Price a Vanilla European Call Option

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Abstract

In this paper, we price a vanilla European call option under the Heston model and then simulate the monthly share price over a year using the Constant Elasticity of Variance (CEV) model, with the assumption of constant volatility each month. Monte Carlo simulations with varying sample sizes are run and the results are plotted against the closed form value for comparison.

1 Introduction

In this paper we go beyond the constant volatility assumption in the Black-Scholes model. Most of code implemented in this submission is derived from Module 5 [7] of the course.

We initialize most variables as given by the question.

- Option maturity is one year
- The option is struck at-the-money
- The current share price is $100
- The risk-free continuously compounded interest rate is 8%
- The volatility for the underlying share is 30%

2 Fourier pricing technique under Heston model dynamics

Black-Scholes option pricing models assumed volatility of the underlying assets to be constant or a predetermined function of time, we will now implement a model which features instantaneous variance of asset price using volatility that evolves stochastically in time. Although there are several models incorporating stochastic volatility including introduced by Hull and White [1], Stein and Stein [2] and Heston [3] [4], we will implement Heston's constant interest rate model.

With the assumption that the underlying share follows the Heston model dynamics, the additional parameters required are specified as follows:

- $\nu_0 = 0.06$
- $\kappa = 9$
- $\theta = 0.06$
- $\rho = -0.4$

The Characteristic function is implemented using a function presented by Albrecher et al [5]. The function is written as:

$$\phi_{S_T} = exp(C(\tau;u) + D(\tau;u)v_t + iu\log(S_t))$$

Where,

$$C(\tau;u) = ri\tau u + \theta\kappa[\tau x_{-} - \frac{1}{a}\log(\frac{1-ge^{d\tau}}{1-g})],$$

$$D(\tau;u) = (\frac{1-e^{d\tau}}{1-ge^{d\tau}})x_{-},$$

$$\tau = T - t,$$

$$g = \frac{x_{-}}{x_{+}},$$

$$x_{\pm} = \frac{b \pm d}{2a},$$

$$d = \sqrt{b^2 - 4ac},$$

$$c = -\frac{u^2 + ui}{2},$$

$$b = \kappa - \rho\sigma iu,$$

$$a = \frac{\sigma^2}{2}$$

[ ]:

```
#Characteristic function code

a = sigma**2/2

def b(u):
```

```python
    return kappa - rho*sigma*1j*u

def c(u):
    return -(u**2+1j*u)/2

def d(u):
    return np.sqrt(b(u)**2-4*a*c(u))

def xminus(u):
    return (b(u)-d(u))/(2*a)

def xplus(u):
    return (b(u)+d(u))/(2*a)

def g(u):
    return xminus(u)/xplus(u)

def C(u):
    val1 = T*xminus(u)-np.log((1-g(u)*np.exp(-T*d(u)))/(1-g(u)))/a
    return r*T*1j*u + theta*kappa*val1

def D(u):
    val1 = 1-np.exp(-T*d(u))
    val2 = 1-g(u)*np.exp(-T*d(u))
    return (val1/val2)*xminus(u)

def log_char(u):
    return np.exp(C(u) + D(u)*v0 + 1j*u*np.log(S0))

def adj_char(u):
    return log_char(u-1j)/log_char(-1j)
```

Now we vectorize the code, calculate an estimate for integrals and calculate the Fourier estimate of our call price.

[]:
```python
delta_t = t_max/N
from_1_to_N = np.linspace(1,N,N)
t_n = (from_1_to_N-1/2)*delta_t

#Integral calculations
first_integral = sum((((np.exp(-1j*t_n*k_log)*adj_char(t_n)).imag)/t_n)*delta_t)
second_integral = sum((((np.exp(-1j*t_n*k_log)*log_char(t_n)).imag)/t_n)*delta_t)

#Call value
fourier_call_val = S0*(1/2 + first_integral/np.pi)-np.exp(-r*T)*K*(1/2 +␣
↪second_integral/np.pi)
fourier_call_val
```

[]: 13.734895692109077

To see the effectiveness of the pricing option under Heston dynamics we will also price the call option under Black-Scholes assumption.

