Mathematical Physics 202's Big Book of Problems

Jonas Rademacker

Book of weekly problems for Mathematical Physics 202, University of Bristol.

Contents

Chapter 1

Practice Problems Week 1

1.1 Math basics

1.1.1 Derivatives

a) $f(x) = (\cos x - \sin x) \cdot e^x$. Show that $\frac{df}{dx} = -2\,\sin x\; e^x$

b) $g(t) = 2e^{at}\arcsin t$. What is $\frac{dg}{dt}$?

c) $g(t) = 2e^{at}\arcsin t$. What is $\frac{dg}{dt}(0)$?

d) $g(t) = 2e^{at}\arcsin t$. What is $\frac{dg}{da}$?

e) $y(\theta) = \frac{\ln\theta}{\sqrt{\theta}}$ What is $\frac{dy}{d\theta}$?

f) $y(x) = \ln\sqrt{4x - x^2}$. Calculate y'.

1.1.2 Integrals

Remember the difference between indefinite integrals (=anti derivatives) and definite integrals:

- Calculate the indefinite integral $\int x\, dx$. Answer: $\frac{1}{2}x^2 + C$. Note that the notation $\int f(x)dx$ sometimes means the indefinite integral / anti-derivative of $f(x)$, and sometimes means the (definite) integral over all values, i.e. usually $-\infty$ and $+\infty$ (for angles $0, \ldots, 2\pi$).
- Calculate $\int\limits_a^b x\, dx$. Answer: $\frac{1}{2}b^2 - \frac{1}{2}a^2$ (definite integral)
- Calculate $\int\limits_0^2 x\, dx$. Answer: 2 (definite integral)

a) Calculate the indefinite integral $I = \int (-2\,\sin x\; e^x)\; dx$.

b) Calculate $\int\limits_0^1 \frac{x}{(1+x^2)^2}\, dx$ Hint: Substitute $u = 1 + x^2$.

c) Calculate $\int \frac{e^{2x}}{1+e^x} dx$. (Hint: substitute $u = 1 + e^x$)

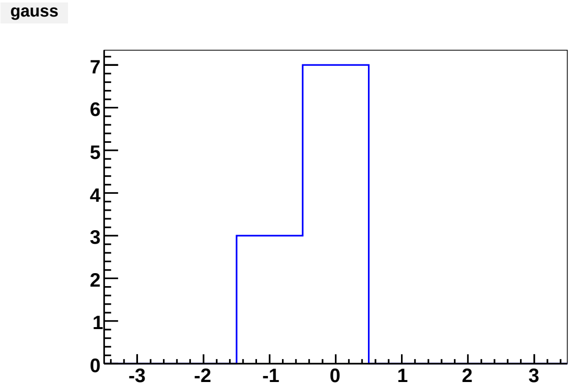
d) Calculate $I = \int (1 + 2x)\, e^{-x}\, dx$. Hint: Use integration by parts: $\int u \cdot v'\, dx = u \cdot v - \int u' \cdot v\, dx$

e) Calculate $I = \int\limits_{x=0}^{1} \int\limits_{y=0}^{-x+1} x^2 y^2 dy\, dx$

1.2 Statistics

1.2.1 Means, variances, width

1) Calculate the mean, variance and standard deviation of the following histogram:

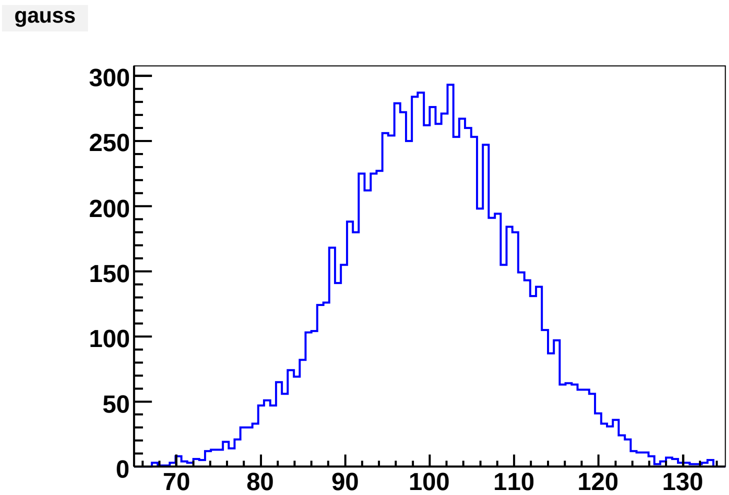


2) What is the 3rd central moment of the histogram in the prev. question? What would it be for a completely symmetric distribution?

1.2.2 Standard Deviation and FWHF

Estimate the standard deviation σ of the following histogram, assuming it is roughly Gaussian (as many distributions in practice are), and using

$$FWHM \approx 2.35\sigma.$$



1.2.3 Means, variances, standard deviations, covariances

For the following datasets, calculate the mean and standard deviation of each, x and y. Also calculate the covariance and the correlation coefficient.

a)

x	2	4	12
y	4	8	24

b)

x	4	1	4
y	2	4	6

1.2.4 Dice

(a) Suppose you throw two dice, what is the probability that the dice add up to 4 *given* that the first shows a 2?

(b) Suppose you throw two dice, what is the probability that the dice add up to 4 *and* that the first shows a 2?

1.2.5 Outbreak

Classic question using Bayes theorem A new strain of techno-prions slowly turns humans into tamagotchi toys over a time period of 20 years. The first 10 years usually go un-noticed. The University of Bristol's Medical School has been quick to develop a test to identify those who carry this horrific disease. The test identifies correctly 100% of those who carry the prions, and misidentifies 0.2% of uninfected people as carriers. Assume that 0.01% of the population carry the prions.

a) What is the probability that somebody identified as a carrier by the test does in fact carry the prion?

b) How much does the test need to be improved so that a positive test implies a probability of carrying the prions of at least 95%?

1.2.6 Doping

Victoria plays a sport in which 1% of participants take performance enhancing drugs. She is randomly called for a drug test. The drug test identifies correctly 98% of real drug users, and mis-identifies 3% of non-users as drug users.

a) Victoria tests positive. What is the probability that she actually took drugs?

b) After the first (positive) test, she is tested immediately again. She tests positive again. What is the probability that she actually took drugs, now (assuming that the probability that the 2nd test returns a correct result is independent of the first test)?

Chapter 2

Practice Problems Week 2

2.1 Theoretical Distributions

2.1.1 Coins

Level: easy Recall the distribution of obtaining N_H heads when flipping a coin 4 times? If not, look it up in the lecture notes.

