CHAPTER 1

Infinite Series, Power Series

1. THE GEOMETRIC SERIES

As a simple example of many of the ideas involved in series, we are going to consider the geometric series. You may recall that in a geometric progression we multiply each term by some fixed number to get the next term. For example, the *sequences*

$$2,\ 4,\ 8,\ 16,\ 32, \ldots, \tag{1.1a}$$

$$1,\ \tfrac{2}{3},\ \tfrac{4}{9},\ \tfrac{8}{27},\ \tfrac{16}{81}, \ldots, \tag{1.1b}$$

$$a,\ ar,\ ar^2,\ ar^3, \ldots, \tag{1.1c}$$

are geometric progressions. It is easy to think of examples of such progressions. Suppose the number of bacteria in a culture doubles every hour. Then the terms of (1.1a) represent the number by which the bacteria population has been multiplied after 1 hr, 2 hr, and so on. Or suppose a bouncing ball rises each time to $\frac{2}{3}$ of the height of the previous bounce. Then (1.1b) would represent the heights of the successive bounces in yards if the ball is originally dropped from a height of 1 yd.

In our first example it is clear that the bacteria population would increase without limit as time went on (mathematically, anyway; that is, assuming that nothing like lack of food prevented the assumed doubling each hour). In the second example, however, the height of bounce of the ball decreases with successive bounces, and we might ask for the total distance the ball goes. The ball falls a distance 1 yd, rises a distance $\frac{2}{3}$ yd and falls a distance $\frac{2}{3}$ yd, rises a distance $\frac{4}{9}$ yd and falls a distance $\frac{4}{9}$ yd, and so on. Thus it seems reasonable to write the following expression for the total distance the ball goes:

$$1 + 2 \cdot \tfrac{2}{3} + 2 \cdot \tfrac{4}{9} + 2 \cdot \tfrac{8}{27} + \cdots = 1 + 2\left(\tfrac{2}{3} + \tfrac{4}{9} + \tfrac{8}{27} + \cdots\right), \tag{1.2}$$

where the three dots mean that the terms continue as they have started (each one being $\frac{2}{3}$ the preceding one), and there is never a last term. Let us consider the expression in parentheses in (1.2), namely

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots. \tag{1.3}$$

This expression is an example of an *infinite series*, and we are asked to find its sum. Not all infinite series have sums; you can see that the series formed by adding the terms in (1.1a) does not have a finite sum. However, even when an infinite series does have a finite sum, we cannot find it by adding the terms because no matter how many we add there are always more. Thus we must find another method. (It is actually deeper than this; what we really have to do is to *define* what we mean by the sum of the series.)

Let us first find the sum of n terms in (1.3). The formula (Problem 2) for the sum of n terms of the geometric progression (1.1c) is

$$S_n = \frac{a(1-r^n)}{1-r}. \tag{1.4}$$

Using (1.4) in (1.3), we find

$$S_n = \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}[1-(\frac{2}{3})^n]}{1-\frac{2}{3}} = 2\left[1-\left(\frac{2}{3}\right)^n\right]. \tag{1.5}$$

As n increases, $(\frac{2}{3})^n$ decreases and approaches zero. Then the sum of n terms approaches 2 as n increases, and we say that the sum of the series is 2. (This is really a definition: The sum of an infinite series is the limit of the sum of n terms as $n \to \infty$.) Then from (1.2), the total distance traveled by the ball is $1 + 2 \cdot 2 = 5$. This is an answer to a mathematical problem. A physicist might well object that a bounce the size of an atom is nonsense! However, after a number of bounces, the remaining infinite number of small terms contribute very little to the final answer (see Problem 1). Thus it makes little difference (in our answer for the total distance) whether we insist that the ball rolls after a certain number of bounces or whether we include the entire series, and it is easier to find the sum of the series than to find the sum of, say, twenty terms.

Series such as (1.3) whose terms form a geometric progression are called *geometric series*. We can write a geometric series in the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots. \tag{1.6}$$

The sum of the geometric series (if it has one) is by definition

$$S = \lim_{n\to\infty} S_n, \tag{1.7}$$

where S_n is the sum of n terms of the series. By following the method of the example above, you can show (Problem 2) that a geometric series has a sum if and only if $|r| < 1$, and in this case the sum is

$$S = \frac{a}{1-r}. \tag{1.8}$$

The series is then called *convergent*.

Here is an interesting use of (1.8). We can write $0.3333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = \frac{3/10}{1-1/10} = \frac{1}{3}$ by (1.8). Now of course you knew that, but how about $0.785714285714\cdots$? We can write this as $0.5 + 0.285714285714\cdots = \frac{1}{2} + \frac{0.285714}{1-10^{-6}} = \frac{1}{2} + \frac{285714}{999999} = \frac{1}{2} + \frac{2}{7} = \frac{11}{14}$. (Note that any repeating decimal is equivalent to a fraction which can be found by this method.) If you want to use a computer to do the arithmetic, be sure to tell it to give you an exact answer or it may hand you back the decimal you started with! You can also use a computer to sum the series, but using (1.8) may be simpler. (Also see Problem 14.)

PROBLEMS, SECTION 1

1. In the bouncing ball example above, find the height of the tenth rebound, and the distance traveled by the ball after it touches the ground the tenth time. Compare this distance with the total distance traveled.

2. Derive the formula (1.4) for the sum S_n of the geometric progression $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. *Hint:* Multiply S_n by r and subtract the result from S_n; then solve for S_n. Show that the geometric series (1.6) converges if and only if $|r| < 1$; also show that if $|r| < 1$, the sum is given by equation (1.8).

Use equation (1.8) to find the fractions that are equivalent to the following repeating decimals:

3. $0.55555\cdots$ **4.** $0.818181\cdots$ **5.** $0.583333\cdots$

6. $0.61111\cdots$ **7.** $0.185185\cdots$ **8.** $0.694444\cdots$

9. $0.857142857142\cdots$ **10.** $0.576923076923076923\cdots$

11. $0.678571428571428571\cdots$

12. In a water purification process, one-nth of the impurity is removed in the first stage. In each succeeding stage, the amount of impurity removed is one-nth of that removed in the preceding stage. Show that if $n = 2$, the water can be made as pure as you like, but that if $n = 3$, at least one-half of the impurity will remain no matter how many stages are used.

13. If you invest a dollar at "6% interest compounded monthly," it amounts to $(1.005)^n$ dollars after n months. If you invest \$10 at the beginning of each month for 10 years (120 months), how much will you have at the end of the 10 years?

14. A computer program gives the result 1/6 for the sum of the series $\sum_{n=0}^{\infty}(-5)^n$. Show that this series is divergent. Do you see what happened? *Warning hint:* Always consider whether an answer is reasonable, whether it's a computer answer or your work by hand.

15. Connect the midpoints of the sides of an equilateral triangle to form 4 smaller equilateral triangles. Leave the middle small triangle blank, but for each of the other 3 small triangles, draw lines connecting the midpoints of the sides to create 4 tiny triangles. Again leave each middle tiny triangle blank and draw the lines to divide the others into 4 parts. Find the infinite series for the total area left blank if this process is continued indefinitely. (Suggestion: Let the area of the original triangle be 1; then the area of the first blank triangle is 1/4.) Sum the series to find the total area left blank. Is the answer what you expect? *Hint:* What is the "area" of a straight line? (Comment: You have constructed a *fractal* called the Sierpiński gasket. A fractal has the property that a magnified view of a small part of it looks very much like the original.)

16. Suppose a large number of particles are bouncing back and forth between $x = 0$ and $x = 1$, except that at each endpoint some escape. Let r be the fraction reflected each time; then $(1 - r)$ is the fraction escaping. Suppose the particles start at $x = 0$ heading toward $x = 1$; eventually all particles will escape. Write an infinite series for the fraction which escape at $x = 1$ and similarly for the fraction which escape at $x = 0$. Sum both the series. What is the largest fraction of the particles which can escape at $x = 0$? (Remember that r must be between 0 and 1.)

2. DEFINITIONS AND NOTATION

There are many other infinite series besides geometric series. Here are some examples:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots, \tag{2.1a}$$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots, \tag{2.1b}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots. \tag{2.1c}$$

In general, an infinite series means an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots, \tag{2.2}$$

where the a_n's (one for each positive integer n) are numbers or functions given by some formula or rule. The three dots in each case mean that the series never ends. The terms continue according to the law of formation, which is supposed to be evident to you by the time you reach the three dots. If there is apt to be doubt about how the terms are formed, a general or nth term is written like this:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots, \tag{2.3a}$$

$$x - x^2 + \frac{x^3}{2} + \cdots + \frac{(-1)^{n-1}x^n}{(n-1)!} + \cdots. \tag{2.3b}$$

(The quantity $n!$, read n factorial, means, for integral n, the product of all integers from 1 to n; for example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. The quantity $0!$ is defined to be 1.) In (2.3a), it is easy to see without the general term that each term is just the square of the number of the term, that is, n^2. However, in (2.3b), if the formula for the general term were missing, you could probably make several reasonable guesses for the next term. To be sure of the law of formation, we must either know a good many more terms or have the formula for the general term. You should verify that the fourth term in (2.3b) is $-x^4/6$.

We can also write series in a shorter abbreviated form using a summation sign $\sum$ followed by the formula for the nth term. For example, (2.3a) would be written

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots = \sum_{n=1}^{\infty} n^2 \tag{2.4}$$

(read "the sum of n^2 from $n = 1$ to ∞"). The series (2.3b) would be written

$$x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(n-1)!}$$

For printing convenience, sums like (2.4) are often written $\sum_{n=1}^{\infty} n^2$.

In Section 1, we have mentioned both sequences and series. The lists in (1.1) are sequences; a *sequence* is simply a set of quantities, one for each n. A *series* is an indicated sum of such quantities, as in (1.3) or (1.6). We will be interested in various sequences related to a series: for example, the sequence a_n of terms of the series, the sequence S_n of partial sums [see (1.5) and (4.5)], the sequence R_n [see (4.7)], and the sequence ρ_n [see (6.2)]. In all these examples, we want to find the limit of a sequence as $n \to \infty$ (if the sequence has a limit). Although limits can be found by computer, many simple limits can be done faster by hand.

Example 1. Find the limit as $n \to \infty$ of the sequence

$$\frac{(2n-1)^4 + \sqrt{1+9n^8}}{1 - n^3 - 7n^4}.$$

We divide numerator and denominator by n^4 and take the limit as $n \to \infty$. Then all terms go to zero except

$$\frac{2^4 + \sqrt{9}}{-7} = -\frac{19}{7}.$$

Example 2. Find $\lim_{n\to\infty} \frac{\ln n}{n}$. By L'Hôpital's rule (see Section 15)

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

Comment: Strictly speaking, we can't differentiate a function of n if n is an integer, but we can consider $f(x) = (\ln x)/x$, and the limit of the sequence is the same as the limit of $f(x)$.

Example 3. Find $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n}$. We first find

$$\ln \left(\frac{1}{n}\right)^{1/n} = -\frac{1}{n} \ln n.$$

Then by Example 2, the limit of $(\ln n)/n$ is 0, so the original limit is $e^0 = 1$.

PROBLEMS, SECTION 2

In the following problems, find the limit of the given sequence as $n \to \infty$.

