Computing Planck's blackbody spectrum, considering index of refraction

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When blackbody radiation is emitted from the opening of a gas filled cavity (e.g. Argon), and/or detected by a detector in a gas (e.g. air), then the index of refraction of the gas must be considered. See this paper by Gaertner, http://www.bipm.org/cc/CCT/Allowed/25/D11_CCTdraftAAG.pdf, which corrects the previous paper by Hartmann, https://www.osapublishing.org/oe/abstract.cfm?uri=oe-14-18-8121. However, both papers do not consider dispersion.

We know:

- Basic spectral radiance as function of frequency, $L_{basic,\nu}(\nu) = \dfrac{\mathrm{d}\Phi}{\mathrm{d}A\, n^2\ \cos(\theta)\, d\Omega\, \mathrm{d}\nu}$ is invariant along a ray when this ray undergoes refraction without Fresnel losses. This is known as the brightness conservation theorem. Another way to see this is to note that $\mathrm{d}U = \mathrm{d}A\, n^2\ \cos(\theta)\, d\Omega$ is the infinitesimal étendue surrounding this ray. Then, conservation of basic radiance results from conservation of étendue and conservation of flux.
- Integrating blackbody spectral basic radiance over frequency, $L_{basic} = \int L_{basic,\nu}(\nu)\,\mathrm{d}\nu$, yields $L_{basic} = \dfrac{\sigma T^4}{\pi}$, with absolute temperature T and Stefan-Boltzmann constant σ, independent of refractive index. This is required by thermodynamics: Think of two blackbody cavities at same temperature, but embedded in different refractive indices, whose small apertures "see" each other. Then, there must be zero net exchange of radiation: Otherwise, heat would flow spontaneously between two bodies at same temperature. The flux Φ emitted from each cavity into the optical channel with étendue U is given by $\Phi = L_{basic}\, U$, and by conservation of étendue Ubetween both ends of the optical channel, everything is consistent.

Note the factor of n^2 in the denominator of $L_{basic,\nu}(\nu)$: this is what makes basic spectral radiance different from standard (geometric) spectral radiance L_ν.

For blackbody radiation, basic spectral radiance as function of frequency is

$$L^{bb}_{basic,\nu}(\nu) = \frac{2h\nu^3}{c^2\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)}$$

There is no refractive index, correctly, because frequency does not change with index, and because of brightness conservation.

However: In practice, we often need spectra as function of wavelength, and that wavelength is often taken within a refractive medium.

The dispersion free case

Let us now change variables from frequency to wavelength λ in units of "wlu meters", e.g. $\mathrm{wlu} = 10^{-9}$ for nanometers, in medium with constant refractive index n:

$$\nu = \frac{c}{n\,\lambda\,\mathrm{wlu}} \qquad \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c}{n\,\lambda^2\,\mathrm{wlu}}$$

or, with $\lambda_{\mathrm{vac,m}} = n\,\lambda\,\mathrm{wlu}$ (the wavelength in vacuum in meters)

$$\nu = \frac{c}{\lambda_{\mathrm{vac,m}}} \qquad \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c}{\lambda_{\mathrm{vac,m}}\,\lambda}$$

Then, basic spectral radiance as function of wavelength in any units in medium is

$$L_{basic,\lambda}(\lambda) = L_{basic,\nu}(\nu(\lambda))\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c_{1L}}{\lambda\,\lambda_{\mathrm{vac,m}}^4\left[\exp\left(\dfrac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]} = \frac{c_{1L}\,n\,\mathrm{wlu}}{\lambda_{\mathrm{vac,m}}^5\left[\exp\left(\dfrac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]}$$

where $c_{1L} = 2hc^2$ and $c_2 = hc/k$ are CODATA's first and second radiation constants.

This result makes it easy to calculate $L_{basic,\lambda}(\lambda)$ numerically in a constant index medium:

First, compute the standard (non-basic) blackbody spectral radiance for wavelength in meters. Then, multiply with $n\,\mathrm{wlu}$. Finally, check the result by integrating over λ, verifying the result is $L_{basic} = \dfrac{\sigma T^4}{\pi}$.

