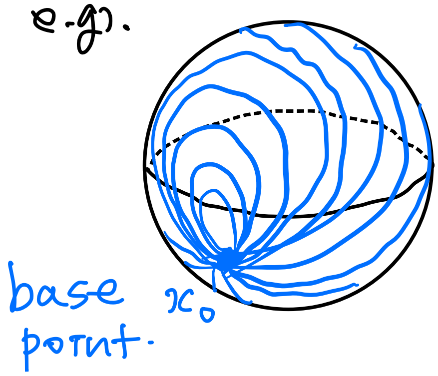
20220908(목).

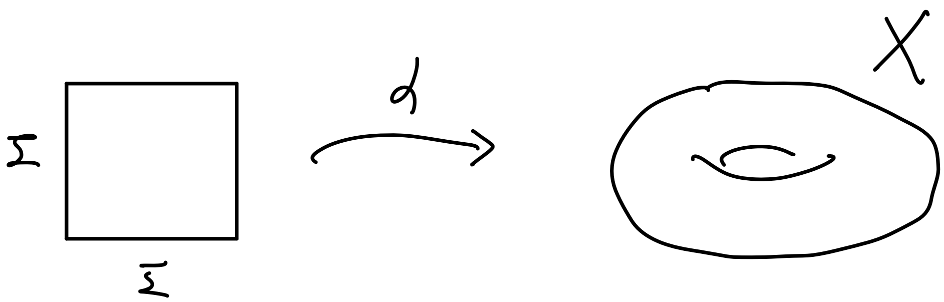
"Homology, Computations."

Introduction

e.g.).



Fundamental group of S^2, $\pi_1(S^2, x_0)$, captures 2-dimensional hole in the sphere S^2.



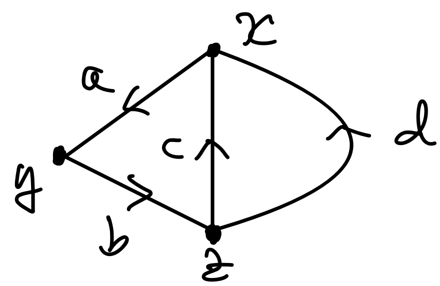
- More complicated & difficult to compute
- $\pi_n(S^k)$, $n > k$ → measuring higher dim'l hole in the k-sphere.
  exist (and quite irregular & bizarre).
- So, what we want an alternative to homotopy gps?

"Homology is a commutative alternative to homotopy."

- X : top. sp. $\Rightarrow$ $H_n(X)$: homology groups. $n = 0, 1, 2, \cdots$ all comm. gps.
- $H_n(X) \approx$ measures # of n-dim'l holes in X. (somehow...)
- We will introduce the ideas via some examples.

$\boxed{1}$ X_1 : graph

$\begin{cases} \text{vertices } x, y, z \\ \text{edges } a, b, c, d. \end{cases}$



① loop, $\begin{cases} abc & : \text{starts \& ends } x. \\ bca & : \quad '' \quad\quad '' \quad y. \\ cab & : \quad '' \quad\quad '' \quad z. \end{cases}$

Then we can deduce that

"Commutativity $\Rightarrow$ doesn't matter where
(implies that) we start.

Also, we prefer to write operation as addition, so loop is represented by

$\boxed{a+b+c} = b+c+a = c+a+b$ etc.

We call this expression a "cycle",
because it is a closed loop, geometrically.

- Another cycle : $\boxed{c-d}$
- Another is : $\boxed{a+b+d}$

- Let C_0 = free abelian group on vertices x, y, z.

  C_1 = free abel. gp. on (directed) edges a, b, c, d.

- Elements of C_0 are integer linear combinations of x, y, z such as $2x - 3y + 4z$.

(We call these 0-dim chains.)

- Elements of C_1 are integer linear combis. of a, b, c, d such as $a+b+d$ or $2a+b-5c+4d$.

(We call these 1-dim chains.)

Q. What do we mean by a cycle algebraically?

i.e., whats special about $a+b+d$?

Not shared by $2a+b-5c+4d$?

A. Relationship between edges & vertices.

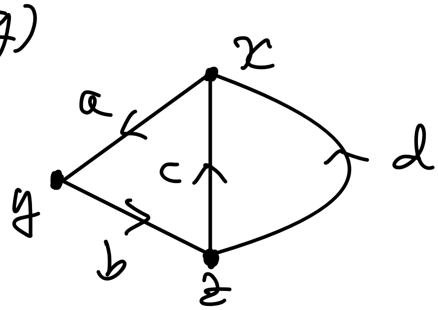
[We are going to express that relationship in terms of a "boundary".

Talking about boundary of an edge]

- Each edge has a $\underbrace{\text{boundary}}$.

" boundary is crucial in homology "

e.g.)



$$\left.\begin{aligned} \partial(a) &= y - x \\ \partial(b) &= z - y \\ \partial(c) &= x - z \\ \partial(d) &= x - z . \end{aligned}\right\} (*).$$

In general, $\partial(\text{edge}) = \text{final point} - \text{initial point}.$

∇ RHS $(= \partial(\text{edge}))$ = combinations of vertices. (i.e., 0-chain) Δ

$\partial : C_1 \longrightarrow C_0$, gp. homomorph.

which extends $(*)$

- $\partial(\alpha a + \beta b + \gamma c + \delta d)$

$\underset{\text{gp. hom.}}{=} \alpha \partial(a) + \beta \partial(b) + \gamma \partial(c) + \delta \partial(d)$

$= \alpha(y - x) + \beta(z - y) + \gamma(x - z) + \delta(x - z)$

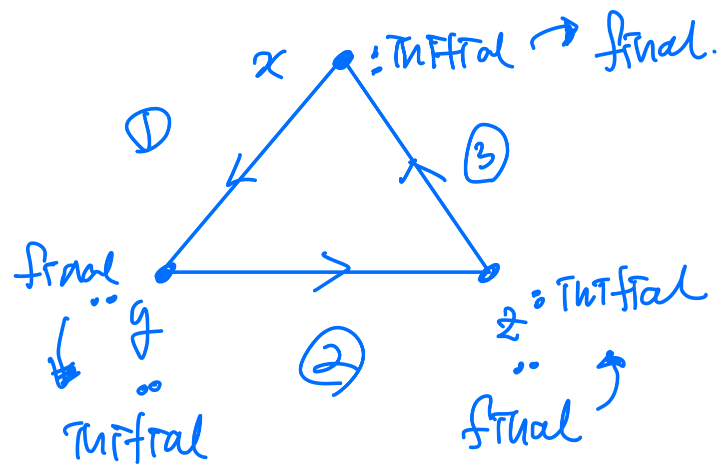
$= (-\alpha + \gamma + \delta) x + (\alpha - \beta) y + (\beta - \gamma - \delta) z$

∵ Vector Gp.는 아니어도 선형대수학에서처럼 계산하면 됨 (주의: 계수들은 모두 정수).

Note.

$$\partial(a+b+c) = (y-x)+(z-y)+(x-z)$$
$$= 0$$

∵



⇒ So, $\partial(\text{cycle})$ 계산하면 vertex들이 연쇄적으로 소거됨

⇒ ∂ : boundary의 계산은 cycle인지 아닌지를 알 수 있는 좋은 도구!!.