1. $\dfrac{n^2 + 5n^3}{2n^3 + 3\sqrt{4+n^6}}$ **2.** $\dfrac{(n+1)^2}{\sqrt{3+5n^2+4n^4}}$ **3.** $\dfrac{(-1)^n\sqrt{n+1}}{n}$

4. $\dfrac{2^n}{n^2}$ **5.** $\dfrac{10^n}{n!}$ **6.** $\dfrac{n^n}{n!}$

7. $(1+n^2)^{1/\ln n}$ **8.** $\dfrac{(n!)^2}{(2n)!}$ **9.** $n \sin(1/n)$

3. APPLICATIONS OF SERIES

In the example of the bouncing ball in Section 1, we saw that it is possible for the sum of an infinite series to be nearly the same as the sum of a fairly small number of terms at the beginning of the series (also see Problem 1.1). Many applied problems cannot be solved exactly, but we may be able to find an answer in terms of an infinite series, and then use only as many terms as necessary to obtain the needed accuracy. We shall see many examples of this both in this chapter and in later chapters. Differential equations (see Chapters 8 and 12) and partial differential equations (see Chapter 13) are frequently solved by using series. We will learn how to find series that represent functions; often a complicated function can be approximated by a few terms of its series (see Section 15).

But there is more to the subject of infinite series than making approximations. We will see (Chapter 2, Section 8) how we can use power series (that is, series whose terms are powers of x) to give meaning to functions of complex numbers, and (Chapter 3, Section 6) how to define a function of a matrix using the power series of the function. Also power series are just a first example of infinite series. In Chapter 7 we will learn about Fourier series (whose terms are sines and cosines). In Chapter 12, we will use power series to solve differential equations, and in Chapters 12 and 13, we will discuss other series such as Legendre and Bessel. Finally, in Chapter 14, we will discover how a study of power series clarifies our understanding of the mathematical functions we use in applications.

4. CONVERGENT AND DIVERGENT SERIES

We have been talking about series which have a finite sum. We have also seen that there are series which do not have finite sums, for example (2.1a). If a series has a finite sum, it is called *convergent*. Otherwise it is called *divergent*. It is important to know whether a series is convergent or divergent. Some weird things can happen if you try to apply ordinary algebra to a divergent series. Suppose we try it with the following series:

$$S = 1 + 2 + 4 + 8 + 16 + \cdots . \tag{4.1}$$

Then,

$$\begin{aligned} 2S &= 2 + 4 + 8 + 16 + \cdots = S - 1, \\ S &= -1. \end{aligned}$$

This is obvious nonsense, and you may laugh at the idea of trying to operate with such a violently divergent series as (4.1). But the same sort of thing can happen in more concealed fashion, and has happened and given wrong answers to people who were not careful enough about the way they used infinite series. At this point you probably would not recognize that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \tag{4.2}$$

is divergent, but it is; and the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{4.3}$$

is convergent as it stands, but can be made to have *any* sum you like by combining the terms in a different order! (See Section 8.) You can see from these examples how essential it is to know whether a series converges, and also to know how to apply algebra to series correctly. There are even cases in which some divergent series can be used (see Chapter 11), but in this chapter we shall be concerned with convergent series.

Before we consider some tests for convergence, let us repeat the definition of convergence more carefully. Let us call the terms of the series a_n so that the series is

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots . \tag{4.4}$$

Remember that the three dots mean that there is never a last term; the series goes on without end. Now consider the sums S_n that we obtain by adding more and more terms of the series. We define

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\cdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n. \end{aligned} \tag{4.5}$$

Each S_n is called a *partial sum*; it is the sum of the first n terms of the series. We had an example of this for a geometric progression in (1.4). The letter n can be any integer; for each n, S_n stops with the nth term. (Since S_n is not an infinite series, there is no question of convergence for it.) As n increases, the partial sums may increase without any limit as in the series (2.1a). They may oscillate as in the series $1-2+3-4+5-\cdots$ (which has partial sums $1, -1, 2, -2, 3, \cdots$) or they may have some more complicated behavior. One possibility is that the S_n's may, after a while, not change very much any more; the a_n's may become very small, and the S_n's come closer and closer to some value S. We are particularly interested in this case in which the S_n's approach a limiting value, say

$$\lim_{n\to\infty} S_n = S. \tag{4.6}$$

(It is understood that S is a finite number.) If this happens, we make the following definitions.

a. If the partial sums S_n of an infinite series tend to a limit S, the series is called *convergent*. Otherwise it is called *divergent*.

b. The limiting value S is called the *sum of the series*.

c. The difference $R_n = S - S_n$ is called the *remainder* (or the remainder after n terms). From (4.6), we see that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (S - S_n) = S - S = 0. \tag{4.7}$$

▶ **Example 1.** We have already (Section 1) found S_n and S for a geometric series. From (1.8) and (1.4), we have for a geometric series, $R_n = \frac{ar^n}{1-r}$ which $\to 0$ as $n \to \infty$ if $|r| < 1$.

▶ **Example 2.** By partial fractions, we can write $\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$. Let's write out a number of terms of the series

$$\begin{aligned}\sum_2^\infty \frac{2}{n^2-1} &= \sum_2^\infty \left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n+2}\right)\\ &= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \cdots\\ &\quad + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2} + \cdots .\end{aligned}$$

Note the cancellation of terms; this kind of series is called a telescoping series. Satisfy yourself that when we have added the nth term $(\frac{1}{n} - \frac{1}{n+2})$, the only terms which have not cancelled are $1, \frac{1}{2}, \frac{-1}{n+1}$, and $\frac{-1}{n+2}$, so we have

$$S_n = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}, \qquad S = \frac{3}{2}, \qquad R_n = \frac{1}{n+1} + \frac{1}{n+2}.$$

▶ **Example 3.** Another interesting series is

$$\begin{aligned}\sum_1^\infty \ln\left(\frac{n}{n+1}\right) &= \sum_1^\infty [\ln n - \ln(n+1)]\\ &= \ln 1 - \ln 2 + \ln 2 - \ln 3 + \ln 3 - \ln 4 + \cdots + \ln n - \ln(n+1) \cdots .\end{aligned}$$

Then $S_n = -\ln(n+1)$ which $\to -\infty$ as $n \to \infty$, so the series diverges. However, note that $a_n = \ln \frac{n}{n+1} \to \ln 1 = 0$ as $n \to \infty$, so we see that even if the terms tend to zero, a series may diverge.

▶ PROBLEMS, SECTION 4

For the following series, write formulas for the sequences a_n, S_n, and R_n, and find the limits of the sequences as $n \to \infty$ (if the limits exist).

1. $\displaystyle\sum_1^\infty \frac{1}{2^n}$

2. $\displaystyle\sum_0^\infty \frac{1}{5^n}$

3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \cdots$

4. $\displaystyle\sum_1^\infty e^{-n\ln 3}$ *Hint:* What is $e^{-\ln 3}$?

5. $\displaystyle\sum_0^\infty e^{2n\ln\sin(\pi/3)}$ *Hint:* Simplify this.

6. $\displaystyle\sum_1^\infty \frac{1}{n(n+1)}$ *Hint:* $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

7. $\frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \cdots$

5. TESTING SERIES FOR CONVERGENCE; THE PRELIMINARY TEST

It is not in general possible to write a simple formula for S_n and find its limit as $n \to \infty$ (as we have done for a few special series), so we need some other way to find out whether a given series converges. Here we shall consider a few simple tests for convergence. These tests will illustrate some of the ideas involved in testing series for convergence and will work for a good many, but not all, cases. There are more complicated tests which you can find in other books. In some cases it may be quite a difficult mathematical problem to investigate the convergence of a complicated series. However, for our purposes the simple tests we give here will be sufficient.

First we discuss a useful *preliminary test.* In most cases you should apply this to a series before you use other tests.

Preliminary test. If the terms of an infinite series do *not* tend to zero (that is, if $\lim_{n\to\infty} a_n \neq 0$), the series diverges. If $\lim_{n\to\infty} a_n = 0$, we must test further.

This is *not* a test for convergence; what it does is to weed out some very badly divergent series which you then do not have to spend time testing by more complicated methods. *Note carefully:* The preliminary test can *never* tell you that a series converges. It does *not* say that series converge if $a_n \to 0$ and, in fact, often they do not. A simple example is the harmonic series (4.2); the nth term certainly tends to zero, but we shall soon show that the series $\sum_{n=1}^{\infty} 1/n$ is divergent. On the other hand, in the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

the terms are tending to 1, so by the preliminary test, this series diverges and no further testing is needed.

PROBLEMS, SECTION 5

Use the preliminary test to decide whether the following series are divergent or require further testing. *Careful:* Do *not* say that a series is convergent; the preliminary test cannot decide this.

1. $\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \frac{25}{26} - \frac{36}{37} + \cdots$

2. $\sqrt{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{4}}{3} + \frac{\sqrt{5}}{4} + \frac{\sqrt{6}}{5} + \cdots$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{n^2+10n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n!+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$

10. $\displaystyle\sum_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$

11. Using (4.6), give a proof of the preliminary test. *Hint:* $S_n - S_{n-1} = a_n$.

6. CONVERGENCE TESTS FOR SERIES OF POSITIVE TERMS; ABSOLUTE CONVERGENCE

We are now going to consider four useful tests for series whose terms are all positive. If some of the terms of a series are negative, we may still want to consider the related series which we get by making all the terms positive; that is, we may consider the series whose terms are the absolute values of the terms of our original series. If this new series converges, we call the original series *absolutely convergent.* It can be proved that if a series converges absolutely, then it converges (Problem 7.9). This means that if the series of absolute values converges, the series is still convergent when you put back the original minus signs. (The sum is different, of course.) The following four tests may be used, then, either for testing series of positive terms, or for testing any series for absolute convergence.

A. The Comparison Test

This test has two parts, (a) and (b).

(a) Let

$$m_1 + m_2 + m_3 + m_4 + \cdots$$

be a series of positive terms which you know converges. Then the series you are testing, namely

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

is absolutely convergent if $|a_n| \le m_n$ (that is, if the absolute value of each term of the a series is no larger than the corresponding term of the m series) for all n from some point on, say after the third term (or the millionth term). See the example and discussion below.

(b) Let

$$d_1 + d_2 + d_3 + d_4 + \cdots$$

be a series of positive terms which you know diverges. Then the series

$$|a_1| + |a_2| + |a_3| + |a_4| + \cdots$$

diverges if $|a_n| \ge d_n$ for all n from some point on.

Warning: Note carefully that neither $|a_n| \ge m_n$ nor $|a_n| \le d_n$ tells us anything. That is, if a series has terms larger than those of a convergent series, it may still converge or it may diverge—we must test it further. Similarly, if a series has terms smaller than those of a divergent series, it may still diverge, or it may converge.

Example. Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$ for convergence.

As a comparison series, we choose the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

Notice that we do not care about the first few terms (or, in fact, any finite number of terms) in a series, because they can affect the sum of the series but *not* whether it converges. When we ask whether a series converges or not, we are asking what happens as we add more and more terms for larger and larger n. Does the sum increase indefinitely, or does it approach a limit? What the first five or hundred or million terms are has no effect on whether the sum eventually increases indefinitely or approaches a limit. Consequently we frequently ignore some of the early terms in testing series for convergence.

In our example, the terms of $\sum_{n=1}^{\infty} 1/n!$ are smaller than the corresponding terms of $\sum_{n=1}^{\infty} 1/2^n$ for all $n > 3$ (Problem 1). We know that the geometric series converges because its ratio is $\frac{1}{2}$. Therefore $\sum_{n=1}^{\infty} 1/n!$ converges also.

PROBLEMS, SECTION 6

1. Show that $n! > 2^n$ for all $n > 3$. *Hint:* Write out a few terms; then consider what you multiply by to go from, say, 5! to 6! and from 2^5 to 2^6.

