20220915 (목) Persistent Homology II..

복습.

1. 위상수학 : 위상공간의 분류

   즉, $\{\text{topological space}\} / \sim$

   → 고차원 이나 다양체의 모양이 복잡한 경우 눈으로 확인 어려움 조작이 너무 복잡해짐.

2. Topological invariant (위상 불변량)

   $I : \{\text{top. sp.}\} \longrightarrow \{\text{some algebraic objects}\}$

   $\mathbb{Z}$
   Group,
   Ring ...

   $$M_1 \cong M_2 \Rightarrow I(M_1) = I(M_2)$$

   as top'l mfd. $\not\Leftarrow$ in general.

   (i.e., $I(M_1) \neq I(M_2) \Rightarrow M_1 \ncong M_2$

   Simplicial cplx. obtained from graph.

3. Topological invariants 예

- Euler characteristic, $\chi(X)$
- fundamental group, $\pi_1(X)$
- higher homotopy group, $\pi_n(X)$
- homology, $H_n(X)$
  cohomology, $H^n(X)$.

⋮

4. Simplicial Homology Group.

- Consider a topological space assembled into something like a LEGO block.

(1) 재료 : n-simplexes (점, 선분, 삼각형, 사면체, ...)

(2) 조립 : 위상공간 K = "simplicial complex"

- 각각의 n-차원으로 나눠서 관찰.

  $C_n(K)$ = a set of n-dim simplexes

(3) boundary map $\partial_n : C_n(K) \to C_{n-1}(K)$

, 각 $C_n(K)$들 간의 관계를 관찰하며

"n-dim cycles"을 찾아내자!!

→ $\partial(\text{cycle}) = 0$ 인 성질 이용,

$\ker \partial_n \ (\leq C_n(K))$ 로 H_n

정의하면 되나? NO!!

→ 

: disk 같은 것의 boundary도 cycle 이지만 1-point로 contractible 되기 때문에 온전한 cycle로 헤아릴 수 없음.

→ 그래서, $\operatorname{Im} \partial_{n+1}$ = boundary 를 zero로 볼 수 있도록 quotient space로 Homology 정의!!

(4) $\partial^2 = 0 \Rightarrow \operatorname{im} \partial \subset \ker \partial$

따라서, $C_n(K)$들의 chain map 얻음.

$$\cdots \to C_{n+1}(K) \to C_n(K) \to C_{n-1}(K) \to \cdots$$

(5) n-th homology

$$H_n(X) := \frac{\text{"cycles"}}{\text{"boundaries"}} = \frac{Z_n(X)}{B_n(X)}.$$

$$= \mathrm{Ker}\,\partial_n / \mathrm{Im}\,\partial_{n+1}$$

Q. Why "simplicial complexes"?

1. Simplicial complexes = the fundamental object to approximate the homology of a point cloud.

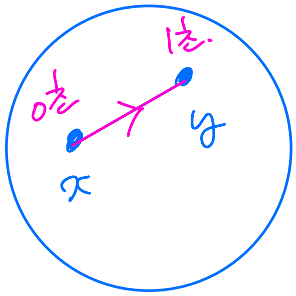
2. Do this by taking a covering of given point cloud.

Def. (Convex set)

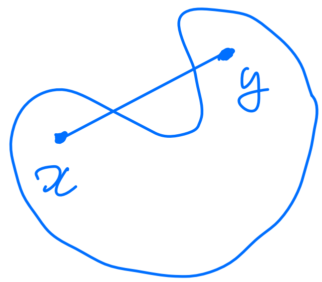
X : a vector sp. (or an affine sp.) $/\mathbb{R}$

$C \subseteq X$: convex subset

$\Longleftrightarrow \forall x, y \in C, \quad \overline{xy} \in C$



$=$ convex.

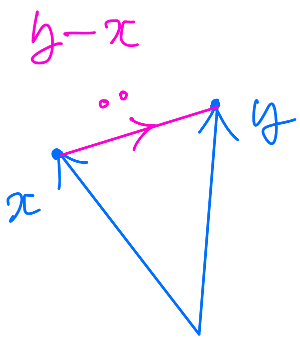


$\neq$ convex.

$\Longleftrightarrow \forall x, y \in C, \quad (1-t)x + ty \in C$

where $t \in [0,1]$

Ex $\mathbb{R}^2$ (or $\mathbb{A}^2$)



$x + t(y-x)$

$= \underset{\sim\sim}{(1-t)}x + \underset{\sim}{t}y$

, $t \in [0,1]$.

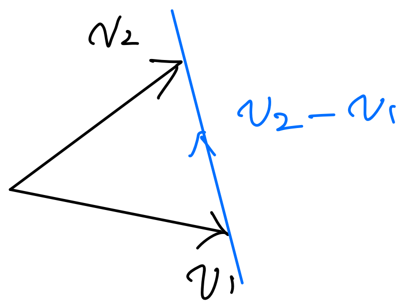
[$((1-t), t)$ = "barycentric coordinate"

Σ barycn coord. $= 1$

Remark. Convex combinations (Convex hull)

n-dim convex hull → n-simplex

$\boxed{1}$ 1-dim ($\subset$ 2-dim. vector sp.)

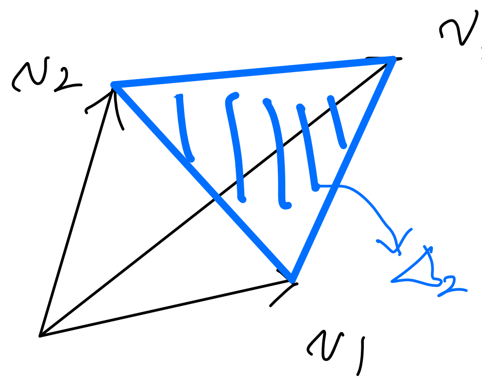


$$v_1 + t(v_2 - v_1)$$
$$= (1-t)v_1 + t v_2$$

(v_1, v_2 : linearly independent)

- sum of coefficients $= 1$
- $((1-t), t)$ = "barycentric coordinate"
- If $t \in [0,1]$, $(1-t)v_1 + t v_2$ describes the line segment between v_1, v_2.
- If $\begin{cases} t \sim 0, & \text{close to } v_1 \\ t \sim 1, & \text{close to } v_2 \end{cases}$
- So, t is some kind of measure of how far apart where we are on the segment.

