The momentum wavefunction

Starting point: $\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(k) e^{ikx} \, dk$

normalised if $\int_{-\infty}^{\infty} |c(k)|^2 \, dk = 1$

Now: sub. de Broglie relation $p = \hbar k$ i.e. $k = \frac{p}{\hbar}$

careful: $\frac{dk}{dp} = \frac{1}{\hbar} \rightarrow dk = \frac{dp}{\hbar}$

$\rightarrow \psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(p/\hbar) \, e^{ipx/\hbar} \, \frac{dp}{\hbar}$

* Corrected Nov 2022 *

$\rightarrow \int_{-\infty}^{\infty} |c(k)|^2 \, dk = 1 \rightarrow \int_{-\infty}^{\infty} |c(p/\hbar)|^2 \frac{1}{\hbar} \, dp = 1$

Notice that this is in the form $\int_{-\infty}^{\infty} P(p,0) \, dp = 1$

Suggests: $P(p,0) = \frac{1}{\hbar}|c(p/\hbar)|^2$

Recall: want $P(p,0) = |\tilde{\psi}(p,0)|^2$

Suggest further: $\tilde{\psi}(p,0) = \frac{1}{\sqrt{\hbar}} c(p/\hbar)$

Want: to lose reference to $c(k)$ $\rightarrow$ want to express this in terms of $\psi(x,0)$

can do this by using $c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0) e^{-ikx} \, dx$

$\rightarrow$ $$\boxed{\tilde{\psi}(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,0) e^{-ipx/\hbar} \, dx}$$

extend to all times t

$$\boxed{\tilde{\psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,t) e^{-ipx/\hbar} \, dx}$$

This is the fully general expression for momentum wavefunction.

Momentum wavefunction is an alternative representation of state of a particle

Recall: $\psi(x,t_0)$ is a complete spec. of state at t_0

$\tilde{\psi}(p,t_0)$ is also a complete spec. of state at t_0.

Say that $\tilde{\psi}(p,t_0)$ is a different representation of state

In above: multiply both sides by $e^{ipx'/\hbar}$ & integrate over p:

$$\int_{-\infty}^{\infty} \tilde{\psi}(p,t) e^{ipx'/\hbar} dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x,t)\, e^{-ipx/\hbar} dx\, e^{ipx'/\hbar}\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, \psi(x,t) \underbrace{\int_{-\infty}^{\infty} dp\, e^{ip(x-x')/\hbar}}_{2\pi\,\delta\left(\frac{x-x'}{\hbar}\right) = 2\pi\hbar\,\delta(x-x')}$$

$$\delta(ay) = \frac{1}{|a|}\delta(y)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, \psi(x,t)\, 2\pi\hbar\, \delta(x-x')$$

$$= \sqrt{2\pi\hbar}\, \psi(x',t)$$

$$\rightarrow \boxed{\psi(x',t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{\psi}(p,t)\, e^{ipx'/\hbar}\, dp}$$

i.e. know $\tilde{\psi}(p,t)$ we can calculate $\psi(x,t)$

$\rightarrow$ $\tilde{\psi}(p,t)$ is a complete specification of state.