MODELING MOUNTAIN BIKE SUSPENSION WITH DIFFERENTIAL EQUATIONS

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ABSTRACT. Characterized by open questions and conflicting opinions lacking scientific backing, the process of fine-tuning mountain bike suspension settings is more a difficult art than a science. This paper applies a second-order nonlinear ordinary differential equation (ODE) to model the dynamics of mountain bike suspension subject to trail forces. We modify the ODE to describe various components of the suspension, including the spring type, rebound, compression, air pressure, volume spacers, and friction. To validate our model, we simulate the performance of four professional riders' suspensions, each on their own preferred terrain. The result is a high-fidelity tool that can be used to predict the suitability of suspension settings to a given terrain.

1. Background/Motivation

Every year, southern Utah's steep redrock mesas host Red Bull Rampage, one of mountain biking's most renowned events. Riders tackle extreme features by hurtling down narrow chutes, soaring over canyons, and back-flipping into vertical drops as high as 95 feet. Traditionally, robust but heavy dual-crown suspension forks were considered mandatory for attempting such technical terrain. However, in 2021, Brandon Semenuk revolutionized Red Bull Rampage by becoming the first athlete to ride out of the starting gate on a much lighter single-crown fork. After a historic run, Semenuk was crowned champion of the competition—exemplifying how recent breakthroughs in mountain bike suspension design have expanded the boundaries of the sport.

Improvements in suspension design have also increased consumer interest. Modern suspension makes bikes easier to handle and more approachable to beginners, helping a wider audience consider mountain biking as their next outdoor hobby. However, a challenge facing many new riders is that realizing the full benefits of advanced suspension requires tuning numerous internal components to accommodate to one's physical attributes, riding style, and intended terrain. Advice for adjusting suspension is often a vague description of how the bike should "feel" or a general heuristic (e.g. "for the compression damping knob, 3 clicks is better than 2"), as opposed to rules derived from the actual suspension dynamics and measurements. In this

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project, we seek to model mountain bike front suspension with a second-order nonlinear ODE. Working from first principles, we demonstrate how different settings can be simulated on various types of terrain. If successful, our model can be used by future bikers to optimize their suspension for their next ride without relying only on hearsay.

A wealth of studies attempt suspension modeling, but many are data-driven rather than derived or investigate different problems. For example, while Nord also uses a second-order ODE to model the front and rear shocks of mountain bikes and motorcycles [NOR19], the study, which partners with a manufacturing company, focuses on improving test and production services rather than optimizing the parameters of existing suspension systems. Other papers, such as Damgaard et. al., only model the rear suspension of a mountain bike, which is mechanically different from the front suspension [BFD09]. In addition, experiments collecting empirical measurements far outnumber those that attempt to mathematically describe suspension phenomena. Waal measures acceleration data at the front and rear axles of a bike during rides and uses this data to create a dynamic system [Waa20]. Although Waal's study has the same objective as us, it differs from our mathematical approach from first principles.

2. Modeling

As mentioned above, many of the models presented for this system use field data for constructing their systems. While these models give insights into the interactions and movements of a suspension system, we choose a mathematical approach utilizing differential equations. We expect a mathematical model would be easier to simulate and deploy as a tool for suspension setup. A model developed from first principles also has the potential to give better mechanistic insight into how each parameter individually contributes to suspension behavior and how they can be improved. Second-order differential equations are the go-to model in math and physics for expressing a spring-mass system. Second-order differential equations take the form:

$$my''(t) + D(y') + S(y) = F(t)$$

where m is the mass (of the rider in our case), $D(y')$ is damping on the spring, $S(y)$ is the force applied by the spring, and $F(t)$ is a function representing external, time-dependent forces on the system.

2.1. **Factors Influencing Suspension Dynamics.**

To define our specific ODE, we will discuss several features that influence suspension dynamics and explain our methods to model them.