$\boxed{2}$ 2-dim (← Give 3-points of general position)



$t_1 v_1 + t_2 v_2 + t_3 v_3$

where $\begin{cases} t_i \in [0, 1] \\ \sum t_i = 1 \end{cases}$

– (t_1, t_2, t_3) : barycentric coordinate of Δ_2

⋮

$\boxed{3}$ n-dim (← (n+1)-points of general position).

$$\Delta_n = \sum_{i=0}^{n} t_i v_i \quad , \text{ n- convex hull}$$

where $\begin{cases} t_i \in [0, 1] \\ \sum t_i = 1. \end{cases}$

- $(t_0, t_1, \ldots, t_n)$: barycentric coord.

→ weight.

$\Downarrow$ In this way

(Abstract) Simplicial complex.

1. $V = \{v_0, \cdots, v_n\}$: vertex set

2. k-dim Simplices $(0 \le k \le n)$

$$\sigma_k = (v_{i_0}, \cdots, v_{i_k})$$

where $i_0, \cdots, i_k \in \{0, \cdots, n\}$: distinct.

e.g.) $\begin{cases} \text{0-dim simplices : } (v_0), \cdots, (v_n) \\ \text{1-dim } \quad \text{'' } \quad : (v_0, v_1) \cdots (v_{n-1}, v_n) \\ \text{n-dim } \quad \text{'' } \quad : (v_0, \cdots, v_n) \end{cases}$

3. Simplicial Complex

- $\Delta = \{\sigma : \text{simplices}\}$, $|\Delta| < \infty$

s.t. $\begin{cases} \sigma \in \Delta, \tau \le \sigma \Rightarrow \tau \in \Delta & (\Delta, \le) : \text{well defined.} \\ \sigma_1, \sigma_2 \in \Delta \Rightarrow \begin{cases} \sigma_1 \cap \sigma_2 = \emptyset \\ \text{or} \\ \sigma_1 \cap \sigma_2 \le \sigma_1 \wedge \sigma_2 \end{cases} \end{cases}$

$\rightarrow$ Δ : top'l sp.

- $C_k(\Delta) = \{\Delta$ 의 k-dim simplices$\}$

$\Downarrow$

$\sigma = (v_0, \cdots, v_k)$

4. Boundary map, ∂

$$\partial_k : C_k(\Delta) \longrightarrow C_{k-1}(\Delta)$$

$$\sigma \longmapsto \partial_k(\sigma) = \sum_{i=0}^{k} (-1)^i (v_0, \cdots, \hat{v_i}, \cdots, v_k)$$

Coefficients $\in \mathbb{Z}, \mathbb{Q}, \mathbb{F}_2, \mathbb{F}_p$.

$\mathbb{F}_2 = \mathbb{Z}_2 = \{0, 1\}$.

(i.e., $-1 \equiv 1 \mod 2$)

$$\underset{(\text{over } \mathbb{F}_2)}{=} \sum_{i=0}^{k} (v_0, \cdots, \hat{v_i}, \cdots, v_k)$$

방향성을 고려하지 않으면 인접성만 남게 됨..

$\Rightarrow$ Homology 계산 = algorithm

Today!!

1. Covering, Nerve, five types of complex
   (Čech, Vietoris-Rips, Delaunay, Alpha, Witness)
   — Voronoi diagram, Delaunay triagulation.

2. Persistent Homology

3. Representations of persistent homology
   - Barcode diagram
   - persistence diagram.

"A roadmap ~ "(2017)" . p.20.

(a) A filtered simplicial complex:

(b) We put a total order on the simplices that is compatible with the filtration:

$\beta_0=1, \beta_1=0$.. $\beta_0=2, \beta_1=0$. $\beta_0=1, \beta_1=1, \beta_2=0$ $\beta_0=1, \beta_1=0, \beta_2=0$

where σ_i denotes the ith simplex in this order.

(c) (Left) The boundary matrix B for the filtered simplicial complex in (a) with respect to order on simplices in (b), and (right) its reduction $\overline{B}$ given by applying Algorithm 1 (one first adds column 5 to column 6, and then column 4 to column 6):

$$B = \begin{pmatrix} 0&0&0&1&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix} \qquad \overline{B} = \begin{pmatrix} 0&0&0&1&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}.$$

$H_0 = \mathbb{Z}$.
$H_1 = 0$
$H_2 = 0$.

$\mathrm{Im}\,\partial_1$ $\mathrm{Im}\,\partial_2$ $\mathrm{Im}\,\partial_1 \cong \mathbb{Z}^2$ $\mathrm{Im}\,\partial_2 \cong \mathbb{Z}$

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$C_2 \cong \langle (1\,2\,3) \rangle$, $C_1 = \langle (1\,2)(2\,3)(3\,1) \rangle$, $C_0 = \langle (1), (2), (3) \rangle$

$(1\,2) \longmapsto (2) - (1) \equiv (2) + (1) \pmod 2$

$(1\,2\,3) \longmapsto (1\,2) - (1\,3) + (2\,3)$
$\equiv (1\,2) + (1\,3) + (2\,3) \pmod 2$

$$\begin{cases} \ker\partial_0 = C_0 \cong \mathbb{Z}^3 \\ \ker\partial_1 = \langle (1\,2)+(2\,3)+(3\,1) \rangle \cong \mathbb{Z} \\ \ker\partial_2 = 0 \end{cases}$$

$\operatorname{Im}\partial_1 \cong \mathbb{Z}^2$

$\operatorname{Im}\partial_2 \cong \mathbb{Z}$

$$H_0 = \frac{\ker\partial_0}{\operatorname{Im}\partial_1} = \mathbb{Z}^3/\mathbb{Z}^2 \cong \mathbb{Z}$$

$$H_1 = \frac{\ker\partial_1}{\operatorname{Im}\partial_2} = \mathbb{Z}/\mathbb{Z} \cong 0$$

$$H_2 = \frac{\ker\partial_2}{\operatorname{Im}\partial_3} = 0/0 \cong 0.$$

※ From $\overline{B}$. we can get the values of $\operatorname{Im}\partial_k$ (for each k) //

Def 3.6. (Simplicial complex)

X : a topological space.

A simplicial complex of X, $K(X)$, ($= K$)

is a finite collection of simplices

such that ① $\sigma \in K,\ \tau \leq \sigma \Rightarrow \tau \in K$

② $\begin{cases} \sigma_1 \cap \sigma_2 = \emptyset \\ \quad \text{or} \\ (\sigma_1 \cap \sigma_2 \neq \emptyset),\ \sigma_1 \cap \sigma_2 \leq \sigma_1 \wedge \sigma_2 \end{cases}$

Remark.

① $(K, \leq)$: well-defined

② $(K, \leq)$: top'l sp.

