Conservation of Normalisation

- The wavefunction remains normalised in time

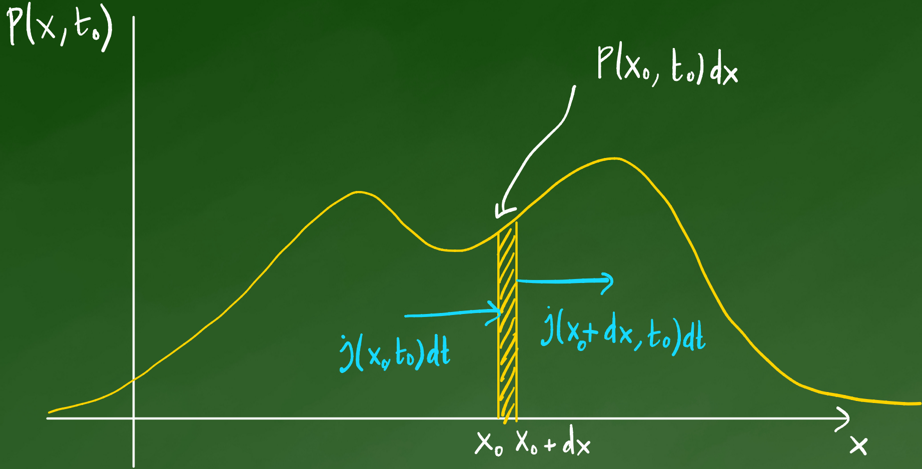
$$\boxed{\int_{-\infty}^{\infty} |\psi(x,t)|^2\, dx = 1 \qquad \text{for all } t}$$

equivalently:
$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} |\psi(x,t)|^2\, dx = 0$$

insightful to look at $\frac{\partial}{\partial t}|\psi(x,t)|^2 \equiv \frac{\partial}{\partial t}P(x,t)$

Can show:
$$\boxed{\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}j(x,t)}$$
CONTINUITY EQ

where
$$\boxed{j(x,t) = \frac{i\hbar}{2M}\left(\frac{\partial\psi^*}{\partial x}\psi(x,t) - \psi^*(x,t)\frac{\partial\psi}{\partial x}\right)}$$
PROBABILITY CURRENT



$$P(x_0, t_0 + dt)\,dx = P(x_0, t_0)\,dx + j(x_0, t_0)\,dt - j(x_0 + dx, t_0)\,dt \qquad \div\, dx\,dt$$

initial — comes in at x_0 — leaves at $x_0 + dx$ — + rearrange

$$\frac{P(x_0, t_0 + dt) - P(x_0, t_0)}{dt} = -\frac{\left(j(x_0 + dx, t_0) - j(x_0, t_0)\right)}{dx}$$

$$\lim_{\substack{dx \to 0 \\ dt \to 0}} \qquad \frac{\partial P}{\partial t} = -\frac{\partial j}{dx}$$

Conservation of norm:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}|\psi(x,t)|^2\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}P(x,t)\,dx$$

$$= \int_{-\infty}^{\infty}-\frac{\partial}{\partial x}j(x,t)\,dx \qquad \text{using continuity eq}^{\underline{n}}$$

$$= \Big[-j(x,t)\Big]_{-\infty}^{\infty}$$

$$= -\underbrace{j(\infty,t)}_{\text{current at infinity.}} + \underbrace{j(-\infty,t)}_{\text{current at minus infinity}}$$

Previously: $\psi(x,t) \rightarrow$ must vanish at $\pm\infty$

$\rightarrow$ $\boxed{j(x,t) \text{ must vanish at } \pm\infty}$ $\rightarrow$ normalisation is conserved.

Derivation of continuity eqⁿ

i.e. show that $\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} j(x,t)$

if $\quad P(x,t) = |\psi(x,t)|^2 = \psi^*(x,t)\,\psi(x,t)$

$$\rightarrow \quad \frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial t}\left(\psi^*(x,t)\,\psi(x,t)\right)$$

$$= \frac{\partial \psi^*}{\partial t}\psi(x,t) + \psi^*(x,t)\frac{\partial \psi}{\partial t}$$

from SE: $\quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi(x,t) \qquad \times -\frac{i}{\hbar}$

$$\rightarrow \quad \frac{\partial \psi}{\partial t} = \frac{i\hbar}{2M}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V(x)\psi(x,t)$$

take complex conjugate: $\quad \frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2M}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar}V(x)\psi^*(x,t) \qquad SE^*$

Sub:

$$\frac{\partial \rho}{\partial t} = \left(-\frac{i\hbar}{2M}\frac{\partial^2 \psi^*}{\partial x^2} + \cancel{\frac{i}{\hbar} V(x)\psi^*(x,t)} \right)\psi(x,t)$$

$$+ \psi^*(x,t)\left(\frac{i\hbar}{2M}\frac{\partial^2 \psi}{\partial x^2} - \cancel{\frac{i}{\hbar} V(x)\psi(x,t)} \right)$$

$$= -\frac{i\hbar}{2M}\left(\frac{\partial^2 \psi^*}{\partial x^2}\psi(x,t) - \psi^*(x,t)\frac{\partial^2 \psi}{\partial x^2} \right)$$

Notice

$$\frac{\partial}{\partial x}\left[\frac{\partial \psi^*}{\partial x}\psi(x,t) - \psi^*(x,t)\frac{\partial \psi}{\partial x} \right]$$

$$= \frac{\partial^2 \psi^*}{\partial x^2}\psi(x,t) + \cancel{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}} - \cancel{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}} - \psi^*(x,t)\frac{\partial^2 \psi}{\partial x^2}$$

$$\longrightarrow \quad \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left[\underbrace{\frac{i\hbar}{2M}\left(\frac{\partial \psi^*}{\partial x}\psi(x,t) - \psi^*(x,t)\frac{\partial \psi}{\partial x} \right)}_{j(x,t)} \right]$$

PROB. CURRENT $j(x,t)$