2. Prove that the harmonic series $\sum_{n=1}^{\infty} 1/n$ is divergent by comparing it with the series

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(8 \text{ terms each equal to } \frac{1}{16}\right) + \cdots,$$

which is $\quad 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.$

3. Prove the convergence of $\sum_{n=1}^{\infty} 1/n^2$ by grouping terms somewhat as in Problem 2.

4. Use the comparison test to prove the convergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$

5. Test the following series for convergence using the comparison test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *Hint:* Which is larger, n or $\sqrt{n}$? (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

6. There are 9 one-digit numbers (1 to 9), 90 two-digit numbers (10 to 99). How many three-digit, four-digit, etc., numbers are there? The first 9 terms of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{9}$ are all greater than $\frac{1}{10}$; similarly consider the next 90 terms, and so on. Thus prove the divergence of the harmonic series by comparison with the series

$$\left[\tfrac{1}{10} + \tfrac{1}{10} + \cdots (9 \text{ terms each } = \tfrac{1}{10})\right] + \left[\ 90 \text{ terms each } = \tfrac{1}{100}\right] + \cdots$$
$$= \tfrac{9}{10} + \tfrac{90}{100} + \cdots = \tfrac{9}{10} + \tfrac{9}{10} + \cdots.$$

The comparison test is really the basic test from which other tests are derived. It is probably the most useful test of all for the experienced mathematician but it is often hard to think of a satisfactory m series until you have had a good deal of experience with series. Consequently, you will probably not use it as often as the next three tests.

B. The Integral Test

We can use this test when the terms of the series are positive and not increasing, that is, when $a_{n+1} \le a_n$. (Again remember that we can ignore any finite number of terms of the series; thus the test can still be used even if the condition $a_{n+1} \le a_n$ does not hold for a finite number of terms.) To apply the test we think of a_n as a function of the variable n, and, forgetting our previous meaning of n, we allow it to take all values, not just integral ones. The test states that:

If $0 < a_{n+1} \leq a_n$ for $n > N$, then $\sum^{\infty} a_n$ converges if $\int^{\infty} a_n \, dn$ is finite and diverges if the integral is infinite. (The integral is to be evaluated *only* at the upper limit; no lower limit is needed.)

To understand this test, imagine a graph sketched of a_n as a function of n. For example, in testing the harmonic series $\sum_{n=1}^{\infty} 1/n$, we consider the graph of the function $y = 1/n$ (similar to Figures 6.1 and 6.2) letting n have all values, not just integral ones. Then the values of y on the graph at $n = 1, 2, 3, \cdots$, are the terms of the series. In Figures 6.1 and 6.2, the areas of the rectangles are just the terms of the series. Notice that in Figure 6.1 the top edge of each rectangle is above the curve, so that the area of the rectangles is greater than the corresponding area under the curve. On the other hand, in Figure 6.2 the rectangles lie below the curve, so their area is less than the corresponding area under the curve. Now the areas of the rectangles are just the terms of the series, and the area under the curve is an integral of $y \, dn$ or $a_n \, dn$. The upper limit on the integrals is ∞ and the lower limit could be made to correspond to any term of the series we wanted to start with. For example (see Figure 6.1), $\int_3^{\infty} a_n \, dn$ is less than the sum of the series from a_3 on, but (see Figure 6.2) greater than the sum of the series from a_4 on. If the integral is finite, then the sum of the series from a_4 on is finite, that is, the series converges. Note again that the terms at the beginning of a series have nothing to do with convergence. On the other hand, if the integral is infinite, then the sum of the series from a_3 on is infinite and the series diverges. Since the beginning terms are of no interest, you should simply evaluate $\int^{\infty} a_n \, dn$. (Also see Problem 16.)

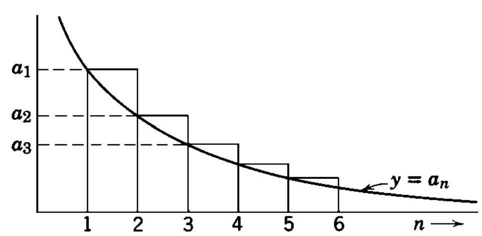


Figure 6.1

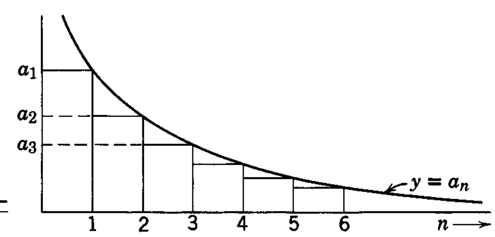


Figure 6.2

▶ **Example.** Test for convergence the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . \tag{6.1}$$

Using the integral test, we evaluate

$$\int^{\infty} \frac{1}{n} \, dn = \ln n \Big|^{\infty} = \infty.$$

(We use the symbol ln to mean a natural logarithm, that is, a logarithm to the base e.) Since the integral is infinite, the series diverges.

PROBLEMS, SECTION 6

Use the integral test to find whether the following series converge or diverge. *Hint* and *warning:* Do *not* use lower limits on your integrals (see Problem 16).

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ **8.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ **9.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-4}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+9}$ **11.** $\displaystyle\sum_{1}^{\infty} \frac{1}{n(1+\ln n)^{3/2}}$ **12.** $\displaystyle\sum_{1}^{\infty} \frac{n}{(n^2+1)^2}$

13. $\displaystyle\sum_{1}^{\infty} \frac{n^2}{n^3+1}$ **14.** $\displaystyle\sum_{1}^{\infty} \frac{1}{\sqrt{n^2+9}}$

15. Use the integral test to prove the following so-called p-series test. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } p > 1, \\ \text{divergent} & \text{if } p \le 1. \end{cases}$$

Caution: Do $p = 1$ separately.

16. In testing $\sum 1/n^2$ for convergence, a student evaluates $\int_0^\infty n^{-2} dn = -n^{-1}\Big|_0^\infty = 0 + \infty = \infty$ and concludes (erroneously) that the series diverges. What is wrong? *Hint:* Consider the area under the curve in a diagram such as Figure 6.1 or 6.2. This example shows the danger of using a lower limit in the integral test.

17. Use the integral test to show that $\sum_{n=0}^{\infty} e^{-n^2}$ converges. *Hint:* Although you cannot *evaluate* the integral, you can show that it is finite (which is all that is necessary) by comparing it with $\int^\infty e^{-n} dn$.

C. The Ratio Test

The integral test depends on your being able to integrate $a_n dn$; this is not always easy! We consider another test which will handle many cases in which we cannot evaluate the integral. Recall that in the geometric series each term could be obtained by multiplying the one before it by the ratio r, that is, $a_{n+1} = ra_n$ or $a_{n+1}/a_n = r$. For other series the ratio a_{n+1}/a_n is not constant but depends on n; let us call the absolute value of this ratio ρ_n. Let us also find the limit (if there is one) of the sequence ρ_n as $n \to \infty$ and call this limit ρ. Thus we define ρ_n and ρ by the equations

$$\rho_n = \left| \frac{a_{n+1}}{a_n} \right|, \qquad \rho = \lim_{n\to\infty} \rho_n. \tag{6.2}$$

If you recall that a geometric series converges if $|r| < 1$, it may seem plausible that a series with $\rho < 1$ should converge and this is true. This statement can be proved (Problem 30) by comparing the series to be tested with a geometric series. Like a geometric series with $|r| > 1$, a series with $\rho > 1$ also diverges (Problem 30). However, if $\rho = 1$, the ratio test does not tell us anything; some series with $\rho = 1$ converge and some diverge, so we must find another test (say one of the two preceding tests). To summarize the ratio test:

$$\text{If } \begin{cases} \rho < 1, & \text{the series converges;} \\ \rho = 1, & \text{use a different test;} \\ \rho > 1, & \text{the series diverges.} \end{cases} \tag{6.3}$$

▶ **Example 1.** Test for convergence the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$$

Using (6.2), we have

$$\begin{aligned} \rho_n &= \left| \frac{1}{(n+1)!} \div \frac{1}{n!} \right| \\ &= \frac{n!}{(n+1)!} = \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{(n+1)(n)(n-1)\cdots 3\cdot 2\cdot 1} = \frac{1}{n+1}, \\ \rho &= \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \frac{1}{n+1} = 0. \end{aligned}$$

Since $\rho < 1$, the series converges.

▶ **Example 2.** Test for convergence the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We find

$$\begin{aligned} \rho_n &= \left| \frac{1}{n+1} \div \frac{1}{n} \right| = \frac{n}{n+1}, \\ \rho &= \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1. \end{aligned}$$

Here the test tells us nothing and we must use some different test. A word of warning from this example: Notice that $\rho_n = n/(n+1)$ is always less than 1. Be careful not to confuse this ratio with ρ and conclude incorrectly that this series converges. (It is actually divergent as we proved by the integral test.) Remember that ρ is *not* the same as the ratio $\rho_n = |a_{n+1}/a_n|$, but is the *limit* of this ratio as $n \to \infty$.

▶ PROBLEMS, SECTION 6

Use the ratio test to find whether the following series converge or diverge:

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ **19.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^{2n}}$ **20.** $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n(n!)^2}{(2n)!}$ **22.** $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n!)^2}$ **23.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$ **25.** $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{\sqrt{n!}}$ **26.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{100^n}{n^{200}}$ **28.** $\displaystyle\sum_{n=0}^{\infty} \frac{n!(2n)!}{(3n)!}$ **29.** $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!}$

30. Prove the ratio test. *Hint:* If $|a_{n+1}/a_n| \to \rho < 1$, take σ so that $\rho < \sigma < 1$. Then $|a_{n+1}/a_n| < \sigma$ if n is large, say $n \geq N$. This means that we have $|a_{N+1}| < \sigma|a_N|$, $|a_{N+2}| < \sigma|a_{N+1}| < \sigma^2|a_N|$, and so on. Compare with the geometric series

$$\sum_{n=1}^{\infty} \sigma^n |a_N|.$$

Also prove that a series with $\rho > 1$ diverges. *Hint:* Take $\rho > \sigma > 1$, and use the preliminary test.

D. A Special Comparison Test

This test has two parts: (a) a convergence test, and (b) a divergence test. (See Problem 37.)

> (a) If $\sum_{n=1}^{\infty} b_n$ is a convergent series of positive terms and $a_n \geq 0$ and a_n/b_n tends to a (finite) limit, then $\sum_{n=1}^{\infty} a_n$ converges.
>
> (b) If $\sum_{n=1}^{\infty} d_n$ is a divergent series of positive terms and $a_n \geq 0$ and a_n/d_n tends to a limit greater than 0 (or tends to $+\infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.

There are really two steps in using either of these tests, namely, to decide on a comparison series, and then to compute the required limit. The first part is the most important; given a good comparison series it is a routine process to find the needed limit. The method of finding the comparison series is best shown by examples.

Example 1. Test for convergence

$$\sum_{n=3}^{\infty} \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}.$$

Remember that whether a series converges or diverges depends on what the terms are as n becomes larger and larger. We are interested in the nth term as $n \to \infty$. Think of $n = 10^{10}$ or 10^{100}, say; a little calculation should convince you that as n increases, $2n^2 - 5n + 1$ is $2n^2$ to quite high accuracy. Similarly, the denominator in our example is nearly $4n^3$ for large n. By Section 9, fact 1, we see that the factor $\sqrt{2}/4$ in every term does not affect convergence. So we consider as a comparison series just

$$\sum_{n=3}^{\infty} \frac{\sqrt{n^2}}{n^3} = \sum_{n=3}^{\infty} \frac{1}{n^2}$$

which we recognize (say by integral test) as a convergent series. Hence we use test (a) to try to show that the given series converges. We have:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2} \div \frac{1}{n^2}\right)$$
$$= \lim_{n\to\infty}\frac{n^2\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}$$
$$= \lim_{n\to\infty}\frac{\sqrt{2-\frac{5}{n}+\frac{1}{n^2}}}{4-\frac{7}{n}+\frac{2}{n^3}} = \frac{\sqrt{2}}{4}.$$

Since this is a finite limit, the given series converges. (With practice, you won't need to do all this algebra! You should be able to look at the original problem and see that, for large n, the terms are essentially $1/n^2$, so the series converges.)