2.1.1. *Spring Characteristics.*

Suspension systems use different types of springs to achieve desired behavior. The primary difference is whether the shock is coil-sprung, using a metal coil to provide resistance, or air-sprung, using a piston of compressed air. The resistive force from the spring ($S(y)$) is often referred to as the spring curve or progression of the spring. Most

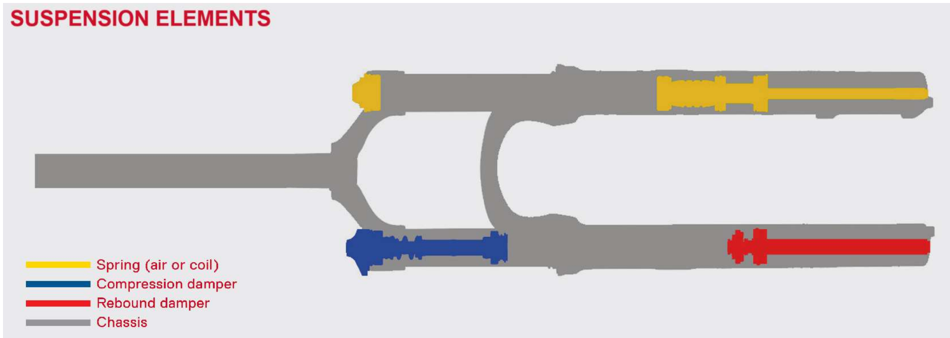

FIGURE 1. Key elements of a mountain bike shock. [ROC]

mountain bikes use cleverly engineered springs to achieve desired spring curves, but in our project, we assume coil suspension provides a resistive force according to Hooke's law: $S(y) = ky$ where k is the spring constant. Investigating and designing spring curves is an interesting topic that warrants further research.

On the other hand, air-sprung suspension provides a force that ramps up as the compression increases. The progression of an air spring can also be increased by adding volume spacers which reduce the volume in the air chamber of the shock by nV_s where n is the number of spacers. We now derive the force from an air spring from Boyle's law. Consider a cylindrical piston with a diameter of $2r$ (called the stanchion diameter) and uncompressed height h (called maximum travel). When uncompressed, this piston has volume $V_0 = \pi r^2 h - nV_s$. The initial pressure P_0 is set using a shock pump. Boyle's law states that at any height y where $0 \leq y \leq h$, the pressure in the piston satisfies $P_y V_y = P_0 V_0$. Since P_y is the total force F_y applied to the area of the piston head at y, we have $P_y = \frac{F_y}{\pi r^2}$. The volume at height y is given by $V_y = \pi r^2(h - y)$. Thus we can solve for F_y:

$$P_y V_y = P_0 V_0$$

$$\frac{F_y}{\pi r^2}\pi r^2(h - y) = P_0(\pi r^2 h - nV_s)$$

$$F_y = \frac{P_0(\pi r^2 h - nV_s)}{h - y}$$

So we have obtained an expression for the spring force of an air spring at height y. Notice that as y approaches h, F_y goes to infinity, so we have a finite-time blowup at $y = h$ corresponding to when the shock is fully compressed. We discuss the consequences of this phenomenon herewith.

We also need to model the mechanical forces experienced when the spring is fully compressed (called a bottom-out) and fully extended. Our fork has a hard stop on both the bottom and top, and we need to express this in our model so the spring does not burst through the bottom of the shock or explode out the top. In reality, these forces are instantaneous impulses and should be modeled with the Dirac delta function. However, this results in a discontinuous spring force that can not be integrated by a solver. Instead, we add a mollification to our spring forces to model this behavior:

$$F_{coil} = k\left(y + \left(\frac{y}{h}\right)^{100} - \left(\frac{y}{h} - 1\right)^{100}\right)$$

$$F_{air} = \frac{P_0(\pi r^2 h - nV_s)}{h - y} - \exp\left(1 - \frac{1000y}{h + 0.01}\right)$$

Note that we do not need a mollifier for the air spring when the shock is fully compressed since the physical forces already exhibit the desired behavior with a finite time blowup. Near $y = 0$ and $y = h$ the ODE becomes very stiff as these forces get large, as visualized in Figure 2. Solving the ODE with SciPy's `solve_ivp` function causes unnatural skips and edges in the solutions since it uses a solver with an adaptive step size. In the stiff regions, the solver can not adequately determine a sufficiently small step size for smooth solutions and produces jagged edges. Instead, we opted for our own implementation of RK4 with a small, fixed step size and got much better results.

