Paper review : Topological reconstruction of grayscale images (L. M. Betthauser, 2018)

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Introduction

1. classification of topological space (or topological manifold) Let X be a topological manifold and $x_0 \in X$ be a point of X.
   (1) Euler characteristic, $\chi(X)$
   (2) fundamental group, $\pi_1(X, x_0)$: 정의 쉽고 계산 어렵거나 불가능
   (3) higher homotopy group, $\pi_n(X, x_0)$
   (4) homology group, $H_n(X)$: 정의는 복잡하지만 계산 가능하고 비교적 쉬운 편 (예제 생략)
   (5) cohomology group, $H^n(X)$

   $\vdots$

Preliminaries

1. cube, cubical complexes, cubical homology
   (cf. simplex, simplicial cplxs., homology)
   (1) Since the pixel of image is the square shape, we choose cubical complexes.
   (2) Respectively, the corresponding concepts – cube, cubical complex, and cubical homology – are redefined in a similar way.
   (3) The topological features can be identified as a result of algebraic calculation.
2. Möbius inversion
   (1) (In combinatoric theory) this function can be thought of as a generalization of the inclusion–exclusion principle.
   (2) Or, this may be viewed as a reduced Euler characteristic.
   (3) And, this has ties to homology group.

Preliminaries

3. Euler calculus
   (1) Definition : The Euler integral of a constructible function $\psi : X \to \mathbb{Z}$ is the sum of the Euler characteristics of each of its level-sets,
   $$\int \psi d\chi := \sum_{-\infty}^{\infty} \chi(\psi^{-1}(n)) \tag{1}$$
   (2) Definition : Let $S \subset X \times Y$ and $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$. Let $\psi : X \to \mathbb{Z}$ be a constructible function.
   The Radon transform with respect to S is the group homomorphism $\mathcal{R}_S : CF(X) \to CF(Y)$,
   $$\mathcal{R}_S(\psi) := \pi_{Y*}[(\pi_X{}^*\psi)1_S] \tag{2}$$
   (3) That is, under special circumstances, one can recover a constructible function from its Radon transform.

Preliminaries

4. Persistent homology
5. Discrete Morse Theory
   (1) Theorem(Morse) : Let $f : M^2 \to \mathbb{R}$ be a differential function on a compact oriented surface M^2 such that all its critical points are nondegenerate. Let us denote by M, m, and s the number of points of maximum, minimum, and saddle respectively of f. Then $M - s + m$ does not depend on f, i.e.,

$$M - s + m = \chi(M^2) \tag{3}$$

   (2) $2 - 2g = \chi = v - e + f$ (g:genus)
   (3) $H_1(X) = \mathbb{Z}^{e-v+1}$ where X is connected graph with a spanning tree T and $e - v + 1 = 2g$ (g:genus of T).

Main result : grayscale images and weighted Euler characteristic

1. Definition : Given a finite full elementary cubical cplx. $\mathcal{K}$ and a positive integer valued function $\mathcal{G} : \mathcal{K} \to \mathbb{N}$ with the property that $\mathcal{G}(\sigma) = \max\{\mathcal{G}(\tau) \in \mathbb{N} | \sigma \subset \tau, \tau \in \mathcal{K}\}$. Define a grayscale digital image to be the pair $(\mathcal{K}, \mathcal{G})$.
(의미 : cubical simplex σ 하나 가져와서 이걸 덮는 simplex 들 중에, 함수 $\mathcal{G}$의 값이 가장 큰 걸 $\mathcal{G}(\sigma)$로 정의)

Main result : grayscale images and weighted Euler characteristic

2. Definition : Given a grayscale digital image $(\mathcal{K}, \mathcal{G})$, the weighted Euler characteristic is defined as the sum

$$\chi_{\mathcal{G}}(\mathcal{K}) = \sum_{i=0}^{d} \sum_{\sigma \in \mathcal{K}^i} (-1)^i \mathcal{G}(\sigma) \tag{4}$$