▶ **Example 2.** Test for convergence

$$\sum_{n=2}^{\infty}\frac{3^n-n^3}{n^5-5n^2}.$$

Here we must first decide which is the important term as $n\to\infty$; is it 3^n or n^3? We can find out by comparing their logarithms since $\ln N$ and N increase or decrease together. We have $\ln 3^n = n\ln 3$, and $\ln n^3 = 3\ln n$. Now $\ln n$ is much smaller than n, so for large n we have $n\ln 3 > 3\ln n$, and $3^n > n^3$. (You might like to compute $100^3 = 10^6$, and $3^{100} > 5\times 10^{47}$.) The denominator of the given series is approximately n^5. Thus the comparison series is $\sum_{n=2}^{\infty} 3^n/n^5$. It is easy to prove this divergent by the ratio test. Now by test (b)

$$\lim_{n\to\infty}\left(\frac{3^n-n^3}{n^5-5n^2} \div \frac{3^n}{n^5}\right) = \lim_{n\to\infty}\frac{1-\frac{n^3}{3^n}}{1-\frac{5}{n^3}} = 1$$

which is greater than zero, so the given series diverges.

▶ PROBLEMS, SECTION 6

Use the special comparison test to find whether the following series converge or diverge.

31. $\displaystyle\sum_{n=9}^{\infty}\frac{(2n+1)(3n-5)}{\sqrt{n^2-73}}$

32. $\displaystyle\sum_{n=0}^{\infty}\frac{n(n+1)}{(n+2)^2(n+3)}$

33. $\displaystyle\sum_{n=5}^{\infty}\frac{1}{2^n-n^2}$

34. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+3n+4}{n^4+7n^3+6n-3}$

35. $\displaystyle\sum_{n=3}^{\infty}\frac{(n-\ln n)^2}{5n^4-3n^2+1}$

36. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^3+5n-1}}{n^2-\sin n^3}$

37. Prove the special comparison test. *Hint* (part a): If $a_n/b_n\to L$ and $M>L$, then $a_n < Mb_n$ for large n. Compare $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{\infty} Mb_n$.

7. ALTERNATING SERIES

So far we have been talking about series of positive terms (including series of absolute values). Now we want to consider one important case of a series whose terms have mixed signs. An *alternating series* is a series whose terms are alternately plus and minus; for example,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots \tag{7.1}$$

is an alternating series. We ask two questions about an alternating series. Does it converge? Does it converge absolutely (that is, when we make all signs positive)? Let us consider the second question first. In this example the series of absolute values

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$$

is the harmonic series (6.1), which diverges. We say that the series (7.1) is not absolutely convergent. Next we must ask whether (7.1) converges as it stands. If it had turned out to be absolutely convergent, we would not have to ask this question since an absolutely convergent series is also convergent (Problem 9). However, a series which is not absolutely convergent may converge or it may diverge; we must test it further. For alternating series the test is very simple:

Test for alternating series. An alternating series converges if the absolute value of the terms decreases steadily to zero, that is, if $|a_{n+1}| \le |a_n|$ and $\lim_{n\to\infty} a_n = 0$.

In our example $\dfrac{1}{n+1} < \dfrac{1}{n}$, and $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$, so (7.1) converges.

PROBLEMS, SECTION 7

Test the following series for convergence.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ **2.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ **3.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n}{n!}$ **5.** $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ **6.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+5}$

7. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n}{1+n^2}$ **8.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{10n}}{n+2}$

9. Prove that an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ is convergent. *Hint:* Put $b_n = a_n + |a_n|$. Then the b_n are nonnegative; we have $|b_n| \le 2|a_n|$ and $a_n = b_n - |a_n|$.

10. The following alternating series are divergent (but you are not asked to prove this). Show that $a_n \to 0$. Why doesn't the alternating series test prove (incorrectly) that these series converge?

(a) $2-\frac{1}{2}+\frac{2}{3}-\frac{1}{4}+\frac{2}{5}-\frac{1}{6}+\frac{2}{7}-\frac{1}{8}\cdots$

(b) $\frac{1}{\sqrt{2}}-\frac{1}{2}+\frac{1}{\sqrt{3}}-\frac{1}{3}+\frac{1}{\sqrt{4}}-\frac{1}{4}+\frac{1}{\sqrt{5}}-\frac{1}{5}\cdots$

8. CONDITIONALLY CONVERGENT SERIES

A series like (7.1) which converges, but does not converge absolutely, is called *conditionally convergent*. You have to use special care in handling conditionally convergent series because the positive terms alone form a divergent series and so do the negative terms alone. If you rearrange the terms, you will probably change the sum of the series, and you may even make it diverge! It is possible to rearrange the terms to make the sum any number you wish. Let us do this with the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. Suppose we want to make the sum equal to 1.5. First we take enough positive terms to add to just over 1.5. The first three positive terms do this:

$$1 + \frac{1}{3} + \frac{1}{5} = 1\frac{8}{15} > 1.5.$$

Then we take enough negative terms to bring the partial sum back under 1.5; the one term $-\frac{1}{2}$ does this. Again we add positive terms until we have a little more than 1.5, and so on. Since the terms of the series are decreasing in absolute value, we are able (as we continue this process) to get partial sums just a little more or a little less than 1.5 but always nearer and nearer to 1.5. But this is what convergence of the series to the sum 1.5 means: that the partial sums should approach 1.5. You should see that we could pick in advance *any* sum that we want, and rearrange the terms of this series to get it. Thus, we must not rearrange the terms of a conditionally convergent series since its convergence and its sum depend on the fact that the terms are added in a particular order.

Here is a physical example of such a series which emphasizes the care needed in applying mathematical approximations in physical problems. Coulomb's law in electricity says that the force between two charges is equal to the product of the charges divided by the square of the distance between them (in electrostatic units; to use other units, say SI, we need only multiply by a numerical constant). Suppose there are unit positive charges at $x = 0, \sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}, \cdots$, and unit negative charges at $x = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}, \cdots$. We want to know the total force acting on the unit positive charge at $x = 0$ due to all the other charges. The negative charges attract the charge at $x = 0$ and try to pull it to the right; we call the forces exerted by them positive, since they are in the direction of the positive x axis. The forces due to the positive charges are in the negative x direction, and we call them negative. For example, the force due to the positive charge at $x = \sqrt{2}$ is $-(1 \cdot 1)/\left(\sqrt{2}\right)^2 = -1/2$. The total force on the charge at $x = 0$ is, then,

$$F = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots. \tag{8.1}$$

Now we know that this series converges as it stands (Section 7). But we have also seen that its sum (even the fact that it converges) can be changed by rearranging the terms. Physically this means that the force on the charge at the origin depends not only on the size and position of the charges, but also on the *order* in which we place them in their positions! This may very well go strongly against your physical intuition. You feel that a physical problem like this should have a definite answer. Think of it this way. Suppose there are two crews of workers, one crew placing the positive charges and one placing the negative. If one crew works faster than the other, it is clear that the force at any stage may be far from the F of equation (8.1) because there are many extra charges of one sign. The crews can never place *all* the charges because there are an infinite number of them. At any stage the forces which would arise from the positive charges that are not yet in place, form a divergent series; similarly, the forces due to the unplaced negative charges form a divergent series of the opposite sign. We cannot then stop at some point and say that the rest of the series is negligible as we could in the bouncing ball problem in Section 1. But if we specify the *order* in which the charges are to be placed, then the sum S of the series is determined (S is probably different from F in (8.1) unless the charges are placed alternately). Physically this means that the value of the force as the crews proceed comes closer and closer to S, and we can use the sum of the (properly arranged) *infinite* series as a good approximation to the force.

9. USEFUL FACTS ABOUT SERIES

We state the following facts for reference:

1. The convergence or divergence of a series is not affected by multiplying every term of the series by the same nonzero constant. Neither is it affected by changing a finite number of terms (for example, omitting the first few terms).

2. Two convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ may be added (or subtracted) term by term. (Adding "term by term" means that the nth term of the sum is $a_n + b_n$.) The resulting series is convergent, and its sum is obtained by adding (subtracting) the sums of the two given series.

3. The terms of an *absolutely convergent series* may be rearranged in any order without affecting either the convergence or the sum. This is *not true* of conditionally convergent series as we have seen in Section 8.

PROBLEMS, SECTION 9

Test the following series for convergence or divergence. Decide for yourself which test is easiest to use, but don't forget the preliminary test. Use the facts stated above when they apply.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{(n+2)(n+3)}$ **2.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ **3.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln 3}}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+4}$ **5.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3-4}$ **6.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{3^n (n!)^2}$ **8.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{5^n}$ **9.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

10. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n-1}$ **11.** $\displaystyle\sum_{n=4}^{\infty} \frac{2n}{n^2-9}$ **12.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-n}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n^2+4)^{3/2}}$ **14.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-n}$ **15.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{10^n}$

16. $\displaystyle\sum_{n=0}^{\infty} \frac{2+(-1)^n}{n^2+7}$ **17.** $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ **18.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{\ln n}}$

19. $\dfrac{1}{2^2} - \dfrac{1}{3^2} + \dfrac{1}{2^3} - \dfrac{1}{3^3} + \dfrac{1}{2^4} - \dfrac{1}{3^4} + \cdots$

20. $\dfrac{1}{2} + \dfrac{1}{2^2} - \dfrac{1}{3} - \dfrac{1}{3^2} + \dfrac{1}{4} + \dfrac{1}{4^2} - \dfrac{1}{5} - \dfrac{1}{5^2} + \cdots$

21. $\displaystyle\sum_{n=1}^{\infty} a_n$ if $a_{n+1} = \dfrac{n}{2n+3} a_n$

22. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

(c) For what values of k is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k^{\ln n}}$ convergent?

10. POWER SERIES; INTERVAL OF CONVERGENCE

We have been discussing series whose terms were constants. Even more important and useful are series whose terms are functions of x. There are many such series, but in this chapter we shall consider series in which the nth term is a constant times x^n or a constant times $(x-a)^n$ where a is a constant. These are called *power series*, because the terms are multiples of powers of x or of $(x-a)$. In later chapters we shall consider Fourier series whose terms involve sines and cosines, and other series (Legendre, Bessel, etc.) in which the terms may be polynomials or other functions.

By definition, a power series is of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \text{or} \tag{10.1}$$
$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots ,$$

where the coefficients a_n are constants. Here are some examples:

$$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots + \frac{(-x)^n}{2^n} + \cdots , \tag{10.2a}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots , \tag{10.2b}$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \cdots , \tag{10.2c}$$

$$1 + \frac{(x+2)}{\sqrt{2}} + \frac{(x+2)^2}{\sqrt{3}} + \cdots + \frac{(x+2)^n}{\sqrt{n+1}} + \cdots . \tag{10.2d}$$

Whether a power series converges or not depends on the value of x we are considering. We often use the ratio test to find the values of x for which a series converges. We illustrate this by testing each of the four series (10.2). Recall that in the ratio test we divide term $n+1$ by term n and take the absolute value of this ratio to get ρ_n, and then take the limit of ρ_n as $n \to \infty$ to get ρ.