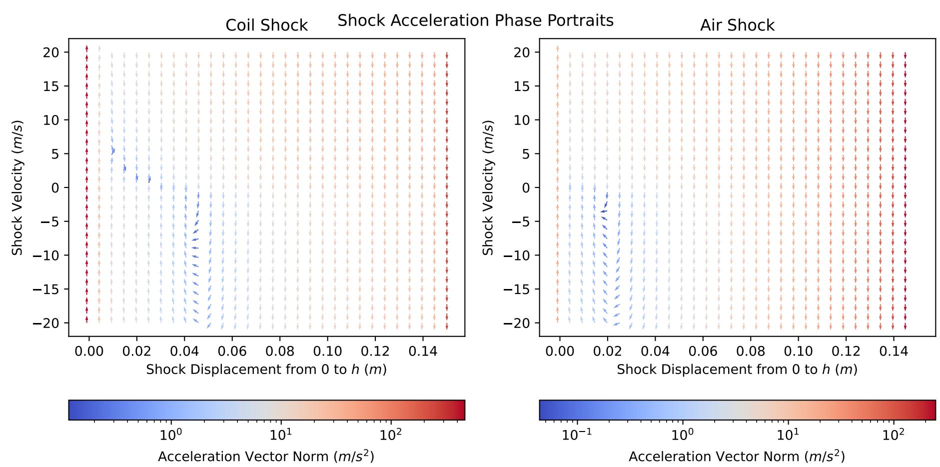

FIGURE 2. Acceleration phase portraits for coil and air shocks. The effects of mollification can be seen in the saturated vectors at $y = 0$ and $y = h$ imposing a strong force on the boundaries. The blue regions correspond to scenarios where the shock is not forced in any direction and has low acceleration. In the coil shock, it is easy to see the different influences of compression and rebound damping in the two blue regions on either side where the velocity is 0.

2.1.2. *Damping.* Damping is where the magic happens in suspension, but it is also the hardest part to model accurately. Shock dampers consist of several oil-flow circuits that regulate resistive force. Most modeling approaches treat dampers as black-box systems and predict behavior using regression on data. Since a full and accurate model of dampers requires difficult and complex fluid flow analysis through changing surfaces, we approximate damper behavior with heuristic functions in our model. We model five types of damping: high-speed rebound (HSR), high-speed compression (HSC), low-speed rebound (LSR), low-speed compression (LSC), and friction.

Rebound affects how quickly the spring returns to its initial position after displacement and is only applied when the shock is extending. Too little rebound causes the spring to get "packed up" by repeated compressions while too much rebound provides an undesirable pogo-stick feeling on the return. HSR is activated when there is sufficient pressure on the shock to open extra oil bypass valves. Given scaling constants $\mathcal{D}_{\text{HSR}}$ and $\mathcal{D}_{\text{LSR}}$, we represent rebound as:

$$R(y, y', t) = \begin{cases} 0 \text{ if } y' \leq 0 \\ \mathcal{D}_{\text{LSR}}\left(\frac{1}{\max(1, |y'|)}\right) + \mathcal{D}_{\text{HSR}} \cdot \min(1, |y'|) \text{ if } y' > 0 \end{cases}$$

The intuition behind this equation is that LSR influences the slow shock movements, but quickly has little influence after the LSR valve is fully open. At this point, the HSR valve begins to open and provide damping. In other words, the LSR parameter influences the rebound rate at the beginning and the HSR parameter influences the rate at high speed when the LSR has a negligible effect.

Compression damping applies resistance as the shock is compressed, opposite the direction of rebound damping. HSC provides resistance when the shock moves quickly through its stroke and dampens high-force impacts, such as hard landings from jumps. LSC provides resistance when the shock moves slowly through its stroke handling chattery irregularities such as rock gardens in the trail. When working well together, the suspension can glide smoothly over small bumps without bottoming out on larger hits. Our equation for compression is opposite that for rebound. Given scaling constants $\mathcal{D}_{\text{HSC}}$ and $\mathcal{D}_{\text{HSC}}$ we have:

$$C(y, y', t) = \begin{cases} 0 \text{ if } y' \geq 0 \\ \mathcal{D}_{\text{LSC}}\left(\frac{1}{\max(1, |y'|)}\right) + \mathcal{D}_{\text{HSC}} \cdot \min(1, |y'|) \text{ if } y' < 0 \end{cases}$$

Developing these equations took several iterations of experimentation and analysis. Despite simplifying assumptions, they provide an adequate approximation of how each parameter changes damping behavior.

