> $d := 7$

$$d := 7 \tag{1}$$

> $n := 6$

$$n := 6 \tag{2}$$

> $n\_min := evalf\left(d^{\frac{1}{4}} \cdot 3 - 1\right)$

$$n\_min := 3.879729686 \tag{3}$$

>

>

> $beta\_plus := \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) + d - 3) + I \cdot \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) - d + 3)$

$$beta\_plus := \sqrt{2\sqrt{7}+4} + \mathrm{I}\sqrt{2\sqrt{7}-4} \tag{4}$$

> $beta\_minus := \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) + d - 3) - I \cdot \mathrm{sqrt}(2 \cdot \mathrm{sqrt}(d) - d + 3)$

$$beta\_minus := \sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4} \tag{5}$$

> $eta\_plus := d + 3 + \mathrm{sqrt}\left(d^2 - 10 \cdot d + 9\right)$

$$eta\_plus := 10 + 2\,\mathrm{I}\sqrt{3} \tag{6}$$

> $eta\_minus := d + 3 - \mathrm{sqrt}\left(d^2 - 10 \cdot d + 9\right)$

$$eta\_minus := 10 - 2\,\mathrm{I}\sqrt{3} \tag{7}$$

Part 1: Let's investage the denominator.

> $E\_denom := \mathrm{Im}\left(eta\_minus \cdot beta\_minus \cdot \mathrm{BesselJ}\left(\frac{d}{2} - 1, \frac{beta\_minus \cdot \mathrm{lambda}}{2}\right) \cdot \mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus \cdot \mathrm{lambda}}{2}\right)\right)$

$$E\_denom := \tag{8}$$

$$-\frac{1}{\pi}\Bigg(8\,\Im\Bigg(\Bigg(\Bigg(\Bigg(\mathrm{I}\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\sqrt{2\sqrt{7}-4}\,\lambda^2$$

$$+3\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda$$

$$-4\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\lambda^2$$

$$-3\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda$$

$$+6\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4} - \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\Bigg)\Bigg)$$

$$\Bigg(\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4} + \mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\left(2\sqrt{7}-4\right)^{3/2}\lambda^3$$

$$- 6\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda^3\sqrt{7}$$
$$-\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\left(2\sqrt{7}+4\right)^{3/2}\lambda^3$$
$$+6\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda^3\sqrt{7}$$
$$-12\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda^3$$
$$+24\,\mathrm{I}\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\sqrt{2\sqrt{7}-4}\,\lambda^2$$
$$-12\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda^3$$
$$+60\,\mathrm{I}\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}-4}\,\lambda$$
$$+60\cos\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\sqrt{2\sqrt{7}+4}\,\lambda$$
$$+96\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\lambda^2$$
$$\left.\left.\left.-120\sin\left(\frac{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}{2}\right)\right)\right)\left(10-2\,\mathrm{I}\sqrt{3}\right)\right)\Big/$$
$$\left(\sqrt{\left(\sqrt{2\sqrt{7}+4}-\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}\,\lambda^5\sqrt{\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\lambda}\,\left(\right.\right.$$
$$\left.\left.\left.\sqrt{2\sqrt{7}+4}-\mathrm{I}\sqrt{2\sqrt{7}-4}\right)\left(\sqrt{2\sqrt{7}+4}+\mathrm{I}\sqrt{2\sqrt{7}-4}\right)^3\right)\right)$$

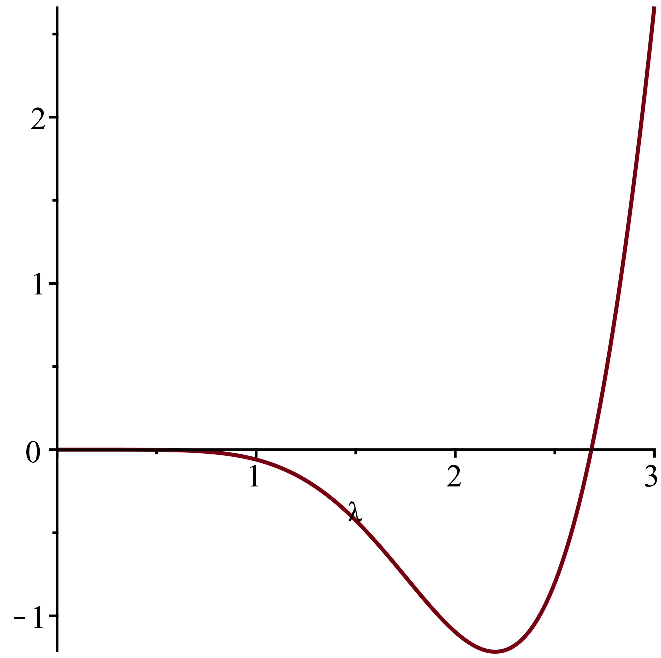
> $true\_deno := evalf(eval(E\_denom, \mathrm{lambda} = lambda0))$