[]:
```
# Code for analytical solution for vanilla European Call option
d_1_stock = (np.log(S0/K)+(r + sigma**2/2)*(T))/(sigma*np.sqrt(T))
d_2_stock = d_1_stock - sigma*np.sqrt(T)

analytic_callprice = S0*norm.cdf(d_1_stock)-K*np.exp(-r*(T))*norm.cdf(d_2_stock)
analytic_callprice
```

[]: 15.711312547892973

3 Simulate a share price path using CEV Model

Cox [6] developed the constant elasticity of variance (CEV) option pricing model, it attempts to capture stochastic volatility and is given by:

$$dS_t = \mu S_t dt + \sigma S_t^{\gamma} dW_t$$

If $\gamma = 1$ this model return the same value as Black-Scholes model, however if the value of $\gamma < 1$ we experience an effect called leverage effect where the volatility increases as the price decreases over subsequent time periods.

Based on the assumption that $\sigma(t_i, t_{i+1}) = \sigma(S_{ti})^{\gamma-1}$, where $\sigma = 0.3$ and $\gamma = 0.75$. We can simulate the next step in a share price path using the following formula:

$$S_{t_{i+1}} = S_{t_i} e^{(r - \frac{\sigma^2(t_i,t_{i+1})}{2})(t_{i+1}-t_i)+\sigma(t_i,t_{i+1})\sqrt{t_{i+1}-t_i}Z}$$

where S_{ti} is the share price at time t_i, $\sigma(t_i, t_{i+1})$ is the volatility for the period $[t_i, t_{i+1}]$, r is the risk-free interest rate, and Z N(0, 1)

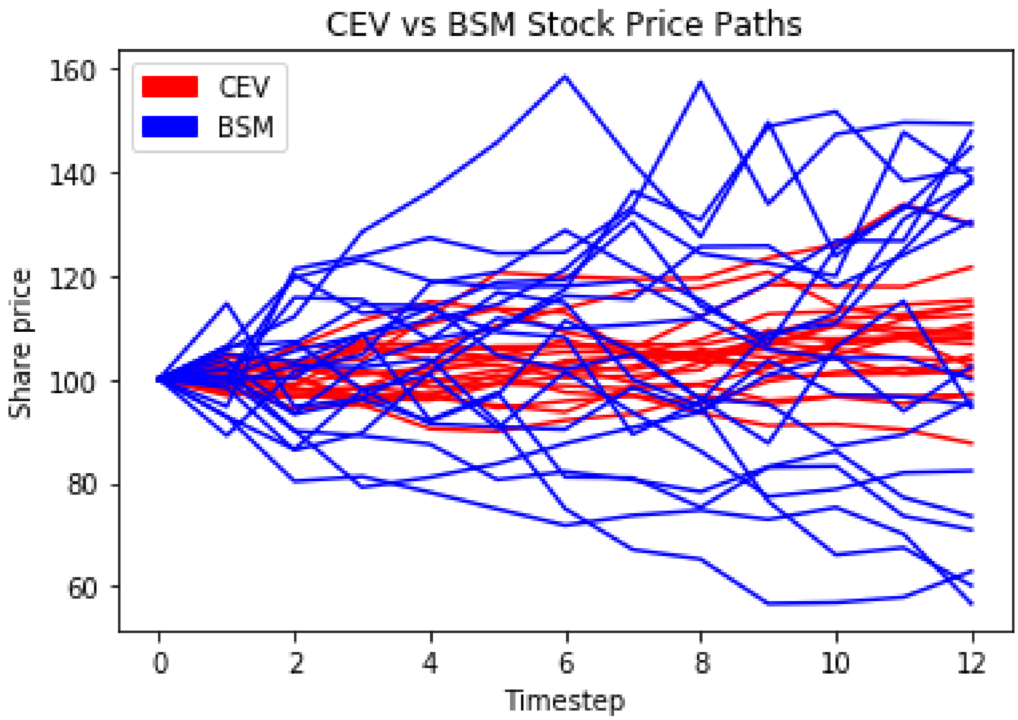
First, we define our helper functions. The next_share_price function is used to calculate the evolution of the share price at t+1, from the share price at t. We generate the random variable Z from within this function. The effective sigma is also written as a function of the share price at t.