Finally, we represent friction in our shock system with the parameter μ. Unfortunately, we have no good estimate for the friction in the shock, so we set it to a low positive number in all of our experiments since the force due to friction is negligible compared to other forces in the system. Adding all of our damping terms together gives us the total damping equation

$$D(y') = R(y') + C(y') + \mu.$$

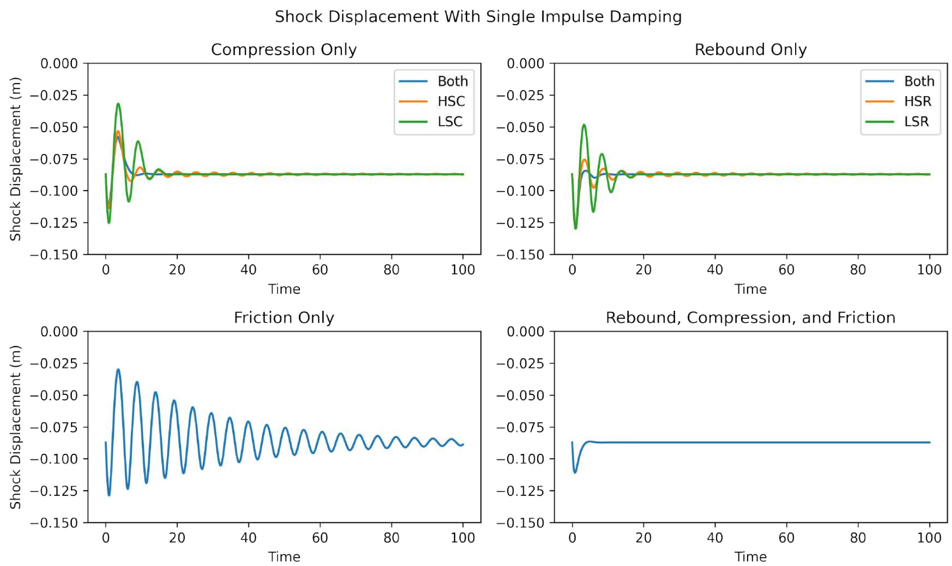

FIGURE 3. This plot demonstrates the effects of components of damping after a bike experiences an impulse force. Adding damping factors enables the shock to quickly and smoothly return to the initial height rather than oscillate wildly. The parameters used to produce these plots were exaggerated from their typical values to illustrate each type of damping.

2.1.3. *External Forces.* The two main forces experienced by the shock are the force exerted by gravity on the rider and the force from impacts with obstacles on the trail.

One of the most important settings to adjust is the resistive force of the spring to comply with rider weight. The force due to the rider is mg where m is the rider's mass and g is the acceleration due to gravity. A heavier rider requires a stiffer spring to avoid harshly bottoming out the suspension while a lightweight rider needs a lighter spring. The initial position y_0 of the spring when weighted by the rider is called sag. To find the sag we set the force due to the rider weight equal to the force due to the spring and solve for the displacement y. For a coil fork, we have $y_0 = \frac{mg}{k}$ and for an air-sprung fork, we have $y_0 = h - \frac{P_0(\pi r^2 h - nV_s)}{mg}$. We will use this point as our initial value when we solve the ODE.

The more exciting force to model is the force on the shock from various trail types that we will simulate. We will attempt to model several different conditions to determine which suspension settings are best for each. After experimentation, we approximate the discontinuous nature of trail forces with quartic functions scaled to realistic ranges. We simulate different types of terrain by randomly combining obstacles with parameters drawn from hand-chosen probability distributions. Adding our forces from rider weight and the trail gives $F(t) = mg + T(t)$ where $T(t)$ is the function describing the force from the trail at time t.