$$true\_deno := -2.546479089\,\Im\left(\left(\left(0.04724555915\right.\right.\right.$$ **(9)**
$$-0.08183170884\,\mathrm{I}\right)\left(3.464101615\,\mathrm{I}\sin\left(\left(1.524098309-0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0^2\right.$$
$$+3.409328907\,\mathrm{I}\cos\left(\left(1.524098309-0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0-4.\sin\left(\left(1.524098309\right.\right.$$
$$\left.\left.-0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0^2-9.144589854\cos\left(\left(1.524098309-0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0$$
$$\left.+6.\sin\left(\left(1.524098309-0.5682214845\,\mathrm{I}\right)\lambda0\right)\right)\left(-30.21006941\,\mathrm{I}\cos\left(\left(1.524098309\right.\right.\right.$$
$$\left.\left.+0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0^3-16.51206510\cos\left(\left(1.524098309\right.\right.$$
$$\left.\left.+0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0^3+83.13843876\,\mathrm{I}\sin\left(\left(1.524098309\right.\right.$$
$$\left.\left.+0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0^2+68.18657814\,\mathrm{I}\cos\left(\left(1.524098309\right.\right.$$
$$\left.\left.+0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0+182.8917971\cos\left(\left(1.524098309+0.5682214845\,\mathrm{I}\right)\lambda0\right)\lambda0$$

$+ 96.\ \sin\big((1.524098309 + 0.5682214845\ I)\ \lambda 0\big)\ \lambda 0^2 - 120.\ \sin\big((1.524098309$
$+ 0.5682214845\ I)\ \lambda 0)\big)\big) \Big/$
$\left(\sqrt{(3.048196618 - 1.136442969\ I)\ \lambda 0}\ \lambda 0^5 \sqrt{(3.048196618 + 1.136442969\ I)\ \lambda 0}\,\right)\big)$

>

> $plot(E\_denom, \text{lambda} = 0\,..3)$



>

> $J\_appx := (v, x, n) \rightarrow sum\left(\dfrac{\left(\frac{x}{2}\right)^v \cdot (-1)^k}{k!\cdot \text{GAMMA}(k+v+1)} \cdot \left(\frac{x^2}{4}\right)^k, k = 0\,..n\right)$

$$J\_appx := (v, x, n) \mapsto \sum_{k=0}^{n} \frac{\left(\frac{x}{2}\right)^v \cdot (-1)^k \cdot \left(\frac{x^2}{4}\right)^k}{k!\cdot \Gamma(k+v+1)} \qquad (10)$$

> $error\_bar := (x, n, v) \rightarrow \dfrac{\left|\frac{x}{2}\right|^{2\,n+v}}{\text{factorial}(n)\cdot \text{GAMMA}(n+v+1)\cdot \left(1 - \frac{|x|^2}{(2\,n+2)^2}\right)}$

$$error\_bar := (x, n, v) \mapsto \frac{\left|\frac{x}{2}\right|^{2\cdot n+v}}{n!\cdot \Gamma(n+v+1)\cdot \left(1 - \frac{|x|^2}{(2\cdot n+2)^2}\right)} \qquad (11)$$

>

> *deno_appx_func* := **proc**(*d*, *n*, *lambda_star*)
local *J1*, *beta_minus*, *beta_plus*, *eta_minus*, *eta_plus*, *J1_appx*, *J2*, *J2_appx*, *err1*, *err2*, *err*, *deno_appx*, *acum_err*, *E_denom*, *true_deno*;