Def 3.7. (Abstract Simplicial complex)

An abstract simplicial complex

is a finite collection of sets A

s.t. $\alpha \in A,\ \beta \subset \alpha \Rightarrow \beta \in A$

※ Construction of different types of complexes

Def 3.8. (Cover)

X : top'l sp.

$\mathcal{U} = \{ U_i \mid i \in I \}$: a cover of X

$\overset{\text{def}}{\iff}$ $\mathcal{U}$ is a collection of sets, U_i,

s.t. $X \subset \bigcup_{i \in I} U_i$

Notation.

① $\mathcal{U} = \{ U_i \}_{i \in I}$ 또는..

② $Cov(X) = \{ U_i \}_{i \in I}$.

Ref) 「Foundations of Algebraic Topology」 — by Samuel Eilenberg, Norman Steenrod.

Note.

지금 우리는 막 흩어져 있는 data를 나름의 규칙을 갖고 연결시켜서 보는 게 목적임

그렇다면 덮개로 data들을 잘 붙여서 연결하고자 하는 data가 흩어져 있는 모양을 그럴듯하게 살려내길 원함.

서로소 뜨듯이...

(참고) A, B가 붙어있다는 걸 수학적으로 표현 : $A \cap B \neq \emptyset$

Def 3.9. (Nerve of a Cover)

X : top'l sp.

$\mathcal{U} = \{ U_i \mid i \in I \}$: any cover of X.

$\mathcal{N}(\mathcal{U}) :=$ the abstract simplicial complex with vertex set I

, where $\underbrace{\operatorname{Span}(i_0, \cdots, i_n)}_{\text{i.e., } n\text{-simplex}} \in C_n(K)$

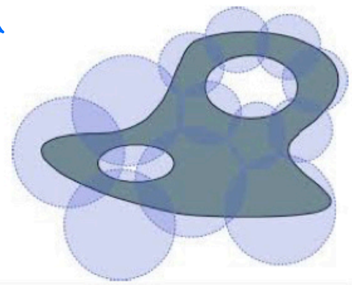
$$\Leftrightarrow \bigcap_{k=1}^{n} U_{i_k} \neq \emptyset$$

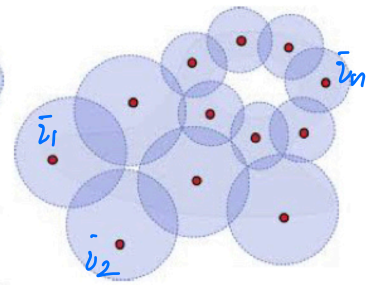
Remark.

$$\mathcal{N}(\mathcal{U}) = \{ \underbrace{(i)_{i \in I}}_{0\text{-simplexes}}, \underbrace{(i_0, i_1)}_{1\text{-simplexes}}, \cdots \underbrace{(i_0, \cdots, i_n)}_{n\text{-simplexes.}}, \cdots \}$$

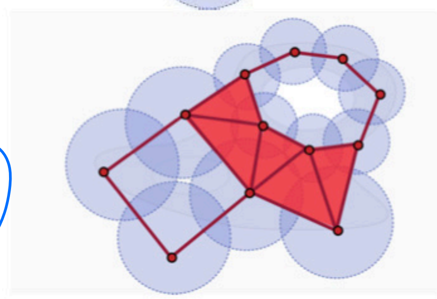
① $X \subset \bigcup_{i \in I} U_i$

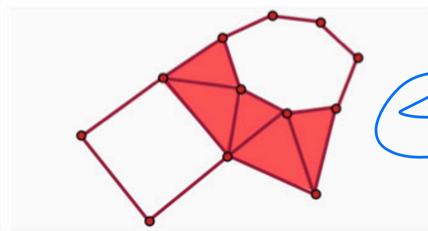
② vertices $= \{i\}_{i \in I}$.





③





④

$\shortparallel$

" Nerve of cover X "

Recall.

(1) Span$(i_0, \cdots, i_k)$: k-simplex

$\Leftrightarrow$ $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$

(2) 2개의 cover를 만나는 vertex를 이어서 edge!!
(중심)
3개의 cover를 만나는 회로의 interior는 면으로 채우고!!

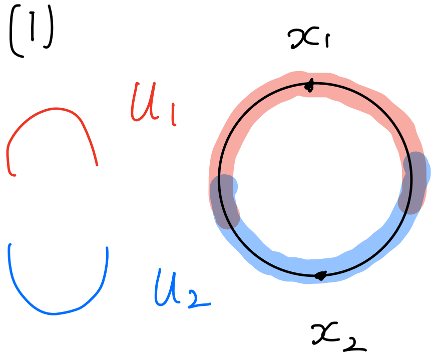
⋮

i.e., 2개 만나면 edge 생성
3개 " face "

Nerve 구하기!!

Example

(1)



① $S^1 \subset U_1 \cup U_2$

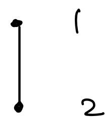
$\left(\text{i.e., } \mathcal{U} = \{U_i \mid i=1,2\} : \text{cover of } S^1 \right)$

② $U_i \neq \emptyset$ $(\forall i)$

③ $U_1 \cap U_2 \neq \emptyset$

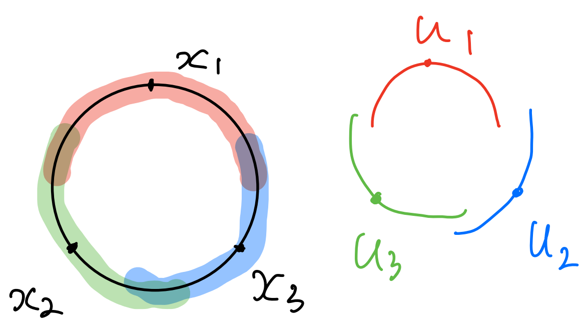
$\Rightarrow \mathcal{N}(\mathcal{U}) = \{ \{1\}, \{2\}, \{1,2\} \}$

$S^1 \not\simeq$



: abstract simplicial complex (1-simplex)

(2)



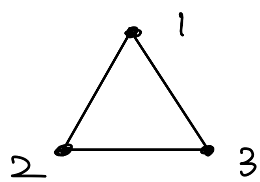
① $S^1 \subset \bigcup_{i=1}^{3} U_i$

② $\forall i,\ U_i \neq \emptyset$

③ $U_i \cap U_j \neq \emptyset$, $\forall i \neq j$. But!! $\bigcap_{i=1}^{3} U_i = \emptyset$

$\Rightarrow \mathcal{N}(\mathcal{U}) = \{ \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\} \}$

$S^1 \simeq$



2-simplex (without interior)

THEOREM 3.1. (Nerve Theorem)

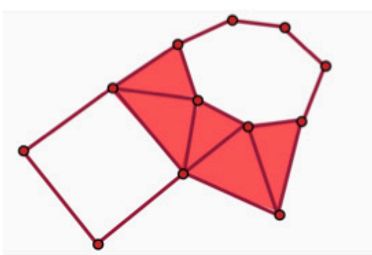
$\mathcal{U}$ = a finite collection of closed, convex sets in Euclidean sp.