Def. A chain t in C_1 is a cycle.
$$\iff \partial(t) = 0$$

So... $\alpha a + \beta b + \gamma c + \delta d$ is cycle !!

$$\Longleftrightarrow \begin{cases} -\alpha + \gamma + \delta = 0 \\ \alpha - \beta = 0 \\ \beta - \gamma - \delta = 0 \end{cases}$$

, 연립방정식의 해집합.

Q. What are all solutions? $Ax = 0$

(just augmented matrix computations).

Aug. matrix $A = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \alpha = r + s = \beta, \quad \gamma = r, \quad \delta = s.$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} r + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} s$$

null sp. of A. $= \underbrace{\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle}_{\text{basis.}}.$

i.e., $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \xleftrightarrow[\text{corresp.}]{} a+b+c$

$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \xleftrightarrow[\text{corresp.}]{} a+b+d.$

these cycles are generators of C_1

$\partial : C_1 \longrightarrow C_0$, we've found that

$\ker \partial =$ "cycles"

$= \langle a+b+c,\ a+b+d \rangle$

the cycles as a gp. $\cong \mathbb{Z} \oplus \mathbb{Z}$.

$H_1(X) =$ "cycles" $= \mathbb{Z} \oplus \mathbb{Z}.\ (\mathbb{Z}^2)$

↳ measures the # of 1-dim holes

(: cycle들의 결합으로 만들 수 있는 최대한의 개수).

(Note that $\underset{\substack{\parallel \\ (a+b+c)-(a+b+d)}}{\underline{c-d}}$ is in this ker, too.)

Definition (tree, spanning tree).

$G = (V, E)$. 그래프
(V: vertex, E: edge)

(1) G = 연결그래프 (connected graph)

$\overset{def}{\Longleftrightarrow}$ $\forall x, y \in V$: distinct, $\exists$ (x, y)-path.

(2) $G_1 = (V_1, E_1)$: 부분그래프 (subgraph)

$\Longleftrightarrow$ $V_1 \subseteq V$ and $E_1 \subseteq E$

(3) G_1 : 생성부분그래프 (spanning subgraph)

$\Longleftrightarrow$ G = subgraph s.t. $V_1 = V$

(즉, G의 모든 vertex들을 포함하는 부분그래프)

(4) T : 수형도 (tree)

$\Longleftrightarrow$ T : connected

s.t. $\forall T_1 \subseteq T$: subgraph, $T_1 \neq$ cycle.

(즉, 회로를 부분그래프로 갖지 않는 연결 그래프)

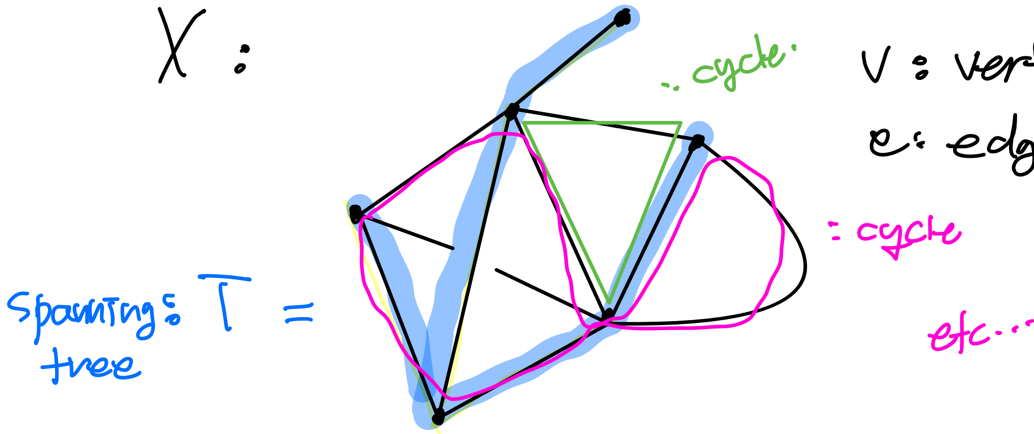
(5) T : 생성수형도 (spanning tree)

$\Longleftrightarrow$ $T \subseteq G$: spanning subgraph

s.t. T = tree.

Q. What about a more general graph?

X :



V : vertices
e : edges.

How many cycles we will expect in such graph?

Q. What is $H_1(X)$?

: $V = 6$, $e = 10$

$\Rightarrow$ Big augmentation matrix.

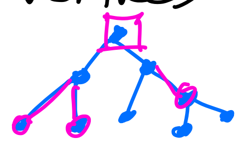
Thm. (Graph theory)

Every connected graph X has a spanning tree T which is a subgraph which includes all vertices.

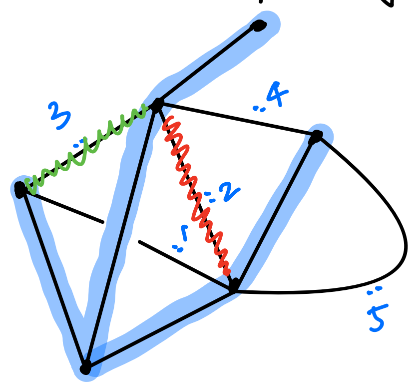
$$\Rightarrow \quad H_1(X) = \mathbb{Z}^{e - v + 1}$$

Note.

1. A tree with V vertices always has $V-1$ edges.



2. There are $e-(V-1)$ edges not in the spanning tree.



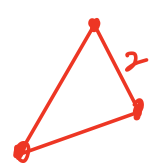
$1 \sim 5$ edges $\notin T$

spanning tree.

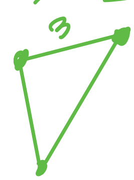
$\therefore$ The edges not in T gives us cycles!!

How? For example, 2-edge case..

sub T + edge ② = cycle



sub$T$ + edge ③ = cycle



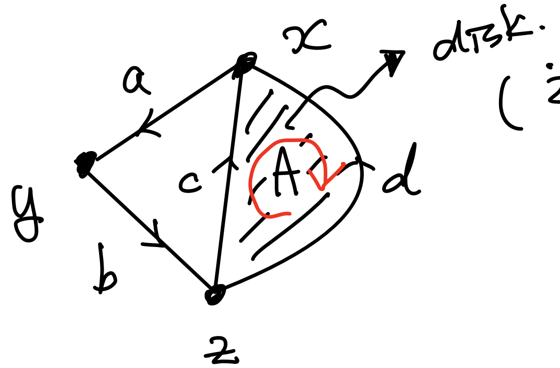
- Every non-used edge gives us a cycle in the original graph X.

- And these cycles are all sort of obviously independent, because each one uses different on used edge.

$\therefore$ # of cycles $= \underbrace{e - v + 1}$

spanning tree T에 포함되지 않은 edge의 개수.

$\boxed{2}$ X_2 :



Adding a 2-dim'l cell A between edges c, d. Oriented A clockwise.

So, $\partial(A) = c - d$: The cycle $c-d$ is now the boundary of this 2-cell (or disk).

C_2 : formal integer combs. of A : 3A, −5A (2-dim'l chains).

$$C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_1} C_0$$

$\partial_1 : C_1 \to C_0$

$\partial_2 : C_2 \to C_1$

- The presence of the 2-cell A means that the cycle $C-d$ is now homotopic to a constant (0) : no longer captures a hole.

