Δ– and σ–Hedging under Stochastic Volatility

Erdös Quantitative Finance – Mini Project 4

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Agenda

Why revisit delta hedging?

- Classic short–call strategy: sell European call, hold Δ shares each day.
- Works if volatility σ is *constant*; otherwise residual P&L $\approx -\text{Vega}\, d\sigma$.
- Goal: quantify that residual risk and test a σ-hedge (second option) under realistic σ_t dynamics.

Three models for σ_t

- **Discrete σ (toy)**

$$\sigma_t \in \{0.20,\ 0.30,\ 0.45\} \quad \text{i.i.d. each day}$$

- **Heston stochastic volatility**

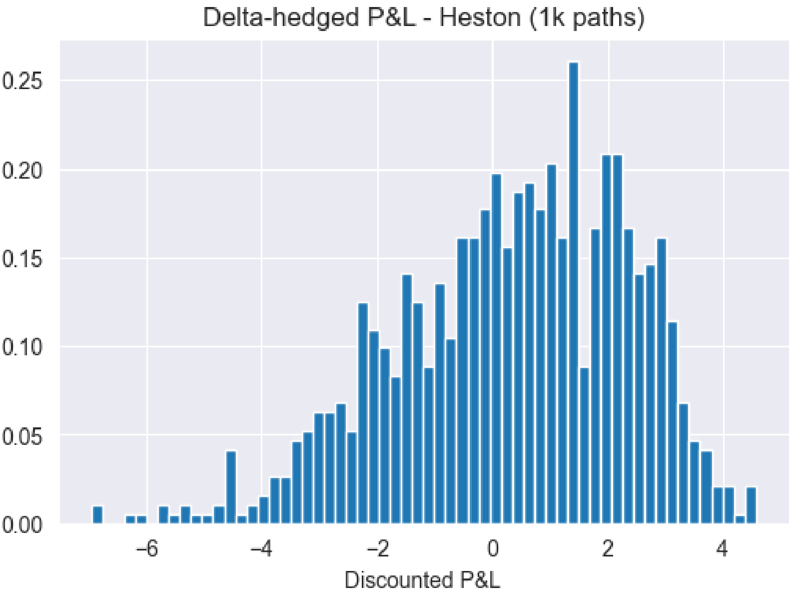
$$dS_t = \mu S_t\, dt + \sqrt{v_t}\, S_t\, dW_t^{(1)},$$
$$dv_t = \kappa(\theta - v_t)\, dt + \xi\sqrt{v_t}\, dW_t^{(2)}.$$

- **GARCH(1,1)**

$$\sigma_t^2 = \omega + \alpha\, \varepsilon_{t-1}^2 + \beta\, \sigma_{t-1}^2$$

- Daily re-hedge; $S_0 = K = 100$, $r = 2\%$, $T = 1$ yr.
- 1 000 paths for quick demo, 30 000 for production stats.

Heston – P&L distribution



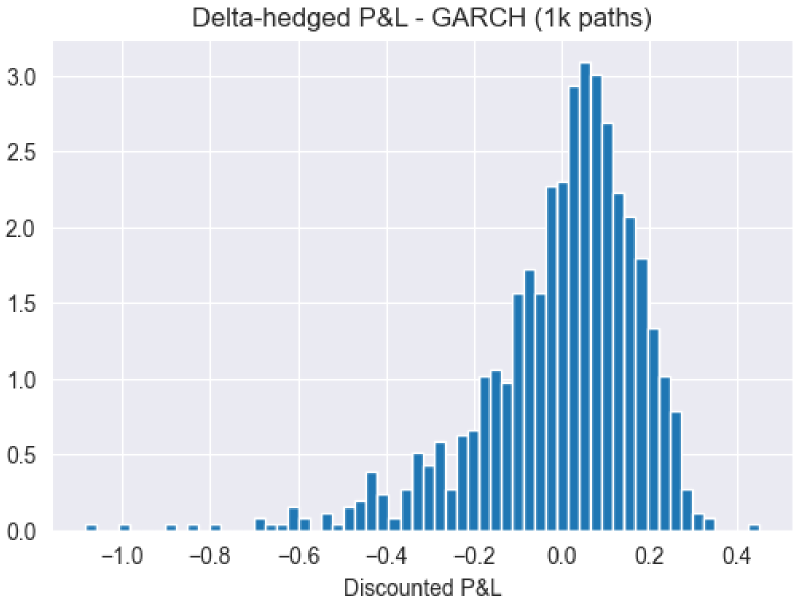
Heston – P&L distribution

Summary (1 000 paths)

Mean $= +0.33$ Std $= 2.03$ Fat left tail ($\sim$–6)

- High vol-of-vol ($\xi = 0.35$) $\rightarrow$ wide dispersion.
- Positive mean here is sampling noise; 30 k paths $\rightarrow$ small *negative* drift (short-vega bias).

GARCH – P&L distribution



GARCH – P&L distribution

Summary (1 000 paths)

Mean $= -0.003$ Std $= 0.19$ Moderate skew

- Variance shocks decay ($\beta = 0.93$) $\rightarrow$ much tighter risk.
- Left tail visible only with larger sample.

Key take-aways

- **Residual risk = Vega $\times$ vol-of-vol.** Heston (large ξ) $\rightarrow$ big P&L swings; GARCH (smaller shocks) $\rightarrow$ tame.
- **Mean P&L sign** depends on realised vs. implied σ:

$$\mathrm{E}[P\&L] \approx (\sigma_{\text{imp}} - \sigma_{\text{realised}}) \times \text{Vega}$$

- σ-**hedge** (add one more option, solve Vega=0 each rebalance) shrinks std-dev by $\sim$ 70–80%, at the cost of trading another instrument.

What we can explore next

- *Parameter sweep*: κ, ξ in Heston; α, β in GARCH.
- *Leverage effect*: EGARCH or correlated price/vol shocks.
- *Transaction costs*: add bid–ask spreads, evaluate breakeven of σ-hedge.
- *Intraday hedging*: lower Δt for high-frequency strategies.