4 The free particle

Goal: Solve SE when no forces

$V(x) = 0 \qquad F(x) = 0$

Classical:

Newton: $\frac{dp}{dt} = 0 \qquad \frac{dx}{dt} = \frac{p(t)}{M}$

Initial condition

$x(0) = x_0$

$p(0) = p_0$

$$\boxed{p(t) = p_0 \qquad x(t) = \frac{p_0}{M} t + x_0}$$

constant momentum.

Step 1 Find stationary states i.e. Solve TISE w/ $V(x) = 0$

TISE: $-\frac{\hbar^2}{2M}\frac{d^2u}{dx^2} = E u(x)$ $\qquad \times -\frac{2M}{\hbar^2}$

$$\rightarrow \quad \frac{d^2u}{dx^2} = -\frac{2ME}{\hbar^2} u(x)$$

$E > 0$

useful defⁿ: $\frac{2ME}{\hbar^2} = k^2 \rightarrow k = \frac{\sqrt{2ME}}{\hbar}$

$$E = \frac{\hbar^2 k^2}{2M}$$

$$\rightarrow \frac{d^2 u}{dx^2} = -k^2 u(x)$$

old friend

Same eqⁿ from SHO!

write down solⁿs: e^{ikx} & e^{-ikx}

[Note: $\cos kx$ & $\sin kx$ are also solⁿs]

check: $\frac{d}{dx} e^{ikx} = (ik) e^{ikx}$

$$\frac{d^2}{dx^2} e^{ikx} \quad (ik)^2 e^{ikx} = -k^2 e^{ikx}$$

general solⁿ: $u(x) = Ae^{ikx} + Be^{-ikx}$

integration constants (complex)

$\hookrightarrow$ Stationary states of a free particle are of the form

$$\psi(x,t) = A' e^{-iEt/\hbar}\left(A e^{ikx} + B e^{-ikx}\right)$$

$$= e^{-i\hbar k^2 t/2M}\left(A'' e^{ikx} + B'' e^{-ikx}\right)$$

$A'' = AA'$
$B'' = BA'$

Done! X

Problems: 1. Want to be able to specify an arbitrary initial state, i.e. $\psi(x,0)$

2. Stationary states of a free particle are unnormalisable!

Problems are resolved using Superposition principle

Check norm:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = \int_{-\infty}^{\infty} \underbrace{\cancel{e^{+i\hbar k^2 t/2M}} \left(A''^{*} e^{-ikx} + B''^{*} e^{+ikx}\right)}_{\psi^*(x,t)}$$

$$\times \underbrace{\cancel{e^{-i\hbar k^2 t/2M}} \left(A'' e^{ikx} + B'' e^{-ikx}\right)}_{\psi(x,t)}$$

$$= \int_{-\infty}^{\infty} \Big(\underbrace{|A''|^2 + |B''|^2}_{\infty} + A''^{*} B'' e^{-2ikx}$$

$$\underbrace{+ A'' B''^{*} e^{2ikx}}_{\approx \text{Small}} \Big) dx$$

$$= \infty \quad \text{i.e. unnormalisable} \quad :($$