

Math 417: Abstract Algebra

Deu et a tea

University of Illinois at Urbana-Champaign Abelian Groups

finite Abdian group.

$$G = \langle a_1, ..., a_n \rangle$$
 - finitely generated group $N \triangleleft G \rightarrow G/N = \langle a_1N, ..., a_nN \rangle$ wife as wirels \Rightarrow quotients of f.gen groups are finitely generated. Fact: If $a_n \in N$, then $G/N = \langle a_1N, ..., a_{n-1}N \rangle$ throw away the identity.

Gall-eN can be generated demands.

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If H \leq G-f.gen is H also f.gen?
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NO it does not have to be f.gen.

Prop: Thm. G is f.gen and abelian, then every subgroup $H \leq G$ is also f.gen.



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>> not solly required Lemma: If **H**-(abelian) group, $N \leq H$ (normal) subgroup If **N** and H/N are f.gen $\Rightarrow H$ is f.gen. Proof: Suppose $N = \mathbb{Z}\{x_1,...,x_m\}$ π induction $\pi: H \to H/N$ $\pi(y_i) = \bar{y}_i$ $extbf{\textit{H}/N} = \mathbb{Z}\{ar{y}_1,...,ar{y}_n\}$

<u>Pick.</u> For each \bar{y}_i pick $y_i \in \boldsymbol{H}$ such that $\bar{y}_i = y_i + \boldsymbol{N}$

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Claim: \mathbf{H} = \mathbb{Z}\{x_1,...,x_m,y_1,...,y_n\} is thate generated
Suppose: u \in \mathbf{H}
     Consider v := \pi(u) = b_1 \bar{y}_1 + ... + b_n \bar{y}_n \in H/N for some b_1 \in \mathbb{Z}
     \underline{\mathsf{Define}}: \ \widehat{\mathsf{v}} := 1 y_1 + \ldots + b_{\mathbf{h}} y_{\mathbf{h}} \in \mathbf{H}; \ \pi(\widehat{\mathsf{v}}) = \mathbf{v} - b_{\mathbf{h}} y_{\mathbf{h}} + \cdots + b_{\mathbf{h}} y_{\mathbf{h}} = \mathbf{v} 
Have: \pi(u) = v = \pi(\hat{v}) \dashv \pi(u - \hat{v}) = 0 \dashv u - \hat{v} \in \mathbb{N}
so u - \hat{v} = a_1x_1 + ... + a_mx_m for some a_1 \in \mathbb{Z}
   \exists u = \sum a_i x_i + \sum b_j y_j = \underline{\mathsf{done}}
                  H is a generating set!
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Proving The Subgroup H Is Finitely Generated

Pf. of thm:
$$G = \mathbb{Z}\{a_1, ..., a_n\}_{\mathcal{H}} \leq G$$
Use induction an $n = \text{size of a generating set of } G$
Base case: $n = 0$, $G = \mathbb{Z}\{\} = \{0\}$

$$n = 1, G = \mathbb{Z}\{a\}$$
we shared $\exists H$ is cyclic

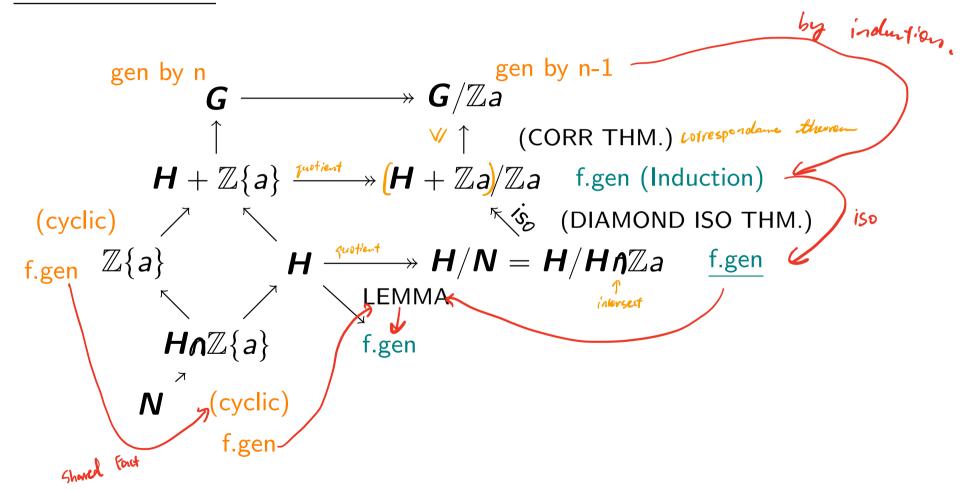
for the content of G

Basically, $G = \langle 8 \rangle \cong \mathbb{Z}_n$ has the subgraps to be cyclic.

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Induction Step

Induction Step: $n \ge 2$



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Consequence of Proof

Consequence.
$$SPIJ$$

Fact: G-f.gen abelian $= \exists m \neq 0, \Phi : \mathbb{Z}^m \rightarrow G$ sury homomorphism

 $N = \ker(\Psi) \triangleleft G$

Thm: N is f.gen, so $\exists n \neq 0, \bar{\alpha} : \mathbb{Z}^n \rightarrow N$

Sury hom.

Sury hom.