- <u>Algebraically</u>: want $c-d=0$.

But then, $a+b+c = a+b+d$.

We need to quotient the gp. of 1-dim'l. cycles (generated by $a+b+c$, $a+b+d$) by the subgp. of 1-dim'l boundaries (generated by $c-d$).

– Homology is not just "cycles"

(boundary도 고려해야 할 필요 有).

$$H_1(X_2) = Z_1 / B_1$$

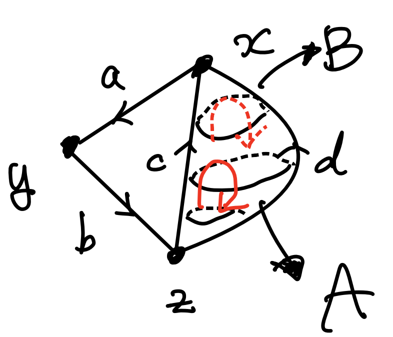
$Z_1 \to$ 1-dim cycles

$B_1 \to$ 1-dim boundaries.

$$H_1(X_2) \cong \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z} \quad \left(= \langle a+b+c,\ a+b+d \rangle / \langle c-d \rangle \right)$$

$$\cong \mathbb{Z}.$$

$\boxed{3}$ X_3



Introduce another 2-cell B between c, d.

$$\partial(B) = c - d = \partial(A)$$

Now with X_3 : $C_2 = \langle A, B \rangle$ integer linear combs. e.g. $3A - 4B$.

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

$$\partial_2(A) = c - d = \partial_2(B)$$

$$H_1(X_3) = \frac{Z_1}{B_1} = \frac{\operatorname{Ker} \partial_1}{\operatorname{Im} \partial_2}$$

← cycles ($\partial(t)=0$ 인 t들)

← boundaries ($\partial(\cdot) = t$ 로 표현되는 t들).

$$\cong \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z}$$

$$\cong \mathbb{Z}$$

same as $H_1(X_2)$.

Now, ∂_2 has a non-trivial kernel.

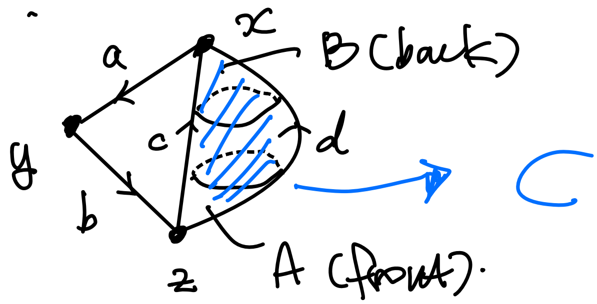
$$\ker \partial_2 = \langle A - B \rangle.$$

$\therefore$ $A - B$ is a 2-dim cycle (in C_2) generating second homology $H_2(X_3)$.

i.e., $H_2(X_3) = \langle A - B \rangle$
$\cong \mathbb{Z}$

Algebraically "$A-B$" is capturing that 2-dim hole.

$\boxed{4}$ X_4.



Let's attach a 3-cell C $\underbrace{\text{ball}}_{=\text{ solid ball.}}$ (in 3-dim sp.) C along the 2-sphere formed by A & B.

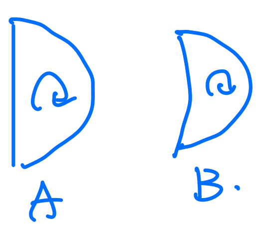
$$\begin{array}{ccccccc} C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\ \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\ \langle C \rangle & & \langle A, B \rangle & & \langle a, b, c, d \rangle & & \langle x, y, z \rangle . \end{array}$$

Orient C so $\partial(C) = A - B$.

We want to choose this

이렇게 주듯이..

(예)



각 orientation 이렇는데

서로 맞닿는 곳은 상쇄되도록 향 정하니까...

(this ensure that the boundary is a cycle).

$$H_1(X_4) = H_1(X_3) \cong \mathbb{Z}$$

$$H_2(X_4) = Z_2 / B_2 = \frac{\ker \partial_2}{\operatorname{Im} \partial_3}$$

$$= \frac{\langle A-B \rangle}{\langle A-B \rangle} = 0$$

(gp. w/ 1-elem.)

안을 채워버리면, homology shrunk 됨. (안 채워진 아이는 2-dim'l hole 아니지...)

지금까지 했던 작업들을 more formally ..

18, 19 C ~

(1) <u>Simplicial homology</u>

(: where we deal with spaces of a relatively prescribe kind.)

= spaces built from simplices

(triangles + ... in higher dims.).

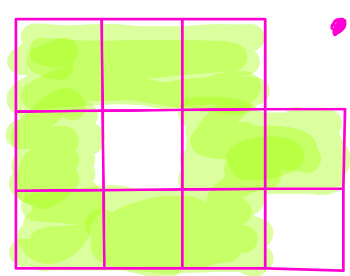
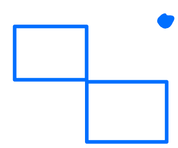
$\Downarrow$ more flexible kind of homology that doesn't oriented self towards such rigid spaces called a "singular homology"

(2) <u>Singular homology</u>

: more flexible, complicated, but less combinatorial...

But for most spaces, the two theories end up giving the same homology groups.

<u>Ex.</u> (Image pixels)

 $\simeq$ 

$H_0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$H_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$H_n \cong 0 \quad (n \geq 2)$

Lec. 32. Simplices & simplicial complexes

are generalizations of triangles.

are naturally very simple basic objects
& we can use to construct a space

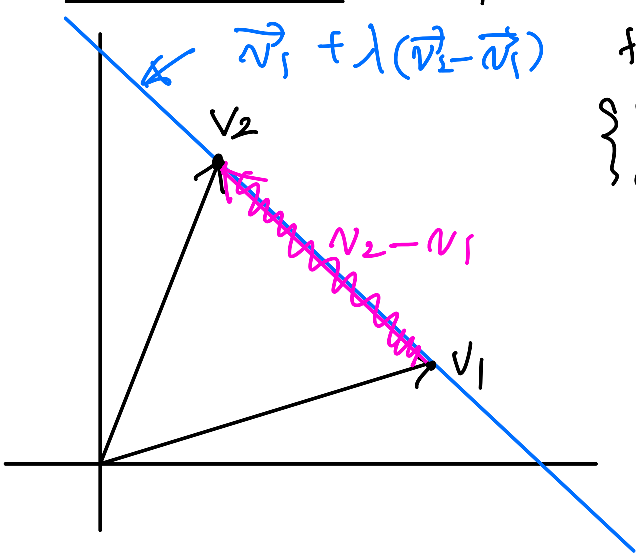
(higher dim'l generalization of a triangle)

- Triangles → higher dimensions → Simplices.

Motivation.

Convex combinations & hull

$\boxed{1}$ 1 - dim. ($\subset$ 2-dim v. sp)



$\exists$ 2 end pts. of two vectors v_1, v_2.

then we know that
① v_1, v_2 : linearly indep.
② we can write every vector in the plane as a linear combs. of v_1, v_2.

Consider special linear combination. the line through the v_1, v_2. (two pts.)

the line determined by two vectors v_1, v_2.

That is characterized by certain types of linear combinations.