Example 1. For (10.2a), we have

$$\rho_n = \left| \frac{(-x)^{n+1}}{2^{n+1}} \div \frac{(-x)^n}{2^n} \right| = \left| \frac{x}{2} \right| ,$$
$$\rho = \left| \frac{x}{2} \right| .$$

The series converges for $\rho < 1$, that is, for $|x/2| < 1$ or $|x| < 2$, and it diverges for $|x| > 2$ (see Problem 6.30). Graphically we consider the interval on the x axis between $x = -2$ and $x = 2$; for any x in this interval the series (10.2a) converges. The endpoints of the interval, $x = 2$ and $x = -2$, must be considered separately. When $x = 2$, (10.2a) is

$$1 - 1 + 1 - 1 + \cdots ,$$

which is divergent; when $x = -2$, (10.2a) is $1 + 1 + 1 + 1 + \cdots$, which is divergent. Then the interval of convergence of (10.2a) is stated as $-2 < x < 2$.

Example 2. For (10.2b) we find

$$\rho_n = \left|\frac{x^{n+1}}{n+1} \div \frac{x^n}{n}\right| = \left|\frac{nx}{n+1}\right|,$$
$$\rho = \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = |x|.$$

The series converges for $|x| < 1$. Again we must consider the endpoints of the interval of convergence, $x = 1$ and $x = -1$. For $x = 1$, the series (10.2b) is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$; this is the alternating harmonic series and is convergent. For $x = -1$, (10.2b) is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$; this is the harmonic series (times -1) and is divergent. Then we state the interval of convergence of (10.2b) as $-1 < x \le 1$. Notice carefully how this differs from our result for (10.2a). Series (10.2a) did not converge at either endpoint and we used only $<$ signs in stating its interval of convergence. Series (10.2b) converges at $x = 1$, so we use the sign $\le$ to include $x = 1$. You must always test a series at its endpoints and include the results in your statement of the interval of convergence. A series may converge at neither, either one, or both of the endpoints.

Example 3. In (10.2c), the absolute value of the nth term is $|x^{2n-1}/(2n-1)!|$. To get term $n+1$ we replace n by $n+1$; then $2n-1$ is replaced by $2(n+1)-1 = 2n+1$, and the absolute value of term $n+1$ is

$$\left|\frac{x^{2n+1}}{(2n+1)!}\right|.$$

Thus we get

$$\rho_n = \left|\frac{x^{2n+1}}{(2n+1)!} \div \frac{x^{2n-1}}{(2n-1)!}\right| = \left|\frac{x^2}{(2n+1)(2n)}\right|,$$
$$\rho = \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n)}\right| = 0.$$

Since $\rho < 1$ for all values of x, this series converges for all x.

Example 4. In (10.2d), we find

$$\rho_n = \left|\frac{(x+2)^{n+1}}{\sqrt{n+2}} \div \frac{(x+2)^n}{\sqrt{n+1}}\right|,$$
$$\rho = \lim_{n\to\infty}\left|(x+2)\frac{\sqrt{n+1}}{\sqrt{n+2}}\right| = |x+2|.$$

The series converges for $|x+2| < 1$; that is, for $-1 < x+2 < 1$, or $-3 < x < -1$. If $x = -3$, (10.2d) is

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

which is convergent by the alternating series test. For $x = -1$, the series is

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

which is divergent by the integral test. Thus, the series converges for $-3 \le x < 1$.

▶ PROBLEMS, SECTION 10

Find the interval of convergence of each of the following power series; be sure to investigate the endpoints of the interval in each case.

1. $\sum_{n=0}^{\infty} (-1)^n x^n$

2. $\sum_{n=0}^{\infty} \frac{(2x)^n}{3^n}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+1)}$

4. $\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n^2}$

5. $\sum_{n=1}^{\infty} \frac{x^n}{(n!)^2}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

7. $\sum_{n=1}^{\infty} \frac{x^{3n}}{n}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$

9. $\sum_{n=1}^{\infty} (-1)^n n^3 x^n$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)^{3/2}}$

11. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{5}\right)^n$

12. $\sum_{n=1}^{\infty} n(-2x)^n$

13. $\sum_{n=1}^{\infty} \frac{n(-x)^n}{n^2+1}$

14. $\sum_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{x}{3}\right)^n$

15. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$

16. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n}$

17. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}$

18. $\sum_{n=1}^{\infty} \frac{(-2)^n (2x+1)^n}{n^2}$

The following series are *not* power series, but you can transform each one into a power series by a change of variable and so find out where it converges.

19. $\sum_0^\infty 8^{-n}(x^2-1)^n$ *Method:* Let $y = x^2 - 1$. The power series $\sum_0^\infty 8^{-n} y^n$ converges for $|y| < 8$, so the original series converges for $|x^2 - 1| < 8$, which means $|x| < 3$.

20. $\sum_0^{\infty} (-1)^n \frac{2^n}{n!}(x^2+1)^{2n}$

21. $\sum_2^{\infty} \frac{(-1)^n x^{n/2}}{n \ln n}$

22. $\sum_0^{\infty} \frac{n!(-1)^n}{x^n}$

23. $\sum_0^{\infty} \frac{3^n(n+1)}{(x+1)^n}$

24. $\sum_0^{\infty} \left(\sqrt{x^2+1}\right)^n \frac{2^n}{3^n+n^3}$

25. $\sum_0^{\infty} (\sin x)^n (-1)^n 2^n$

11. THEOREMS ABOUT POWER SERIES

We have seen that a power series $\sum_{n=0}^{\infty} a_n x^n$ converges in some interval with center at the origin. For each value of x (in the interval of convergence) the series has a finite sum whose value depends, of course, on the value of x. Thus we can write the sum of the series as $S(x) = \sum_{n=0}^{\infty} a_n x^n$. We see then that a power series (within its interval of convergence) defines a function of x, namely $S(x)$. In describing the relation of the series and the function $S(x)$, we may say that the series converges to the function $S(x)$, or that the function $S(x)$ is represented by the series, or that the series is the power series of the function. Here we have thought of obtaining the function from a given series. We shall also (Section 12) be interested in finding a power series that converges to a given function. When we are working with power series and the functions they represent, it is useful to know the following theorems (which we state without proof; see advanced calculus texts). Power series are very useful and convenient because within their interval of convergence they can be handled much like polynomials.

1. A power series may be differentiated or integrated term by term; the resulting series converges to the derivative or integral of the function represented by the original series within the same interval of convergence as the original series (that is, not necessarily at the endpoints of the interval).

2. Two power series may be added, subtracted, or multiplied; the resultant series converges at least in the common interval of convergence. You may divide two series if the denominator series is not zero at $x = 0$, or if it is and the zero is canceled by the numerator [as, for example, in $(\sin x)/x$; see (13.1)]. The resulting series will have *some* interval of convergence (which can be found by the ratio test or more simply by complex variable theory—see Chapter 2, Section 7).

3. One series may be substituted in another provided that the values of the substituted series are in the interval of convergence of the other series.

4. The power series of a function is unique, that is, there is just one power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which converges to a given function.

12. EXPANDING FUNCTIONS IN POWER SERIES

Very often in applied work, it is useful to find power series that represent given functions. We illustrate one method of obtaining such series by finding the series for $\sin x$. In this method we *assume* that there *is* such a series (see Section 14 for discussion of this point) and set out to find what the coefficients in the series must be. Thus we write

$$\sin x = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{12.1}$$

and try to find numerical values of the coefficients a_n to make (12.1) an identity (within the interval of convergence of the series). Since the interval of convergence of a power series contains the origin, (12.1) must hold when $x = 0$. If we substitute $x = 0$ into (12.1), we get $0 = a_0$ since $\sin 0 = 0$ and all the terms except a_0 on the right-hand side of the equation contain the factor x. Then to make (12.1) valid at $x = 0$, we must have $a_0 = 0$. Next we differentiate (12.1) term by term to get

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots . \tag{12.2}$$

(This is justified by Theorem 1 of Section 11.) Again putting $x = 0$, we get $1 = a_1$. We differentiate again, and put $x = 0$ to get

$$\begin{aligned} -\sin x &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots , \\ 0 &= 2a_2. \end{aligned} \tag{12.3}$$

Continuing the process of taking successive derivatives of (12.1) and putting $x = 0$, we get

$$\begin{aligned} -\cos x &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots , \\ -1 &= 3!\, a_3, \qquad a_3 = -\frac{1}{3!}; \\ \sin x &= 4 \cdot 3 \cdot 2 \cdot a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 x + \cdots , \\ 0 &= a_4; \\ \cos x &= 5 \cdot 4 \cdot 3 \cdot 2a_5 + \cdots , \\ 1 &= 5!\, a_5, \cdots . \end{aligned} \tag{12.4}$$

We substitute these values back into (12.1) and get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots . \tag{12.5}$$

You can probably see how to write more terms of this series without further computation. The $\sin x$ series converges for all x; see Example 3, Section 10.

Series obtained in this way are called *Maclaurin series* or *Taylor series about the origin*. A Taylor series in general means a series of powers of $(x - a)$, where a is some constant. It is found by writing $(x - a)$ instead of x on the right-hand side of an equation like (12.1), differentiating just as we have done, but substituting $x = a$ instead of $x = 0$ at each step. Let us carry out this process in general for a function $f(x)$. As above, we assume that there is a Taylor series for $f(x)$, and write

$$\begin{aligned} f(x) &= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots \\ &\qquad + a_n(x-a)^n + \cdots , \\ f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots \\ &\qquad + na_n(x-a)^{n-1} + \cdots , \\ f''(x) &= 2a_2 + 3 \cdot 2a_3(x-a) + 4 \cdot 3a_4(x-a)^2 + \cdots \\ &\qquad + n(n-1)a_n(x-a)^{n-2} + \cdots , \\ f'''(x) &= 3!\, a_3 + 4 \cdot 3 \cdot 2a_4(x-a) + \cdots \\ &\qquad + n(n-1)(n-2)a_n(x-a)^{n-3} + \cdots , \\ &\;\;\vdots \\ f^{(n)}(x) &= n(n-1)(n-2)\cdots 1 \cdot a_n + \text{terms containing powers of } (x-a). \end{aligned} \tag{12.6}$$

[The symbol $f^{(n)}(x)$ means the nth derivative of $f(x)$.] We now put $x = a$ in each equation of (12.6) and obtain

$$f(a) = a_0, \quad f'(a) = a_1, \quad f''(a) = 2a_2, \\ f'''(a) = 3!\,a_3, \quad \cdots, \quad f^{(n)}(a) = n!\,a_n. \tag{12.7}$$

[Remember that $f'(a)$ means to differentiate $f(x)$ and then put $x = a$; $f''(a)$ means to find $f''(x)$ and then put $x = a$, and so on.]

We can then write the Taylor series for $f(x)$ about $x = a$:

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \cdots. \tag{12.8}$$

The Maclaurin series for $f(x)$ is the Taylor series about the origin. Putting $a = 0$ in (12.8), we obtain the Maclaurin series for $f(x)$:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots. \tag{12.9}$$

We have written this in general because it is sometimes convenient to have the formulas for the coefficients. However, finding the higher order derivatives in (12.9) for any but the simplest functions is unnecessarily complicated (try it for, say, $e^{\tan x}$). In Section 13, we shall discuss much easier ways of getting Maclaurin and Taylor series by combining a few basic series. Meanwhile, you should verify (Problem 1, below) the basic series (13.1) to (13.5) and memorize them.

