Operators

9.1. Consider a particle whose wavefunction is the Gaussian wavepacket from Example 4.1 in the lecture notes,

$$\Psi(x) = \left(\frac{1}{a\sqrt{2\pi}}\right)^{1/2} e^{-x^2/4a^2} e^{ip_0x/\hbar},$$

where a and p_0 are arbitrary real constants.

(a) Write down the quantum mechanical prediction for the average momentum, $\langle p\rangle$, in terms of the momentum operator $\hat{P}$.

(b) Evaluate this expression for the Gaussian wavepacket $\Psi(x)$.

***Hint**: You may want to use the following results for Gaussian integrals in order to help carry out this calculation:*

$$\int_{-\infty}^{\infty} e^{-by^2}\,dy = \sqrt{\frac{\pi}{b}}, \qquad \int_{-\infty}^{\infty} ye^{-by^2}\,dy = 0.$$

In Example 5.1 we showed that the associated momentum wavefunction of the particle is

$$\tilde{\Psi}(p) = \left(\frac{2a^2}{\pi\hbar^2}\right)^{1/4} e^{-a^2(p-p_0)^2/\hbar^2},$$

(c) Write down the definition of the average momentum $\langle p\rangle$ in terms of $P(p)$, the probability density for the particle to have momentum p.

(d) Evaluate this expression for the probability density $P(p) = |\tilde{\Psi}(p)|^2$ associated to the Gaussian wavepacket.

***Hint**: You may again want to make use of the above results for Gaussian integrals.*

(e) Do your answers to part (b) and (d) agree with each other?

9.2. Consider a particle whose wavefunction is the Gaussian wavepacket $\Psi(x)$ from Problem 9.1.

(a) Calculate the action of the momentum operator followed by the position operator acting on this wavefunction, that is, calculate $\hat{X}\hat{P}\Psi(x)$.

(b) Calculate the action of the position operator followed by the momentum operator acting on this wavefunction, that is, calculate $\hat{P}\hat{X}\Psi(x)$.

(c) Write down the difference between applying the operators in the two different orders, that is, write down $\hat{X}\hat{P}\Psi(x) - \hat{P}\hat{X}\Psi(x)$.

(d) Does your result to part (c) confirm the **canonical commutation relation**?

The Uncertainty Principle

9.3. Consider a particle with spatial wavefunction $\Psi(x,t_0)$ and momentum wavefunction $\tilde{\Psi}(p,t_0)$ from Problem 8.3:

$$\Psi(x,t_0) = \begin{cases} \frac{\sqrt{15}}{4}(1-x^2) & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \tilde{\Psi}(p,t_0) = \sqrt{\frac{15\hbar^3}{2\pi}} \left(\frac{\hbar \sin(p/\hbar)}{p^3} - \frac{\cos(p/\hbar)}{p^2} \right).$$

(a) Show that both $\Psi(x,t_0)$ and $\tilde{\Psi}(p,t_0)$ are even functions.
Hint*: Recall that an even function is one such that $f(-y) = f(y)$.*

(b) Explain why this implies that both the average position and average momentum are zero,

$$\langle x \rangle = 0, \qquad \langle p \rangle = 0.$$

(c) Calculate the average square position of the particle, $\langle x^2 \rangle$, and the standard deviation, $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

It can be shown that the average square momentum of the particle is

$$\langle p^2 \rangle = \frac{5\hbar^2}{2}.$$

(d) Calculate the standard deviation in the momentum of the particle, $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$.

(e) Show that the particle satisfies the **Uncertainty Principle**. How close is the particle to saturating the bound of the uncertainty principle?