2.2. **Model Analysis.** Now that we have defined our model, we investigate its attributes to determine if it is realistic.

2.2.1. *Stability Analysis.* First, we want to show that its stability properties align with our expectations. We can rewrite our equation as a system of first-order equations. Let $P = y$, $Q = \dot{y}$, and assume we have only the constant force of the rider on the bike. Then we have

$$\mathbf{V} = \begin{bmatrix} \dot{P} \\ \dot{Q} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} Q \\ -\frac{1}{m}\left(D(Q) + S(P)\right) + g) \end{bmatrix}$$

Applying Lyapunov's indirect method gives us the linearization

$$\frac{d}{dt}\begin{bmatrix} \dot{P} \\ \dot{Q} \end{bmatrix}\Big|_{\mathbf{V}=\mathbf{0}} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{m}\frac{dS}{dP}\big|_{P=0} & -\frac{1}{m}\frac{dD}{dQ}\big|_{Q=0} \end{bmatrix}$$

Let $\delta = \frac{1}{m}\frac{dS}{dP}\big|_{P=0}$ and $\sigma = \frac{1}{m}\frac{dD}{dQ}\big|_{P=0}$. Then we can compute the eigenvalues.

$$\lambda = -\frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 - \sigma}$$

Note that if $\delta = 0$ and $\sigma > 0$, we have imaginary eigenvalues representing the oscillating solution of the rider bobbing up and down on shock. For us to have the desired damped, stable solution (two negative eigenvalues), we need to have $\delta > 0$ and $\sigma > 0$. So D and S should be increasing functions at the equilibrium point. Interestingly, after analyzing stability, we realized that our initial damping functions were not always increasing, prompting us to re-engineer them.

2.2.2. *Dimensional Analysis.* Since we expect our model to give us applicable insights about tuning mountain bike suspension and not only describe its general behavior, we need to be careful with our units to get realistic, interpretable values. Some shock components are usually measured with Imperial units while others use SI units, so we ensure everything is converted to standard SI units with meters for length. Furthermore, we try to use values compliant with expected real-world measurements for our model parameters. Table 1 gives a complete breakdown of the model's parameters, their typical units of measurement, ranges of values, and conversions to the units used in the model.

Parameter	**Units**	**Expected value range**
Rider mass (m)	kg	Any realistic human body weight
Coil spring constant (k)	N/m	5,429 to 10,508 N/m [eva19]
Initial air pressure (P_0)	Pa	172,369 to 1.827e+6 Pa [SRA22]
Volume spacer volume (V_s)	m^3	10 cm^3
Stanchion diameter ($2r$)	mm	(32, 34, 36, 38, 40) mm
Travel Length (h)	mm	100 mm to 205 mm
Damping HSC ($\mathcal{D}_{\text{HSC}}$)	NA	To determine by simulation
Damping LSC ($\mathcal{D}_{\text{LSC}}$)	NA	To determine by simulation
Damping R ($\mathcal{D}_{\text{R}}$)	NA	To determine by simulation
Damping HSR ($\mathcal{D}_{\text{HSR}}$)	NA	To determine by simulation
Friction (μ)	$kg * m/s^2$	To determine by simulation

Table 1. Suspension model parameters. Note that Damping units are undetermined. This is because a physical solution for the Damping equation goes beyond the scope of this paper, so our equations and values for damping are simply approximations of how these systems are supposed to behave.

3. Results

To demonstrate the effectiveness of our model, we simulated the shocks and terrains for four professional riders in different mountain biking disciplines.

- **Tom Pidcock:** 2-time Olympic cross-country gold medalist. Uses SR Suntour 34 Axon (34 mm stanchions) with 120 mm travel. Rides smooth trails with chatter from small obstacles. Weighs 58 kg.
- **Isabeau Courdurier:** 3-time Enduro World Cup winner. Uses Rockshox Zeb (38 mm stanchions) with 170 mm travel. Rides trails with consistent big impacts from obstacles. Weighs 52 kg.
- **Brandon Semenuk:** 5-time Red Bull Rampage winner. Uses Rockshox Zeb (38 mm stanchions) with 190 mm travel. Rides trails with no chatter, but big impacts from jumps and drops. Weighs 80 kg.
- **Amaury Pierron:** 2-time Downhill World Cup champion. Uses Fox 40 (40 mm stanchions) with 203 mm travel. Rides trails with a combination of big impacts from obstacles and jumps. Weighs 81 kg.