(a) We call the mean of a predicted or theoretical distribution the "expectation value". What is the expectation value $\langle N_H \rangle$ and the standard deviation σ_{N_H}, of this distribution? Mean:

$$\begin{aligned}
\langle N_H \rangle &= \sum P(N_{H\ i}) N_{H\ i} \\
&= \frac{1}{1} \sum P(N_{H\ i}) N_{H\ i} \\
&= \frac{1}{1} \left(0 \cdot \frac{1}{16} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{16} \right) \\
&= 2
\end{aligned}$$

Variance:

$$\begin{aligned}
V &= \sum P(N_{H\ i}) \left(N_{H\ i} - \langle N_H \rangle \right)^2 \\
&= \frac{1}{1} \sum P(N_{H\ i}) \left(N_{H\ i} - \langle N_H \rangle \right)^2 \\
&= (0-2)^2 \cdot \frac{1}{16} + (1-2)^2 \cdot \frac{1}{4} + (2-2)^2 \cdot \frac{3}{8} + (3-2)^2 \cdot \frac{1}{4} + (4-2)^2 \cdot \frac{1}{16} \\
&= 1
\end{aligned}$$

Standard deviation $\sigma = \sqrt{V} = 1$. Also possible to use

$$V = \langle N_H^2 \rangle - \langle N_H \rangle^2$$

Try it!

2.1.2 PDFs

Even PDF's

Level: easy-medium Show that the expectation value of any even PDF is 0. An even function is one that has $f(x) = f(-x)$.

Flat

Level: medium Show that the standard diviation of a flat distribution is

$$\sigma_{\text{flat}} = \frac{\text{width}}{\sqrt{12}}$$

Hints: Your distribution is:

$$F(x) = \left\{ \begin{array}{lll} 1/(b-a) & \text{if} & x \in [a,b] \\ 0 & & \text{else} \end{array} \right\}$$

with width $= (b-a)$. Calculate $V_x = \langle x^2 \rangle - \langle x \rangle^2$. It's obvious what $\langle x \rangle$ should be, but do check that you can calculate it.

Decay

Level: medium The probability density for an unstable nucleus that exits at time $t_0 = 0$ to decay at a later time t is given by

$$p(t) = \frac{1}{\tau} e^{-t/\tau}$$

(a) Show that this distribution is properly normalised. Keep in mind that the particle cannot have decayed in the past, so $p(t) = 0$ for $t < 0$.

(b) The cumulative probability distribution F for a pdf p is defined as

$$F(t) \equiv \int_{-\infty}^{t} p(t')\, dt'$$

What is the cumulative probability distribution for $p(t) = \frac{1}{\tau} e^{-t/\tau}$

(c) What is the expectation value and the standard deviation of $p(t)$?

(d) Find the co-ordinate transformation that transforms a flat distribution between 0 and 1 (PDF in question 2.1.2 with $a = 0$, $b = 1$) to the above exponential. If we call the flatly-distributed random variable x, you want to find $t(x)$ such that

$$p_{\text{flat}}(t)\, dt = p(x)\, dx$$

where $p(t) = \frac{1}{\tau} e^{-t/\tau}$ and

$$p_{\text{flat}}(x) = \left\{ \begin{array}{ll} 1 & \text{if} \quad x \in [0,1) \\ 0 & \text{else} \end{array} \right\}$$

The way to solve this is to first find the inverse transformation, $x(t)$, and then solve this for t to find what you really want, $t(x)$. The first step is to put in $p(t)$ and $p_{\text{flat}}(x)$ and re-arrange things a bit.

$$\frac{dx}{dt} = \frac{1}{\tau} e^{-t/\tau}$$

You should be able to take it from here. This will give you $x(t)$. Now solve for t to get $t(x)$. Use your integration constants wisely - so that you map a distribution between 0 and 1 to a distribution between 0 and ∞.

Gaussian

Level: medium to hard The Gaussian PDF is given by:

$$g(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This notation means that $g(x;\mu,\sigma)$ is a function of x, with the parameters μ,σ. You could also simply write $g(x)$.

You can assume that this is properly normalised, i.e. $\int\limits_{-\infty}^{+\infty} g(x)\, dx = 1$ (you might or might not need this information).

(a) Show that all integrals over $g(x;\mu,\sigma)$ can be expressed in terms of the gaussian with $\mu = 0$ and $\sigma = 1$:

$$\int\limits_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \int\limits_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du$$

(b) Show that its mean is μ. Hint: Use the result from the previous question and an appropriate co-ordinate transformation.

(c) Advanced question: Show that the standard deviation of the Gaussian is (as the variable name suggests) σ

Convolution

Level: fairly tricky - but you will meet convolutions again in part 2 of the course, so why not familiarise yourself with them, now. Consider random variable $z = x + y$, where x is distributed according to the PDF p_x and y is distributed according to the PDF p_y. The PDF of z is given by:

$$p_z(z) = \int\limits_{-\infty}^{\infty} p_x(x) p_y(z-x) dx = \int\limits_{-\infty}^{\infty} p_y(x) p_x(z-x) dx$$

If x and y are both distributed according to a flat distribution between 0 and 1 (see question 2.1.2), what is p_z?

2.1.3 Calculating Probabilities

Life insurance

Level: Easy-medium once you've done the Binomial distribution The sort of thing that goes through the mind of an insurance agent when he asks you how old you are: If the probability that any person thirty years old will be dead within 10 years is $p = 0.01$, find the probability that out of a group of 10 such people

a) none

b) exactly one

c) not more than one

d) more than one

e) at least one

will be dead within 10 years.