PROBLEMS, SECTION 12

1. By the method used to obtain (12.5) [which is the series (13.1) below], verify each of the other series (13.2) to (13.5) below.

13. TECHNIQUES FOR OBTAINING POWER SERIES EXPANSIONS

There are often simpler ways for finding the power series of a function than the successive differentiation process in Section 12. Theorem 4 in Section 11 tells us that for a given function there is *just one* power series, that is, just one series of the form $\sum_{n=0}^{\infty} a_n x^n$. Therefore we can obtain it by any correct method and be sure that it is the same Maclaurin series we would get by using the method of Section 12. We shall illustrate a variety of methods for obtaining power series. First of all, it is a great timesaver for you to verify (Problem 12.1) and then memorize the basic series (13.1) to (13.5). We shall use these series without further derivation when we need them.

convergent for

(13.1) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad \text{all } x;$$

(13.2) $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad \text{all } x;$$

(13.3) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \qquad \text{all } x;$$

(13.4)
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \le 1;$$

(13.5) $$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots, \qquad |x| < 1,$$

(binomial series; p is any real number, positive or negative and $\binom{p}{n}$ is called a binomial coefficient—see method C below.)

When we use a series to approximate a function, we may want only the first few terms, but in derivations, we may want the formula for the general term so that we can write the series in summation form. Let's look at some methods of obtaining either or both of these results.

A. Multiplying a Series by a Polynomial or by Another Series

▶ **Example 1.** To find the series for $(x+1)\sin x$, we multiply $(x+1)$ times the series (13.1) and collect terms:

$$\begin{aligned}(x+1)\sin x &= (x+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\\ &= x + x^2 - \frac{x^3}{3!} - \frac{x^4}{3!} + \cdots .\end{aligned}$$

You can see that this is easier to do than taking the successive derivatives of the product $(x+1)\sin x$, and Theorem 4 assures us that the results are the same.

Example 2. To find the series for $e^x \cos x$, we multiply (13.2) by (13.3):

$$\begin{aligned}
e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\\
&\qquad\quad - \frac{x^2}{2!} - \frac{x^3}{2!} - \frac{x^4}{2!\,2!} \cdots\\
&\qquad\qquad\qquad\qquad + \frac{x^4}{4!} \cdots\\
\hline
&= 1 + x + 0x^2 - \frac{x^3}{3} - \frac{x^4}{6} \cdots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} \cdots .
\end{aligned}$$

There are two points to note here. First, as you multiply, line up the terms involving each power of x in a column; this makes it easier to combine them. Second, be careful to include *all* the terms in the product out to the power you intend to stop with, but don't include *any* higher powers. In the above example, note that we did not include the $x^3 \cdot x^2$ terms; if we wanted the x^5 term in the answer, we would have to include *all* products giving x^5 (namely, $x \cdot x^4, x^3 \cdot x^2$, and $x^5 \cdot 1$).

Also see Chapter 2, Problem 17.30, for a simple way of getting the general term of this series.

B. Division of Two Series or of a Series by a Polynomial

Example 1. To find the series for $(1/x)\ln(1 + x)$, we divide (13.4) by x. You should be able to do this in your head and just write down the answer.

$$\frac{1}{x}\ln(1 + x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots .$$

To obtain the summation form, we again just divide (13.4) by x. We can simplify the result by changing the limits to start at $n = 0$, that is, replace n by $n + 1$.

$$\frac{1}{x}\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n-1}}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n + 1}.$$

▶ **Example 2.** To find the series for $\tan x$, we divide the series for $\sin x$ by the series for $\cos x$ by long division:

$$
\begin{array}{r|l}
 & x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 \cdots \\
\hline
1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \cdots & x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \cdots \\
 & x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} \cdots \\
\cline{2-2}
 & \qquad \dfrac{x^3}{3} - \dfrac{x^5}{30} \cdots \\
 & \qquad \dfrac{x^3}{3} - \dfrac{x^5}{6} \cdots \\
\cline{2-2}
 & \qquad\qquad \dfrac{2x^5}{15} \cdots \text{, etc.}
\end{array}
$$

C. Binomial Series

If you recall the binomial theorem, you may see that (13.5) looks just like the beginning of the binomial theorem for the expansion of $(a + b)^n$ if we put $a = 1$, $b = x$, and $n = p$. The difference here is that we allow p to be negative or fractional, and in these cases the expansion is an infinite series. The series converges for $|x| < 1$ as you can verify by the ratio test. (See Problem 1.)

From (13.5), we see that the binomial coefficients are:

$$
\begin{aligned}
\binom{p}{0} &= 1, \\
\binom{p}{1} &= p, \\
\binom{p}{2} &= \frac{p(p-1)}{2!}, \\
\binom{p}{3} &= \frac{p(p-1)(p-2)}{3!}, \cdots, \\
\binom{p}{n} &= \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}.
\end{aligned}
\tag{13.6}
$$

▶ **Example 1.** To find the series for $1/(1 + x)$, we use the binomial series (13.5) to write

$$
\begin{aligned}
\frac{1}{1+x} = (1+x)^{-1} &= 1 - x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\
&= 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-x)^n.
\end{aligned}
$$

Example 2. The series for $\sqrt{1+x}$ is (13.5) with $p = 1/2$.

$$\begin{aligned}\sqrt{1+x} = (1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n \\ &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 \cdots .\end{aligned}$$

From (13.6) we can see that the binomial coefficients when $n = 0$ and $n = 1$ are $\binom{1/2}{0} = 1$ and $\binom{1/2}{1} = 1/2$. For $n \geq 2$, we can write

$$\begin{aligned}\binom{\frac{1}{2}}{n} &= \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2} - n + 1)}{n!} = \frac{(-1)^{n-1} 3 \cdot 5 \cdot 7 \cdots (2n-3)}{n!\, 2^n} \\ &= \frac{(-1)^{n-1}(2n-3)!!}{(2n)!!}\end{aligned}$$

where the double factorial of an odd number means the product of that number times all smaller odd numbers, and a similar definition for even numbers. For example, $7!! = 7 \cdot 5 \cdot 3$, and $8!! = 8 \cdot 6 \cdot 4 \cdot 2$.

PROBLEMS, SECTION 13

1. Use the ratio test to show that a binomial series converges for $|x| < 1$.

2. Show that the binomial coefficients $\binom{-1}{n} = (-1)^n$.

3. Show that if p is a positive integer, then $\binom{p}{n} = 0$ when $n > p$, so $(1+x)^p = \sum \binom{p}{n} x^n$ is just a sum of $p+1$ terms, from $n = 0$ to $n = p$. For example, $(1+x)^2$ has 3 terms, $(1+x)^3$ has 4 terms, etc. This is just the familiar binomial theorem.

4. Write the Maclaurin series for $1/\sqrt{1+x}$ in $\sum$ form using the binomial coefficient notation. Then find a formula for the binomial coefficients in terms of n as we did in Example 2 above.

D. Substitution of a Polynomial or a Series for the Variable in Another Series

Example 1. Find the series for e^{-x^2}. Since we know the series (13.3) for e^x, we simply replace the x there by $-x^2$ to get

$$\begin{aligned}e^{-x^2} &= 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots \\ &= 1 - x^2 + \frac{(x^4)}{2!} - \frac{x^6}{3!} + \cdots .\end{aligned}$$

Example 2. Find the series for $e^{\tan x}$. Here we must replace the x in (13.3) by the series of Example 2 in method B. Let us agree in advance to keep terms only as far as x^4; we then write only terms which can give rise to powers of x up to 4, and neglect any higher powers:

$$
\begin{aligned}
e^{\tan x} &= 1 + \left(x + \frac{x^3}{3} + \cdots\right) + \frac{1}{2!}\left(x + \frac{x^3}{3} + \cdots\right)^2 \\
&\quad + \frac{1}{3!}\left(x + \frac{x^3}{3} + \cdots\right)^3 + \frac{1}{4!}(x + \cdots)^4 + \cdots \\
&= 1 + x \qquad + \frac{x^3}{3} \qquad + \cdots \\
&\qquad + \frac{x^2}{2!} \qquad + \frac{2x^4}{3\cdot 2!} + \cdots \\
&\qquad\qquad + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\hline
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3}{8}x^4 + \cdots .
\end{aligned}
$$

E. Combination of Methods

▶ **Example.** Find the series for $\arctan x$. Since

$$\int_0^x \frac{dt}{1+t^2} = \arctan t\Big|_0^x = \arctan x,$$

we first write out (as a binomial series) $(1+t^2)^{-1}$ and then integrate term by term:

$$
\begin{aligned}
&(1+t^2)^{-1} = 1 - t^2 + t^4 - t^6 + \cdots ; \\
&\int_0^x \frac{dt}{1+t^2} = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots \Big|_0^x .
\end{aligned}
$$

Thus, we have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \tag{13.7}$$

Compare this simple way of getting the series with the method in Section 12 of finding successive derivatives of $\arctan x$.

F. Taylor Series Using the Basic Maclaurin Series

In many simple cases it is possible to obtain a Taylor series using the basic memorized Maclaurin series instead of the formulas or method of Section 12.

▶ **Example 1.** Find the first few terms of the Taylor series for $\ln x$ about $x = 1$. [This means a series of powers of $(x-1)$ rather than powers of x.] We write

$$\ln x = \ln[1 + (x-1)]$$

and use (13.4) with x replaced by $(x-1)$:

$$\ln x = \ln[1 + (x-1)] = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \cdots .$$

Example 2. Expand $\cos x$ about $x = 3\pi/2$. We write

$$\cos x = \cos\left[\frac{3\pi}{2} + \left(x - \frac{3\pi}{2}\right)\right] = \sin\left(x - \frac{3\pi}{2}\right)$$
$$= \left(x - \frac{3\pi}{2}\right) - \frac{1}{3!}\left(x - \frac{3\pi}{2}\right)^3 + \frac{1}{5!}\left(x - \frac{3\pi}{2}\right)^5 \cdots$$

using (13.1) with x replaced by $(x - 3\pi/2)$.

G. Using a Computer

You can also do problems like these using a computer. This is a good method for complicated functions where it saves you a lot of algebra. However, you're not saving time if it takes longer to type a problem into the computer than to do it in your head! For example, you should be able to just write down the first few terms of $(\sin x)/x$ or $(1 - \cos x)/x^2$. A good method of study is to practice doing problems by hand and also check your results using the computer. This will turn up errors you are making by hand, and also let you discover what the computer will do and what it won't do! It is very illuminating to computer plot the function you are expanding, along with several partial sums of the series, in order to see how accurately the partial sums represent the function—see the following example.

