Final - Part III

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We examine the linear homogeneous second order ordinary differential Bessel's equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0 \qquad \text{where } v = 1.0.$$

We employ uniform step sizes h and $n = 19$ subintervals such that $h = (x_r - x_l)/n = (20 - 19)/19 = 1$. Our subintervals are defined as a set of $n + 1$ points x_i where $x_0 = 1$ and $x_n = 20$. Using centered finite differences, we can approximate the derivatives for $i = 1 \ldots n - 1$.

$$y' = \frac{y_{i+1} - y_{i-1}}{2h} \qquad y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad \text{where } y_i = y(x_i).$$

To solve for $y(x)$ in the interval $[x_0, x_n]$, we first substitute the original equation with the finite difference formulas.

$$(x_i^2)\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + (x_i)\frac{y_{i+1} - y_{i-1}}{2h} + (x_i^2 - 1.0)y_i = 0$$

Multiply through by h^2 and collect like terms:

$$(x_i^2)(y_{i+1} - 2y_i + y_{i-1}) + \frac{h}{2}(x_i)(y_{i+1} - y_{i-1}) + h^2(x_i^2 - 1.0)y_i = 0$$

$$\left[x_i^2 + \frac{h}{2}(x_i)\right] y_{i+1} + \left[x_i^2 - \frac{h}{2}(x_i)\right] y_{i-1} + \left[h^2x_i^2 - h^2 - 2x_i^2\right] y_i = 0$$

And coerce the coefficients into the form:

$$h^2x_i^2 - h^2 - 2x_i^2 = a_i, \qquad x_i^2 - \frac{h}{2}(x_i) = b_i, \qquad x_i^2 + \frac{h}{2}(x_i) = c_i$$

To get the following system of linear equations:

$$b_i(y_{i-1}) + a_i(y_i) + c_i(y_{i+1}) = 0$$

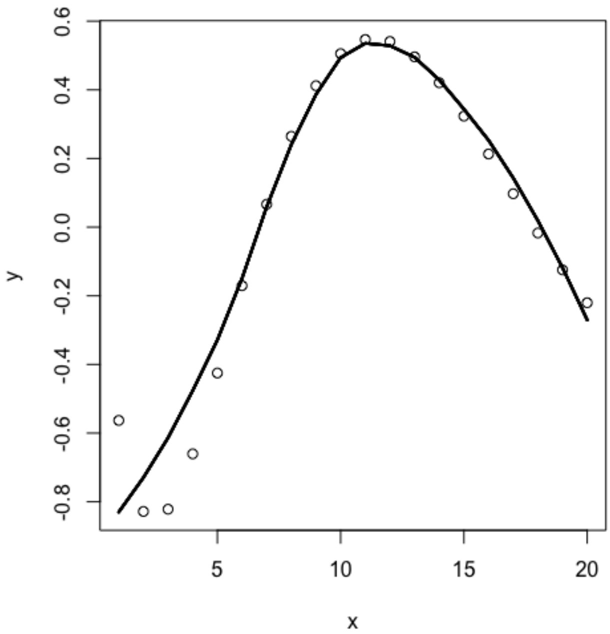
With, with the provided boundary conditions:

$$y_0 = .2324 \qquad \left[\frac{dy}{dx}\right]_{x=x_n} = .4314$$

Can be represented in the following tridiagonal matrix:

$$\begin{bmatrix} 1 & 0 & 0 & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & b_{i-1} & a_{i-1} & c_{i-1} & & & & \\ & & & b_i & a_i & c_i & & & \\ & & & & b_{i+1} & a_{i+1} & c_{i+1} & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_{i-1} \\ y_i \\ y_{i+1} \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} .2324 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ .4314 \end{bmatrix}$$

The solution to the system is plotted below at the interval $[1, 20]$.



Since we have only an approximation for $y(x)$, to find the root of $y(x)$ to a tolerance of .0001, I chose to reproduce the `tridiagonal` algorithm at the interval $[a, b] = [6, 7]$, where $y(6) = -0.170576$ and $y(7) = 0.0666176$. I took the interval $[a, b]$ and split it into $m = 10{,}000$ equally spaced points, and created an `x_root` populated with these new subintervals. I also created new `a_root`, `b_root`, `c_root`, and `rhs_root` vectors and called the tridiagonal matrix with these updated inputs.

To find x such that $y(x) = 0$, I employed the `bisection` algorithm, which iterated only once to return $x = 6.5$ as the root.