Just for exploration purposes, we also added a varying_vol flag, to allow us to switch between a constant volatility and varying volatility. We use the initial stock price instead of the previous price when assuming a contant volatility.

The other function is the generate_share_price_path function. We first create an empty numpy array of shape (sample_size x timesteps+1). Note the addition of one element to the timestep, as the first element is equal to the initial stock price. We then iterate through each path, and each timestep, applying the next_share_price function against the previous share price. We then convert the result into a pandas DataFrame and return the results.

[7]:
```python
def next_share_price(prev_price, r, dT, sigma_const, gamma, sample_size,␣
 ↪varying_vol = True):
    Z = stats.norm.rvs(size=sample_size)
    if varying_vol:
        sigma = sigma_const*(prev_price)**(gamma-1)
    else:
        sigma = sigma_const*(S0)**(gamma-1)

    return prev_price*np.exp((r-(sigma**2)/2)*(dT)+(sigma)*(np.sqrt(dT))*Z)

def generate_share_price_path(S0, r, T, sigma_const, gamma, sample_size,␣
 ↪timesteps, varying_vol = True):
    df = pd.DataFrame([S0]*sample_size)
    for t in range(1, timesteps+1):
        df[t] = next_share_price(df[t-1], r, 1/timesteps, sigma_const, gamma,␣
 ↪sample_size, varying_vol)
    return df.T
```

[8]:
```python
import matplotlib.patches as mpatches
T = 10
sample_size = 20

share_price_path_cev = generate_share_price_path(S0, r, T, sigma_const, gamma,␣
 ↪sample_size, timesteps)
share_price_path_black_scholes = generate_share_price_path(S0, r, T,␣
 ↪sigma_const, 1.0, sample_size, timesteps, varying_vol=False)

plt.plot(share_price_path_cev, color='red')
plt.plot(share_price_path_black_scholes, color='blue')
plt.xlabel("Timestep")
plt.ylabel("Share price")
red_patch = mpatches.Patch(color='red', label='CEV')
blue_patch = mpatches.Patch(color='blue', label='BSM')
plt.legend(handles=[red_patch, blue_patch], loc='upper left')
plt.title("CEV vs BSM Stock Price Paths")
plt.show()
```



Next, we create a python dictionary called share_price_paths, to hold our results for part 2. The key of this dictionary would be the number of sample price paths, while the values would be the dataframes containing the price paths. We also track the rough processing time required at each step, by printing the time when each iteration completes.

[9]:
```
import time

T = 1
sample_sizes = range(1000, 50001, 1000)

share_price_paths = {}

print("Start generating share price path")
start = time.time()
for sample_size in sample_sizes:
    share_val = generate_share_price_path(S0, r, T, sigma_const, gamma,␣
 ↪sample_size, timesteps, varying_vol=False)

    share_price_paths[sample_size] = share_val
    #print("Updated for sample size {} at {}".format(sample_size, datetime.
 ↪datetime.now().strftime('%H:%M')))
end = time.time()
print(f"Generating all samples paths takes {(end - start):.2f}s")
```

```
Start generating share price path
Generating all samples paths takes 2.58s
```