Parity Bit

Level: medium (once you've done the binomial distribution Let's say that the probability that a single bit (which can have the values "I" or "0") is transmitted wrongly (i.e. what is sent as "I" arrives as "0" and vice versa) over the internet is $p = 10^{-6}$ (completely made-up number).

a) What is the probability that an ASCII character of 7 bits will have at least one wrong bit?

b) How many characters do you expect to be wrong in a $1\mathrm{MB} = 2^{20}$ bytes of text?

c) One way to reduce the error rate in such transmissions is the so-called parity bit. The sender adds to each 7-bit ASCII character one more bit. This is done such that to over-all number of bits set to "I" is even. So the 7-bit character "0III000" would have a partity bit of "I" added, giving "0III000I":

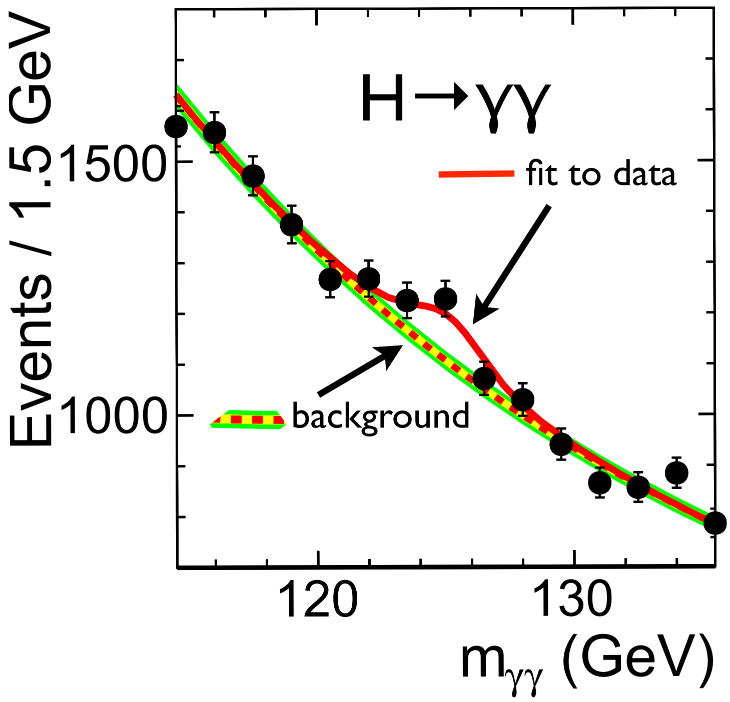
   0III000
   →0III000I ,

   while "0II0000" would have a "0" added, giving "0II00000":

   0II0000
   →0II00000 .

   The recipient will check the number of "I" bits and request re-transmission if it finds an odd number. This method will find single-bit failures. How many wrong characters to you expect in a $1\mathrm{MB}$ text, now?

2.1.4 Confidence Levels: Higgs Search



The plot above shows the result of the CMS experiment's search for the decay of a Higgs meson to two photons. It shows the number of events found in bins of the reconstructed Higgs mass, given in GeV. Apart from the data themselves, the plot also shows a fit to the data (which you can ignore) and the expected background distribution. Around $125\,\mathrm{GeV}$, you can see an excess of events compared to the expected background distribution.

Estimate the probability that the excess above the exponential background distribution is a statistical fluctuation. Please ignore the fit and the error bars. Note that, near $125\,\mathrm{GeV}$, there is only one bin which significantly deviates from the background curve. Assume this bin has 1258 events, while we expect 1156 events from background processes.

a) What is the probability to find such a fluctuation in a given bin?

b) Now calculate the probability of seeing such a fluctuation given that the researchers searched in 100 bins.

2.2 Supplementary Statistics Questions

These are for additional practice

2.2.1 Combinatorics

1. The Zimbabwean Cabinet has 30 ministries to be divided between three parties: Zanu-PF (14), Main MDC (13), and MDC (Mutambara) (3).

   i) How many ways are there to allocate the various ministries to the different parties, given these constraints?

   ii) If neither Robert Mugabe's Zanu-PF nor the opposition (Main MDC and MDC (Mutambara)) are allowed to have both, defence and home affairs (which includes police), how many ways are there now to divide the ministries between the parties?

2.2.2 Molecules*

Level: medium-hard

1) According to the equipartition theorem, the average kinetic Energy of gas molecules at temperature T is $\overline{E_{kin}} =$ ____________. (Look up the equipartition theorem if necessary and don't forget that the speed of the particle corresponds to 3 degrees of freedom). What is the mean speed squared, $\overline{|v|^2}$, for a given molecular mass m?

2) From Maxwell's speed distribution, we know that the mean speed of a molecule at temperature T is $\overline{|v|} = \sqrt{\frac{8k_BT}{\pi m}}$. Compare this to the previous result. Why is it OK to calculate the $\overline{|v|^2}$ from the kinetic energy $\overline{E_{kin}}$, while it is not OK to calculate $\overline{|v|}$ from it?

3) Show that if, and only if the fluctuations about the mean $|v|^2$ are small, then $\overline{|v|} \approx \sqrt{\overline{|v|^2}}$. Instructions:

   a) Obviously, $v = \sqrt{v^2}$. Express v^2 as the $\overline{|v|^2} + \Delta v^2$. Perform a Taylor expansion of $v = \sqrt{|v|^2}$ around $|v|^2 = \overline{|v|^2}$ up to second order (i.e. get sth that looks like $a + b\Delta v^2 + c\left(\Delta v^2\right)^2$)

   b) Now take the mean of that. Note that $\overline{\Delta|v|^2}$ is the 1st central moment of $|v|^2$ - what happens to that term? What does the $\overline{\left(\Delta|v|^2\right)^2}$ term represent?

   c) Hence, if the Variance of $|v|^2$ is small compared to $\overline{|v|^2}$, and *only* then, is $\overline{|v|} \approx \sqrt{\overline{|v|^2}}$ an acceptable approximation.

2.2.3 CLT

A random variable X is the sum of N independent random variables y_i with $i = 1, \ldots, N$:

$$X = \sum_{i=1}^{N} y_i$$

i) Prove that

$$\langle X \rangle = \sum \langle y_i \rangle$$

(this is easy - use linearity of expectation values)

ii) Prove that the standard deviation of X is given by

$$\sigma_x = \sqrt{\sum \sigma^2(y_i)}$$

where $\sigma(y_i)$ is the standard deviation of the random variable y_i. Hint: Start with

$$V(X) = \langle (X - \langle X \rangle)^2 \rangle$$

Chapter 3

Practice Problems Week 3

3.1 Vector Basics

3.1.1 What animal is it?

We denote vectors with arrows on top, like this: $\vec{A}$. Everything else is just a simple number.

(a) Which of the expressions below is

- a vector
- a scalar (simply a number)
- nonsense?

Remember that the dot: $\cdot$, between vectors is the dot-product. The same symbol between scalars is the usual product between to numbers, and between scalars and vectors it is normal scalar multiplicaton of vectors (which changes the length, but not the direction of vectors). In the latter two cases you can omit the dot - you cannot between vectors, to avoid confusion with the cross product. The cross $\times$ indicates the cross-product and only makes sense between vectors.