- 즉, 위상에서 오일러 표수의 정의대로 계산. image를 각 차원 (i)에 따른 simplexes의 조합인 i-complexes, $\mathcal{K}^i$로 볼 때, 각 컴플렉스의 원소 개수 세어서 alternating sum 한 것
- the i-th betti number $\beta_i := \mathrm{rank} H_i(K)$, i.e., the rank of i-th homology.
- $\chi =$ alternating sum of betti numbers $(= v - e + f)$

Main result : grayscale images and weighted Euler characteristic

3. Definition : Given a grayscale digital image $(\mathcal{K}, \mathcal{G})$. Define the Euler characteristic curve of $(\mathcal{K}, \mathcal{G})$ with respect to $f \in \mathbb{R}^d$, $\mathrm{ECC}(\mathcal{K}, f)$ to be the function

$$\begin{aligned} \mathrm{ECC}(\mathcal{K}, f) :& \mathbb{R} \to \mathbb{Z} \\ & t \mapsto \chi_{\mathcal{G}}(\{\sigma \in \mathcal{K} | max\{f \cdot x | x \in \sigma \subset \mathbb{R}^d\}\}). \end{aligned}$$

- $ECC(\mathcal{K}, f)(t) = \chi_{\mathcal{G}}(\{\sigma \in \mathcal{K} | max(f \cdot x) \leq t, x \in \sigma\})$?? (다시 생각해 볼 것!)

Main result : grayscale images and weighted Euler characteristic

4. Corollary(Theorem 4.16) :

- Given a grayscale digital image $(\mathcal{K}, \mathcal{G})$.
- $\mathcal{T} := \{v \in \mathcal{K}^0, i.e., vertices\}$: the poset (partially ordered set) under the product order $(\mathbb{Z}, \leq)^d$
- $\mathcal{F}$: a collection of generic vectors with one vector pointing in direction of each orthant of $\mathbb{R}^d$. Then

$$\mathcal{G}(\mathcal{C}_v) B_{\mathcal{K}} * \mu(\underline{0}, v) = \sum_{f \in \mathcal{F}} \operatorname{sgn}(f) \chi_{\mathcal{G}}(st^f_{\leq}(v)) \tag{5}$$

Main result : grayscale images and weighted Euler characteristic

5. Corollary(Theorem 4.17) :
   - Given a grayscale digital image $(\mathcal{K}, \mathcal{G})$ in $\mathbb{R}^d$.
   - $\mathcal{F}$: a collection of 2^d generic vectors with one vector pointing in direction of each orthant of $\mathbb{R}^d$.

   We may reconstruct $(\mathcal{K}, \mathcal{G})$ from the set of weighted Euler characteristic curve $\mathrm{ECC}(\mathcal{K}, f)$ of $(\mathcal{K}, \mathcal{G})$ generated by the lower star filtrations of the filtration vectors in $\mathcal{F}$.

