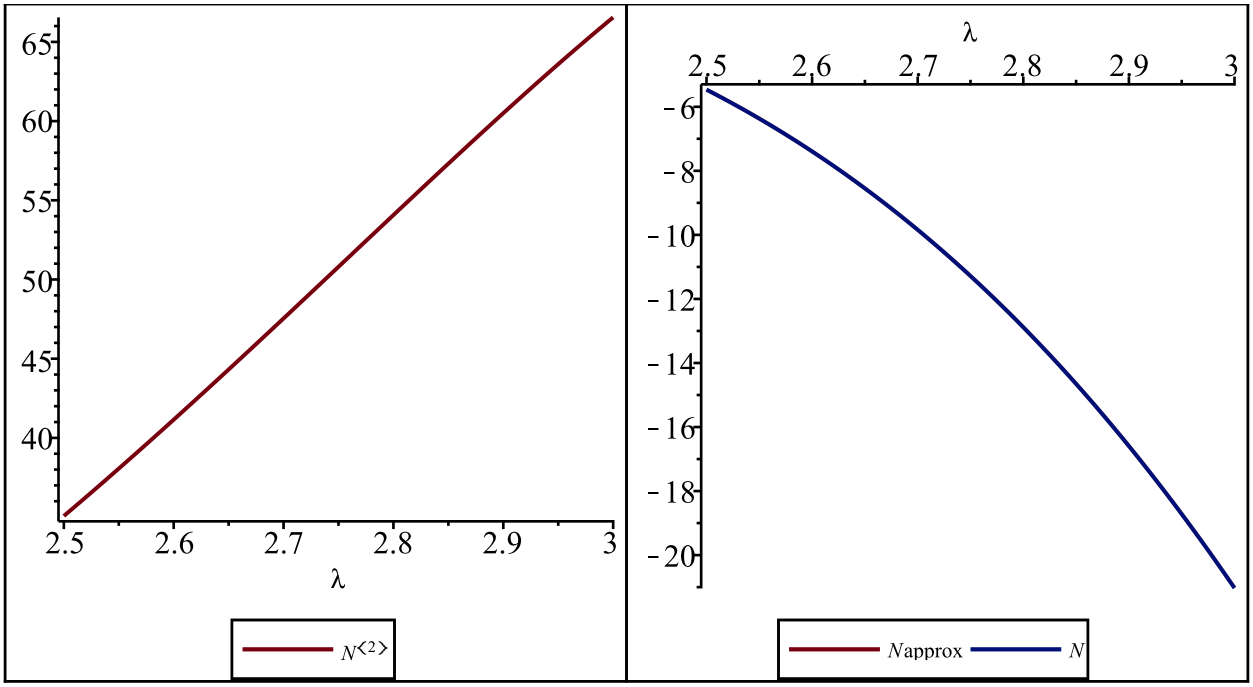

```
d = 4
the error bounds of polynomail approximation= 0.249450
the truncated polynomail of the numerator  at lambda 2.5=
-5.367159
```



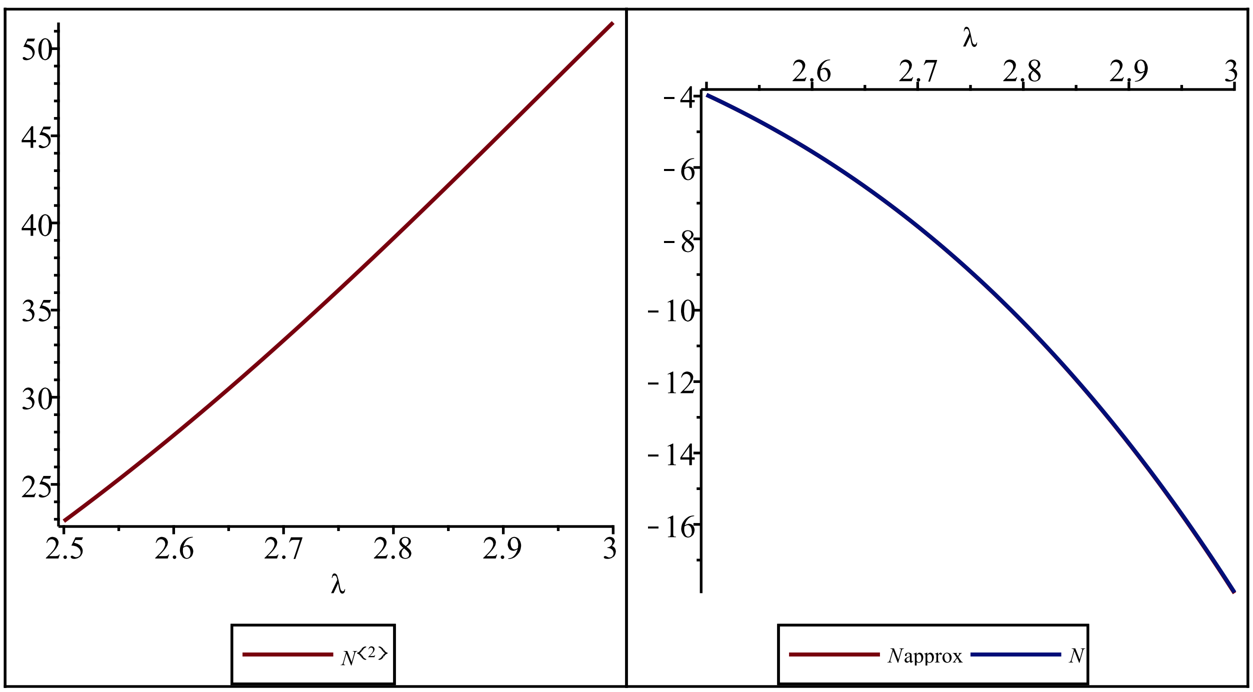
d = 5
the error bounds of polynomail approximation= 0.378808
the truncated polynomail of the numerator at lambda 2.5=
-6.082985