(i) $\lambda\vec{A}$

(ii) $\vec{A}\times\vec{B}$

(iii) $\vec{A}\cdot\vec{B}$

(iv) $\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$

(v) $\left(\vec{C}\cdot\vec{A}\right)\times\vec{B}$

(vi) $\vec{C}\cdot\vec{A}\times\vec{B}$

(vii) $\left(\vec{C}\cdot\vec{A}\right)\vec{B}$

(viii) $\left(\vec{A}\times\vec{B}\right)\times\vec{C}$

(ix) Why is this nonsense: $\vec{A}\times\vec{B}+\vec{A}\cdot\vec{B}$

(b) Which of the below is zero (either scalar-zero 0 or vector zero $\vec{0}$), which one is not zero, which one can be either, depending on the value of $\vec{A}$ and $\vec{B}$. You can assume that neither $\vec{A}$ nor $\vec{B}$ themselves are $\vec{0}$:

(i) $\vec{A}\cdot\vec{A}$

(ii) $\vec{A}\times\vec{A}$

(iii) $\left(\vec{A}\times\vec{B}\right)\cdot\vec{A}$

(iv) $\left(\vec{A}\times\vec{B}\right)\times\vec{A}$

(v) $\lambda\vec{A}+\mu\vec{B}$ for $\mu\neq 0,\ \lambda\neq 0$

(vi) $\vec{A}\times\vec{B}+\left(\vec{A}\cdot\vec{B}\right)\vec{B}$ Hint: The answer to this one is *not* "can be either".

3.1.2 Vector operations

(a) Calculate $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}3\\2\\1\end{pmatrix}$

(b) Draw the vectors $\vec{A}=\begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\vec{B}=\begin{pmatrix}2\\1\\0\end{pmatrix}$. These "span" a paralellogram (diamond shape). What is its area?

(c) Calculate $\vec{A}\cdot\vec{B}$ for $\vec{A}=\begin{pmatrix}0\\1\\2\end{pmatrix}$ and $\vec{B}=\begin{pmatrix}1\\4\\0\end{pmatrix}$

(d) Calculate the length of $\vec{A}$ and $\vec{B}$ in the previous questions

(e) Calculate the angle between $\vec{A}$ and $\vec{B}$ from the previous two questions.

(f) What this the unit vector in the direction of $\begin{pmatrix}3\\4\\0\end{pmatrix}$?

(g) Applying a constant force $\vec{F}=\begin{pmatrix}1\\2\\3\end{pmatrix}N$ to move, in a straight line an object from point $\vec{A}=\begin{pmatrix}10\\11\\12\end{pmatrix}m$ to point $\vec{B}=\begin{pmatrix}10\\11\\13\end{pmatrix}m$: How much work is done?

3.2 Other Math Basics

(a) Use a Taylor/McLauren expansion to show that

$$\frac{1}{1+x}\approx 1-x+x^2$$

(b) Use a Taylor/McLauren expansion to show that, for values of x near 1,

$$\ln(x)\approx\frac{-3}{2}+2x-\frac{1}{2}x^2$$

Hint: Expand around $x_0=1$.

(c) Calculate the following integral

$$\iint\limits_{\text{unit disk}}(x^2+y^2)dx\,dy$$

over the unit disk. Hint: Think carefully about the integration limits, especially if you do it in cartesian co-ordinates. Better: Change the co-ordinate system to one that is more suitable to a disk.

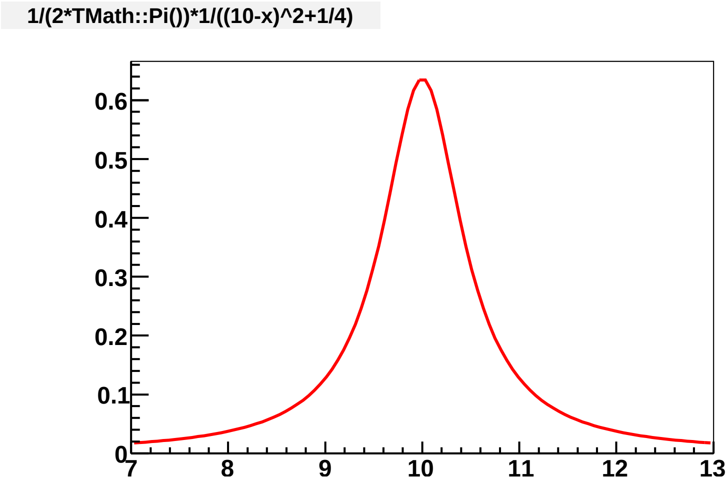
3.3 Additional Statistics Questions

3.3.1 Breit Wigner

The Breit Wigner function describes the probability density to find that a short-lived particle with mean mass M_0 and lifetime $\tau = \frac{\hbar}{\Gamma c^2}$ actually has the mass M.

$$\mathrm{B}_{\mathrm{M}_0}(M) = \frac{1}{2\pi} \frac{\Gamma}{(M - M_0)^2 + (\Gamma/2)^2}$$

An example for $M_0 = 10$ and $\Gamma = 1$ (arbitrary units, usually one would use MeV or GeV), is given below:



i) Show that the maximum of the Breit Wigner is at $M = M_0$

ii) Show that the FWHM of the Breit Wigner is Γ

Note that the Breit Wigner function neither has a well-defined mean, nor a standard deviation.

Chapter 4

Practice Problems Week 4

4.1 Lines and Gradients

4.1.1 $d\vec{r}$

Calculate $d\vec{r}$ and $ds = |d\vec{r}|$ expressed in

(a) in polar coordinates and the polar basis $\vec{e}_r$, $\vec{e}_\phi$

(b) in cylindrical coordinates the cylindrical basis $\vec{e}_\rho$, $\vec{e}_\phi$, $\vec{e}_z$,

(c) the spherical polar coordinates and the corresponding basis $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$,

Hint: in any orthonormal basis, $\vec{u}_1, \vec{u}_2, \vec{u}_3$, any vector $\vec{A}$ and be expressed as

