Oscillations - Revision

1.1. Consider a harmonic oscillator who's movement is described in the form

$$x(t) = A\cos(\omega t + \varphi)$$

with $A = 2\ m$, $\omega = \frac{2\pi}{3}\frac{1}{s}$ and $\varphi = \frac{\pi}{6}$, where m stands for meters and s is seconds.

(a) What is the period T of oscillation.?

(b) What is the physical meaning of A?

(c) Express $x(t)$ in the form

$$A\cos(\omega(t - t_0))\ .$$

(d) Make a reasonably careful graph of x as a function of t (with t on the x-axis and x on the y-axis).

(e) From looking at the graph, tell whether at $t = 0$, the velocity is positive or negative.

(f) Find $x(0)$ and $v(0)$, the position and velocity at $t = 0$.

(g) Express $x(t)$ in the form

$$x(t) = B\cos(\omega t) + C\sin(\omega t)\ .$$

What is the physical meaning of B and C?

(h) Consider another oscillator whose initial position and velocity are $x(0) = 2\ m$, $v(0) = 6\ \frac{m}{s}$ and its angular frequency is $\omega = \frac{3}{s}$. Write the solution in the form

$$x(t) = B\cos(\omega t) + C\sin(\omega t)$$

and

$$x(t) = A\cos(\omega t + \varphi)\ .$$

1.2. Consider the electric circuit illustrated in Fig. 1, composed of a capacitor C and an inductance L (so called "LC resonant circuit"). Suppose that the capacitor is initially charged, so that a current will start flowing in the circuit.

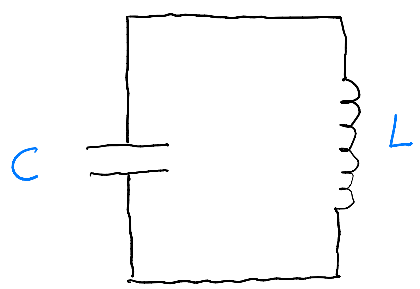

Figure 1: LC circuit

(a) Write the Kirchhoff equation for the current I, capacitor voltage V_c as functions of time and show that I and the capacitor charge Q obey harmonic oscillator equations

(b) Find the angular frequency ω for this system.

(c) Discuss the correspondence between the circuit parameters C, L and properties I, Q with those of a mechanical mass-spring oscillator M, k and x, v.

1.3. Consider an object of mass M connected to two walls by an elastic cord under tension T. The object's rest position is in the middle of the distance from the two walls, at distance a from each wall, as illustrated in Fig. 2. (No gravity!) Suppose that the mass is undergoing small transverse oscillations, small meaning that the displacement y is much smaller than a so that the angle made by the cord with the horizontal is small.

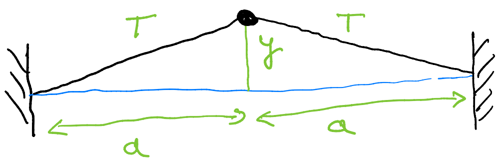

Figure 2: Transverse oscillations

(a) Draw clear drawings to illustrate the forces.
(b) Write the Newton equation for the transverse movement to first order in y and show that it represents a harmonic oscillator.
(c) Find the angular frequency ω for this system.

1.4. Consider a simple pendulum composed by a point-like mass M connected to the ceiling by a massless, inextensible string of length l, as shown in Fig. 3.

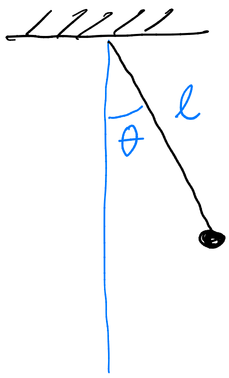

Figure 3: Simple pendulum

(a) Make a clear drawing of the forces acting on the mass M.
(b) Write the equation of motion in the limit of small oscillation amplitude.
(c) Find the general solution.

Dimensional Analysis

1.5. Consider a particle of mass m acted upon by a force $F = -kx^3$. It is difficult to solve the problem exactly, but try to say as much as you can about it. In particular:

(a) What is the potential?
(b) Argue qualitatively that the movement is oscillatory.
(c) Let A be the amplitude of the oscillation. Using dimensional, analysis determine the dependence of the period T of the parameters of the problem.

Coupled Oscillators

1.6. Consider the case of two coupled identical mass-spring harmonic oscillators, as shown in Fig. 4.

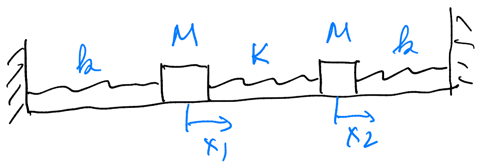

Figure 4: Coupled Longitudinal Oscillators

The general solution can be written in the form:

$$x_1(t) = B_1 \cos(\omega_1 t) + C_1 \sin(\omega_1 t) + B_2 \cos(\omega_2 t) + C_2 \sin(\omega_2 t) \quad (1)$$
$$x_2(t) = B_1 \cos(\omega_1 t) + C_1 \sin(\omega_1 t) - B_2 \cos(\omega_2 t) - C_2 \sin(\omega_2 t) \quad (2)$$

where $\omega_1 = \sqrt{\frac{k}{M}}$ and $\omega_2 = \sqrt{\frac{k+2K}{M}}$ are the two normal mode angular frequencies. Write the solution in terms of the time $t = 0$ initial conditions of the masses $x_1(0)$, $v_1(0)$, $x_2(0)$,$v_2(0)$.