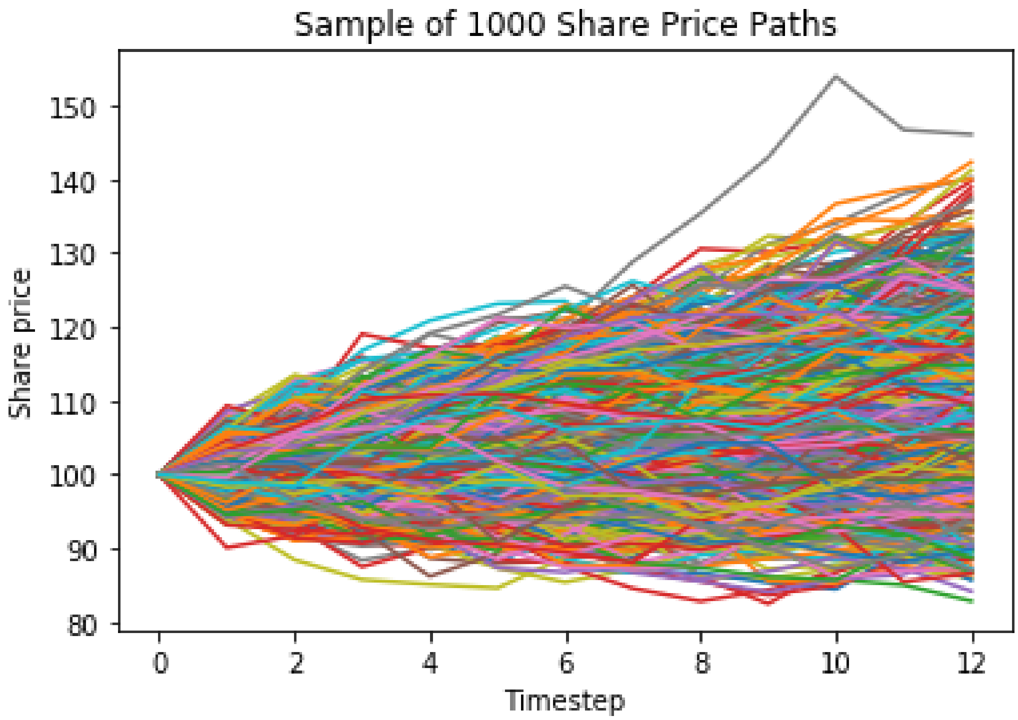
To display our output, we show the first 10 price paths generated by our iteration with 1000 samples:

[10]:
```python
share_price_paths[1000].iloc[:, 0:10]
```

[10]:
```
             0           1           2           3           4           5  \
0   100.000000  100.000000  100.000000  100.000000  100.000000  100.000000
1    99.626771   99.374526  100.689569   97.065445  106.862386   99.918343
2   103.745565   96.967926  102.622183   96.451414  106.332275   96.640428
3   107.154631   96.585243  105.974778   93.207852  100.177288   99.129417
4   111.716818   95.838855  101.540594   94.146336   98.646601  102.592136
5   120.620494   93.477848  100.930914   94.502805  100.662067  103.175996
6   122.217586   87.674152  101.799340   91.076602  101.876289  102.336328
7   123.138256   87.108925  103.904210   94.061221  106.060742  103.426255
8   129.359695   87.871374  100.925245   92.948402  115.010833   98.206543
9   123.575048   87.571414  105.883449   95.799071  117.769685   97.640631
10  125.486476   88.836040  104.337774   96.829970  119.866089  100.623917
11  124.538067   92.114326  106.173600   98.493852  118.497793   99.093147
12  123.548415   92.619346  104.478810   98.081928  114.719969   94.669329

             6           7           8           9
0   100.000000  100.000000  100.000000  100.000000
1   106.483031  100.796117  100.030802  102.593647
2   105.512423  101.627748   98.448810  103.179244
3   107.821291   98.518154   97.598741  103.563761
4   108.394844   97.918995   99.489053  110.865271
5   108.425203   99.326116   97.269739  108.916593
6   107.844372  103.199284   98.432527  108.089622
7   108.269775  100.286497  101.084604  107.860547
8   108.268731   99.113198  101.552266  103.792957
9   103.400905   98.076554  100.633537  106.623775
10  107.548810   98.062335   97.583078  110.004438
11  104.394412   98.481303   94.325492  107.980799
12  103.306627   95.171645   94.012006  108.960908
```