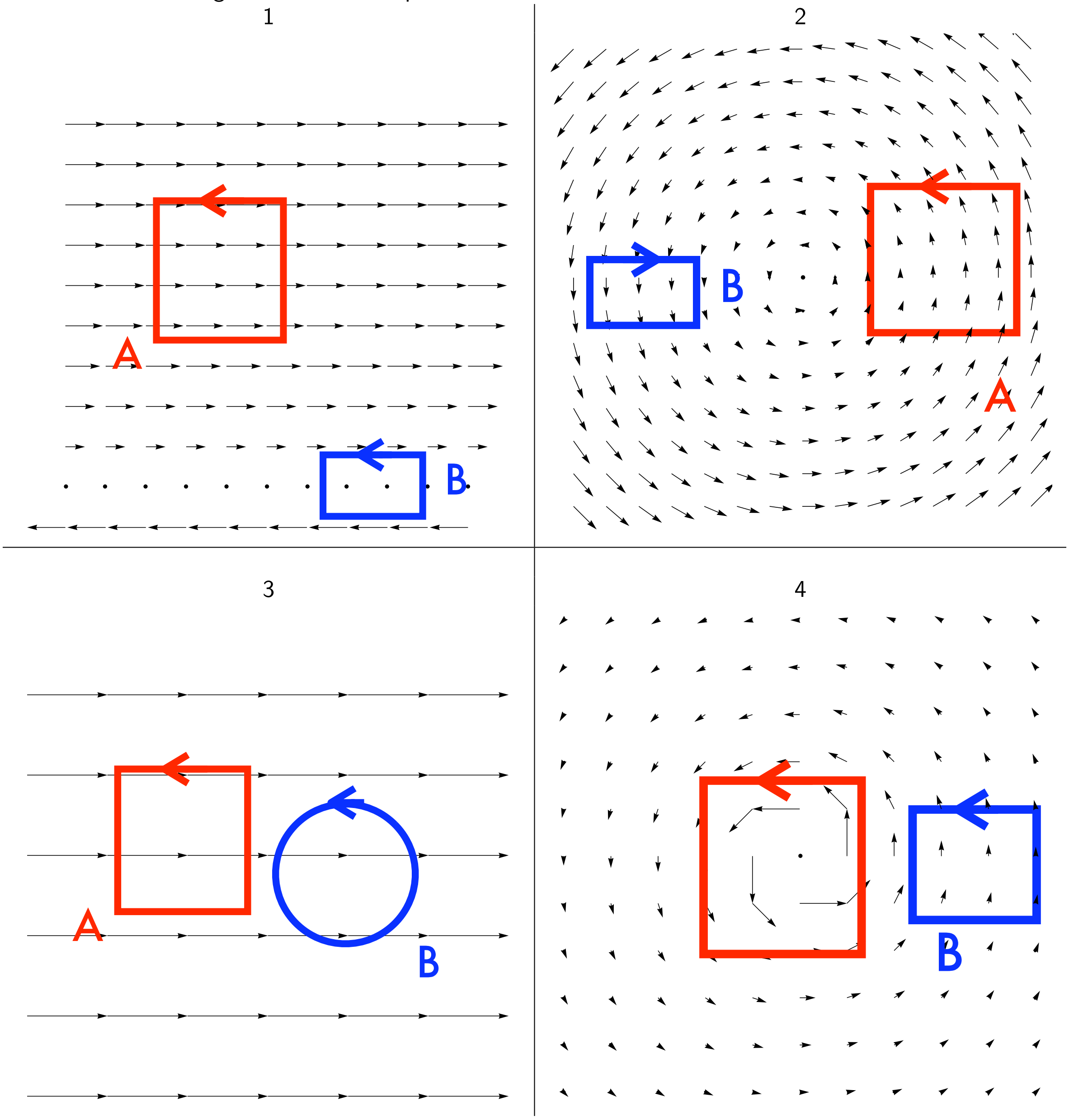
$$\vec{A} = \left(\vec{A}\cdot\vec{u}_1\right)\vec{u}_1 + \left(\vec{A}\cdot\vec{u}_2\right)\vec{u}_2 + \left(\vec{A}\cdot\vec{u}_3\right)\vec{u}_3$$

Recall the expressions for $\vec{e}_r, \ldots$ from the notes and use the formula above.

One way to approach this (there are others): First express $d\vec{r}$ and $\vec{e}_r, \ldots$ in the cartesian basis vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$, and the appropriate curve-linear coordinates. Then calculate the dot products. Compare your result with those in the lecture notes.

4.1.2 Multiple Choice Path & field guessing

Consider the following vector fields and paths:



(a) Which of the 8 closed paths is positive, negative, zero? Hint: **three** of the paths are zero (one might not be obvious).

(b) Which field matches which plot?

$$\vec{F} = \begin{pmatrix} 1 - e^{-y/a} \\ 0 \end{pmatrix}, \quad \vec{G} = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad \vec{K} = \frac{1}{x^2 + y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

where $a > 0$, $b > 0$

4.1.3 Three Paths

For

$$\vec{F} = \begin{pmatrix} 3x^2 + 67y \\ -14yz \\ 20xz^2 \end{pmatrix}$$

Evaluate

$$\int_\gamma \vec{F} \cdot d\vec{s}$$

from $(0, 0, 0)$ to $(1, 1, 1)$ along the following paths γ:

a) $x = t, y = t^2, z = t^3$

b) the straight lines from (0,0,0) to (1,0.0), then to (1,1,0), and then to (1,1,1).

c) the straight line joining $(0, 0, 0)$ and $(1, 1, 1)$.

d) Is this vector field conservative?

4.1.4 Conservative Fields

1) a) Show that

$$\vec{F} = \begin{pmatrix} 2xy + z^3 \\ x^2 \\ 3xz^2 \end{pmatrix}$$

   is a conservative force field

   b) Find the scalar potential

   c) Find the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$

2) In two dimensions:

   a) show that

$$\vec{F} = \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix}$$

   is a conservative field

   b) Calculate its potential ϕ

   c) Parameterise the equipotential lines, i.e. find an expression

$$\vec{\ell}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

   such that $\phi(\vec{\ell}(t)) = \mathrm{const}$. The section in the lecture notes on paramterising paths might be helpful for this.

   d) Using your parameterisation, prove that the equipotential lines are always perpendicular to the force field.

   e) Sketch the field and equipotential lines for $\alpha = 2$ and $\beta = \frac{1}{2}$. (Equipotential lines are drawn for equal spacings in potential, i.e. for ϕ_0, $\phi_0 + \Delta\phi$, $\phi_0 + 2\Delta\phi$ for constant, sensibly chosen $\Delta\phi$.)

4.2 Multiple Integrals Review

Consider the 2-D region B enclosed by the following lines:

- $y = x$
- $xy = 1$
- $y = 2$

Hint: Draw this before you continue.

a) Calculate the area of the enclosed region

$$A = \iint\limits_B dx\, dy$$

b) Calculate the volume enclosed by this area and the surface defined by $z = \frac{y^2}{x^2}$

$$V = \iint\limits_B \frac{y^2}{x^2}\, dx\, dy$$

Chapter 5

Practice Problems Week 5

5.1 Surface Integrals

5.1.1 Find $d\vec{S}$

Find $d\vec{S}$ for each of the following surfaces. Sketch the intersections of the surfaces with the $x-y$ plane, the $x-z$ plane and the $y-z$ plane. Add the projection of $d\vec{S}$ at several points on the surface to your graph.

