Final - Part I

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My approach to the problem was to find the simplest/naive way to calculate the answer for the specific parameters (x_1 and x_2) provided in the question prompt. The reason I chose simplicity over robustness is because the computation of the integrals of ellipses is not simple and usually requires double or triple integration, which I found unnecessary for finding a solution for the provided equations.

$$e_1 := (x - 0.2324)^2 + 2y^2 = 1$$
$$e_2 := (x - 0.4314)^2 + 4y^2 = 1$$

1. Through visual inspection, I could see that the area of intersection, A, was bounded by $e_1 \leq x_A$ and $e_2 > x_A$ where $x_A := e_1 = e_2 = 1.0722$. The domain of integration was then calculated to be:
$$x_{min} = -0.5686$$
$$x_{max} = 1.2324$$
2. I then derived e_1 and e_2 in terms of y and created two functions, `e_1` and `e_2`. These functions represent the ellipses in the positive y-axis only.
3. To find A, I used the `trapezoid` function to integrate `e_1` from $x_{min} \leq x_i \leq x_A$ and `e_2` from $x_A < x_i < x_{max}$ with $n = 5000$.
4. Finally, I multiplied the result by 2 (to account for the area of the ellipse in the negative y-axis) and calculated the area to be:
$$A = 1.4626$$

The area of e_1 is ≈ 2.2212.
The area of e_2 is ≈ 1.5706.

The functions `closed` and `integrate` were used to derive the sums for the closed form proof.

Proof in Closed Form

We solve for e_2 in terms of y_2 (in the positive y-axis) and integrate y_2 from $a = -0.5686$ to $b = 1.0722$.

$$
\begin{aligned}
y_2 &= (.25)^{\frac{1}{2}}(1-(x-.4314)^2)^{\frac{1}{2}} \\
\int y_2 dx &= \int (.25)^{\frac{1}{2}}(1-(x-.4314)^2)^{\frac{1}{2}} dx \\
&= (.25)^{\frac{1}{2}} \int (1-(x-.4314)^2)^{\frac{1}{2}} dx
\end{aligned}
$$

$u = (x - .4314)$ and $du = dx$

$$
= \frac{1}{2}\int (1-(u)^2)^{\frac{1}{2}} du
$$

$u = sin(\theta)$ and $du = cos(\theta)d\theta$

$$
\begin{aligned}
&= \frac{1}{2}\int (1-sin^2(\theta))^{\frac{1}{2}} cos(\theta)d\theta \\
&= \frac{1}{2}\int (cos^2(\theta))^{\frac{1}{2}} cos(\theta)d\theta \\
&= \frac{1}{2}\int cos^2(\theta)d\theta
\end{aligned}
$$

$$
cos^2(\theta) = \frac{1+cos(2\theta)}{2}
$$

$$
\begin{aligned}
&= \frac{1}{4}\int 1 + cos(2\theta)d\theta \\
&= \frac{1}{4}(\theta + \frac{1}{2}sin(2\theta) \\
&= \frac{1}{4}sin^{-1}(u) + \frac{1}{4}u\sqrt{1-u^2} \\
&= \left[\frac{1}{4}sin^{-1}(x-.4314) + \frac{1}{4}(x-.4314)\sqrt{1-(x-.4314)^2}\right]_{-0.5686}^{1.0722} \\
&\approx 0.6896
\end{aligned}
$$

Then, we solve for e_1 in terms of y_1 (in the positive y-axis) and integrate y from $a = 1.0722$ to $b = 1.2324$.

$$
\begin{aligned}
y_2 &= \sqrt{.5}(1-(x-.2324)^2)^{\frac{1}{2}} \\
\int y_2 dx &= \sqrt{.5}\int (1-(x-.2324)^2)^{\frac{1}{2}} dx \\
&\vdots \\
&= \left[\frac{\sqrt{.5}}{2}sin^{-1}(x-.2324) + \frac{\sqrt{.5}}{2}(x-.2324)\sqrt{1-(x-.2324)^2}\right]_{1.0722}^{1.2324} \\
&\approx 0.0417
\end{aligned}
$$

We add the sums and multiply by two (to account for the area in the negative y-axis), which results in the answer:

$$
A = 1.4626
$$