The line through the end pts. of v_1, v_2

: $v_1 + \lambda(v_2 - v_1)$

$= \underline{(1-\lambda)v_1 + \lambda v_2}$

That we recognized are we're getting a linear combination of a particular kind.

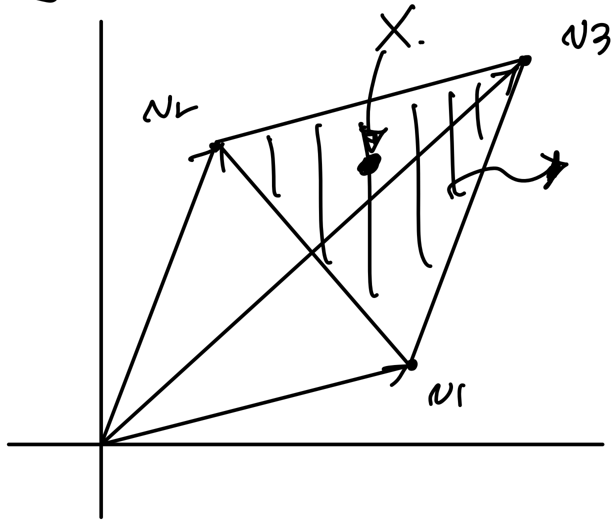
where the $\boxed{\text{sum of coefficients} = 1.}$

That's the characterization of the line

- $\underline{\text{If}}$ $0 \le \lambda \le 1$ $\underline{\text{then}}$ we describe the line segment between $\vec{v}_1, \vec{v}_2$.
- $\underline{\text{If}}$ $\lambda \sim 0$ close to v_1.
  $\lambda \sim 1$ close to v_2.

So, λ is some kind of measure of how far apart where we are on the segment.

$\boxed{2}$ 2-dim $\leftarrow$ 3-pts. (= the end pts. of vectors v_1, v_2, v_3).



$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

where $\begin{cases} 0 \le \lambda_i \le 1 \\ \sum \lambda_i = 1 \end{cases}$

- $(\lambda_1, \lambda_2, \lambda_3)$ called barycentric coordinates for point in $\triangle$.

$\rightarrow$ This is a one way that think about creating higher dimensional analogues of triangles

: 2-pts of linear combination.

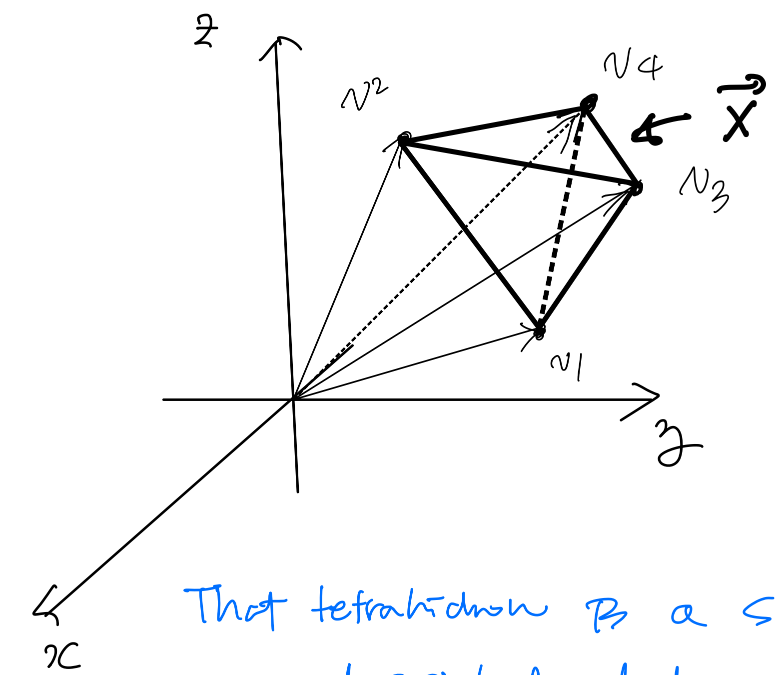
$\Downarrow$

3-pts of Convex combination.

always coeff $\in [0, 1]$, $\sum$ coeff $= 1$.

: This is a one way of generalizing what a triangle is to a notion of simplex in a higher dim'l sp..

$\boxed{3}$ In A^3, 4-pts. in general position.
(i.e., not all lying in a plane)



$\downarrow$

these 4-pts. determine a solid figure, "tetrahedron".

That tetrahedron is a simplex can be described algebraically exact same way described over there.

$\boxed{\text{Tetrahedron}}$ (or $\boxed{\text{Simplex}}$)

formed by 4 general vectors $v_1, \ldots, v_4$ in A^3 (can be described by ..)

: $\vec{x} = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4$.

convex hull.

$$\begin{cases} 0 \leq \lambda_i \leq 1 \\ \sum \lambda_i = 1. \end{cases}$$

$\boxed{4}$ In $\mathbb{A}^n$, (we need $(n+1)$-pts. (or vectors)).

$v_0, \ldots, v_n$: $(n+1)$ pts. in <u>general position</u>

(i.e., not all lying in some hyperplane)

Then we can describe (we can define) sort of n-dim'l simplex in terms of a similar linear combination of those $(n+1)$ vectors

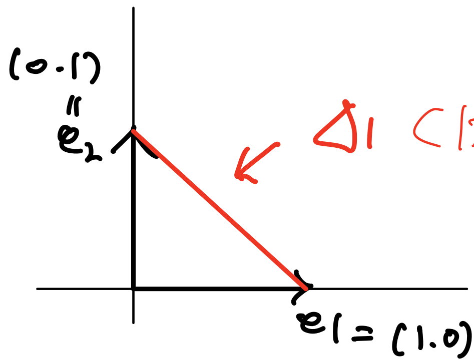
<u>n-simplex</u>, $\Delta_n = \sum_{i=0}^{n} \lambda_i v_i$

where $\begin{cases} 0 \le \lambda_i \le 1 \\ \sum_0^1 \lambda_i = 1. \end{cases}$

∴ This is a concrete way of talking about what a n-dim simplex is.

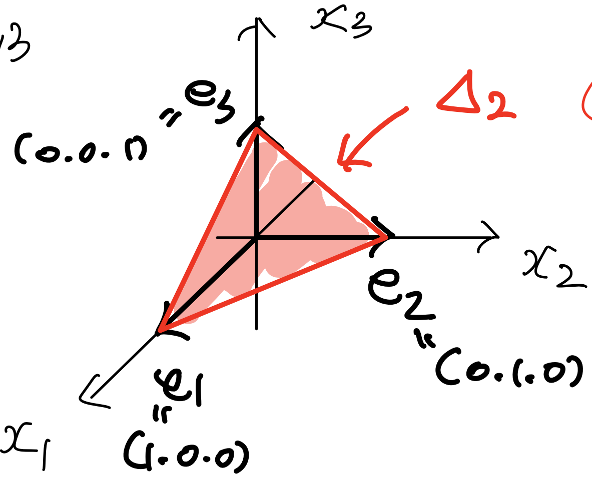
[5] Standard form for simplices.