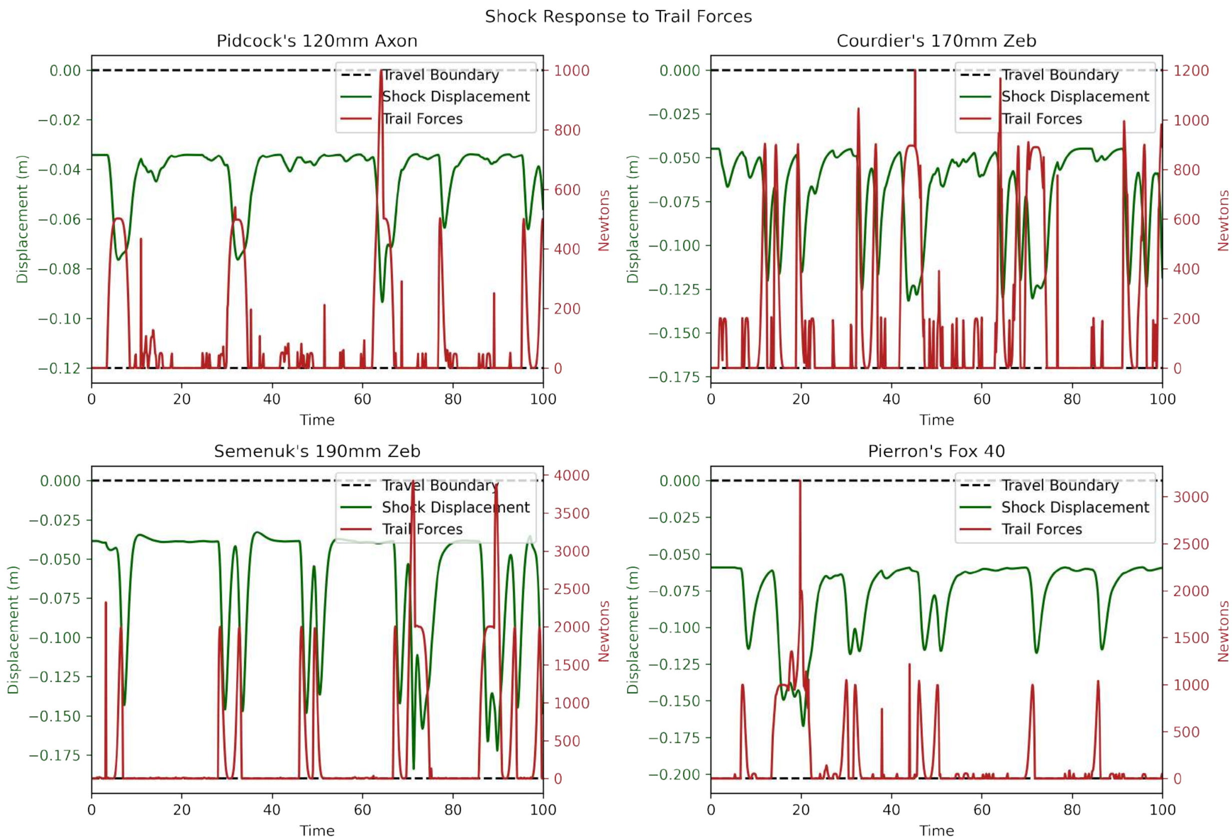

FIGURE 4. This plot demonstrates the effects of a simulated trail for each of the four professional riders with their individual shock setups and respective trail conditions.

Each trail is tailored to the rider's discipline, and we build a model of each rider's specific shock as seen in Figure 4. Note that the amount of force provided by the trail is different in each case. Each shock has been tuned to the expected trails for each riding style. These shocks react well to the trail, avoiding bottom-outs and remaining supple on small bumps. This demonstrates the ability of our model to handle a wide variety of scenarios and shock parameters.