 $\alpha = i \cdot \bar{\alpha}$
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So $\alpha = \mathbb{Z}^m / \alpha(\mathbb{Z}^n) = \mathbb{Z}^n$

For some $\alpha \in Mat_{m \times n}(\mathbb{Z})$
 $\alpha = i \cdot \bar{\alpha}$
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Studying the Homomorphism from \mathbb{Z}^n to \mathbb{Z}^m

Observe: $\mathbf{A} \in \mathsf{Mat}_{m \times n}(\mathbb{Z})$, define

$$L_{m{A}}: \mathbb{Z}^n o \mathbb{Z}^m ext{ by } m{A} = (a_{ij}), a_{ij} \in \mathbb{Z}$$

$$L_{m{A}}ig((c_1,...,c_n)ig) := ig(\sum_{j=1}^n a_{1j}c_j, \sum_{j=1}^n a_{2j}c_j, ..., \sum_{j=1}^n a_{mj}c_jig)$$

i.e.
$$L_{m{A}} \left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right) = m{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} \in \mathbb{Z}^m$$

Claim: L_A is a homomorphism.

Studying the Homomorphism from \mathbb{Z}^n to \mathbb{Z}^m

To find out what this matrix is, we can pluy in sun

A

Also, any homomorphism $\alpha: \mathbb{Z}^n \to \mathbb{Z}^m$

is equal to $L_{\mathbf{A}}$ for a unique $\mathbf{A} \in \mathsf{Mat}_{m \times n}(\mathbb{Z})$

Furthermore
$$\mathbf{A} \in \mathsf{Mat}_{m \times n}(\mathbb{Z})$$

$$oldsymbol{B} \in \operatorname{\mathsf{Mat}}^n_{oldsymbol{n} imes oldsymbol{p}}(\mathbb{Z})$$

$$L_{B} \xrightarrow{L_{A}} L_{A}$$

$$\underline{Then:} \ L_{AB} = L_{A} \circ L_{B} \ \mathbb{Z}^{p} \xrightarrow{L_{AB}} \mathbb{Z}^{m}$$

Studying the Homomorphism from \mathbb{Z}^n to \mathbb{Z}^m

previously its just home.

Rem: Suppose
$$L_{\boldsymbol{A}}: \mathbb{Z}^n \to \mathbb{Z}^m$$
 is an isomorphism $[\boldsymbol{A} \in \mathsf{Mat}_{m \times n}(\mathbb{Z})]$
Then $(L_{\boldsymbol{A}})^{-1}: \mathbb{Z}^m \to \mathbb{Z}^n$ is also an isomorphism $L_{\boldsymbol{B}}$ for some $\boldsymbol{B} \in \mathsf{Mat}_{n \times m}(\mathbb{Z})$

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Smith Normal Form

[When $\mathbf{A} \in \mathsf{Mat}_{m \times n}(\mathbb{Z})$], \mathbf{A} is in Smith normal form if

$$\mathbf{A} = diag(d_1, d_2, ..., d_s), d_1 \leq 0, d_i | d_{i+1}$$

$$S = min(m, n)$$

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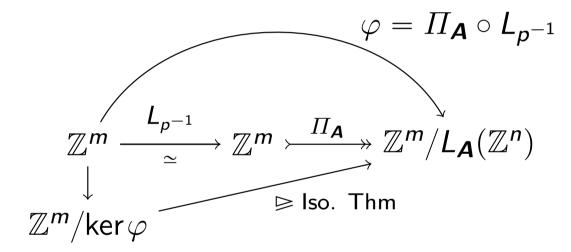
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Prop: If \mathbf{A} = diag(d_1, ..., d_s) \in \mathsf{Mat}_{m \times n}(\mathbb{Z}) [Smith Normal Form]
   \mathbb{Z}^m/L_{\mathbf{A}}(\mathbb{Z}^n) \simeq \mathbb{Z}/\mathbb{Z}_{d_1} \times ... \times \mathbb{Z}/\mathbb{Z}_{d_s} \times \mathbb{Z}^{m-s} = \mathbf{G} \  \, \boldsymbol{\leqslant} \,
   Proof: \mathbb{Z}^m \xrightarrow{\varphi} \mathbf{G} by
                  (x_1,...,x_s,x_{s+1},...,x_m) \mapsto ([x_1]_{d_1},...,[x_s]_{d_s},\underline{x_{s+1},...,x_m})
   This is surjective, \ker \varphi = L_{\mathbf{A}}(\mathbb{Z}^n) by (so
                             means: XSII..., I'm are all zero.
                                   and Kildi for i=1,...,s,
                          So [cer(Q) = linear comb of 2^n = L_A(2^n)
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Prop: If
$$B = PAQ$$
, A , $B \in \operatorname{Mat}_{m \times n}(\mathbb{Z})$

$$P, Q \quad \mathbb{Z} - \text{invertible}$$
then $\mathbb{Z}^m/L_A(\mathbb{Z}^n) \cong \mathbb{Z}^m/L_B(\mathbb{Z}^n)$.
Say that $A \backsim B$ "equivalent".
$$\Pi_A : \mathbb{Z}^m \to \mathbb{Z}^m/L_A(\mathbb{Z}^n)$$

$$\Pi_B : \mathbb{Z}^m \to \mathbb{Z}^m/L_B(\mathbb{Z}^n)$$

Pf: Consider



Claim: $\ker \varphi = L_{\mathbf{B}}(\mathbb{Z}^n)$

$$x \in \ker \varphi \iff P_x^{-1} \in \ker \Pi_A$$
 $\iff P_x^{-1} = Ay \text{ for some } y \in \mathbb{Z}^n$
 $\iff P_x^{-1} = AQz, \text{ for some } z \in \mathbb{Z}^n \ [z = Q^{-1}y]$
 $\iff x = PAQz = Bz \text{ for some } z \in \mathbb{Z}^n$
 $\iff x \in L_B(\mathbb{Z}^n)$

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