$= \{ C_i \subset \mathbb{E}^n \mid \overline{C_i} = C_i : \text{convex},\ i \in I : \text{finite} \}$

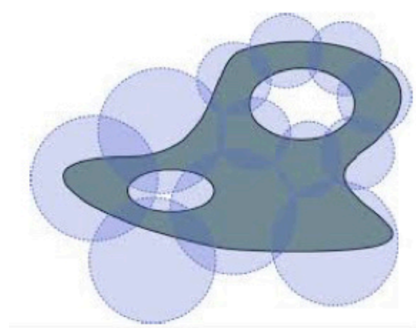
"closed"

(i.e., all limit points of C_i $\in C_i$)

$$\Rightarrow \quad \mathcal{N}(\mathcal{U}) \simeq \bigcup_{i \in I} U_i$$



$\simeq$



: Same homotopy type

Rmk.

X의 covering이 위의 세 조건 (finite / closed convex set들의 collection / in Euc. sp.) 만족하면 Nerve는 cover의 top.를 어느정도 반영.

Note.

$$\text{diffeo} \Rightarrow \text{homeo} \Rightarrow \text{homotopy eq} \Rightarrow \text{(co)homology isom.}$$

$$\not\Leftarrow \qquad \not\Leftarrow \qquad \not\Leftarrow$$

$\because$ Poincaré conj.

$\boxed{1}$

Def. 3.10 (Čech Complex)

metric sp.

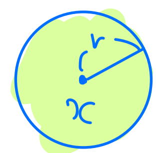
Let $X = \{ x_0, x_1, \cdots, x_n \mid x_i \in \mathbb{R}^d \}$

finite set.

For $x \in X$,

$$B(x,r) := \{ y \in \mathbb{R}^d \mid d(x,y) \leq r \}$$

: the closed ball centered at x w/ radius r $(r \geq 0)$.



$$\mathrm{Cech}(X,r) = \mathcal{N}(\underbrace{B(X,r)})$$

$\{ B_r(x) \mid x \in X \}$: closed cover of X

abstract simplicial complex ..

$$\underset{\cdot\cdot}{=} \{ \sigma \subset X \mid \bigcap_{x \in \sigma} B(x,r) \neq \emptyset \}$$

def. of Nerve

Remark.

(1) $\mathrm{Vertex}(\check{C}(X,r)) = \{ \{x\} \mid x \in X \} = X$

(2) closed ball $B(x,r)$ in Euclidean sp. $\mathbb{R}^d$

= closed, convex set.

Since X : finite, $B(X,r)$: finite collection (finite covering)

$\Rightarrow$ By Nerve thm.

$$\check{C}(X,r) \simeq B(X,r)$$

$\boxed{2}$ **Def 3-11.** (Vietoris Rips Complex)

$$X = \{x_0, \cdots, x_n \mid x_i \in \mathbb{R}^d\}$$

: a finite set of points in $\mathbb{R}^d$

$$B(x, r) = \{y \in \mathbb{R}^d \mid d(x, y) \le r\}, \ x \in X$$

: closed ball of x

$x_i \neq x_j$ 이므로 vertex set 포함됨.

$$VR(X, r) := \{\sigma \subset X \mid B(x_i, r) \cap B(x_j, r) \neq \emptyset, \ \boxed{\forall x_i, x_j \in \sigma}\}$$

: "The Vietoris Rips complex of X & r"

$$= \{\sigma \subset X \mid \mathrm{diam}(\sigma) \le 2r\}$$

Note. X : a metric space, finite. $r \geq 0$.

(1) Čech cplx.

- vertex set : X
- simplex $(x_0, \cdots, x_n)$ when
$$\bigcap_{i=0}^{n} B(x_i, r) \neq \emptyset$$

(2) VR cplx.

- Vertex set : X
- simplex $(x_0, \cdots, x_n)$ when
$$d(x_i, x_j) \le 2r, \ \forall i, j.$$

(3) $\check{C}(X,r) \subseteq VR(X,r) \subseteq \check{C}(X,\sqrt{2}r)$

$\uparrow$ Same edge set in $\mathbb{R}^d$

$\uparrow$ Jung's Thm.

$\check{C}(X,r) = \mathcal{N}(B(X,r))$

$\simeq$: homotopy eq. $B(X,r)$ by Nerve Thm.

$\check{C}(X,\sqrt{2}r) = \mathcal{N}(B(X,\sqrt{2}r))$

$\simeq$: homotopy eq. $B(X,\sqrt{2}r)$ by NT.

(4) $VR(X,r)$는 Nerve로 정의되지 않음.

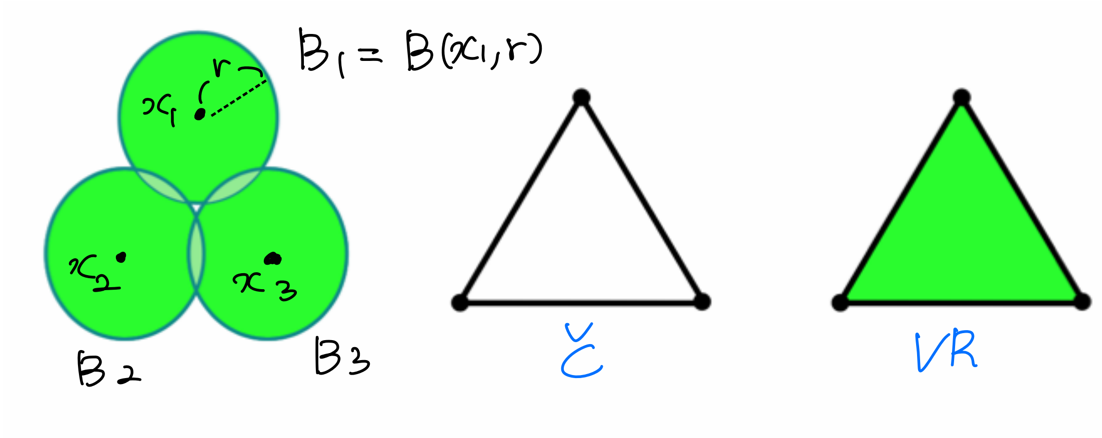
→ Nerve Thm. 적용 불가.

→ But!! 두 Čech cplx. 사이에 有

So.. good approximation.