V^2



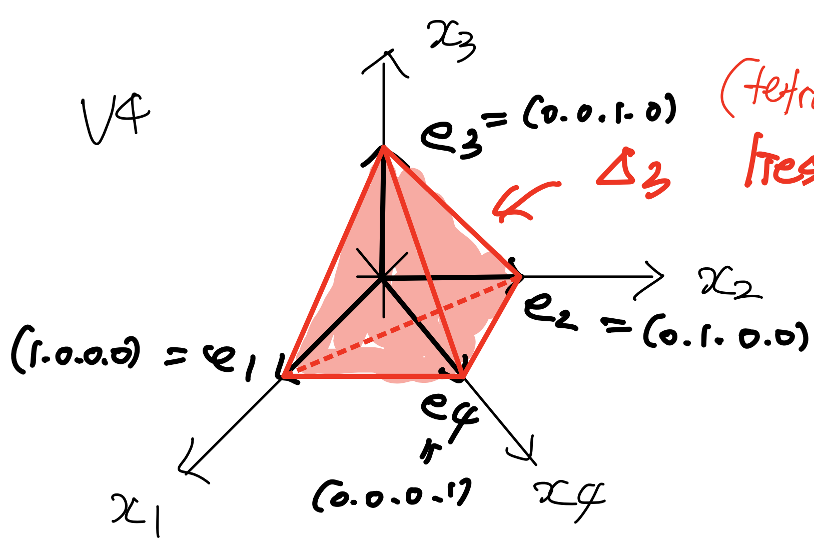
Δ_1 (line segment)
or
1-simplex.
lies in a line
$x_1 + x_2 = 1.$

V^3



Δ_2 (triangle) canonical
or
2-simplex.
lies in plane
$x_1 + x_2 + x_3 = 1.$

$V4$

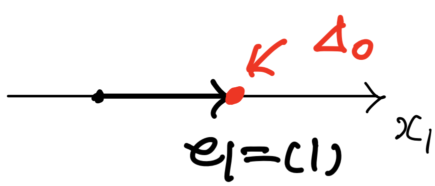


(tetrahedron) 3-dim convex hull. (Simplex)

Δ_3

lies in hyperplane
$x_1 + x_2 + x_3 + x_4 = 1$

V^1

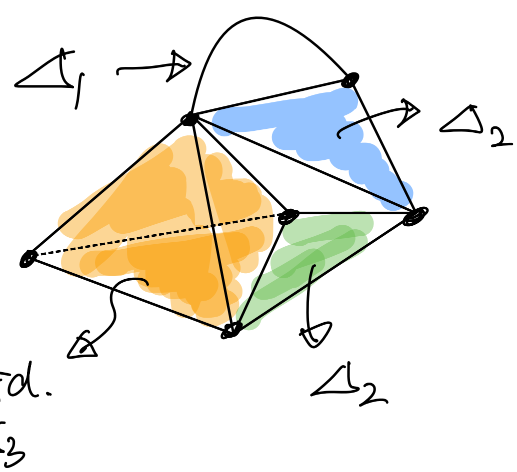


(point) or 0-simplex.
lies in point $x_1 = 1$

In higher dimensions so we can talk about equilateral simplices of any dimension of taking a convex hull of the basis vectors in sp. of n-dim higher all of those you're getting

Simplicial complexes

Build up a space using simplices.

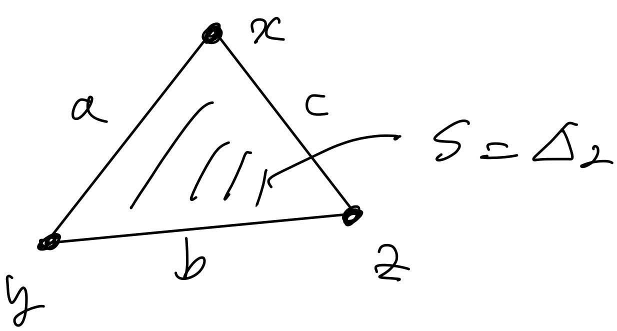


Any two simplices are either disjoint, or meet in a common face. .

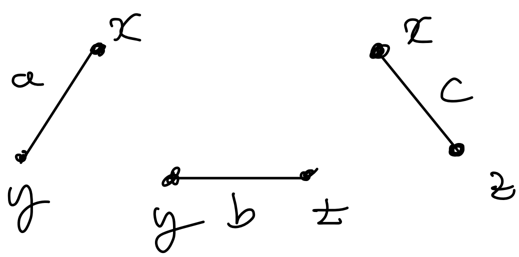
Q. What is a face of a simplex S?

A. Another simplex $P \subseteq S$ whose vertices are also vectors of S.

Ex1. 2-dim. example. (triangle).



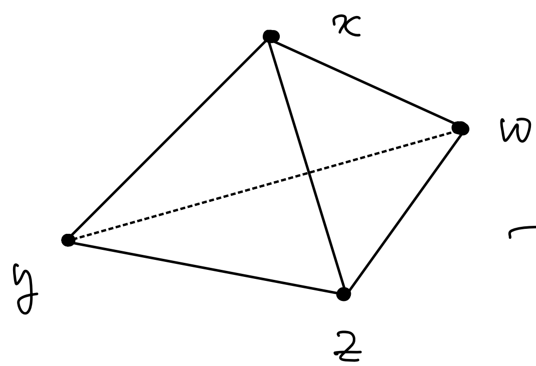
① 1-dim faces.



② 0-dim faces.

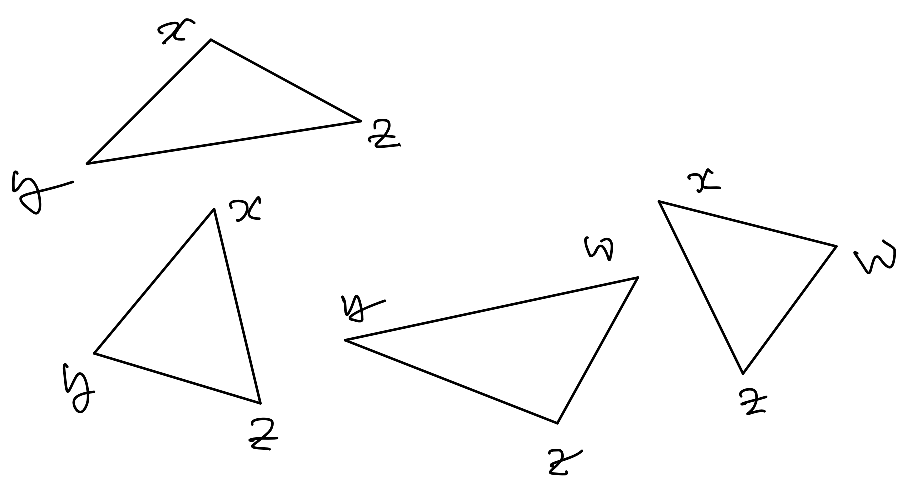
x, y, z,

Ex2. 3-dim example (tetrahedron).

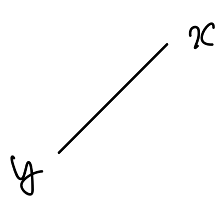


— Δ_3

① Δ_3 has 2-dim faces. (4).



② has 1-dim faces (6)



etc. ...

③ has 0-dim faces (4)

x, y, z, w.

- A face is determined by a subset of vertices.