One interesting observation is on Brandon Semenuk's ride near time step 70 where the trail forces reach 4,000 N, the equivalent of a 30 foot drop—no wonder he won Red Bull Rampage! The model shows good bottom-out performance, which is a hard phenomena to model since at this point the forces are very high, and other models could be prone to allowing the shock stanchions to punch through the shock housing and proceed to negative infinity. At the same time as handling large impact forces, we are able to maintain small-bump sensitivty as visualized in Thomas Pidcock's graph of shock displacement.

An especially exciting aspect of our model is that it outputs real, interpretable values and responds to realistic forces experienced on the trail. This was not easy to achieve. Overall, our model reflects reasonable behavior and interactions with the forces applied to the shock.

4. Analysis/Conclusions

Our mathematical model of mountain bike suspension from first principles is demonstrably capable of replicating suspension dynamics. It carefully considers an impressive number of shock characteristics and tunable parameters and accurately calculates how each influences shock behavior. However, there are several ways our research can be elaborated to provide better results. First, as mentioned earlier, we make rudimentary approximations for complex damping systems which could be made more rigorous by thorough analysis. Using physics, we could better approximate how these parameters exactly influence shock behavior. We could also get an estimate for friction by measuring an actual shock. Using these improved damping estimates, our model could make more interesting predictions about damping, which is one of the most interesting aspects of the suspension problem.

Originally, we also had hoped to develop an optimization algorithm to adjust parameters, simulate, and find the best fit for a rider. If we had more time we could also experiment with different progression curves for coil shocks and even invent custom progression curves on coil shocks. Finally, we recognize that beginner riders have very different styles from advanced riders. Usually, advanced riders are better at anticipating obstacles and weighting or unweighting the front shock of the bike to accommodate. We could make ability-specific shock recommendations by letting the rider mass be a function of time. Furthermore, we could give coaching on how riders could improve their technique by changing how they position themselves for specific obstacles.

In this project we learned that modeling real-world phenomena can be complicated but rewarding. Developing a model with many interactive subsystems that realistically handles forces from a trail simulator provided lots of learning opportunities. We had to obtain sufficient domain knowledge, translate suspension components to mathematical expressions, and troubleshoot unexpected system behaviors. The project gave us lots of valuable hands-on experience with properties of differential equations and numerical methods. In the end, it was satisfying to see math-inspired modeling replicate something we are passionate about.

References

[BFD09] Nicolai Christian Bloch Rasmus Bruus Nielsen Jonas Morsbol Jesper Kann Brian Frydensberg Damgaard, Gustav Winther Svendsen. Modelling and dimensioning the rear suspension of a mountain bike. Bachelor project, Aalborg Universitet, 2009. https://kann.dk/files/53b_I6.pdf.

[eva19] evan9r. Mrp ribbon *coil* master thread. https://www.mtbr.com/threads/mrp-ribbon-coil-master-thread.1105265/, 2019. Accessed: Dec 5 2024.

[NOR19] RASMUS NORD. Modelling of a motorcycle /mountain bike suspension and digitization of a cam drum motor control. Master's thesis, KTH Royal Institute of Technology School of Engineering Sciences, City, Country, 2019. https://kth.diva-portal.org/smash/get/diva2:1380258/FULLTEXT01.pdf.

[ROC] ROCKSHOX. Suspension theory guide. https://www.sram.com/globalassets/document-hierarchy/service-manuals/rockshox/suspension-theory-guide.pdf. Accessed: 2024-11-20.

[SRA22] SRAM. 2022 front suspension specifications. https://www.sram.com/globalassets/document-hierarchy/service-manuals/rockshox/front-suspension/2022-front-suspension-oil-air-coil-token-and-specifications-english.pdf, 2022. Accessed: Dec 5 2024.

[Waa20] Steven R. Waal. A quantitative approach for tuning a mountain bike suspension. Master's thesis, California Polytechnic State University, City, Country, 2020. https://digitalcommons.calpoly.edu/cgi/viewcontent.cgi?article=3821context=theses.