1.7. Consider two bodies of mass M_1 and M_2 respectively, connected by elastic cords, all with tension T as illustrated in Fig. 5. The masses move transversally. Let the horizontal distance between the masses, and between the masses and the walls, be equal to a. Denote y_1 and y_2 their transverse displacement.

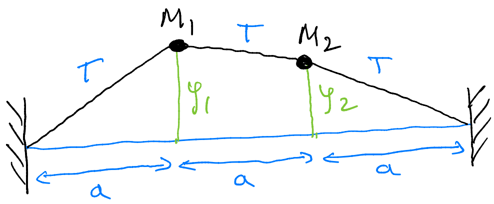

Figure 5: Coupled Transverse Oscillators

(a) Write the equations of motion, in the the case of very small displacements (small with respect to a), i.e. first order in y_1 and y_2 and show that they are coupled harmonic oscillators.
(b) Show that the horizontal forces are zero in first order in y_1 and y_2, so that in this approximation the movement is transversal only.
(c) For $M_1 = M_2 = M$ find the normal coordinates, normal modes and a general solution.
(d) Make drawings of the configuration of the system when mass 1 is at its maximal amplitude.
(e) Let the initial conditions be $y_1(0)$, $v_1(0)$, $y_2(0)$, $v_2(0)$. Write the solution corresponding to these initial conditions

1.8. Consider two simple pendulums of mass M_1 and M_2 and lengths l_1 and l_2 that are also connected between them by a spring of spring constant K, as shown in Fig. 6.

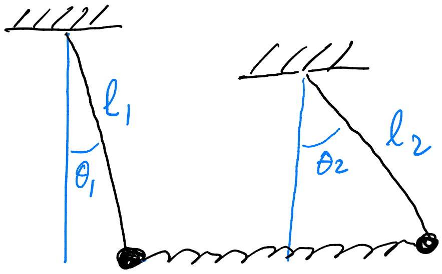

Figure 6: Coupled Pendulums

(a) Make a clear drawing of the forces.
(b) Write the equations of motion in the limit of small oscillation amplitudes.

1.9. Consider two objects of mass $M_1 = M_2 = M$ connected to walls by springs of constants $k_1 = k_2 = k$, and coupled by a spring of spring constant K as illustrated earlier in Fig. 4. Let x_1 and x_2 denote the displacement of the masses from their equilibrium position.

(a) Write the total energy of the system in terms of the displacements x_1 and x_2 and of the velocities $\dot{x}_1$ and $\dot{x}_2$.
(b) Consider the "normalised" normal coordinates

$$\tilde{y}_1 = \frac{x_1 + x_2}{\sqrt{2}} \tag{3}$$

$$\tilde{y}_2 = \frac{x_1 - x_2}{\sqrt{2}}. \tag{4}$$

Show that also these "normalised" normal coordinates behave as two uncoupled simple harmonic oscillators.
(c) You can think of them as two fictitious particles. Show that the total energy of these two fictitious particles is the same as that of the coupled oscillators.

Linear Equations

1.10. Consider two coupled oscillators with distinct masses and spring constants, as illustrated in Fig. 7.

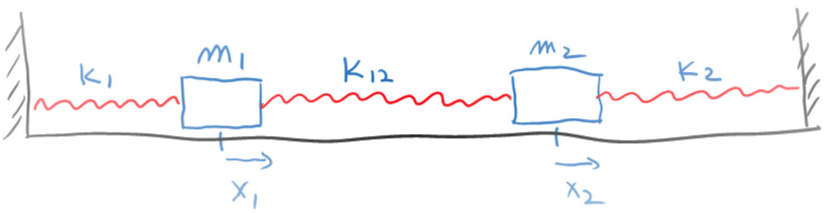

Figure 7: Distinct Coupled Oscillators

(a) Let the displacement of the masses from their equilibrium positions be denoted by x_1 and x_2. Write the Newton equations for the two coupled oscillators .
(b) Argue that they form a system of two linear homogeneous equations with the unknowns x_1 and x_2.

(c) Suppose that the pair $\{x_1^{(1)}, x_2^{(1)}\}$ is a solution for the equations, and that $\{x_1^{(2)}, x_2^{(2)}\}$ is a different solution. Prove the superposition principle, i.e. show that $\{\tilde{x}_1 = a x_1^{(1)} + b x_1^{(2)}, \tilde{x}_2 = a x_2^{(1)} + b x_2^{(2)}\}$ where a and b are arbitrary constants is also a solution.

Note: A superposition of a solution for two oscillators means that both contributions to solution (1), given by the pair $\{x_1^{(1)}, x_2^{(1)}\}$, are multiplied by the **same** constant a, and similarly both contributions to solution (2), given by the pair $\{x_1^{(2)}, x_2^{(2)}\}$, are multiplied by the **same** constant b.

1.11. Consider (i) a damped harmonic oscillator and (ii) a damped harmonic oscillator with a time depending driving force. For each of these two cases state the type of equation of motion : (1) linear non-homogeneous, (2) linear homogeneous, or (3) non-linear.

1.12. Consider again an object of mass m connected to a spring that applies a force

$$F = -kx^3 \,, \tag{5}$$

where x is the displacement of the mass from the equilibrium position.

(a) Write down the equation of motion.

(b) Show that if $x'(t)$ and $x''(t)$ are solutions to this equation then $x(t) = \alpha x'(t) + \beta x''(t)$, where α and β are arbitrary constants, is in general *not* a solution.

(c) Using this information state what type of the equation of motion you have found, namely (1) linear non-homogeneous, (2) linear homogeneous or (3) non-linear.