Final - Part II

Michelle Chung

August 9, 2018

Arc Length

To find an a and b that obeyed the provided constraints: $a < b$, (a, b) and (b, a) pass through the curve, and the arc length of the curve the points (a, b) and $(b, a) = 1 + \beta$, I began with creating the following functions:

- `I` is the function of the curve expressed in terms of y, or

$$y = \frac{1}{\sqrt[0.2324]{1 - \frac{1}{x^{0.2324}}}}$$

- `I_prime` returns $\frac{dy}{dx}$ at the inputted x-value
- `arc_length` computes the value of the at the provided x-value
- `sum` is used to compute the arc length over the provided interval $[a, b]$
- `test` is used to test the arc length based on a test iterate value that produces an $[a, b]$ interval (explained in detail below)
- `trapezoid` is used to integrate a provided function at a provided interval.

Using these functions, I:

1. Chose an initial iterate, 19.7830 and stored it in the `double` type variable, `test_iterate`. The way I chose this iterate was by recognizing that all pairs of points p_1, p_2 along the curve that obey the constraint $p_1 = (a, b)$ and $p_1 = (b, a)$ can be derived from the line $y = x$. At the point x_i, $a = \mathtt{I}(x)$ and $b = x$. 19.7830 was the first x value to produce a pair.
2. Chose a step size, .0001.
3. In my main function, I created a while loop. Inside the while loop, I tested the arc length beginning at the initial iterate. I then added .0001 to that iterate value until the arc length converged to $1 + \beta$ (1.4314).

My answers were as follows:

$$a = 16.4470 \qquad b = 23.8804 \qquad n = 40,974 \qquad t = 1.6s$$

Where n and t are the number of iterations and time until convergence.

Proof in Closed Form

$$s = \int \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$u = (\frac{dy}{dx}) \text{ and } du = dx$$

$$= \int (1 + u^2)^{\frac{1}{2}} du$$

$$u = sinh(\theta) \text{ and } du = cosh(\theta) d\theta$$

$$= \int (1 + sinh^2(\theta))^{\frac{1}{2}} cosh(\theta) d\theta$$

$$= \int cosh^2(\theta) d\theta$$

$$cosh^2(\theta) = \frac{1 + cosh(2\theta)}{2}$$

$$= \frac{1}{2} \int 1 + cosh(2\theta) d\theta$$

$$= \frac{1}{2}(\theta + \frac{1}{2} sinh(2\theta)$$

$$= \frac{1}{2} sinh^{-1}(u) + \frac{1}{2} u\sqrt{1 - u^2}$$

$$= \left[\frac{1}{2} sinh^{-1}(\frac{dy}{dx}) + \frac{1}{2}(\frac{dy}{dx})\sqrt{1 + (\frac{dy}{dx})^2} \right]_{16.4470}^{23.8804}$$

$$\approx 1.4314$$

Area

To find the area, A of the intersection between the "square" and the curve, I took the area of the rectangle and subtracted from it the integral of the curve at the interval of their intersection, $[16.4470, 23.8804]$.

$$A_{square} = 177.5126 \qquad A_{curve} = 145.6800$$

$$A_{square} - A_{curve} = 31.8326$$

Proof in Closed Form

While the computation of the area of a rectangle is easy, there is no closed form integral of the curve function.