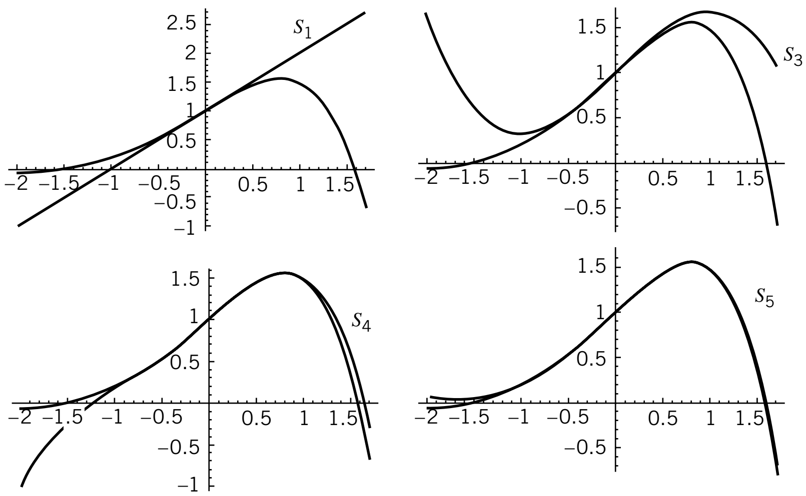


Figure 13.1

Example. Plot the function $e^x \cos x$ together with several partial sums of its Maclaurin series. Using Example 2 in 13A or a computer, we have

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} \cdots .$$

Figure 13.1 shows plots of the function along with each of the partial sums $S_1 = 1+x$, $S_3 = 1 + x - \frac{x^3}{3}$, $S_4 = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$, $S_5 = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30}$. We can see from the graphs the values of x for which an approximation is fairly good. Also see Section 14.

PROBLEMS, SECTION 13

Using the methods of this section:

(a) Find the first few terms of the Maclaurin series for each of the following functions.

(b) Find the general term and write the series in summation form.

(c) Check your results in (a) by computer.

(d) Use a computer to plot the function and several approximating partial sums of the series.

5. $x^2 \ln(1-x)$ **6.** $x\sqrt{1+x}$ **7.** $\dfrac{1}{x}\sin x$

8. $\dfrac{1}{\sqrt{1-x^2}}$ **9.** $\dfrac{1+x}{1-x}$ **10.** $\sin x^2$

11. $\dfrac{\sin\sqrt{x}}{\sqrt{x}}, \quad x > 0$ **12.** $\displaystyle\int_0^x \cos t^2\, dt$ **13.** $\displaystyle\int_0^x e^{-t^2}\, dt$

14. $\ln\sqrt{\dfrac{1+x}{1-x}} = \displaystyle\int_0^x \frac{dt}{1-t^2}$ **15.** $\arcsin x = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$

16. $\cosh x = \dfrac{e^x + e^{-x}}{2}$ **17.** $\ln\dfrac{1+x}{1-x}$

18. $\displaystyle\int_0^x \frac{\sin t\, dt}{t}$ **19.** $\ln(x+\sqrt{1+x^2}\,) = \displaystyle\int_0^x \frac{dt}{\sqrt{1+t^2}}$

Find the first few terms of the Maclaurin series for each of the following functions and check your results by computer.

20. $e^x \sin x$ **21.** $\tan^2 x$ **22.** $\dfrac{e^x}{1-x}$

23. $\dfrac{1}{1+x+x^2}$ **24.** $\sec x = \dfrac{1}{\cos x}$ **25.** $\dfrac{2x}{e^{2x}-1}$

26. $\dfrac{1}{\sqrt{\cos x}}$ **27.** $e^{\sin x}$ **28.** $\sin[\ln(1+x)]$

29. $\sqrt{1+\ln(1+x)}$ **30.** $\sqrt{\dfrac{1-x}{1+x}}$ **31.** $\cos(e^x - 1)$

32. $\ln(1+xe^x)$ **33.** $\dfrac{1-\sin x}{1-x}$ **34.** $\ln(2-e^{-x})$

35. $\dfrac{x}{\sin x}$ **36.** $\displaystyle\int_0^u \frac{\sin x\, dx}{\sqrt{1-x^2}}$

37. $\ln\cos x$ *Hints:* Method 1: Write $\cos x = 1 + (\cos x - 1) = 1 + u$; use the series you know for $\ln(1+u)$; replace u by the Maclaurin series for $(\cos x - 1)$. Method 2: $\ln\cos x = -\int_0^x \tan u\, du$. Use the series of Example 2 in method B.

38. $e^{\cos x}$ *Hint:* $e^{\cos x} = e \cdot e^{\cos x - 1}$.

Using method F above, find the first few terms of the Taylor series for the following functions about the given points.

39. $f(x) = \sin x, \quad a = \pi/2$ **40.** $f(x) = \dfrac{1}{x}, \quad a = 1$

41. $f(x) = e^x, \quad a = 3$ **42.** $f(x) = \cos x, \quad a = \pi$

43. $f(x) = \cot x, \quad a = \pi/2$ **44.** $f(x) = \sqrt{x} \quad a = 25$

14. ACCURACY OF SERIES APPROXIMATIONS

The thoughtful student might well be disturbed about the mathematical manipulations we have been doing. How do we know whether these processes we have shown really give us series that approximate the functions being expanded? Certainly *some* functions cannot be expanded in a power series; since a power series becomes just a_0 when $x = 0$, it cannot be equal to any function (like $1/x$ or $\ln x$) which is infinite at the origin. So we might ask whether there are other functions (besides those that become infinite at the origin) which cannot be expanded in a power series. All we have done so far is to show methods of finding the power series for a function *if it has one*. Now is there a chance that there might be some functions which do not have series expansions, but for which our formal methods would give us a spurious series? Unfortunately, the answer is "Yes"; fortunately, this is not a very common difficulty in practice. However, you should know of the possibility and what to do about it. You may first think of the fact that, say, the equation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

is not valid for $|x| \geq 1$. This is a fairly easy restriction to determine; from the beginning we recognized that we could use our series expansions only when they converged. But there is another difficulty which can arise. It is possible for a series found by the above methods to converge and still not represent the function being expanded! A simple example of this is $e^{-(1/x^2)}$ for which the formal series is $0+0+0+\cdots$ because $e^{-(1/x^2)}$ and all its derivatives are zero at the origin (Problem 15.26). It is clear that $e^{-(1/x^2)}$ is not zero for $x^2 > 0$, so the series is certainly not correct. You can startle your friends with the following physical interpretation of this. Suppose that at $t = 0$ a car is at rest (zero velocity), and has zero acceleration, zero rate of change of acceleration, etc. (all derivatives of distance with respect to time are zero at $t = 0$). Then according to Newton's second law (force equals mass times acceleration), the instantaneous force acting on the car is also zero (and, in fact, so are all the derivatives of the force). Now we ask "Is it possible for the car to be moving immediately after $t = 0$?" The answer is "Yes"! For example, let its distance from the origin as a function of time be $e^{-(1/t^2)}$.

This strange behavior is really the fault of the function itself and not of our method of finding series. The most satisfactory way of avoiding the difficulty is to recognize, by complex variable theory, when functions can or cannot have power series. We shall consider this in Chapter 14, Section 2. Meanwhile, let us consider two important questions: (1) Does the Taylor or Maclaurin series in (12.8) or (12.9) actually converge to the function being expanded? (2) In a computation problem, if we know that a series converges to a given function, how rapidly does it converge? That is, how many terms must we use to get the accuracy we require? We take up these questions in order.

The *remainder* $R_n(x)$ in a Taylor series is the difference between the value of the function and the sum of $n + 1$ terms of the series:

$$R_n(x) = f(x) - \left[f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a)\right]. \tag{14.1}$$

Saying that the series converges to the function means that $\lim_{n\to\infty} |R_n(x)| = 0$. There are many different formulas for $R_n(x)$ which are useful for special purposes; you can find these in calculus books. One such formula is

$$R_n(x) = \frac{(x-a)^{n+1} f^{(n+1)}(c)}{(n+1)!} \tag{14.2}$$

where c is some point between a and x. You can use this formula in some simple cases to prove that the Taylor or Maclaurin series for a function does converge to the function (Problems 11 to 13).

Error in Series Approximations Now suppose that we know in advance that the power series of a function does converge to the function (within the interval of convergence), and we want to use a series approximation for the function. We would like to estimate the error caused by using only a few terms of the series.

There is an easy way to estimate this error when the series is alternating and meets the alternating series test for convergence (Section 7). In this case the error is (in absolute value) less than the absolute value of the first neglected term (see Problem 1).

(14.3) If $S = \sum_{n=1}^{\infty} a_n$ is an alternating series with $|a_{n+1}| < |a_n|$, and $\lim_{n\to\infty} a_n = 0$, then $|S - (a_1 + a_2 + \cdots + a_n)| \le |a_{n+1}|$.

▶ **Example 1.** Consider the series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} \cdots .$$

The sum of this series [see (1.8), $a = 1, r = -\frac{1}{2}$] is $S = \frac{2}{3} = 0.666\cdots$. The sum of the terms through $-\frac{1}{32}$ is 0.656+, which differs from S by about 0.01. This is less than the next term $= \frac{1}{64} = 0.015+$.

Estimating the error by the first neglected term may be quite misleading for convergent series that are not alternating.

▶ **Example 2.** Suppose we approximate $\sum_{n=1}^{\infty} 1/n^2$ by the sum of the first five terms; the error is then about 0.18 [see problem 2(a)]. But the first neglected term is $1/6^2 = 0.028$ which is much less than the error. However, note that we are finding the sum of the power series $\sum_{n=1}^{\infty} x^n/n^2$ when $x = 1$, which is the largest x for which the series converges. If, instead, we ask for the sum of the series when $x = 1/2$, we find [see Problem 2(b)]:

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{1}{2}\right)^n = 0.5822+ .$$

The sum of the first five terms of the series is 0.5815+, so the error is about 0.0007. The next term is $(\frac{1}{6})^2/6^2 = 0.0004$, which is less than the error but still of the same order of magnitude. We can state the following theorem [Problem 2(c)] which covers many practical problems.

$$\text{If } S = \sum_{n=0}^{\infty} a_n x^n \text{ converges for } |x| < 1, \text{ and if}$$
$$|a_{n+1}| < |a_n| \text{ for } n > N, \text{ then} \tag{14.4}$$
$$\left| S - \sum_{n=0}^{N} a_n x^n \right| < |a_{N+1} x^{N+1}| \div (1 - |x|).$$

That is, as in (14.3), the error may be estimated by the first neglected term, but here the error may be a few times as large as the first neglected term instead of smaller. In the example of $\sum x^n/n^2$ with $x = \frac{1}{2}$, we have $1 - x = \frac{1}{2}$, so (14.4) says that the error is less than two times the next term. We observe that the error 0.0007 is less than 2(0.0004) as (14.4) says.

For values of $|x|$ much less than 1, $1 - |x|$ is about 1, so the next term gives a good error estimate in this case. If the interval of convergence is not $|x| < 1$, but, for example, $|x| < 2$ as in

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{x}{2}\right)^n,$$

we can easily let $x/2 = y$, and apply the theorem in terms of y.

PROBLEMS, SECTION 14

1. Prove theorem (14.3). *Hint:* Group the terms in the error as $(a_{n+1} + a_{n+2}) + (a_{n+3} + a_{n+4}) + \cdots$ to show that the error has the same sign as a_{n+1}. Then group them as $a_{n+1} + (a_{n+2} + a_{n+3}) + (a_{n+4} + a_{n+5}) + \cdots$ to show that the error has magnitude less than $|a_{n+1}|$.

2. (a) Using computer or tables (or see Chapter 7, Section 11), verify that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6 = 1.6449+$, and also verify that the error in approximating the sum of the series by the first five terms is approximately 0.1813.

(b) By computer or tables verify that $\sum_{n=1}^{\infty}(1/n^2)(1/2)^n = \pi^2/12 - (1/2)(\ln 2)^2 = 0.5822+$, and that the sum of the first five terms is 0.5815+.

(c) Prove theorem (14.4). *Hint:* The error is $|\sum_{N+1}^{\infty} a_n x^n|$. Use the fact that the absolute value of a sum is less than or equal to the sum of the absolute values. Then use the fact that $|a_{n+1}| \le |a_n|$ to replace all a_n by a_{N+1}, and write the appropriate inequality. Sum the geometric series to get the result.