$\Delta_3 = (x, y, z, w)$

$\binom{4}{3}$ Δ_2-faces $= (x,y,z)\ (x,y,w), (y,z,w), (x,z,w).$

$\binom{4}{2}$ Δ_1-faces $= (x,y), (x,z), \ldots, (z,w).$

$\binom{4}{1}$ Δ_0-faces $= (x), (y), (z), (w).$

- lattice of faces ← in the sense of partially ordered set.

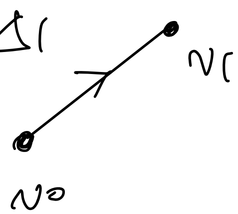
$\cong$

lattice of subsets of vertex set $\{x, y, z, w\}$.

⟨Orientation⟩.

$\boxed{1}$ 1-dim

Δ_1



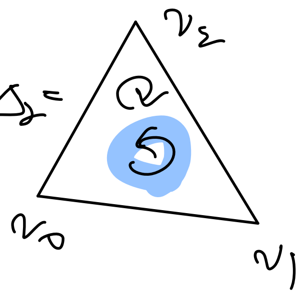
oriented simplex.

$S = (v_0\ v_1)$

has boundary

$\partial(S) = v_1 - v_0$

$\boxed{2}$ 2-dim simplex.

$\Delta_2 =$ 

$\circlearrowleft = (v_0\ v_1\ v_2) \sim (v_1\ v_2\ v_0) \sim (v_2\ v_0\ v_1)$: one orientation.

$\circlearrowright = (v_0\ v_2\ v_1) \sim (v_2\ v_1\ v_0) \sim (v_1\ v_0\ v_2)$

(clockwise) : another orientation

→ Choose orientation $(v_0\ v_1\ v_2) = S$ of simplex.
(by labeling).

has boundary

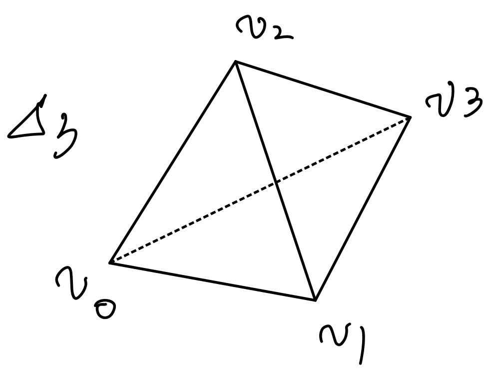
$$\partial(S) = (v_0 v_1) + (v_1 v_2) + (v_2 v_0)$$

$$= (v_0 v_1) - (v_0 v_2) + (v_1 v_2)$$

$$= \sum_{i=0}^{2} (-1)^i (v_0, \cdots, \hat{v_i}, \cdots, v_2)$$

hat means omit !!

$\boxed{3}$ 3- simplex.



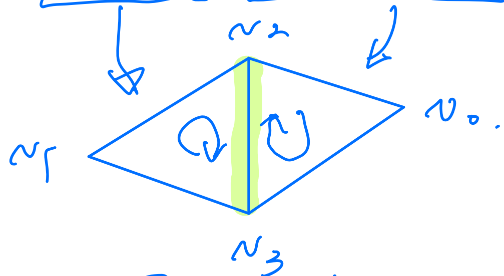
(oriented) simplex

$S = (v_0\, v_1\, v_2\, v_3)$

define boundary :

$$\partial(S) = \sum_{i=0}^{3} (-1)^i (v_0, \cdots, \hat{v_i}, \cdots v_3)$$

$$= (1\,2\,3) - (0\,2\,3) + (0\,1\,3) - (0\,1\,2)$$



(두 face가 공유하는 line에서 상쇄되어야하니까 0되도록 orientation 시킨다.)

하나 면의 orientation 정해지는 순간 각면 방향은 자동으로 정해진다.

Only 2 - orientations,

any one determined by a single 1-dim faces orientation.

Thm. (Fundamental formula in alg. top.).

$$\partial^2 = 0.$$

Pf. (For $S = \Delta_3$)

$$\partial(S) = (123) - (023) + (013) - (012)$$

$$\partial(\partial(S)) = \partial(123) - \partial(023) + \partial(013) - \partial(012)$$

$$= \left[\cancel{(23)} - \cancel{(13)} + \cancel{(12)}\right] - \left[\cancel{(23)} - \cancel{(03)} + \cancel{(02)}\right]$$

$$+ \left[\cancel{(13)} - \cancel{(03)} + \cancel{(01)}\right] - \left[\cancel{(12)} - \cancel{(02)} + \cancel{(01)}\right]$$

$$= 0$$

Lec. 33. Computing homology groups

X : space

$\rightsquigarrow$ sequence of groups. (= "chains")

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

3-di chains. 2-di chains 1-di chains 0-di chains

- $\boxed{\partial^2 = 0}$ implies $\boxed{\mathrm{im}\, \partial_{i+1} \subset \ker \partial_i}$

$\cancel{\triangle}$ $\ker \partial_1 = Z_1$. gp of cycles $\subseteq C_1$

$\mathrm{im}\, \partial_2 = B_1$. " " boundaries $\subseteq C_1$ $\cancel{\triangle}$.

i.e., $B_1 \subseteq Z_1$.

<u>Def.</u> $H_1 := Z_1 / B_1$: quotient gp.

e.g. $H_1 = 0, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}$, or $\mathbb{Z}^n$

or others.

$\cancel{X}$ infinite cyclic. or infinite comm. gp. $\cancel{X}$.

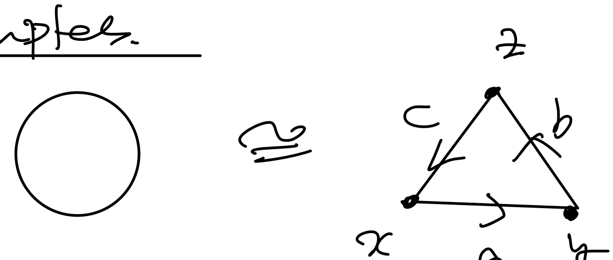
The whole story is repeated in every stage.

- Same story:

$$\boxed{H_n = Z_n / B_n}$$

Some examples.

$\boxed{1}$ $X = S^1$



skeleton of a Δ

Δ 3 - line segments.

- vertices $\{x, y, z\}$.
- 3 1-simplices (line segments).

$a = (x\ y),\ b = (y\ z),\ c = (z\ x)$.

So.. actual object.

$$X = \{\underbrace{a, b, c}_{\substack{\text{1-dim}\\ \text{ones.}}}, \underbrace{x, y, z}_{\substack{\text{0-dim}\\ \text{ones..}}}\}.$$

S^1 $\not\cong$ solid. $\rightarrow$ // 0 $\because$ empty interior.

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

C_1: $\underbrace{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}_{a\ b\ c}$ $\ni$ $la+mb+nc$

C_0: $\underbrace{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}_{x\ y\ z}$ $\ni$ $\alpha x+\beta y+\gamma z$: typical element looks like.

$\partial_0 = 0$, i.e., $x, y, z \longmapsto 0$.

$$\boxed{\begin{aligned} \partial_1 : \quad a &\longmapsto y-x \\ b &\longmapsto z-y \\ c &\longmapsto x-z. \end{aligned}}$$

$\partial_2 : \quad 0 \longmapsto 0.$

H at C_0 :

sub gp. : $\forall$ $Z_0 = \ker \partial_0 = C_0 = \langle x, y, z \rangle$.

