3 Separation of Variables & Time Independent SE

Technique of separation of variables is a general method for solving PDEs

Starting point: Assumption: (strong) $\psi(x,t) = f(t)\,u(x)$ factorises into spatial & temporal parts

$$\frac{\partial}{\partial t}\left(f(t)u(x)\right) = \frac{df(t)}{dt}u(x)$$

partial derivatives = total derivatives for functions of single variable.

$$\frac{\partial^2}{\partial x^2}\left(f(t)u(x)\right) = f(t)\frac{d^2u}{dx^2}$$

SE: $$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi(x,t)$$ sub $\psi(x,t) = f(t)u(x)$

$$\rightarrow\quad i\hbar\frac{df}{dt}u(x) = -\frac{\hbar^2}{2M}f(t)\frac{d^2u}{dx^2} + V(x)f(t)u(x)$$ $\div\, f(t)u(x)$

$$\rightarrow \quad \underbrace{i\hbar \frac{1}{f(t)} \frac{df}{dt}}_{\substack{\text{function of } t \\ \text{constant in } x}} = \underbrace{-\frac{\hbar^2}{2M} \frac{1}{u(x)} \frac{d^2u}{dx^2} + V(x)}_{\substack{\text{function of } x \\ \text{constant in } t}}$$

$\rightarrow$ Both sides must be equal to a constant

Call this constant E

SE is equivalent to pair of eq^{n}s if $\psi(x,t) = f(t)u(x)$

$$\boxed{i\hbar \frac{1}{f(t)} \frac{df}{dt} = E}$$

temporal eq^n

$$\boxed{-\frac{\hbar^2}{2M} \frac{1}{u(x)} \frac{d^2u}{dx^2} + V(x) = E}$$

Time-independent SE (TISE)

Temporal Eq$^{\underline{n}}$

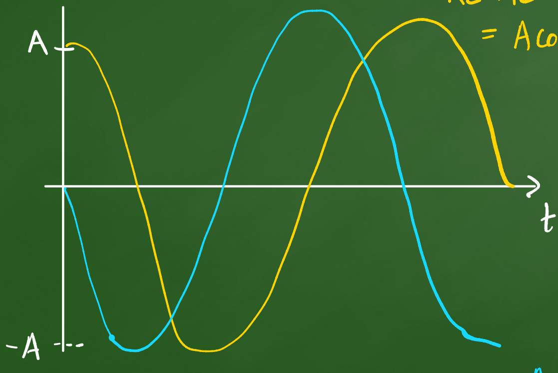
multiple by $\dfrac{i f(t)}{\hbar}$ $\longrightarrow$ $\dfrac{df}{dt} = -\dfrac{iE}{\hbar} f(t)$

write down solution: $\boxed{f(t) = Ae^{-iEt/\hbar}}$ A integration constant

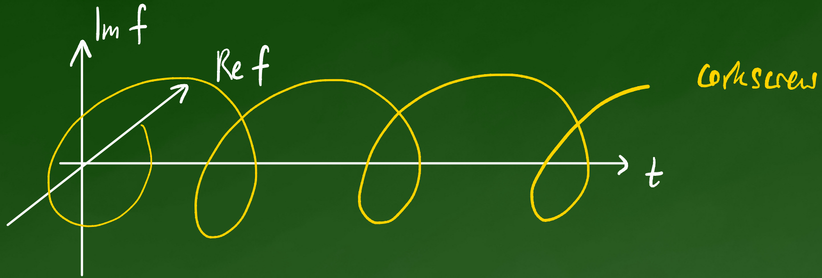
$\mathrm{Re}\, Ae^{-iEt/\hbar}$
$= A\cos Et/\hbar$

[A is real here]

$\cos\theta + i\sin\theta = e^{i\theta}$



$\mathrm{Im}\, Ae^{-iEt/\hbar} = -A\sin Et/\hbar$



Useful to look at the angular frequency $\omega = \frac{2\pi}{T}$ period.

See that $T = \frac{2\pi\hbar}{E}$

$$f(t+T) = Ae^{-iE\left(t + \frac{2\pi\hbar}{E}\right)/\hbar}$$

$$= Ae^{-iEt/\hbar} \underbrace{e^{-i2\pi\cancel{\hbar}\cancel{E}/\cancel{\hbar}\cancel{E}}}_{e^{-i2\pi} = 1}$$

$$= Ae^{-iEt/\hbar}$$

$$= f(t)$$

$$\longrightarrow \quad \omega = \frac{2\pi}{T} = \frac{\cancel{2\pi}E}{\cancel{2\pi}\hbar}$$

or $\quad E = \hbar\omega$

$$= \frac{h}{\cancel{2\pi}}\cancel{2\pi}f = hf$$

first hint that E relates to energy.