Simple Proof of Sylvester's Inertial Theorem

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Here the notation K denotes either the complex number field $\mathbb{C}$ or the entire members of the reals $\mathbb{R}$.

The bilinear forms are defined as below:

Definition 1. *Let V be a vector space over the field K,a bilinear form on V is a function $f : V \times V \to K$ such that for $u, v, w \in V$ and $\lambda \in V$*

- $f(u + \lambda v, w) = f(u, w) + \lambda f(v, w)$
- $f(w, u + \lambda v) = f(w, u) + \lambda f(w, v)$

That is,the function $f(u, v)$ has linearity property for both its argument u, v and it maps a pair of vectors (u, v) to a unique member in the field K.

It is trivial that

$$f(0 + 0, 0) = f(0, 0) + f(0, 0) = f(0, 0) = 0$$

Now we make the assumption that the bilinear forms are all symmetric:

$$f(u, v) = f(v, u)$$

Under such assumption,the quadratic form $q(v)$ is defined as:

$$q(v) = f(v, v)$$

Since:

$$\begin{aligned} q(u + w) &= f(u + w, u + w) \\ &= f(u, u) + 2f(u, w) + f(w, w) \\ &= q(u) + 2f(u, w) + q(w) \end{aligned}$$

There

$$f(u, w) = \frac{1}{2}(q(u + w) - q(u) - q(w))$$

We see that the bilinear form and its quadratic form are intimately related,they are ascertained by each other.

Define the linear transformation T_f associated with f and is between the vector space V and its dual space V':

$$T_f : V \to V' \qquad (T_f(v))\,(w) = f(v,w)$$

The transformation T_f is clearly linear.

Two vectors $u, w \in V$ are said to be f-orthogonal if

$$f(u,w) = 0$$

Consider the set

$$u^{\perp} = \{w \in V | f(u,w) = 0\}$$

Theorem 1. *$u^{\perp}$ is a subspace of V.*

Proof. Absolutely $0 \in u^{\perp}$

Let $v, w \in u^{\perp}$,$\lambda \in K$,because

$$f(u, v + \lambda w) = f(u,v) + \lambda f(u,w)$$

Then $u + \lambda w \in u^{\perp}$ □

Lemma 1. *Let f be a symmetric bilinear form on the vector space V. If $v \in V$ and $q(v) \neq 0$, then*

$$V = \text{span}\{v\} \oplus v^{\perp}$$

Proof. Suppose that $w \in \text{span}\{v\} \cap v^{\perp}$,then $w = \lambda v$, because

$$f(v,w) = \lambda f(v,v) = \lambda q(v) = 0$$

Thus $\lambda = 0$. Also, consider, for any $w \in V$, there

$$w = \frac{f(v,w)}{f(v,v)} v + \left(w - \frac{f(v,w)}{f(v,v)} v\right)$$

□

Using this lemma,we could prove the following important theorem by mathematical induction:

Theorem 2. *For any symmetric bilinear form f on V,there exists an f-orthogonal basis B of V.*

We can also in the same way, provide the proof with *Gram-Schmidt's procedure*. But that involves excessive discussion on the f-orthogonal and the linearly independent properties.

Proof. Assume the theorem holds for the vector space of dimension n,and when $n = 1$ the theorem becomes trivial. Now $n > 1$.

If the corresponding quadratic form $q(v) = 0$ for any $v \in V$,then by

$$f(u, w) = \frac{1}{2}(q(u + w) - q(u) - q(w))$$

any basis will suffice. Assume there exists $v \in V$ that $q(v) \neq 0$.

By Lemma 1

$$V = \text{span}\{v\} \oplus v^{\perp}$$

Then the subspace $v^{\perp}$ is $(n-1)$-dimensional and the only thing to prove is linearly independency, but $\text{span}\{v\} \cap v^{\perp} = \{0\}$. □

Now we have the preparation,the first step to Sylvester's inertial theorem is as below:

Theorem 3. *Let f be a symmetric bilinear form f, let B and B' be two f-orthogonal basis. Called the element v is bilinearly nonzero if $q(v) \neq 0$. Then the number of the bilinearly nonzero elements in B and B' are equal.*

Proof. Once the symmetric bilinear form f is fixed, range T_f as a subspace of the dual space V' has unvaried dimension. Suppose the elements of B are

$$b_1, b_2, \cdots, b_r, \cdots$$

The first r's elements are bilinearly nonzero, and the rest are not.

Similarly for B'

$$b'_1, b'_2, \cdots, b'_s, \cdots$$

If there exist $\alpha_1, \alpha_2, \cdots, \alpha_r$ that

$$\alpha_1 T_f(b_1) + \alpha_2 T_f(b_2) + \cdots + \alpha_r T_f(b_r) = 0$$

Then by $(\alpha_1 T_f(b_1) + \alpha_2 T_f(b_2) + \cdots + \alpha_r T_f(b_r))(b_i) = 0$ we conclude that $\alpha_i q(b_i) = 0$, since b_i is bilinearly nonzero,$\alpha_i = 0$. The linear functionals

$$T_f(b_1), T_f(b_2), \cdots, T_f(b_r)$$

forms a basis for the subspace range T_f, similarly

$$T_f(b'_1), T_f(b'_2), \cdots, T_f(b'_s)$$

also a basis. Because they're for the same subspace, $r = s$. □

A vector v is bilinearly negative if $q(v)$ is a nonzero real negative number, v is bilinearly positive if $q(v)$ is a nonzero real positive number, v is bilinearly complex if $q(v)$ contains a phase factor $e^{i\theta}$ and $\theta \neq k\pi$, where $k \in \mathbb{Z}$.

Theorem 4 (Sylvester's Inertial theorem, 1852)**.** *Let V be a vector space over the field $\mathbb{R}$, let f be a symmetric bilinear form on V. If B and B' are both f-orthogonal bases for the vector space V, then B and B' contains the same number of the bilinearly positive elements,same number of the bilinearly negative elements.*

Proof. By Theorem 3, in both bases, the sum of the numbers of the bilinearly positive and bilinearly negative elements are equal. Assume that B has r bilinearly positive elements, and B' has r' bilinearly positive elements.

Let the bilinearly positive elements in B be

$$b_1, b_2, \cdots, b_r$$

The space they span is denoted by U, and the elements that are not bilinearly positive in B' be

$$b'_{r'+1}, b'_{r'+2}, \cdots, b'_n$$

The space the vectors above span is denoted by W.

Let $u \in U$ and $u \neq 0$, then $u = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_r b_r$, there

$$q(u) = \alpha_1^2 q(b_1) + \alpha_2^2 q(b_2) + \cdots + \alpha_r^2 q(b_r) > 0$$

also, let $w \in W$ and $w \neq 0$, then $w = \beta_{r'+1} b'_{r'+1} + \beta_{r'+2} b'_{r'+2} + \cdots + \beta_n b'_n$, there

$$q(w) = \beta_{r'+1}^2 q(b'_{r'+1}) + \beta_{r'+2}^2 q(b'_{r'+2}) + \cdots + \beta_n^2 q(b'_n) \leqslant 0$$

Thus $U \cap W = \{0\}$, we have

$$U \oplus W \subseteq V$$

Thus

$$r + (n - r') \leqslant n$$

or

$$r \leqslant r'$$

By symmetry,$r' \leqslant r$, hence $r' = r$. The Sylvester's inertial theorem is proved. $\square$