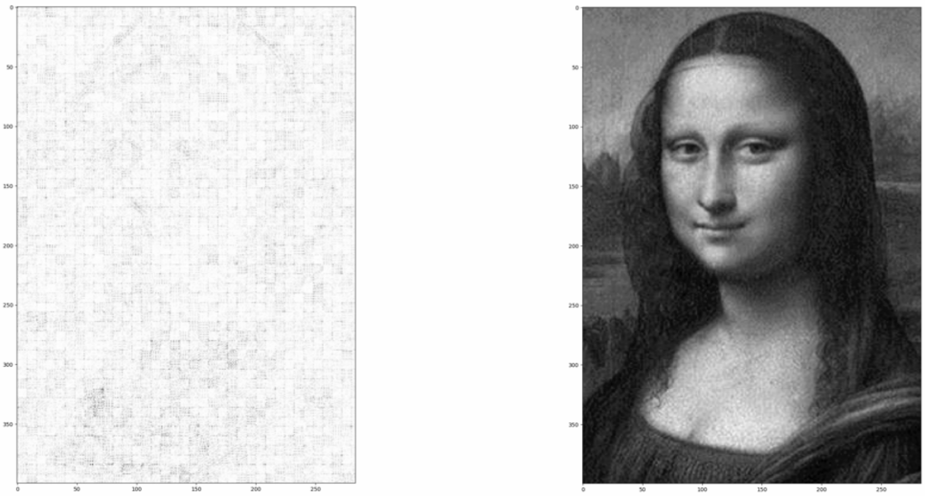

Figure A-1. Example of the input and output of the second half of the reconstruction pipeline described in Algorithm 5.6. A collection of 2^2 Euler characteristic curves which corresponds to the support of the Möbius inversion of a grayscale image (shown above) is inputted. After convolution with the ζ-function, the original gray scale image (displayed below) is returned. The composition of the processes depicted by Figure A-2 and Figure A-1 is lossless.

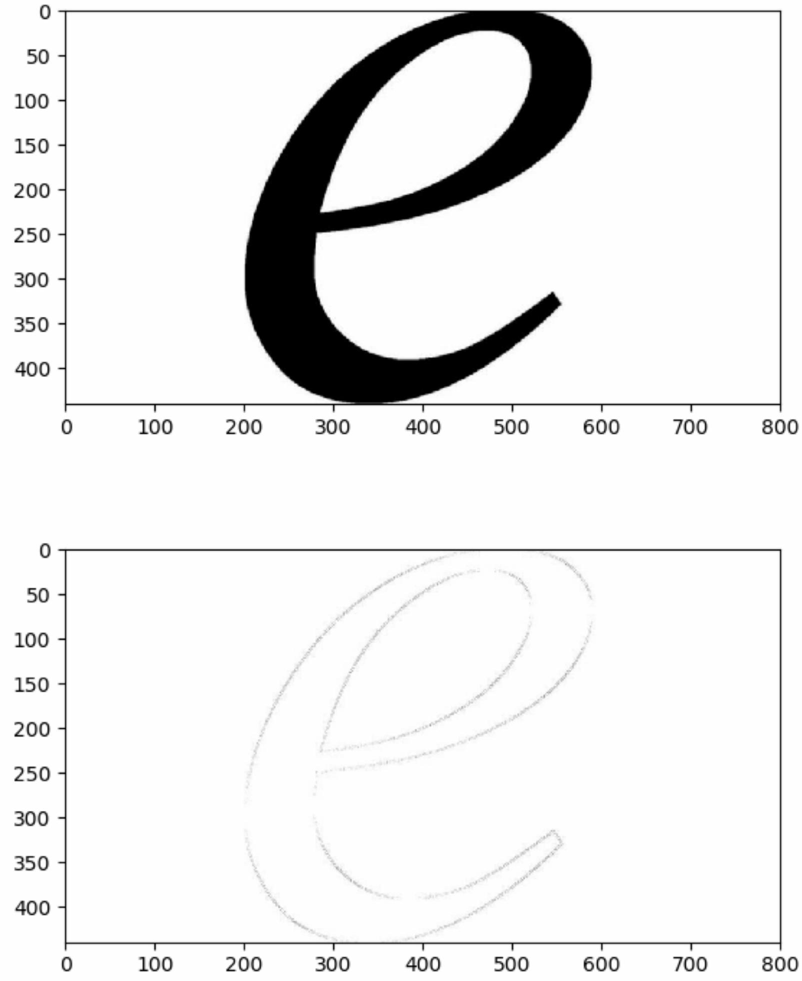

Figure A-2. Example of the input and output of the first half of the reconstruction pipeline. The letter-E (shown above) is compressed by storing the support of the Möbius inversion of the grayscale image (displayed on the bottom) which in turn is stored as a collection of 2^2 Euler characteristic curves (such as the one in Figure 5-1).