And, Čech 보다 구성하기 쉬움.

→ $VR(X,r)$ 구하는 이유로 "Bootcamp note"에 언급된 내용. (무슨 소린지 잘 모르겠음...).



- $X = \{x_1, x_2, x_3\}$
- $\{B(x_i, r) \mid i = 1,2,3\}$: X의 cover.
- $B_1 \cap B_2 \neq \emptyset$, $B_2 \cap B_3 \neq \emptyset$, $B_3 \cap B_1 \neq \emptyset$
- $B_1 \cap B_2 \cap B_3 = \emptyset$

$$\check{C}(X, r) = N(B(X, r))$$
$$= \Big\{ \underbrace{\{x_i\}_{i=1,2,3}}_{3개},\ \underbrace{\{x_i, x_j\}_{i \neq j}}_{3개} \Big\}$$

$$VR(X, r) = \{ \sigma \subset X \mid B(x_i, r) \cap B(x_j, r) \neq \emptyset,\ \forall x_i, x_j \in \sigma \}$$
$$= \Big\{ \underbrace{\{x_i\}_{i=1,2,3}}_{3개},\ \underbrace{\{x_i, x_j\}_{i \neq j}}_{3개},\ \underline{\{x_1, x_2, x_3\}} \Big\}$$

문제점 : $\mathbb{R}^d$ 의 차원 관계없이...

closed ball 이 많이 겹칠수 있으니까 위의 두 cplx.
의 차원은 엄청 커질 수 有.

해결 : Delaunay, Alpha complexes.

① Delaunay complex

- Voronoi cell : 각 vertex의 영역 분할
- Voronoi diagram = { Voronoi cells }
- D-cplx. = nerve of V-diag.

② Alpha complex

- $B(x, r)$ + Voronoi cell
  → new cell !!

$\boxed{3}$ Delaunay Complex.

Def 3.12.

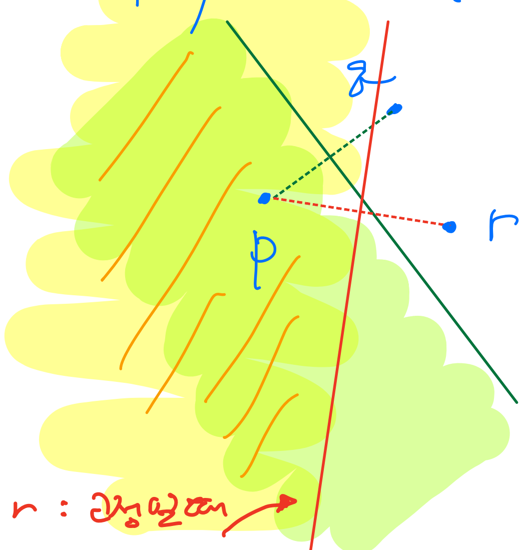
$X \subset \mathbb{R}^d$: finite point set

(1) Voronoi cell of $p \in X$

$$V(p) = \{ x \in \mathbb{R}^d \mid d(x,p) \leq d(x,q) \ \forall q \in X \}$$

$\mathbb{R}^2$

$p \neq q \in X$: fixed. ($\neq r$)



① $\{ x \in \mathbb{R}^2 \mid d(x,p) \leq d(x,q) \}$

$= \{ x \in \mathbb{R}^2 \mid d(x,p) \leq \frac{1}{2} d(p,q) \}$

② $\{ x \in \mathbb{R}^2 \mid d(x,p) \leq d(x,r) \}$

$= \{ x \in \mathbb{R}^2 \mid d(x,p) \leq \frac{1}{2} d(p,r) \}$

$\vdots$

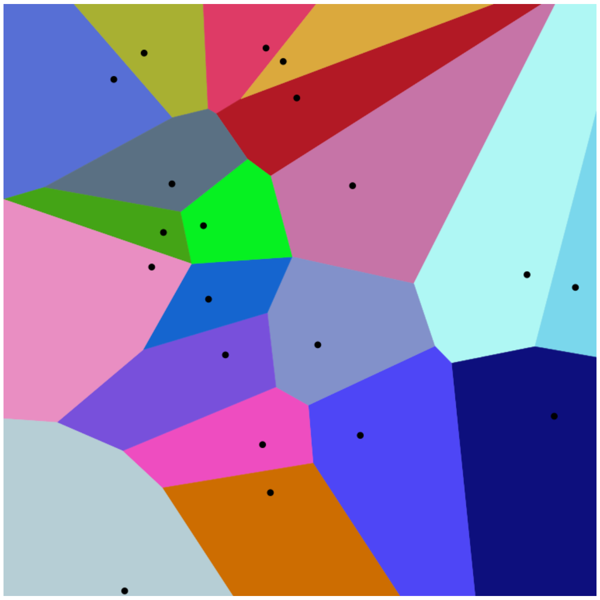
p, r : 고정일때 →
영역을 분할하는 직선은
$d(x,p) = d(x,q)$ 인
모든 x들...

p에 인접하는 vertex 에 대해,

$V(p,q) \cap V(p,r) = V(p)$

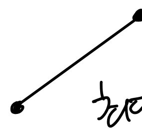
(2) $V(X) = \{ V(p) \mid p \in X \}$

: "Voronoi diagram of X"



: Voronoi diagram

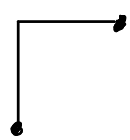
for L_2-norm.



최단거리

: Voronoi diag.

for L_1-norm.



최단거리

Remark.

$V(X)$: the covering of $\mathbb{R}^d$, also.

Def 3.13. (Delaunay Complex)

$$\mathrm{Del}(X) := \mathcal{N}(\underbrace{V(X)}_{\{\underbrace{V(p)}_{\{x \in \mathbb{R}^d \mid d(x,p) \le d(x,q),\ \forall q \in X\}} \mid p \in X\}}) : \text{the nerve of V-diag.}$$

$$= \{\sigma \subset X \mid \bigcap_{p \in \sigma} V(p) \neq \emptyset\}$$

Remark. Inner $\Rightarrow$ norm $\Rightarrow$ metric $\Rightarrow$ top.

(1) $d(x,p) \le d(x,q)$

: 일반화

$\|x-p\| \le \|x-q\|$

(2) radius 조절 $\rightarrow$ get a family of subcplx. of the Delaunay cplx.