$beta\_plus := \mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)+d-3)+I\cdot\mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)-d+3);$
$beta\_minus := \mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)+d-3)-I\cdot\mathrm{sqrt}(2\cdot\mathrm{sqrt}(d)-d+3);$
$eta\_plus := d+3+\mathrm{sqrt}\left(d^2-10\cdot d+9\right);$
$eta\_minus := d+3-\mathrm{sqrt}\left(d^2-10\cdot d+9\right);$
$J1 := evalf\left(eval\left(\mathrm{BesselJ}\left(\frac{d}{2}-1, \frac{beta\_minus\cdot\mathrm{lambda}}{2}\right), \mathrm{lambda}=lambda\_star\right)\right);$
$J1\_appx := evalf\left(eval\left(J\_appx\left(\frac{d}{2}-1, \frac{beta\_minus\cdot\mathrm{lambda}}{2}, n\right), \mathrm{lambda}=lambda\_star\right)\right);$
$J2 := evalf\left(eval\left(\mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus\cdot\mathrm{lambda}}{2}\right), \mathrm{lambda}=lambda\_star\right)\right);$
$J2\_appx := evalf\left(eval\left(J\_appx\left(\frac{d}{2}, \frac{beta\_plus\cdot\mathrm{lambda}}{2}, n\right), \mathrm{lambda}=lambda\_star\right)\right);$
$err1 := evalf\left(eval\left(error\_bar\left(\frac{|beta\_plus|\cdot\mathrm{lambda}}{2}, n, \frac{d}{2}-1\right), \mathrm{lambda}=3\right)\right);$
$err2 := evalf\left(eval\left(error\_bar\left(\frac{|beta\_plus|\cdot\mathrm{lambda}}{2}, n, \frac{d}{2}\right), \mathrm{lambda}=3\right)\right);$
$err := \max(err1, err2);$
$printf$("|J1-J1_appx|= %f; err %f", $|J1-J1\_appx|, err$);
$printf$("|J2-J2_appx|= %f; err %f;", $|J2-J2\_appx|, err$);
$deno\_appx := evalf(eval(\mathrm{Im}(eta\_minus\cdot beta\_minus\cdot J1\_appx\cdot J2\_appx), \mathrm{lambda}=lambda\_star));$
$acum\_err := evalf\left(|eta\_minus\cdot beta\_minus|\cdot err^2+|J2\_appx|\cdot|eta\_minus\cdot beta\_minus|\cdot err+|eta\_minus\cdot beta\_minus|\cdot|J1\_appx|\cdot err\right);$
$E\_denom := \mathrm{Im}\left(eta\_minus\cdot beta\_minus\cdot\mathrm{BesselJ}\left(\frac{d}{2}-1, \frac{beta\_minus\cdot\mathrm{lambda}}{2}\right)\cdot\mathrm{BesselJ}\left(\frac{d}{2}, \frac{beta\_plus\cdot\mathrm{lambda}}{2}\right)\right);$
$true\_deno := evalf\left(eval\left(E\_denom, \mathrm{lambda}=lambda\_star\right)\right);$
$printf$("the bound for denominator for lambda = %f: [%f, %f] (true value %f) \n", $lambda\_star, deno\_appx-acum\_err, deno\_appx+acum\_err, true\_deno$);
end proc