Next, we plot the price paths for the iteration with 1000 samples,

[11]:
```python
plt.plot(share_price_paths[1000])
plt.xlabel('Timestep')
plt.ylabel('Share price')
plt.title('Sample of 1000 Share Price Paths')
plt.show()
```



4 Monte Carlo estimates

Using Monte Carlo, we calculate the price of the vanilla call option as follows:

[12]:
```
price_estimate = []
price_std = []


for size in sample_sizes:
    S_Ts = share_price_paths[size].iloc[12, :]
    payoff = np.maximum(S_Ts - K, 0)
    discounted_price = np.exp(-r*T)*payoff
    price_estimate.append(discounted_price.mean())
    price_std.append(discounted_price.std()/np.sqrt(size))
```

[13]:
```
print("The price estimated by Monte Carlo when using sample size of 50,000 is :␣
 ↪{:.3f}".format(price_estimate[-1]))
```

```
The price estimated by Monte Carlo when using sample size of 50,000 is : 8.694
```

We then compare with calculation of CEV model using noncentralchi-squared

[14]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import ncx2

S0 = 100
sigma = 0.3
gamma = 0.75
r = 0.08
T = 1
```

[15]:
```
z = 2 + 1/(1-gamma)
def C(t,K):
    kappa = 2*r/(sigma**2*(1-gamma)*(np.exp(2*r*(1-gamma)*t)-1))
    x = kappa*S0**(2*(1-gamma))*np.exp(2*r*(1-gamma)*t)
    y = kappa*K**(2*(1-gamma))
    return S0*(1-ncx2.cdf(y,z,x))-K*np.exp(-r*t)*ncx2.cdf(x,z-2,y)
```

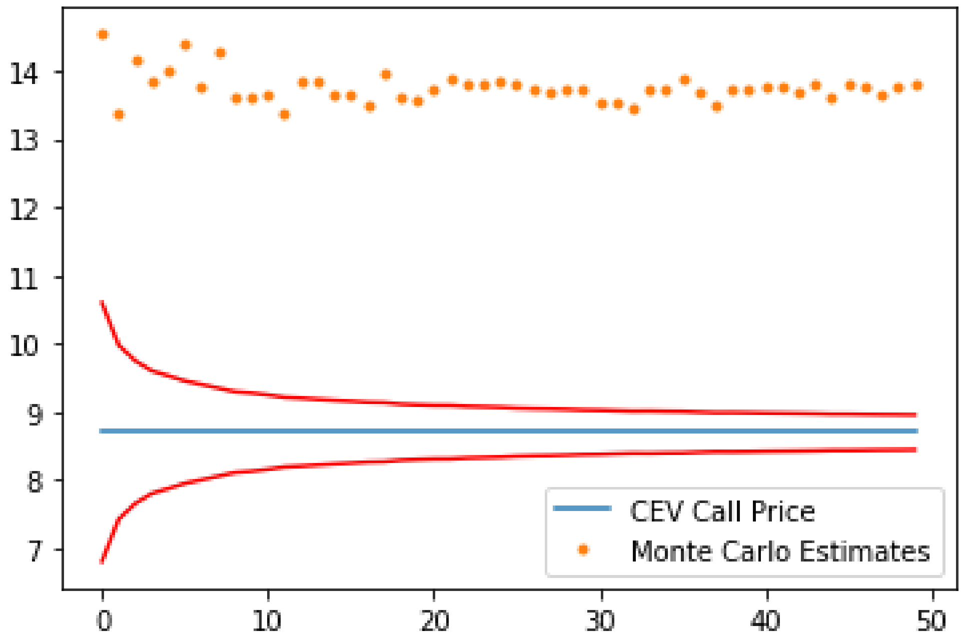
[16]:
```
cev_call_price = C(T, 100)
```

[17]:
```
print("The price calculated via CEV model using noncentral chi-squared␣
 ↪distribution is : {:.3f}".format(cev_call_price))
```