$B_0 = \operatorname{Im} \partial_1 = \langle y-x, z-y, x-z \rangle$.

Q. What is $H_0 = Z_0 / B_0 = \dfrac{\langle x, y, z \rangle}{\langle y-x, z-y, x-z \rangle}$

One way of thinking about what a quotient is ?

Quotient by B_0

= Setting element of B_0 to 0

: (It means)

$$y - x = z - y = x - z = 0$$

$$\Rightarrow \quad x = y = z .$$

($\exists$ only one degree of freedom what's remaining)

$H_0 \cong \mathbb{Z}$, because every element is of form $nx + B_0$.

$\because$ $x + B_0 = y + B_0 = z + B_0$

i.e., $x \equiv y \equiv z \pmod{B_0}$ $\square$

$\langle$ Another thinking... $\rangle$

- $H_0 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}$

- $H_1 \cong Z_1 / B_1 = \ker \partial_1 / \operatorname{Im} \partial_2$

$\cancel{\nabla}$ $\partial(la + mb + nc)$

$= l(y-x) + m(z-y) + n(x-z)$

$= (n-l)x + (l-m)y + (m-n)z$

$= 0$

$\Rightarrow l = m = n$

$\therefore \ker \partial_1 = \langle a+b+c \rangle$

$=$ "cycles" $= Z_1$

$B_1 = \operatorname{Im} \partial_2 = 0$ $\cancel{\Delta}$

$\therefore H_1 \cong \mathbb{Z}/0 \cong \mathbb{Z}$

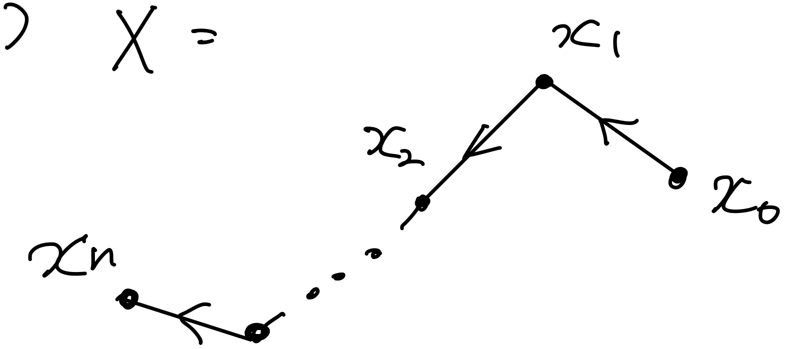
Conclusion.

$$\begin{cases} H_0(S^1) = \mathbb{Z} \\ H_1(S^1) = \mathbb{Z} \\ H_n(S^1) = 0, \quad n \geq 2 \end{cases}$$

Q. What does $H_0(X)$ measure?

A. The # of connected components.

E.g.) $X =$



$$\cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

$Z_0 = C_0 = \ker \partial_0$ ($\because$ ∂_0 : zero map)

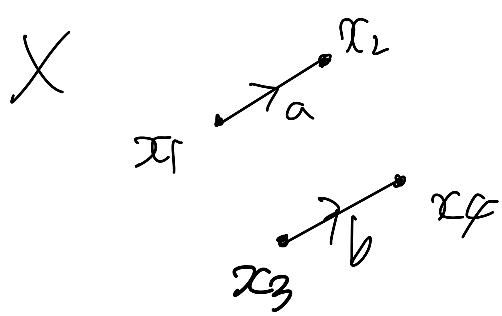
$B_0 = \operatorname{Im} \partial_1 = \langle x_k - x_{k-1} \mid 1 \leq k \leq n \rangle$

i.e., equating all elements of $B_0 = 0$

$\Rightarrow$ Setting all vertices equal to each other, so to x_1.

$$x_0 = x_1 = \cdots = x_n$$

$\underbrace{\text{Connecting vertices}}_{\text{동치류}}$ 를 하나로 취급



$$\operatorname{Im} \partial_1 = \langle \underbrace{x_2 - x_1}_{\partial(a)}, \underbrace{x_4 - x_3}_{\partial(b)} \rangle$$

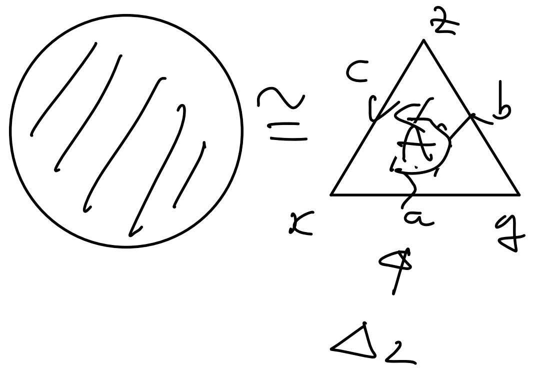
$$Z_0 / B_0 = \langle x_1, x_2, x_3, x_4 \rangle / \langle x_2 - x_1, x_4 - x_3 \rangle$$

$$= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z} \oplus \mathbb{Z}$$

$$\cong \mathbb{Z} \oplus \mathbb{Z} \sim \{ \alpha x_1 + \beta x_2 \}$$

Ex2.

$X = D$. disk.



$$\underset{0}{\overset{\|}{C_3}} \xrightarrow{\partial_3} \underset{A}{\overset{C_2}{\mathbb{Z}}} \xrightarrow{\partial_2} \underbrace{\underset{a\quad b\quad c}{\overset{C_1}{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}} \xrightarrow{\partial_1} \underset{x\quad y\quad z}{\overset{C_0}{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}} \xrightarrow{\partial_0} 0}_{\text{as for } S^1}$$

$\ker\partial_1 = \langle a+b+c \rangle = Z_1$: 1-cycles

$\operatorname{Im}\partial_2 = \langle a+b+c \rangle = B_1$

$\because$ $A = (xyz)$

$\partial(A) = (yz) - (xz) + (xy)$

$= b + c + a$ $\cancel{\text{☆}}$

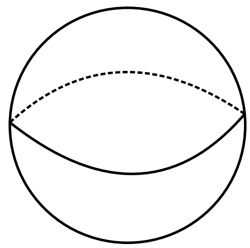
So, $H_1 = Z_1 / B_1 = \langle a+b+c \rangle / \langle a+b+c \rangle = 0$.

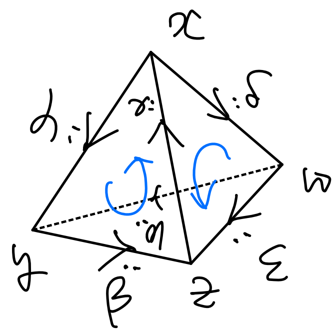
$$\therefore \begin{cases} H_0(D) = \mathbb{Z} \\ H_1(D) = 0 \\ H_n(D) = 0, \quad n \geq 2 \end{cases}$$

$$H_2 = Z_2 / B_2 = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = 0/0 = 0.$$

More homology computations.

S^2





$$X = \{ (xyz)\ (xzw)\ (ywz)\ (xwy) \}.$$

← maximal simplices
the triangular faces.

(of course has the subsimplices
like as vertices, lines.)

