Orbits

4.1. The graph in Fig. 1 describes the effective potential $U_{eff}(r)$ for a particle moving in a central potential with a given angular momentum L around the origin.

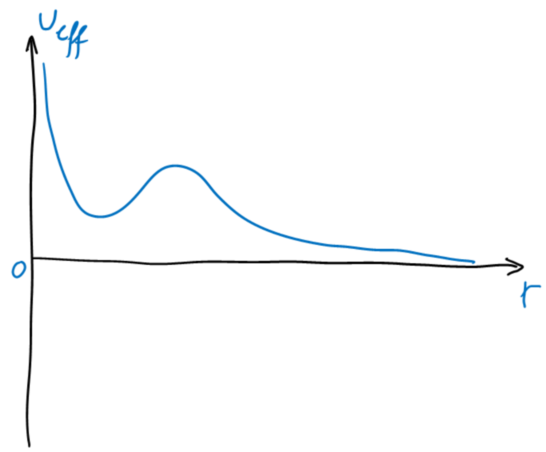


Figure 1: Effective potential for a given L

(a) Identify the radii where the particle has circular orbits.
(b) Which of them are stable and which are unstable?
(c) Suppose L is increased. Qualitatively, how will the effective potential change? (Make a graph).
(d) What will be the consequences? Explain in a few words.

4.2. Consider a particle moving in a central potential $U(r) = -\frac{A}{r^k}$ with $A > 0$. For which values of k are circular orbits stable?
Hint: Consider the radial equation and look at the shape of the effective potential. Circular orbits are at the extremal points of the effective potential, more precisely where $\frac{dU_{eff}}{dr} = 0$. The orbit is stable if it is at the bottom of a potential well, and unstable if it is the top of a potential hill. What is the mathematical condition that tells us which of these situations we are in?

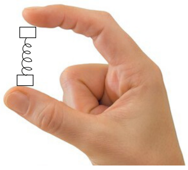
4.3. A particle of mass M moves on a frictionless table attached to a spring with spring constant k, which is attached at its other end on a point on the table Let us take that point as the origin of the coordinate system. Suppose that in its relaxed state the length of the spring is very small, so we can approximate the equilibrium position to be at $r = 0$. Hence, the spring exerts on the mass a radial force of magnitude $f = -kr$, where r is the distance of the particle from the origin. This is the so called 3-D harmonic oscillator problem.
Suppose the particle moves on a circular orbit and has kinetic energy K.

(a) What is the velocity of the particle, the radius of the circle, and the angular momentum?
(b) Show that the circular orbit is stable.
(c) Suppose that we perturb the particle by hitting it and giving it a small radial velocity. What is the frequency of the radial oscillations?
(d) For a larger amplitude, are the radial oscillations harmonic?
(e) If the radial velocity we gave the particle when we hit it is ω, (not necessarily small), find the minimal and maximal radius.
(f) The case of the 3-D harmonic oscillator is one of the few that can be solved exactly. All the orbits are ellipses with the origin O of the central potential situated at the centre of the ellipse (different from the planetary motion, where the Sun is in one of the focal points of the ellipse, not in the centre.) The period of all the orbits is $\omega = \sqrt{\frac{k}{M}}$. Consider again the small perturbations of a circular orbit described at point (iii). Knowing the exact shape of the orbits described above, and their period,

make a drawing of the perturbed orbit. Check if the frequency of the radial movement derived in c) matches the frequency of the orbital movement.

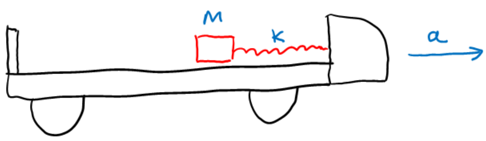
Non-inertial frame

4.4. Consider two objects of mass $m_1 = m_2 = m$ connected by a spring of constant k. The spring in its relaxed state has length L. I hold the two masses and spring in a vertical position, as illustrated in Fig. 2 and I compress the spring by $D < L$. I release the masses and let them fall to the ground.

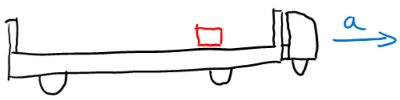


(a) Calculate the movement of the masses.
(b) Make a graph illustrating the hight above the ground of each mass as a function of time.
(c) Is it possible for the upper (or lower) mass to ever go upwards at some time?
(d) In case you decided that a mass can sometimes move upwards, find the form of the function that gives the minimal compression D for which the upper mass moves upwards in exactly three occasions. Argue this without making any explicit calculations but using only dimensional analysis, (There are many parameters in the problem, but perhaps you can argue that one of them is irrelevant enabling you to simplify the problem.)

4.5. Consider a truck having on its platform a mass m connected to a spring of constant k, connected at the other end to the truck front wall, as illustrated in Fig. 3. The mass can move on the truck platform without friction. The truck and the mass are initially at rest. At $t = 0$ the truck starts moving with constant acceleration a. Describe what will happen to the mass. Give its position as a function of time relative to the truck (chose a convenient system of coordinates) and relative to the ground.



4.6. Consider a truck with a mass m that can move without friction on its platform, as illustrated in the figure. Suppose that the front and rear walls of the truck are rigid and if the mass collides with them it will bounce elastically. Initially the truck and the mass are at rest. At $t = 0$ the truck starts moving with acceleration a. What will happen? Describe the motion of the mass as seen by Alice, in the truck and by Bob, on the ground. Give its position and velocity as a function of time relative to the truck (chose a convenient system of coordinates) and relative to the ground.



4.7. The international Space Station orbits the Earth at altitude of about 400km. Calculate the gravitational acceleration g^* at the Space Station altitude, knowing that the radius of the Earth is 6371km and g at the surface of Earth is 9.8m/s^2. As you will see, the gravitational force due to Earth is still considerable at that altitude, yet the astronauts in the space station are floating in the station. Why?