(p.t. bottom)

$$\pi_p(x) = \|x-p\|^2 - w_p$$

where w_p : the weight of p

$$V_p = \{ x \in \mathbb{R}^d \mid \pi_p(x) \le \pi_q(x) \ \forall q \in X \}$$

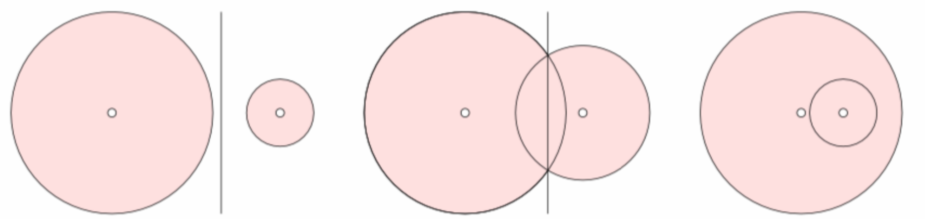
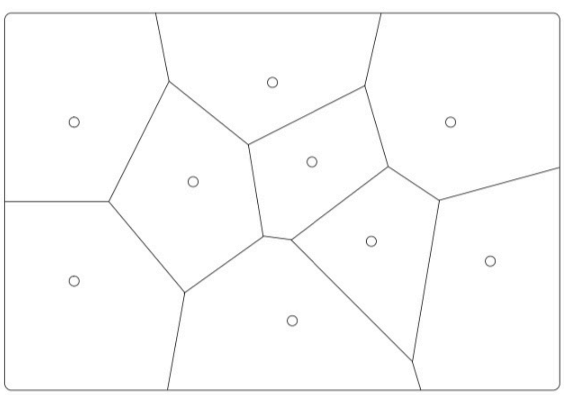
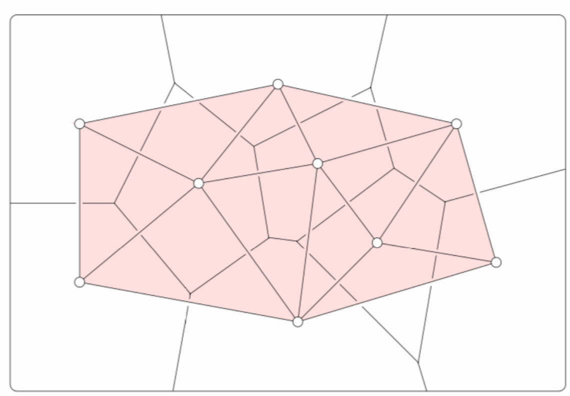


Figure III.12: The bisector of two weighted points. From left to right: two circles side by side, two intersecting circles, and two nested circles.





Voronoi diagram

||

The dual of Delaunay triangulation. = 삼각형 외접원의 중심을 모두 연결

Delaunay Triangulation

$\boxed{4}$ <u>Def 3.14</u>. (Alpha Complex).

$X \subset \mathbb{R}^d$: finite set of pts.

$p \in X$.

$$R(p,r) := B(p,r) \cap V(p)$$

$$\mathrm{Alpha}(X,r) := \{ \sigma \subset X \mid \bigcap_{p \in \sigma} R(p,r) \neq \emptyset \}$$

$\boxed{5}$ Witness Complex

Introduction.

For large dataset

→ too expensive to compute a simplicial complex.

→ "landmark" = a subset of points.

→ Construct a simplicial cplx for landmark points.

= "Witness complex"

(= Weak version of Delaunay complex.)

Recall. (X, d) : a metric sp. $p \in X$.

(1) $V(p) = \{ x \in \mathbb{R}^d \mid d(x, p) \leq d(x, q),\ \forall q \in X \}$

→ $V(X) = \{ V(p) \mid p \in X \}$: Voronoi diagram.

(2) $Del(X) = \{ \sigma \subset X \mid \bigcap_{p \in \sigma} V(p) \neq \emptyset \}$

: Delaunay cplx.

Def. (X, d) : a metric space. ($\subset \mathbb{R}^d$: finite.)

$L \subseteq X$: a finite subset

Suppose that $\emptyset \neq \sigma \subseteq L$.

(1) $x_\sigma \in X$ is a "weak witness for σ" with respect to (w.r.t.) L.

$\overset{def}{\iff}$ $\forall l_\sigma \in \sigma$, $\forall l_{\bar{\sigma}} \in L \setminus \sigma$,

$d(x_\sigma, l_\sigma) \leq d(x_\sigma, l_{\bar{\sigma}})$

// For arbitrary $l_\sigma \in \sigma (\subset L)$: fix,

we consider the restricted subset $L \subseteq X$.

$\rightsquigarrow V(l_\sigma, L) = \{ x \in \mathbb{R}^d \mid d(x, l_\sigma) \leq d(x, l_{\bar{\sigma}}), \forall l_{\bar{\sigma}} \in L \setminus \sigma \}$

$\rightsquigarrow$ If $\displaystyle\bigcap_{l_\sigma \in \sigma} V(l_\sigma, L) \neq \emptyset$,

then $\exists x_\sigma \in \displaystyle\bigcap_{l_\sigma \in \sigma} V(l_\sigma, L)$

s.t. x_σ = "weak witness for σ"

And we can define the Delaunay cpx. w.r.t. L. //

(2) $\mathrm{Del}^{\omega}(X, L) := \{\sigma \subset L \mid \sigma \text{ has a w.w. in } X\}$

$$= \{\sigma \subset L \mid \bigcap_{l_\sigma \in \sigma} V(l_\sigma, L) \neq \emptyset\}$$

$$\left(= \mathrm{Del}(X)\big|_L \right)$$

To obtain nested complexes, one can extend the definition of witness to ε-witness.

<u>Def.</u> (X, d) : a metric sp., X : finite.

(1) $x_\sigma \in X$: a weak ε-witness for σ w.r.t. L.

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall l_\sigma \in \sigma, \forall l_{\bar\sigma} \in L \setminus \sigma,$

$$d(x_\sigma, l_\sigma) \leq d(x_\sigma, l_{\bar\sigma}) + \varepsilon$$

(2) $\mathrm{Del}^{\omega}(X, L, \varepsilon) = \{\sigma \subset L \mid \bigcap_{l_\sigma \in \sigma} V(l_\sigma, L, \varepsilon) \neq \emptyset\}$

Exercises 3.1

metric sp.

#1. $X \subseteq \mathbb{R}^d$: finite.

