Probability Density & Normalisation

7.1. Consider the following wavefunction,

$$\Psi(x,t_0) = \begin{cases} \frac{\sqrt{15}}{4}(1-x^2) & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that this is a normalised wavefunction.

(b) Sketch the wavefunction $\Psi(x,t_0)$ and the probability density $P(x,t_0)$.

(c) What is the probability amplitude at $x = \frac{1}{2}$.

(d) Where is the probability density to find the particle largest? What is the probability density there?

7.2. Consider the following wavefunction,

$$\Psi(x,t_0) = \begin{cases} A\left(1+\frac{x}{2}\right) & \text{if } -2 \leq x \leq 0, \\ A\left(1-\frac{x}{2}\right) & \text{if } 0 \leq x \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

where A is a real constant.

(a) Normalise the wavefunction. That is, find the value of A such that the wavefunction is normalised.

(b) Sketch the wavefunction $\Psi(x,t_0)$, and the probability density $P(x,t_0)$.

(c) What is the probability to find the particle between $x = 1$ and $x = 2$.

7.3. Consider the following wavefunction,

$$\Psi(x,t_0) = \begin{cases} \frac{1}{\sqrt{a}}e^{2\pi x i/a} & \text{if } 0 \leq x \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the wavefunction $\Psi(x,t_0)$, taking care to plot both the real and imaginary parts.

(b) Show that this is a normalised wavefunction.

(c) Sketch the probability density $P(x,t_0)$.

(d) What is the expectation value of the position of the particle?

Hint: *Recall from Mathematical Physics 202 that for a probability density $P(x,t_0)$, the expectation (or expected) value is defined by*

$$\langle x\rangle = \int_{-\infty}^{\infty} xP(x,t_0)dx,$$

which is the average, or mean, position where the particle will be found.

Probability current

7.4. Consider the wavefunction

$$\Psi(x,t_0) = \frac{1}{\pi^{1/4}}e^{-x^2/2}e^{ik_0x},$$

where k_0 is an arbitrary real constant.

(a) Find the probability current $j(x,t_0)$ associated to this wavefunction.

(b) In most contexts where the continuity equation holds, the current can be viewed as being a density times a velocity (for example in fluid dynamics, electromagnetism, thermodynamics, etc).

We can take this view here and write $j(x,t_0) = u(x,t_0)P(x,t_0)$, where $u(x,t_0)$ is a **probability velocity**. What is the probability velocity for the above wavefunction?

7.5. Assume that we have a wavefunction $\Psi(x, t_0)$ with associated probability current $j(x, t_0)$. Consider now a second wavefunction $\Psi'(x, t_0)$, related to $\Psi(x, t_0)$ via

$$\Psi'(x, t_0) = \Psi(x, t_0)e^{ik_0 x},$$

where k_0 is an arbitrary real constant.

(a) Find the probability density $P'(x, t_0)$ associated to $\Psi'(x, t_0)$. How does it relate to the probability density $P(x, t_0)$ associated to $\Psi(x, t_0)$?

(b) Show that the probability current $j'(x, t_0)$ associated to $\Psi'(x, t_0)$ is

$$j'(x, t_0) = j(x, t_0) + \frac{\hbar k_0}{M} P(x, t_0)$$

where $P(x, t_0) = |\Psi(x, t_0)|^2$.

(c) By considering how the probability velocity of $\Psi(x, t_0)$ and $\Psi'(x, t_0)$ are related, explain what changes about a particle when we multiply its wavefunction by $e^{ik_0 x}$.

Factorising functions

7.6. Which of the following functions $f(x, y)$ factorise into $f(x, y) = g(x)h(y)$?

(a) $f(x, y) = xy^2$ (b) $f(x, y) = x^2 + 2y$ (c) $f(x, y) = xy^2 + 1$

(d) $f(x, y) = e^{x+y}$ (e) $f(x, y) = x^2 \cos(y) + 2xe^{iy} + 2xe^{-iy}$ (f) $f(x, y) = \log(xy)$