```
The price calculated via CEV model using noncentral chi-squared distribution is
: 8.702
```

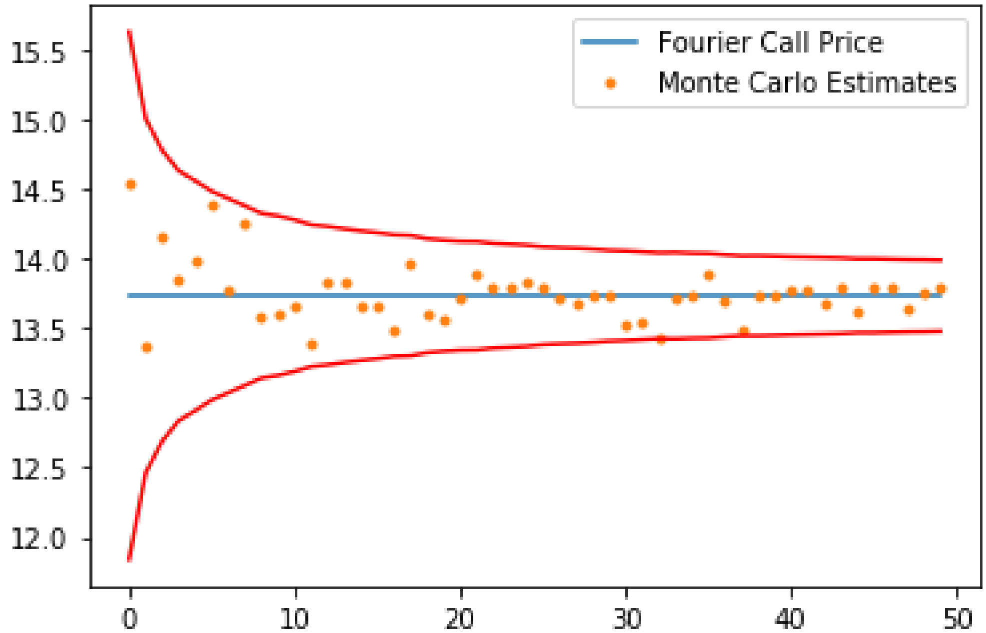
We plot the prices of our Monte Carlo estimates against the CEV noncentral chi-squared distribution prices calculated from Part 1 above:

[18]:
```
plt.plot([cev_call_price]*50, label='CEV Call Price')
plt.plot(price_estimate, '.', label='Monte Carlo Estimates')
plt.plot(price_estimate + 3*np.array(price_std), 'r')
plt.plot(price_estimate - 3*np.array(price_std), 'r')
plt.legend()
plt.show()
```



We also plot the prices of our Monte Carlo estimates against the Fourier Call prices calculated from Part 1 above:

[19]:
```
plt.plot([fourier_call_val]*50, label='Fourier Call Price')
plt.plot(price_estimate, '.', label='Monte Carlo Estimates')
plt.plot(price_estimate + 3*np.array(price_std), 'r')
plt.plot(price_estimate - 3*np.array(price_std), 'r')
plt.legend()
plt.show()
```



We notice that there is a discrepency. Upon further investigation, for the Heston model, v_0 = 0.06 (v_0 = stock volatility ^ 0.5), so stock volatility = .06**0.5= 0.2449. Note that sigma under Heston model refers to volatility of stock volatility

Under our stock price Monte Carlo calculation, the default stock volatility is 0.0948 (given by 0.3*(100)*-.25). Thus its much less than Heston. To have the same initial stock volatility, we find a new value for σ by equating $\sigma(S_{ti})^{\gamma-1} = \sqrt{0.06}$, giving us $\sigma = 0.775$.

We then find that our newly calculated Monte Carlo calculated call prices are aligned with the Fourier prices.

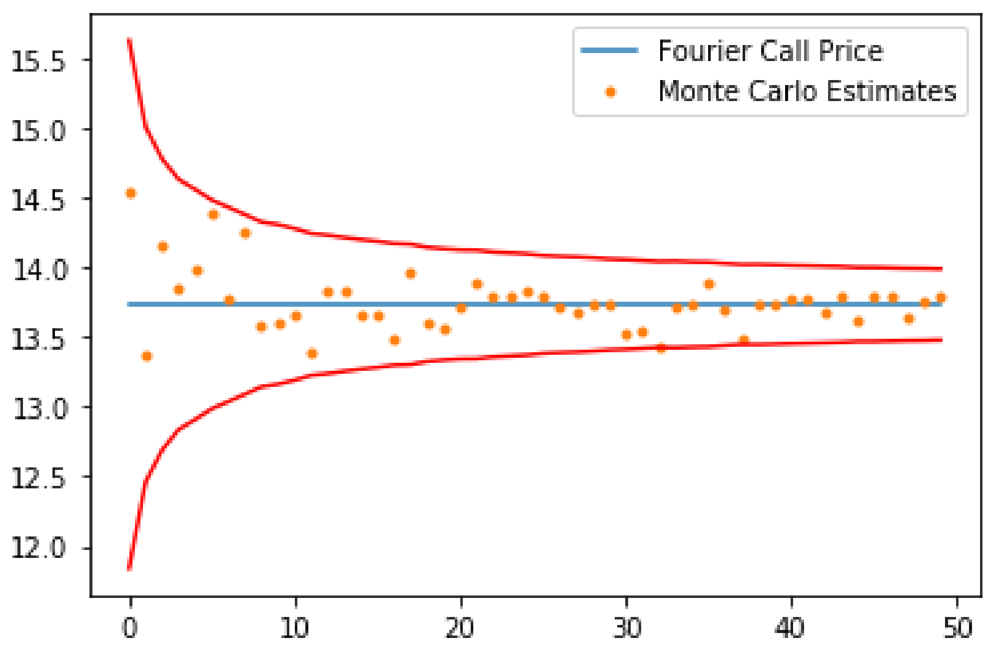
[20]:
```
sigma_const = 0.775