The dispersive case

If there is dispersion, $n = n(\lambda)$:

$$\nu = \frac{c}{n(\lambda)\,\lambda\,\mathrm{wlu}} \qquad \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{\left(n(\lambda) + \lambda\dfrac{\mathrm{d}n}{\mathrm{d}\lambda}\right)c}{(n(\lambda)\,\lambda)^2\,\mathrm{wlu}}.$$

Note the additional $\lambda\dfrac{\mathrm{d}n}{\mathrm{d}\lambda}$ term in the numerator. Then,

$$L_{basic,\lambda}(\lambda) = L_{basic,\nu}(\nu)\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c_{1L}\left(n(\lambda) + \lambda\dfrac{\mathrm{d}n}{\mathrm{d}\lambda}\right)}{n(\lambda)\,\lambda\,\lambda_{\mathrm{vac,m}}^4\left[\exp\left(\dfrac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]} = \frac{c_{1L}\left(n(\lambda) + \lambda\dfrac{\mathrm{d}n}{\mathrm{d}\lambda}\right)\mathrm{wlu}}{\lambda_{\mathrm{vac,m}}^5\left[\exp\left(\dfrac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]}$$

The numerical calculation proceeds simlilar to the constant index case: Compute compute the standard (non-basic) blackbody spectral radiance for wavelength in meters. Then, multiply with $\left(n(\lambda) + \lambda\dfrac{\mathrm{d}n}{\mathrm{d}\lambda}\right)\mathrm{wlu}$. It is the $\dfrac{\mathrm{d}n}{\mathrm{d}\lambda}$ term which may cause some headaches. The dispersion curve $n(\lambda)$ may be given as a function, in e.g. Sellmeier or Cauchy cofficients. Then, it can be readily differentiated analytically. In case of $n(\lambda)$ given by tabulated values, I would recommend to perform a cubic spline interpolation of the tabulated values and then differentiate the cubic polynomials analytically. In Matlab, for example, `pp = spline(lambda, n)` returns a piecewise polynomial structure that can be evaluated using `ppval`, and which can be differentiated analytically. See my implementation of `PlanckSpectrum` for details.

The wavelength in this section is the wavelength within the medium where the basic radiance is investigated, not in vacuum. Accordingly, $n(\lambda)$ is taken as a function of wavelength in medium here. However, in practice, n is given as a function of vacuum wavelength, $n = n(\lambda_{vac})$. When the medium under consideration is a gas, like air or Argon, at near atmospheric pressure, the difference is entirely negligible: The refractive index of such gases, its wavelength dependence (refractiveindex.info) and the error in n is about

```
n_Ar = 1.00028;
dn_Ar_dlam_nm = 1.5e-8; % per nanometer
lam_vac = 500; %nm
lam_Ar = lam_vac / n_Ar;
delta_lam = lam_vac - lam_Ar;
delta_n = dn_Ar_dlam_nm * delta_lam
```

```
delta_n = 2.0994e-09
```

which corresponds to a relative error of about 0.00001 in the difference of refractive index in gas vs. vacuum, way below measurement accuracy of that refractive index.

For the (academic) case of blackbody radiation as function of wavelength in extremely dense gases or other optically dense media, we need to convert $n(\lambda_{vac})$ to the equivalent dispersion curve as function of wavelength in medium before using it in the above formula.

Measuring blackbody radiation in a medium

A practical situation in a laboratory might be like this: A cavity at a high temperature is filled with Argon at atmospheric pressure and same high temperature. Blackbody radiation is emitted from a small opening into the laboratory, passing through a boundary layer with a gradient of both Argon vs. air fraction and temperature, until the radiation is then detected in air, at ambient temperature and atmospheric pressure.

Neglecting the reflection losses at the Ar-air boundary layer is justified, both due to the small index change and the gradient.

Then, the conservation of étendue and the conservation of energy tell us that basic spectral radiance along each ray from inside the cavity until the detector remains unchanged: basic (spectral) radiance is flux divided by étendue. This is why I prefer to use basic radiance whenever I can: The conservation of basic radiance along a ray is an extremely useful mental tool to analyze, for example, what can and cannot be done to extract light out of LEDs (see https://www.degruyter.com/view/journals/aot/2/4/article-p291.xml), to quantitatively analyze the signal level at the image of a microscope with an immersion objective, and more.

If the quantity to be measured is spectral radiance (without the "basic"), however, we can simply use

$$L_\lambda(\lambda) = L_{basic,\lambda}(\lambda)\,(n(\lambda))^2$$