Try to draw a full 3-D graph of the surfaces and $d\vec{S}$.

(Of course, $d\vec{S}$ is infinitesimally small - draw something that is visible and proportional to $d\vec{S}$.)

a) $z = 2 - x - y$

b) $z^2 + x^2 = 1$

c) $z = x^2 + y^2$

d) $z = xy$

e) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

f) Example: $x^2 + y^2 + z^2 = R^2$ (solutions for $d\vec{S}$ using various approaches below.)

- Using spherical co-ordinates Reminder: $\vec{r}$ in spherical coordinates is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix}$$

translate the equation into spherical co-ordinate:

$$\begin{aligned}
R^2 &= x^2 + y^2 + z^2 \\
R^2 &= (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2 \\
R^2 &= r^2
\end{aligned}$$

This eliminates one parameter - the co-ordinate r is replaced with the constant R.

$$\vec{S}(r,\theta,\phi) = R\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

partial derivatives:

$$\frac{\partial}{\partial\phi}\vec{S} = R\begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix}, \qquad \frac{\partial}{\partial\theta}\vec{S} = R\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}$$

Finally calculate $d\vec{S}$:

$$d\vec{S} = \frac{\partial}{\partial\theta}\vec{S} \times \frac{\partial}{\partial\phi}\vec{S}\,d\theta\,d\phi = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \times \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix} R^2 d\theta\,d\phi = \begin{pmatrix} \sin^2\theta\cos\phi \\ \sin^2\theta\sin\phi \\ \sin\theta\cos\theta \end{pmatrix} R^2 d\theta\,d\phi$$

Check correct orientation. Pick an easy point, e.g.: $\theta = \pi/2, \phi = 0$, i.e. where the sphere interects the positive x axis. For $d\vec{S}$ to point outwards, given that the centre of the sphere is at (0/0/0), it should to point along $+x$ at this point i.e along $\begin{pmatrix} +|a| \\ 0 \\ 0 \end{pmatrix}$ where $|a|$ is any positive number. Check:

$$d\vec{S}(\theta = \pi/2, \phi = 0) \quad = \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} R^2 d\theta\,d\phi$$

OK! *(Note: if, unlike here, the surface is not closed, the orientation is your choice. There are two things to consider though: If you are using Stokes' theorem the orientation of the surface is related to the direction in which you go through the circular path by the right-hand-rule. And if you have split your surface into several parts, the orientations should be consistent with each other - so the direction of the surface normal should be the same where the two component surfaces touch.).*

- Cylindrical co-ordinates can sometimes be very useful even when integrating over a sphere. While your sphere is clearly spherically symmetric, it is also cylindrically symmetric. Often the problem seen as a whole (which will include a field) is not spherically symmetric, but it might still exhibit cylindrical symmetry, in which case it tends to be easier to use cylindrical coordinates even with spheres. Anyway, it's a good exercise. Reminder: the position vector in cylindrical co-ordinates is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$$

where, remember, this ρ is the distance from the $z-$axis, a different quantity than r in spherical coordinates. The equation for a sphere

$$x^2 + y^2 + z^2 = R^2$$

becomes in cylindrical coordinates:

$$\rho^2 + z^2 = R^2$$

We have a choice which pair of co-ordinates to pick to parameterise the surface. I can eliminate either ρ or z using this equation. It turns out, eliminating $\rho = \sqrt{R^2 - z^2}$ and parametersing the sphere in terms of z and ϕ is easier than using ρ and ϕ, not least because for ρ, which is positive by definition, we have a unique solution to $\rho^2 + z^2 = R^2$ (see the solution in cartesian coordinates, next, for an illustration why this matters). A sphere in cylindrical coordinates, parameterised in terms of z and ϕ is therefore (simply replacing ρ with $\sqrt{R^2 - z^2}$ in $\begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$):

$$\vec{S} = \begin{pmatrix} \sqrt{R^2 - z^2}\cos\phi \\ \sqrt{R^2 - z^2}\sin\phi \\ z \end{pmatrix}$$

$$
\begin{aligned}
\frac{\partial \vec{S}}{\partial z} &= \begin{pmatrix} \frac{-z}{\sqrt{R^2-z^2}}\cos\phi \\ \frac{-z}{\sqrt{R^2-z^2}}\sin\phi \\ 1 \end{pmatrix} \\
\frac{\partial \vec{S}}{\partial \phi} &= \begin{pmatrix} -\sqrt{R^2-z^2}\sin\phi \\ \sqrt{R^2-z^2}\cos\phi \\ 0 \end{pmatrix} \\
d\vec{S} &= \frac{\partial \vec{S}}{\partial \phi} \times \frac{\partial \vec{S}}{\partial z}\, dz\, d\phi \\
d\vec{S} &= \begin{pmatrix} -\sqrt{R^2-z^2}\sin\phi \\ \sqrt{R^2-z^2}\cos\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{-z}{\sqrt{R^2-z^2}}\cos\phi \\ \frac{-z}{\sqrt{R^2-z^2}}\sin\phi \\ 1 \end{pmatrix} dz\, d\phi \\
&= \begin{pmatrix} \sqrt{R^2-z^2}\cos\phi \\ \sqrt{R^2-z^2}\sin\phi \\ z \end{pmatrix} dz\, d\phi
\end{aligned}
$$

Does $d\vec{S}$ point outwards? Let's check for $z = 0$ and $\phi = 0$, when it should point towards $+x$:

$$
d\vec{S}(z=0, \phi=0) = \begin{pmatrix} R \\ 0 \\ 0 \end{pmatrix} dz\, d\phi
$$

and it does, all fine. Note that $\left|d\vec{S}\right|$ for a sphere is particularly simple in cylindrical co-ordinate: $\left|d\vec{S}\right| = R\, dz\, d\phi$.