- $\check{C}(X, r) = \text{Nerve}(B(X, r))$
$$= \{ \sigma \subset X \mid \bigcap_{x \in \sigma} B(x, r) \neq \emptyset \}$$

- $VR(X, r) = \{ \sigma \subset X \mid B(x_i, r) \cap B(x_j, r) \neq \emptyset, \ \forall x_i \neq x_j \in \sigma \}$
$$= \{ \sigma \subseteq X \mid \text{diam}\, \sigma \leq 2r \}$$

- $\text{Del}(X) = \text{Nerve}(\underbrace{V(X)}_{\text{voronoi diag.}})$

where voronoi cell
$$V(p) = \{ x \in \mathbb{R}^d \mid d(x, p) \leq d(x, q), \ \forall q \in X \}$$

- $\text{Alpha}(X, r) = \text{Nerve}(\{ B(x, r) \cap V(x) \mid x \in X \})$
$$= \{ \sigma \subset X \mid \bigcap_{x \in \sigma} [B(x, r) \cap V(x)] \neq \emptyset \}$$

- $\text{Del}(X, L) := \{ \sigma \subset L \mid \sigma \text{ has a w.w. in } X \}$
$$= \{ \sigma \subset L \mid \bigcap_{l_\sigma \in \sigma} V(l_\sigma, L) \neq \emptyset \}$$
$\left(= \text{Del}(X)\big|_L \right)$: witness cpx.

- $\mathrm{Del}(X, L, \varepsilon) = \{\sigma \subset L \mid \bigcap_{l_\sigma \in \sigma} V(l_\sigma, L, \varepsilon) \neq \emptyset\}$

#2. $\check{C}(X, r) \subseteq VR(X, r) \subseteq \check{C}(X, 2r)$

$\subseteq \check{C}(X, \sqrt{2}\, r)$

#3. $\chi(X) = \boxed{\sum_{i=0}^{n} (-1)^i |\Delta_k|}$

V - e + f 의 일반화.

(where $|\Delta_k|$ = the # of k-simplices)

$= \sum_{i=0}^{n} (-1)^i \beta_i$

(where the betti number $\beta_i(X) = \operatorname{rank} H_i(X)$ $(= \dim H_i(X))$.)

$= 2 - 2g$

5. Persistent Homology

Idea.

D = data cloud, data set

↓

Cov(D) = covering of D

Nerve of Covering.

↓

Several simplicial complexes (until now)

↓

Filter of complexes

↓

To approximate the original homology

, Persistent Homology !!

→ representation of PH.

- Barcode diagram
- Persistence diagram.

Remark.

1. Simplicial complexes defined by nerve depend on r.

e.g.) $\begin{cases} r : \text{small} \rightarrow \beta_0 \uparrow \\ r : \text{large} \rightarrow \beta_1 \downarrow . \end{cases}$

2. Find the optimal r value that captures topological features well.

(Using filter ~ sequence of simplicial cplxes.)

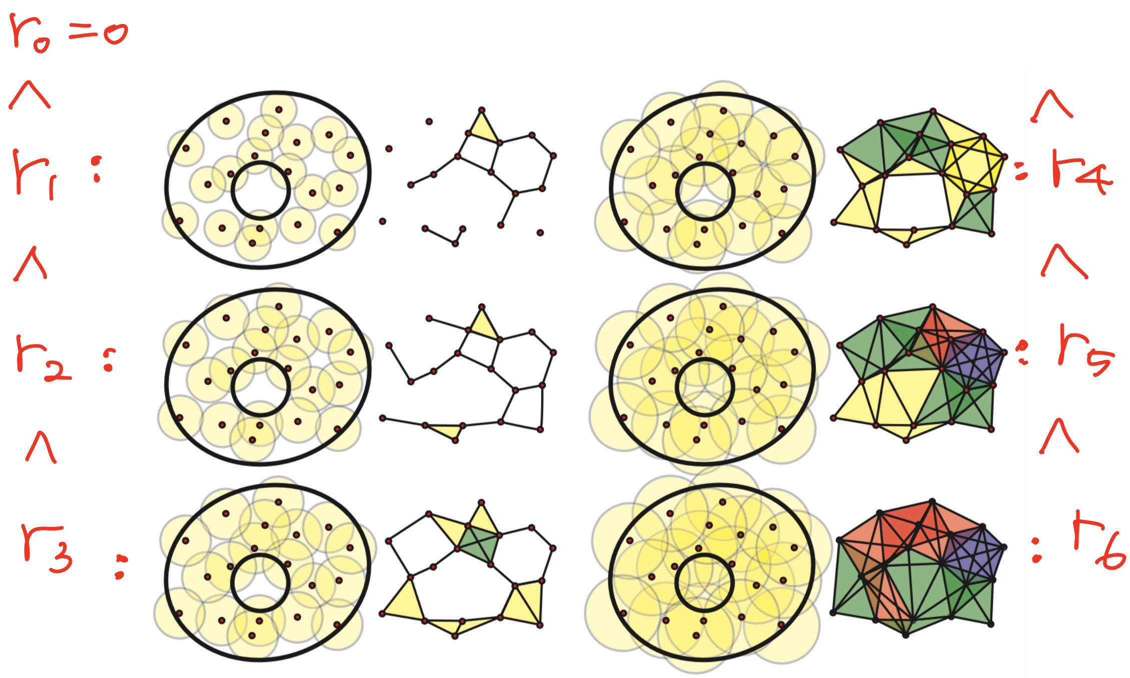


FIGURE 3. A sequence of Rips complexes for a point cloud data set representing an annulus. Upon increasing ϵ, holes appear and disappear. Which holes are real and which are noise?

e.g.) $$\{VR(X,r)\}_{r\in[0,6]}$$

$$VR(X,0) \hookrightarrow VR(X,r_1) \hookrightarrow \cdots \hookrightarrow VR(X,r_6)$$

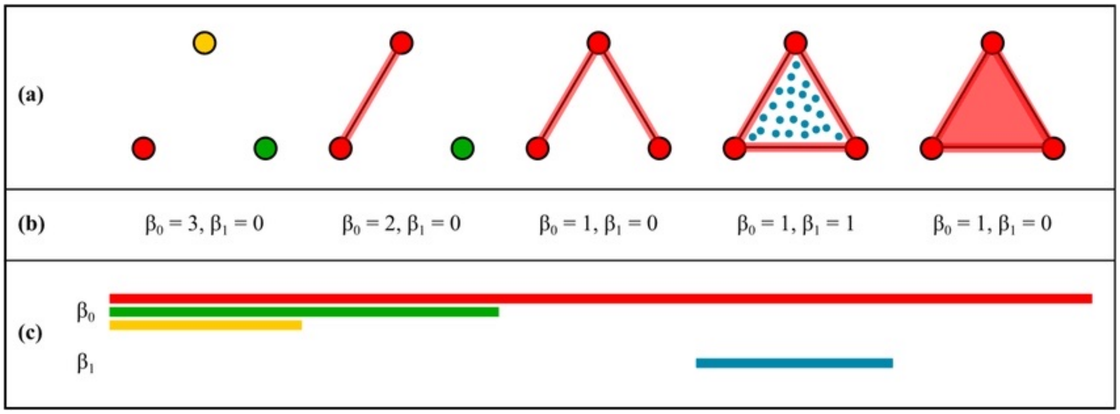
(Simply, we write $V_k = VR(X, r_k)$)

this filter induces

$$H_\bullet(V_i) \longrightarrow H_\bullet(V_j) \quad (i<j)$$

$\Rightarrow$ These maps reveal which features persist!!.

