Homework Set 2

Due Tuesday, 3/12 at 23:59pm.

Problem 1: Answer the following questions about Jordan forms:

a) If $A \in \mathbb{R}^{n\times n}$ with characteristic polynomial $\chi(\lambda) = (x-\lambda_1)^{n_1}\cdots(x-\lambda_r)^{n_r}$ for some $r \in \mathbb{N}$, what is the trace of A?

b) How many possible Jordan forms are there for a 6×6 complex matrix with characteristic polynomial $(x+2)^3(x-1)^2(x+1)$?

c) Let $A \in \mathbb{R}^{n\times n}$ have eigenvalue λ and define

$$V_\lambda \triangleq \{\mathbf{v} \in \mathbb{R}^n : (A-\lambda I)^k \mathbf{v} = 0 \text{ for some } k$$

Show there always exists an $k \in \mathbb{N}$ such that $V_\lambda = \operatorname{Ker}(A-\lambda I)^k$.

d) Show that, for the value of $k \in \mathbb{N}$ whcih satisfies part c, $\operatorname{Im}(A-\lambda I)^k$ and $\operatorname{Ker}(A-\lambda I)^k$ are subspaces of $\mathbb{R}^n$ that are invariant under A. Then use the Range-Nullspace decomposition theorem to show

$$\mathbb{R}^n = \operatorname{Im}(A-\lambda I)^k \oplus \operatorname{Ker}(A-\lambda I)^k$$

is a decomposition of $\mathbb{R}^n$, where $\oplus$ denotes the direct sum.

e) How would you generalize part d to show that

$$\mathbb{R}^n = \operatorname{Ker}(A-\lambda_1 I)^{n_1} \oplus \operatorname{Ker}(A-\lambda_2 I)^{n_2} \oplus \cdots \oplus \operatorname{Ker}(A-\lambda_r I)^{n_r}$$

A few sentences of explanation will suffice; no formal proof is necessary.

Problem 2: Show that if all eigenvalues of $A \in \mathbb{R}^{n\times n}$ are distinct, then $(sI-A)^{-1}$ can be expressed as

$$\sum_{i=1}^{n} \frac{1}{s-\lambda_i}\mathbf{q}_i\mathbf{p}_i$$

where $\mathbf{q}_i, \mathbf{p}_i \in \mathbb{R}^n$ are the right and left eigenvectors, respectively, of A associated with λ_i.

Problem 3: For the LTV system $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$, show that

$$\|\mathbf{x}(t)\| \le \|\mathbf{x}_0\| \exp\left(\int_{t_0}^{t} \|A(s)\|\, ds\right), \quad \forall\, t \ge t_0$$

Problem 4: Consider the uncontrolled LTI system of the form

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} a & 2 \\ -2 & -1 \end{bmatrix} \mathbf{x}(t)$$

where $a \in \mathbb{R}$.

a) Express the trace, determinant, characteristic polynomial, and eigenvalues in terms of a.

b) For specific values of $a \in \{-6, -2, 1, 2, 3, 4, 5\}$, classify the stability of the system modes (stable, marginally stable, unstable).

c) (*MATLAB Coding*) For each value of a in part b), plot the solution trajectories

i. by computing the matrix exponential and directly plotting $\mathbf{x}(t)$.

ii. by discretizing the system with $\Delta t = 0.01$ and simulating the difference equation.

For each plot i. and ii., make two types of figures. The first figure is a plot of $x_1(t)$ and $x_2(t)$ versus time; the second figure is a plot of $x_1(t)$ versus $x_2(t)$. Organize all your plots in a 7×2 collection of subplots, with labeled legends where applicable. Make sure to choose various different initial conditions.
(The second type of figure is called a *phase portrait*, and we will be seeing more on this when discussing stability in future lectures.)

Problem 5: Use the three methods we discussed in class to discretize the following continuous-time system with $\Delta t = 0.1$. You may use MATLAB if you'd like.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

Problem 6: Compute the fundamental and state transition matrices of the following systems:

a) $\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} \mathbf{x}(t)$ b) $\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \mathbf{x}(t)$ c) $\dot{\mathbf{x}}(t) = \begin{bmatrix} \sin t & \cos t & 1 \\ 0 & \sin t & \cos t \\ 0 & 0 & \sin t \end{bmatrix} \mathbf{x}(t)$