Homework 5

Advanced Algorithms

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① Let there be x jobs and y machines.
Out of the x jobs, let $y(y-1)$ be of time equal to 1. This implies that these jobs will be balanced on all ~~a~~ machines.

Now,
let there be a job with time equal to y.
This job will be assigned to only one machine.

Overall,

Makespan of $y(y-1)$ jobs $= \cancel{y}-1 \approx y$ (using pigeon hole)

Makespan of last job $= 2y-1$

Hence,

Approximation ratio $= \frac{2m-1}{m} = 2 - \frac{1}{m}$

$\Rightarrow AR = 2$ when $m \to \infty$

(3) Number of vertices in a FIS is $\geq \frac{|V_d|}{d+1} \geq \frac{|V|(d-5)}{(d+1)^2}$

[Reference - JaJa]

where, V_d is a subset of vertices each of degree less than d.

For, FIS of size at least $n/c \approx \frac{|V|}{c}$

we get, $c = \cancel{\frac{d-5}{(d+1)^2}} \frac{(d+1)^2}{d-5}$

$$\Rightarrow c = -d^2 + (2+c)d - 1$$

(i) For $c=3$

$$3 = -d^2 + 5d - 1$$
$$0 = -d^2 + 5d - 4$$
$$= -d^2 + d + 4d - 4$$
$$= d(1-d) + 4(1-d)$$
$$= (d-4)(d-1)$$

$\Rightarrow$ For $d=1$, FIS has size at least $n/3$.

<u>Graph Class</u>: Leaf nodes of a perfect binary tree

(ii) For $c=2$

$$2 = -d^2 + 4d - 1$$
$$0 = -d^2 + 4d - 3$$
$$= (d-3)(d-1)$$

$\Rightarrow$ For $d=1$, FIS has size at least $n/2$

Again, <u>graph class</u>: leaf nodes of a perfect binary tree

④ For ANY given constant c,

we know that $c \geq 1$

$\because$ FIS is a subset of graph G. (or proper subset)

Whenever c is rational,

i.e. $\frac{|V|}{c} \simeq \frac{|V|}{a} \times b$ where $a < b$ & $a, b \in \mathbb{N}$

In this case, an FIS can be created using the proof of this theorem (given at end)

Theorem: For any planar graph $G = (V, E)$ we can construct a FIS in linear sequential time

where

$$|FIS| \geq \frac{|V|(d+1)^2}{d-5}$$ where d is a constant.

Whenever c is irrational,

if we take a closest rational number which is less than c, we can get as close as possible as,

$$\frac{a}{b}|V| > \frac{|V|}{c}$$

Proof of Theorem

Let V_d be the set of vertices of $G = (V, E)$ that have degree $\leq d$; where d is any integer.

Let $V_h = V - V_d$

For $d \geq 6$, $|V_d| \geq \frac{|V|}{c}$ for some constant c

[using Euler's algorithm]

$$\Sigma \deg(v) = 2|E| \quad (\text{handshake theorem})$$

Now, $\sum_{v} \deg(v) \geq \sum_{v \in V_h} \text{degree}(v)$

$$\geq (d+1)(|V_h|)$$

Now, $(d+1)|V_h| \leq 2(3|V|-6)$

So, $|V_d| \geq |V| - |V_h| \geq \frac{|V|(d-5)}{d+1}$

Now, the number of vertices in FIS is

$$|FIS| \geq \frac{|V_d|}{d+1} \geq \frac{|V|(d-5)}{(d+1)^2}$$

which is a constant fraction of $|V|$ because $d \geq 6$ (constant).