$deno\_appx\_func := \mathbf{proc}(d, n, lambda\_star)$ **(12)**

```
local J1, beta_minus, beta_plus, eta_minus, eta_plus, J1_appx, J2, J2_appx, err1, err2, err,
deno_appx, acum_err, E_denom, true_deno;
beta_plus := sqrt(2 * sqrt(d) + d - 3) + I * sqrt(2 * sqrt(d) - d + 3);
beta_minus := sqrt(2 * sqrt(d) + d - 3) - I * sqrt(2 * sqrt(d) - d + 3);
eta_plus := d + 3 + sqrt(d^2 - 10 * d + 9);
eta_minus := d + 3 - sqrt(d^2 - 10 * d + 9);
J1 := evalf(eval(BesselJ(1/2 * d - 1, 1/2 * beta_minus * lambda), lambda
 = lambda_star));
J1_appx := evalf(eval(J_appx(1/2 * d - 1, 1/2 * beta_minus * lambda, n), lambda
 = lambda_star));
J2 := evalf(eval(BesselJ(1/2 * d, 1/2 * beta_plus * lambda), lambda = lambda_star));
J2_appx := evalf(eval(J_appx(1/2 * d, 1/2 * beta_plus * lambda, n), lambda
 = lambda_star));
```

```
err1 := evalf(eval(error_bar(1/2*abs(beta_plus)*lambda, n, 1/2*d - 1), lambda
 = 3));
err2 := evalf(eval(error_bar(1/2*abs(beta_plus)*lambda, n, 1/2*d), lambda = 3));
err := max(err1, err2);
printf("|J1-J1_appx|= %f; err %f", abs(J1 - J1_appx), err);
printf("|J2-J2_appx|= %f; err %f;", abs(J2 - J2_appx), err);
deno_appx := evalf(eval(Im(eta_minus*beta_minus*J1_appx*J2_appx), lambda
 = lambda_star));
acum_err := evalf(abs(eta_minus*beta_minus)*err^2 + abs(J2_appx)*abs(eta_minus
 *beta_minus)*err + abs(eta_minus*beta_minus)*abs(J1_appx)*err);
E_denom := Im(eta_minus*beta_minus*BesselJ(1/2*d - 1, 1/2*beta_minus
 *lambda)*BesselJ(1/2*d, 1/2*beta_plus*lambda));
true_deno := evalf(eval(E_denom, lambda = lambda_star));
printf("the bound for denominator for lambda = %f: [%f, %f] (true value %f) \n",
lambda_star, deno_appx - acum_err, deno_appx + acum_err, true_deno)
end proc
```

```
> deno_appx_func(d, n, 3)
|J1-J1_appx|= 0.000409; err 0.005484|J2-J2_appx|= 0.000095; err
0.005484;the bound for denominator for lambda = 3.000000:
[2.361337, 2.968798] (true value 2.666108)
```

```
> deno_appx_func(d, n, 2.5)
|J1-J1_appx|= 0.000021; err 0.005484|J2-J2_appx|= 0.000004; err
0.005484;the bound for denominator for lambda = 2.500000:
[-1.039206, -0.562060] (true value -0.800674)
```

```
> for d from 2 to 8 by 1
  do printf("\nd=%g \n", d); deno_appx_func(d, n, 2.5);
  deno_appx_func(d, n, 3)
  end do;

d=2
|J1-J1_appx|= 0.000010; err 0.002141|J2-J2_appx|= 0.000002; err
0.002141;the bound for denominator for lambda = 2.500000:
[-2.668223, -2.370181] (true value -2.519024)
|J1-J1_appx|= 0.000132; err 0.002141|J2-J2_appx|= 0.000029; err
0.002141;the bound for denominator for lambda = 3.000000:
[2.020681, 2.466317] (true value 2.245478)

d=3
|J1-J1_appx|= 0.000019; err 0.003969|J2-J2_appx|= 0.000004; err
0.003969;the bound for denominator for lambda = 2.500000:
[-2.436764, -1.843032] (true value -2.139845)
|J1-J1_appx|= 0.000271; err 0.003969|J2-J2_appx|= 0.000063; err
0.003969;the bound for denominator for lambda = 3.000000:
[6.608817, 7.482410] (true value 7.050610)

d=4
|J1-J1_appx|= 0.000025; err 0.005345|J2-J2_appx|= 0.000005; err
0.005345;the bound for denominator for lambda = 2.500000:
```

```
[-2.096279, -1.336230] (true value -1.716870)
|J1-J1_appx|= 0.000384; err 0.005345|J2-J2_appx|= 0.000091; err
0.005345;the bound for denominator for lambda = 3.000000:
[7.882288, 8.973645] (true value 8.415394)

d=5
|J1-J1_appx|= 0.000026; err 0.006009|J2-J2_appx|= 0.000005; err
0.006009;the bound for denominator for lambda = 2.500000:
[-1.812802, -1.056404] (true value -1.434535)
|J1-J1_appx|= 0.000442; err 0.006009|J2-J2_appx|= 0.000105; err
0.006009;the bound for denominator for lambda = 3.000000:
[6.564977, 7.615899] (true value 7.086824)

d=6
|J1-J1_appx|= 0.000024; err 0.005995|J2-J2_appx|= 0.000005; err
0.005995;the bound for denominator for lambda = 2.500000:
[-1.466306, -0.828475] (true value -1.146975)
|J1-J1_appx|= 0.000446; err 0.005995|J2-J2_appx|= 0.000105; err
0.005995;the bound for denominator for lambda = 3.000000:
[4.351783, 5.202549] (true value 4.787209)

d=7
|J1-J1_appx|= 0.000021; err 0.005484|J2-J2_appx|= 0.000004; err
0.005484;the bound for denominator for lambda = 2.500000:
[-1.039206, -0.562060] (true value -0.800674)
|J1-J1_appx|= 0.000409; err 0.005484|J2-J2_appx|= 0.000095; err
0.005484;the bound for denominator for lambda = 3.000000:
[2.361337, 2.968798] (true value 2.666108)

d=8
|J1-J1_appx|= 0.000016; err 0.004691|J2-J2_appx|= 0.000003; err
0.004691;the bound for denominator for lambda = 2.500000:
[-0.600872, -0.275738] (true value -0.438445)
|J1-J1_appx|= 0.000348; err 0.004691|J2-J2_appx|= 0.000080; err
0.004691;the bound for denominator for lambda = 3.000000:
[0.959400, 1.353657] (true value 1.151773)
```

```
>
```