- Using cartesian coordinates (this is <u>not</u> recommended, just shown as an illustration that it can be done, and that it is trouble!)

$$
\begin{aligned}
& \text{solve for one of the co-ordinates, say } z \\
z &= \pm\sqrt{R^2 - x^2 - y^2} \\
& \\
& \text{2 solutions:} \\
\vec{S}_1(x,y) &= \begin{pmatrix} x \\ y \\ \sqrt{R^2-x^2-y^2} \end{pmatrix} \quad \text{for} \quad z \geq 0 \\
\vec{S}_2(x,y) &= \begin{pmatrix} x \\ y \\ -\sqrt{R^2-x^2-y^2} \end{pmatrix} \quad \text{for} \quad z < 0 \\
& \text{calculate partial derivatives:} \\
\frac{\partial}{\partial x}\vec{S}_1 &= \begin{pmatrix} 1 \\ 0 \\ \frac{-x}{\sqrt{R^2-x^2-y^2}} \end{pmatrix}, \quad \frac{\partial}{\partial y}\vec{S}_1 = \begin{pmatrix} 0 \\ 1 \\ \frac{-y}{\sqrt{R^2-x^2-y^2}} \end{pmatrix} \\
\frac{\partial}{\partial x}\vec{S}_2 &= \begin{pmatrix} 1 \\ 0 \\ \frac{x}{\sqrt{R^2-x^2-y^2}} \end{pmatrix}, \quad \frac{\partial}{\partial y}\vec{S}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{y}{\sqrt{R^2-x^2-y^2}} \end{pmatrix} \\
d\vec{S}_1 &= \frac{\partial}{\partial x}\vec{S}_1 \times \frac{\partial}{\partial y}\vec{S}_1\, dx\, dy = \begin{pmatrix} \frac{x}{\sqrt{R^2-x^2-y^2}} \\ \frac{y}{\sqrt{R^2-x^2-y^2}} \\ 1 \end{pmatrix} dx\, dy \\
d\vec{S}_2 &= \frac{\partial}{\partial x}\vec{S}_2 \times \frac{\partial}{\partial y}\vec{S}_2\, dx\, dy = \begin{pmatrix} \frac{-x}{\sqrt{R^2-x^2-y^2}} \\ \frac{-y}{\sqrt{R^2-x^2-y^2}} \\ 1 \end{pmatrix} dx\, dy
\end{aligned}
$$

Since it's a closed surface, we should check that $d\vec{S}$ points outwards. Obviously the case for $d\vec{S}_1$, but not for $d\vec{S}_2$. Check the z-direction with to convince yourself - for $d\vec{S}$ to point outwards, it should point downwards (to negative z) for the lower ($z < 0$) half of the sphere and upwards (positive z) for the upper ($z \geq 0$) part. So $d\vec{S}_1$ is fine, but we need to take the negative of $d\vec{S}_2$. Final result:

$$\begin{aligned} d\vec{S}_1 &= \begin{pmatrix} \frac{x}{\sqrt{R^2-x^2-y^2}} \\ \frac{y}{\sqrt{R^2-x^2-y^2}} \\ 1 \end{pmatrix} dx\,dy \\ d\vec{S}_{2\ \text{correct orientation}} &= \begin{pmatrix} \frac{x}{\sqrt{R^2-x^2-y^2}} \\ \frac{-}{\sqrt{R^2-x^2-y^2}} \\ -1 \end{pmatrix} dx\,dy \end{aligned}$$

5.1.2 Mass

The distribution of mass on the sphere

$$x^2 + y^2 + z^2 = R^2$$

is given by

$$\sigma(x, y, z) = \frac{\sigma_0}{R^2}\left(x^2 + y^2\right)$$

Find an expresion in terms of σ_0 and R for the total mass of the shell.

5.1.3 Flux through disk

1. Calculate the flux of the field $\vec{C}$

   $$\vec{C} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   trough the unit-disk in the $x - y$ plane (the plane is defined by $z = 0$, and the integration limits by $x^2 + y^2 < 1$, which are easier in polar or cylindrical coordinates).

2. Evaluate

   $$\iint \vec{G} \cdot d\vec{S}$$

   through the disk

   $$(x - 2)^2 + (y - 3)^2 < 9, \quad z = 0$$

   where

   $$\vec{G} = \begin{pmatrix} 0 \\ 2yz \\ 2 - z^2 \end{pmatrix}$$

   Hint: the answer is 18π

5.1.4 Flux through triangular plane section

Evaluate

$$\iint_S \vec{A} \cdot d\vec{S}$$

where

$$\vec{A} = \begin{pmatrix} 18z \\ -12 \\ 3y \end{pmatrix}$$

and S is the part of the plane

$$2x + 3y + 6z = 12$$

which is located in the first octant, i.e. where $x > 0$, $y > 0$ and $z > 0$.

Chapter 6

Practice Problems Week 6

6.1 Divergence

6.1.1 Sum and product rule for divergence

Prove

(a)

$$\vec{\nabla} \cdot \left(\vec{A} + \vec{B}\right) = \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B}$$

(b)

$$\vec{\nabla} \cdot \left(\phi\, \vec{A}\right) = \left(\vec{\nabla}\phi\right) \cdot \vec{A} + \phi \left(\vec{\nabla} \cdot \vec{A}\right)$$

6.1.2 Divergence of $1/r^2$ field

Prove:

$$\mathrm{div}\left(\frac{\vec{r}}{r^3}\right) = 0$$

(for $r \neq 0$)

6.1.3 Incompressible fluid

a) Using the continuity equation, show that for the velocity field $\vec{v}$ of an incompressible liquid, $\vec{\nabla} \cdot \vec{v} = 0$

b) Given this velocity field:

$$\vec{v} = \begin{pmatrix} x + 3y \\ y - 2z \\ x + az \end{pmatrix}$$

For which value of a is $\mathrm{div}\vec{v} = 0$?

6.2 Curl

a) If

$$\vec{A} = \begin{pmatrix} xz^3 \\ -2yz \\ 3yz^4 \end{pmatrix}$$

find $\mathrm{curl}\vec{A} = \vec{\nabla} \times \vec{A}$.

b)

$$\vec{A} = \begin{pmatrix} x^2y \\ -2xz \\ 2yz \end{pmatrix}$$

Calculate $\mathrm{curl}\,\mathrm{curl}\vec{A} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)$

6.3 Integrals and Integral Theorems

Hint: Make sure you use the right integral theorem to simplify the problem where possible.

1) Evaluate

$$\oiint \vec{F} \cdot d\vec{S}$$

where

$$\vec{F} = \begin{pmatrix} 4xz \\ -y^2 \\ yz \end{pmatrix}$$

and is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

2) Evaluate

$$\oiint_{\partial V} \vec{r} \cdot d\vec{S}$$

for and arbitrary volume V of size V_o (i.e V is the collection of points constituting the volume, while V_o is its size, for example in litres, pints of m^3). The vector $\vec{r}$ is the position vector, i.e. $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and ∂V is the surface enclosing the volume V.