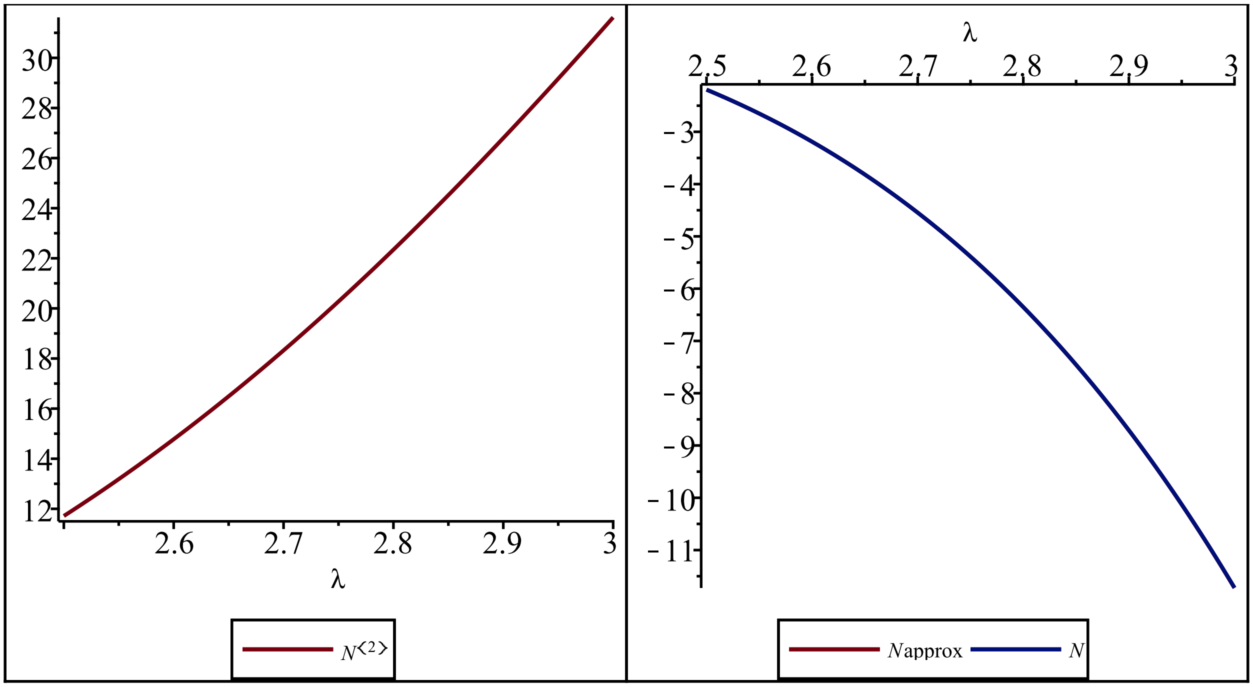
d = 6
the error bounds of polynomail approximation= 0.455876
the truncated polynomail of the numerator at lambda 2.5=
-5.465016



d = 7
the error bounds of polynomail approximation= 0.460456
the truncated polynomail of the numerator at lambda 2.5=
-3.964234



d = 8
the error bounds of polynomail approximation= 0.404525
the truncated polynomail of the numerator at lambda 2.5=
-2.193441



30
28
26
24
22
20
18
16
14
12
2.6
2.7
2.8
2.9
3
λ
$N^{\langle 2 \rangle}$
λ
2,5
2,6
2,7
2,8
2,9
3
−3
−4
−5
−6
−7
−8
−9
−10
−11
Napprox
N