6. Representation of PH : Barcode, P-diag.



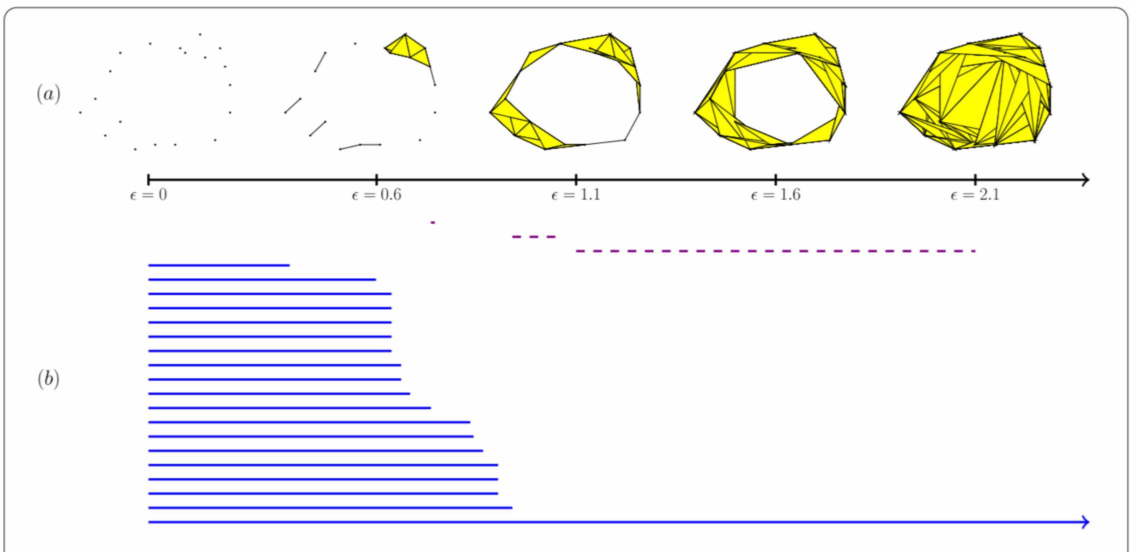


Figure 1 Example of persistent homology for a point cloud. (a) A finite set of points in $\mathbb{R}^2$ (for $\epsilon = 0$) and a nested sequence of spaces obtained from it (from $\epsilon = 0$ to $\epsilon = 2.1$). **(b)** Barcode for the nested sequence of spaces illustrated in (a). Solid lines represent the lifetime of components, and dashed lines represent the lifetime of holes.

← [Roadmap p.3.]

※ Persistence diagram. (??)

7. Functional Persistence

X : top'e sp. with an associate function $f : X \to \mathbb{R}$.

E.g.)

X = 3D image of tumor

$f : X \to \mathbb{R}$ measures of radioactivity of each point.

$\downarrow$

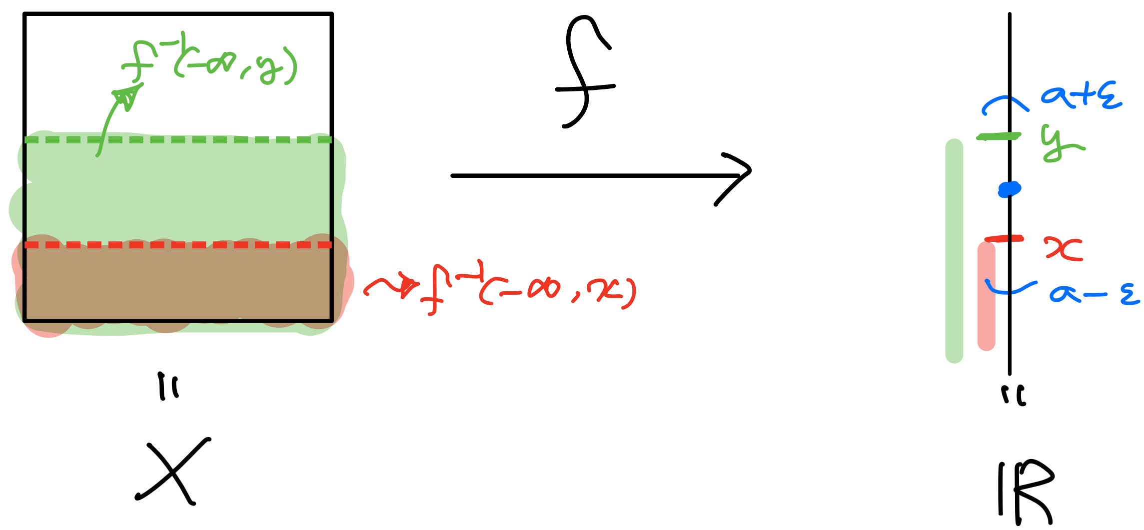
$f(X)$: the image filled by radioactivity

$\downarrow$

For each $a \in \mathbb{R}$,

$S \underset{\text{let}}{=}$ corresponding super / sub level sets of $f^{-1}(-\infty, a)$

Compute $H_0(S)$



$$0 \to f^{-1}(-\infty, x) \to f^{-1}(-\infty, y) \qquad (\%)$$

Let $\begin{cases} FX = f^{-1}(-\infty, x) \\ FY = f^{-1}(-\infty, y) \end{cases}$

Def. 1.1

(1) $a \in \mathbb{R}$: homological regular value

$\overset{\text{def}}{\Longleftrightarrow}$ $(\%)$ induces isomorphism on all homological gps.

i.e. $H_\bullet(FX) \xrightarrow{\cong} H_\bullet(FY)$

(2) $a \in \mathbb{R}$: homological critical value

$\overset{\Longleftrightarrow}{\text{def}}$ $a \neq$ homological regular value.

Def 7.2. (Tame)

A function $f : X \to \mathbb{R}$: "tame"

$\overset{\Longleftrightarrow}{\text{def}}$.
- f has finitely many homological critical value.
- $h_\bullet(f^{-1}(-\infty, a]) < \infty$ $\forall a \in \mathbb{R}$.

Notation

$h_\bullet(X) = \dim H_\bullet(X)$