3) An electrostatic field is given by

$$\vec{E} = \lambda \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

where λ is a constant. Use Gauss' law to find the total charge enclosed by the surface consisting of S_1, the hemisphere

$$z = \sqrt{R^2 - x^2 - y^2}$$

and S_2, its circular base in the xy plane.

4) Verify the Divergence Theorem for the example of the vector field

$$\vec{A} = \begin{pmatrix} 4x \\ -2y^2 \\ z^2 \end{pmatrix}$$

and the volume V bound by

$$x^2 + y^2 = 4, \quad z = 0, \quad \text{and} \quad z = 3$$

i.e. a cylinder section, centered at the z-axis with radius 2, and with lids at $z = 0$ and $z = 3$. Verify the divergence theorem by calculating

$$\oiint_{\partial V} \vec{A} \cdot d\vec{S}$$

a) not using the divergence theorem, i.e. calculating the surface integral directly. Note that you are dealing with three surfaces: The cylinder wall, and two "lids" at $z = 0$ and $z = 3$.

b) using the divergence theorem, i.e. calculating an appropriate volume integral

(Note that this is quite different from a *proof*. All you have to show here is that it works for this example; in a proof you would have to show that it works always.)

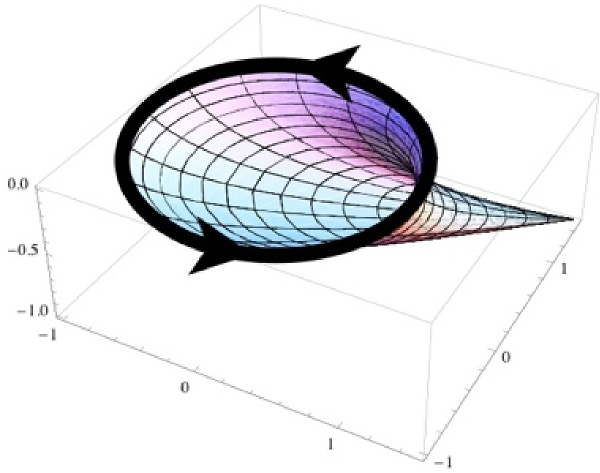
5) For

$$\vec{B} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

Evaluate the surface integral

$$\iint_S \vec{B} \cdot d\vec{S}$$

over the surface depicted below:



Where the circle shown in black is the unit circle in the $x - y$ plane, i.e. the circle is given by

$$x^2 + y^2 = 1, \quad z = 0$$

Hint: If the divergence of a field $\vec{F}$ is $zero$, there are tricks we can play to simplify the integral over a complicated surface by replacing it with a suitable alternative surface.

6) Repeat the calculation for the same surface, but now the field is

$$\vec{C} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

7) Show that, for any simple path γ in the $x - y$ plane, the area enclosed by the path is given by:

$$A = \frac{1}{2} \oint_\gamma \begin{pmatrix} -y \\ x \end{pmatrix} \cdot d\vec{r}$$

Hint: A simple path is one where each point is passed-through only once as you go along the path - a circle is simple, the figure 8 isn't, because the path describing the figure 8 crosses itself. Usually, this is nothing to worry about, because you'd normally deal a simple path.

- You could use Green's theorem. But we'll derive it directly from Stoke's theorem (after all, Green's theorem is just a special case of Stoke's theorem).

- To use Stoke's theorem, let's go into 3-dimensions.

$$A = \frac{1}{2}\oint_\gamma \begin{pmatrix} -y \\ x \end{pmatrix} \cdot d\vec{r}$$

with $d\vec{r} = \begin{pmatrix} dx \\ dy \end{pmatrix}$ is the same as

$$A = \frac{1}{2}\oint_\gamma \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \cdot d\vec{r}$$

with $\vec{\gamma} = \begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix}$ and $d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$.

- The enclosed area is given by

$$A = \iint_S dx\,dy$$

which can be written (somewhat over-complicatedly), as

$$A = \iint_S \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot d\vec{S}$$

where $d\vec{S}$ is the vectorial surface element of the $x - y$ plane:

$$\begin{aligned} \vec{S} &= \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \\ d\vec{S} &= \frac{\partial}{\partial x}\vec{S} \times \frac{\partial}{\partial y}\vec{S}\,dx\,dy \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy \end{aligned}$$

- To use Stoke's theorem, you will want to express

$$A = \iint_S \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot d\vec{S}$$

as

$$A = \iint_S \vec{\nabla} \times \vec{V} \cdot d\vec{S}$$

So that you can fill this into:

$$A = \oint_{\partial S} \vec{V} \cdot d\vec{r}$$

which will then give the result.

- So your task is now to find a vector $\vec{V}$ such that

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{\nabla} \times \vec{V}$$

and combining it all together to prove the claim.

8) Use this result to calculate the area enclosed by the following path:

$$\begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

which describes an ellipse in the $x - y$ plane.

9) Evaluate

$$\oint \vec{F} \cdot d\vec{r}$$

around the circle

$$(x-2)^2 + (y-3)^2 = 9, \quad z = 0$$

where

$$\vec{F} = \begin{pmatrix} x^2 + yz^2 \\ 2x - y^3 \\ 0 \end{pmatrix}$$

Hint: Use Stoke's theorem.

10) Verify Stoke's theorem for

$$\vec{A} = \begin{pmatrix} 2x - y \\ -yz^2 \\ -y^2 z \end{pmatrix}$$

Where $\vec{S}$ is the upper half surface of the sphere given by

$$x^2 + y^2 + z^2 = 1$$

i.e. calculate the surface integral of $\vec{\nabla} \times \vec{A}$ over the half sphere, and the line integral of $\vec{A}$ over its boundary, and compare the result. (This is quite different from a *proof*. All you have to show here is that it works for this example; in a proof you would have to show that it works always.)

11) Curl

(a) If

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where $\vec{\omega}$ is a constant vector, prove that

$$\vec{\omega} = \frac{1}{2}\mathrm{curl}\,\vec{v}$$

Hint: Use the nabla-calculus techniques shown in lecture 10 or 11. If you don't remember the vector triple product rule, memorise it now:

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \cdot \vec{C}\right)\vec{B} - \left(\vec{A} \cdot \vec{B}\right)\vec{C}$$

(b) Interpret the result physically.