(But the space is determined by
specifying the maximal simplices.)

$$\begin{tikzcd}
C_3 \arrow[r, "\partial_3"] \arrow[d, equal] & C_2 \arrow[r, "\partial_2"] \arrow[d, equal] & C_1 \arrow[r, "\partial_1"] \arrow[d, equal] & C_0 \arrow[r, "\partial_0 = 0"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^6 \arrow[r] & \mathbb{Z}^4 \arrow[r] & 0
\end{tikzcd}$$

$\mathbb{Z}^4 \cong \langle (xyz), (xzw), (ywz), (xwy) \rangle$

$\mathbb{Z}^6 \cong \langle (xy), (yz), (zx), (xw), (wz), (wy) \rangle$

$\mathbb{Z}^4 = \langle x, y, z, w \rangle.$

$$H_n = \frac{Z_n}{B_n}, \quad \begin{cases} Z_n = \ker \partial_n, & \text{cycles.} \\ B_n = \operatorname{im} \partial_{n+1}, & \text{boundaries.} \end{cases}$$

H_0 : $Z_0 = C_0 = \langle x, y, z, w \rangle$.

$B_0 = \operatorname{im} \partial_1 = \langle y-x, z-y, x-z,$
$w-x, z-w, y-w \rangle$.

$= \frac{Z_0}{B_0} \cong \mathbb{Z} = \langle x + B_0 \rangle$.

$H_1 = \frac{Z_1}{B_1}$

$\cancel{\partial}\ Z_1 = \ker \partial_1$

$\ker \partial_1 \ni \alpha(xy) + \beta(yz) + \gamma(zx) + \delta(xw) + \varepsilon(wz) + \eta(wy)$

$\downarrow \partial$

$\alpha(y-x) + \beta(z-y) + \gamma(x-z) + \delta(w-x)$
$+ \varepsilon(z-w) + \eta(y-w) = 0$

$\Rightarrow x(-\alpha+\gamma-\delta) + y(\alpha-\beta+\eta) + z(\beta-\gamma+\varepsilon)$
$+ w(\delta-\varepsilon-\eta) = 0$.

$\therefore$ 4 eqns. in 6 unknowns.

(using row reduction
for integer)

$$\begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \varepsilon & \eta \end{array}$$

$$\begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

parameters: r s t

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \\ \eta \end{pmatrix} = \begin{pmatrix} r-s-t \\ r-s \\ r \\ s+t \\ s \\ t \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

"generators of Z_1" $\longrightarrow$ $\alpha+\beta+\gamma$ $\quad$ $-\alpha-\beta+\delta+\varepsilon$ $\quad$ $-\alpha+\delta+\eta$.

(geometrically, these are all cycles.)

$\therefore Z_1 = \mathbb{Z}^3$

$$B_1 = \mathrm{Im}\,\partial_2.$$

$$= \langle \alpha+\beta+\gamma,\ -(\gamma+\delta+\varepsilon),\ -\beta+\varepsilon-\eta,\ \delta+\eta-\alpha \rangle$$

$$\therefore\ Z_1/B_1 = 0$$

$$(\text{since } B_1 = Z_1.)$$

→ S^2 doesn't have any 1-dim'l holes.

$$H_2 = Z_2/B_2.$$

face 의 orientation 정할 때,
겹치는 변에서 반대방향이 엇갈려서 서로 상쇄되도록 정했어..

So... $\partial(\sum \text{faces.}) = 0$

$$\partial\big(\underset{A}{(xyz)} + \underset{B}{(xzw)} + \underset{C}{(ywz)} + \underset{D}{(xwy)}\big) = 0$$

$$Z_2 = \langle A+B+C+D \rangle \cong \mathbb{Z}.$$

$$B_2 = 0$$

$$\therefore\ H_2 = Z_2/B_2 \cong \mathbb{Z}/0 \cong \mathbb{Z}.$$

$$H_0(S^2) = \mathbb{Z}$$
$$H_1(S^2) = 0$$
$$H_2(S^2) = \mathbb{Z}$$
$$H_n(S^2) = 0, \ n \geq 3$$

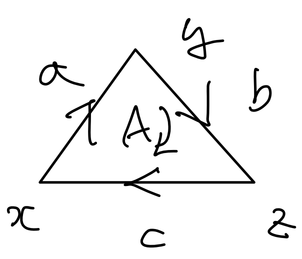
얻 을 받드 댐.

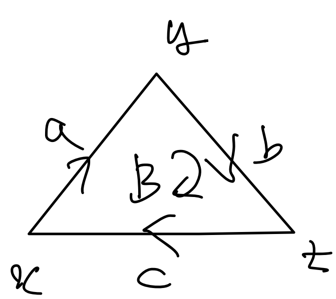
※ (Eilenberg) Semi-simplicial cplxs.

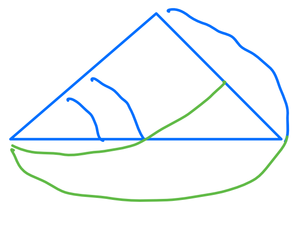
= Δ - complex.

Another computation (← more simpler).

$H_n(S^2)$







$$0 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

$$\langle A, B\rangle \qquad \langle a, b, c\rangle \qquad \langle x, y, z\rangle$$

① $H_1 = Z_1 / B_1$

Z_1 : $\alpha a + \beta b + \gamma c$

$$\big\downarrow \partial$$

$$\alpha(y-x) + \beta(z-y) + \gamma(x-z) = 0$$

$$\parallel$$

$$x(-\alpha+\gamma) + y(\alpha-\beta) + z(\beta-\gamma)$$

$\Leftrightarrow \quad \alpha = \beta = \gamma.$

$\therefore \quad Z_1 = \langle a+b+c\rangle.$

$$B_1 = \operatorname{Im} \partial_2 = \langle a+b+c\rangle \quad \left\{ \begin{array}{l} \partial(A) = a+b+c \\ \partial(B) = a+b+c. \end{array} \right.$$

$\therefore \quad \boxed{H_1 = 0.}$

② $H_2 = Z_2 / B_2$.

$Z_2 = \ker \partial_2 = \langle A - B \rangle \cong \mathbb{Z}$.

$B_2 = \operatorname{Im} \partial_3 = 0$.

$\therefore \boxed{H_2 = \mathbb{Z}.}$

* Simplicial set is flexible ∂'s..

∘%∘ Betti numbers of space X.

$b_n = \operatorname{rank} H_n$

where $H_n = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\substack{\parallel \\ \text{the \# of these} \\ \parallel \\ b_n}} \oplus \underbrace{\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}}_{\substack{\text{finite subgroups} \\ \text{(torsion coefficients)}}}$

Theorem.

$\chi(X) =$ "Euler characteristic of X"

$$= \sum_{i=0}^{n} (-1)^i b_i = b_0 - b_1 + b_2 - \cdots.$$

$\boxed{\text{Ex}}$ S^2

$H_0(S^2) \cong \mathbb{Z} \longrightarrow b_0 = 1$

$H_1(S^2) \cong 0 \longrightarrow b_1 = 0$

$H_2(S^2) \cong \mathbb{Z} \longrightarrow b_2 = 1$

$H_n(S^2) \cong 0 \quad (n \geq 3).$ $b_n = 0$

$(n \geq 3).$

$$\therefore \chi(S^2) = b_0 - b_1 + b_2$$

$$= 1 - 0 + 1 = 2.$$

$$(= 2 - 2g).$$