T = 1
sample_sizes = range(1000, 50001, 1000)

share_price_paths = {}

print("Start generating share price path")
start = time.time()
for sample_size in sample_sizes:
    share_val = generate_share_price_path(S0, r, T, sigma_const, gamma,␣
 ↪sample_size, timesteps, varying_vol=False)

    share_price_paths[sample_size] = share_val
    #print("Updated for sample size {} at {}".format(sample_size, datetime.
 ↪datetime.now().strftime('%H:%M')))
```

```python
end = time.time()
print(f"Generating all samples paths takes {(end - start):.2f}s")
```

```
Start generating share price path
Generating all samples paths takes 2.10s
```

[21]:
```python
price_estimate = []
price_std = []


for size in sample_sizes:
    S_Ts = share_price_paths[size].iloc[12, :]
    payoff = np.maximum(S_Ts - K, 0)
    discounted_price = np.exp(-r*T)*payoff
    price_estimate.append(discounted_price.mean())
    price_std.append(discounted_price.std()/np.sqrt(size))
```

[22]:
```python
plt.plot([fourier_call_val]*50, label='Fourier Call Price')
plt.plot(price_estimate, '.', label='Monte Carlo Estimates')
plt.plot(price_estimate + 3*np.array(price_std), 'r')
plt.plot(price_estimate - 3*np.array(price_std), 'r')
plt.legend()
plt.show()
```



5 Conclusion

A vanilla European call option was priced for a fluctuating volatility condition, using the Heston model. The call price was evaluated as $13.73, which was verified to be similar to - but lower than - a constant volatility estimate using the Black-Scholes model at $15.71.

Initially the values calculated for the underlying share price did not agree as expected, with the estimates well outside the expected error range. This was because the volatility term was not equivalent in the two different calculation methods. After resolving this, the CEV model calculated a price of $8.70 for the underlying share, which agreed closely with the Monte Carlo estimate of $8.69. This represents an absolute error of 1c and a relative discrepancy of approximately 0.1

The agreement between the results from various methods, although expected, provides confidence to choose the most suitable method for a situation with the knowledge that the results are accurate within a small margin of error.

References

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