In Problems 3 to 7, assume that the Maclaurin series converges to the function.

3. If $0 < x < \frac{1}{2}$, show [using theorem (14.3)] that $\sqrt{1+x} = 1 + \frac{1}{2}x$ with an error less than 0.032. *Hint:* Note that the series is alternating after the first term.

4. Show that $\sin x = x$ with an error less than 0.021 for $0 < x < \frac{1}{2}$, and with an error less than 0.0002 for $0 < x < 0.1$. *Hint:* Use theorem (14.3) and note that the "next" term is the x^3 term.

5. Show that $1 - \cos x = x^2/2$ with an error less than 0.003 for $|x| < \frac{1}{2}$.

6. Show that $\ln(1-x) = -x$ with an error less than 0.0056 for $|x| < 0.1$. *Hint:* Use theorem (14.4).

7. Show that $2/\sqrt{4-x} = 1 + \frac{1}{8}x$ with an error less than $\frac{1}{32}$ for $0 < x < 1$. *Hint:* Let $x = 4y$, and use theorem (14.4).

8. Estimate the error if $\sum_{n=1}^{\infty} x^n/n^3$ is approximated by the sum of its first three terms for $|x| < \frac{1}{2}$.

9. Consider the series in Problem 4.6 and show that the remainder after n terms is $R_n = 1/(n+1)$. Compare the value of term $n+1$ with R_n for $n = 3, n = 10, n = 100, n = 500$ to see that the first neglected term is not a useful estimate of the error.

10. Show that the interval of convergence of the series $\sum_{n=1}^{\infty} x^n/(n^2+n)$ is $|x| \le 1$. (For $x = 1$, this is the series of Problem 9.) Using theorem (14.4), show that for $x = \frac{1}{2}$, four terms will give two decimal place accuracy.

11. Show that the Maclaurin series for $\sin x$ converges to $\sin x$. *Hint:* If $f(x) = \sin x$, $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, and so $|f^{(n+1)}(x)| \le 1$ for all x and all n. Let $n \to \infty$ in (14.2).

12. Show as in Problem 11 that the Maclaurin series for e^x converges to e^x.

13. Show that the Maclaurin series for $(1+x)^p$ converges to $(1+x)^p$ when $0 < x < 1$.

15. SOME USES OF SERIES

In this chapter we are going to consider a few rather straightforward uses of series. In later chapters there will also be many other cases where we need them.

Numerical Computation With computers and calculators so available, you may wonder why we would ever want to use series for numerical computation. Here is an example to warn you of the pitfalls of blind computation.

Example 1. Evaluate $f(x) = \ln\sqrt{(1+x)/(1-x)} - \tan x$ at $x = 0.0015$.

Here are answers from several calculators and computers: -9×10^{-16}, 3×10^{-10}, 6.06×10^{-16}, 5.5×10^{-16}. All of these are wrong! Let's use series to see what's going on. By Section 13 methods we find, for $x = 0.0015$:

$$\begin{aligned}
\ln\sqrt{(1+x)/(1-x)} &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7}\cdots &&= 0.0015000011250015187524 41,\\
\tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}\cdots &&= 0.0015000011250010125009 22,\\
f(x) &= \frac{x^5}{15} + \frac{4x^7}{45}\cdots &&= 5.0625 \times 10^{-16}
\end{aligned}$$

with an error of the order of x^7 or 10^{-21}. Now we see that the answer is the difference of two numbers which are identical until the 16th decimal place, so any computer carrying fewer digits will lose all accuracy in the subtraction. It may also be necessary to tell your computer that the value of x is an exact number and not a 4 decimal place approximation. The moral here is that a computer is a tool—a very useful tool, yes—but you need to be constantly aware of whether an answer is reasonable when you are doing problems either by hand or by computer. A final point is that in an applied problem you may want, not a numerical value, but a simple approximation for a complicated function. Here we might approximate $f(x)$ by $x^5/15$ for small x.

Example 2. Evaluate

$$\frac{d^5}{dx^5}\left(\frac{1}{x}\sin x^2\right)\bigg|_{x=0}.$$

We can do this by computer, but it's probably faster to use $\sin x^2 = x^2-(x^2)^3/3!\cdots$, and observe that when we divide this by x and take 5 derivatives, the x^2 term is gone. The second term divided by x is an x^5 term and the fifth derivative of x^5 is $5!$. Any further terms will have a power of x which is zero at $x=0$. Thus we have

$$\frac{d^5}{dx^5}\left(\frac{1}{x}\cdot\frac{-(x^2)^3}{3!}\right)_{x=0} = -\frac{5!}{3!} = -20.$$

Summing series We have seen a few numerical series which we could sum exactly (see Sections 1 and 4) and we will see some others later (see Chapter 7, Section 11). Here it is interesting to note that if $f(x)=\sum a_n x^n$, and we let x have a particular value (within the interval of convergence), then we get a numerical series whose sum is the value of the function for that x. For example, if we substitute $x=1$ in (13.4), we get

$$\ln(1+1)=\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$$

so the sum of the alternating harmonic series is $\ln 2$.

We can also find sums of series from tables or computer, either the exact sum if that is known, or a numerical approximation (see Problems 20 to 22, and also Problems 14.2, 16.1, 16.30, and 16.31).

Integrals By Theorem 1 of Section 11, we may integrate a power series term by term. Then we can find an approximation for an integral when the indefinite integral cannot be found in terms of elementary functions. As an example, consider the Fresnel integrals (integrals of $\sin x^2$ and $\cos x^2$) which occur in the problem of Fresnel diffraction in optics. We find

$$\begin{aligned}\int_0^t \sin x^2\,dx &= \int_0^t\left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\cdots\right)dx\\ &= \frac{t^3}{3}-\frac{t^7}{7\cdot 3!}+\frac{t^{11}}{11\cdot 5!}-\cdots\end{aligned}$$

so for $t<1$, the integral is approximately $\frac{t^3}{3}-\frac{t^7}{42}$ with an error < 0.00076 since this is an alternating series (see (14.3)).

Evaluation of Indeterminate Forms Suppose we want to find

$$\lim_{x\to 0}\frac{1-e^x}{x}.$$

If we try to substitute $x=0$, we get $0/0$. Expressions that lead us to such meaningless results when we substitute are called indeterminate forms. You can evaluate these by computer, but simple ones can often be done quickly by series. For example,

$$\begin{aligned}\lim_{x\to 0}\frac{1-e^x}{x} &= \lim_{x\to 0}\frac{1-(1+x+(x^2/2!)+\cdots)}{x}\\ &= \lim_{x\to 0}\left(-1-\frac{x}{2!}-\cdots\right) = -1.\end{aligned}$$

You may recall L'Hôpital's rule which says that

$$\lim_{x\to a}\frac{f(x)}{\phi(x)} = \lim_{x\to a}\frac{f'(x)}{\phi'(x)}$$

when $f(a)$ and $\phi(a)$ are both zero, and f'/ϕ' approaches a limit or tends to infinity (that is, does not oscillate) as $x \to a$. Let's use power series to see why this is true. We consider functions $f(x)$ and $\phi(x)$ which are expandable in a Taylor series about $x = a$, and assume that $\phi'(a) \neq 0$. Using (12.8), we have

$$\lim_{x\to a}\frac{f(x)}{\phi(x)} = \lim_{x\to a}\frac{f(a) + (x-a)f'(a) + (x-a)^2 f''(a)/2! + \cdots}{\phi(a) + (x-a)\phi'(a) + (x-a)^2\phi''(a)/2! + \cdots}.$$

If $f(a) = 0$ and $\phi(a) = 0$, and we cancel one $(x - a)$ factor, this becomes

$$\lim_{x\to a}\frac{f'(a) + (x-a)f''(a)/2! + \cdots}{\phi'(a) + (x-a)\phi''(a)/2! + \cdots} = \frac{f'(a)}{\phi'(a)} = \lim_{x\to a}\frac{f'(x)}{\phi'(x)}$$

as L'Hôpital's rule says. If $f'(a) = 0$ and $\phi'(a) = 0$, and $\phi''(a) \neq 0$, then a repetition of the rule gives the limit as $f''(a)/\phi''(a)$, and so on.

There are other indeterminate forms besides 0/0, for example, ∞/∞, $0 \cdot \infty$, etc. L'Hôpital's rule holds for the ∞/∞ form as well as the 0/0 form. Series are most useful for the 0/0 form or others which can easily be put into the 0/0 form. For example, the limit $\lim_{x\to 0}(1/x)\sin x$ is an $\infty \cdot 0$ form, but is easily written as $\lim_{x\to 0}(\sin x)/x$ which is a 0/0 form. Also *note carefully:* Series (of powers of x) are useful mainly in finding limits as $x \to 0$, because for $x = 0$ such a series collapses to the constant term; for any other value of x we have an infinite series whose sum we probably do not know (see Problem 25, however).

Series Approximations When a problem in, say, differential equations or physics is too difficult in its exact form, we often can get an approximate answer by replacing one or more of the functions in the problem by a few terms of its infinite series. We shall illustrate this idea by two examples.

▶ **Example 3.** In elementary physics we find that the equation of motion of a simple pendulum is (see Chapter 11, Section 8, or a physics textbook):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

This differential equation cannot be solved for θ in terms of elementary functions (see Chapter 11, Section 8), and you may recall that what is usually done is to approximate $\sin\theta$ by θ. Recall the infinite series (13.1) for $\sin\theta$; θ is simply the first term of the series for $\sin\theta$. (Remember that θ is in radians; see discussion in Chapter 2, end of Section 3.) For small values of θ (say $\theta < \frac{1}{2}$ radian or about 30°), this series converges rapidly, and using the first term gives a good approximation (see Problem 14.4). The solutions of the differential equation are then $\theta = A\sin\sqrt{g/l}\,t$ and $\theta = B\cos\sqrt{g/l}\,t$ (A and B constants) as you can verify; we say that the pendulum is executing simple harmonic motion (see Chapter 7, Section 2).

Example 4. Let us consider a radioactive substance containing N_0 atoms at $t = 0$. It is known that the number of atoms remaining at a later time t is given by the formula (see Chapter 8, Section 3):

$$N = N_0 e^{-\lambda t} \tag{15.1}$$

where λ is a constant which is characteristic of the radioactive substance. To find λ for a given substance, a physicist measures in the laboratory the number of decays ΔN during the time interval Δt for a succession of Δt intervals. It is customary to plot each value of $\Delta N/\Delta t$ at the midpoint of the corresponding time interval Δt. If $\lambda\,\Delta t$ is small, this graph is a good approximation to the exact dN/dt graph. A better approximation can be obtained by plotting $\Delta N/\Delta t$ a little to the left of the midpoint. Let us show that the midpoint *does* give a good approximation and also find the more accurate t value. (An approximate value of λ, good enough for calculating the correction, is assumed known from a rough preliminary graph.)

What we should *like* to plot is the graph of dN/dt, that is, the graph of the slope of the curve in Figure 15.1. What we *measure* is the value of $\Delta N/\Delta t$ for each Δt interval. Consider one such Δt interval in Figure 15.1, from t_1 to t_2. To get an accurate graph we should plot the measured value of the quotient $\Delta N/\Delta t$ at the point between t_1 and t_2 where $\Delta N/\Delta t = dN/dt$. Let us write this condition and find the t which satisfies it. The quantity ΔN is the change in N, that is, $N(t_2) - N(t_1)$; the value of dN/dt we get from (15.1). Then $dN/dt = \Delta N/\Delta t$ becomes

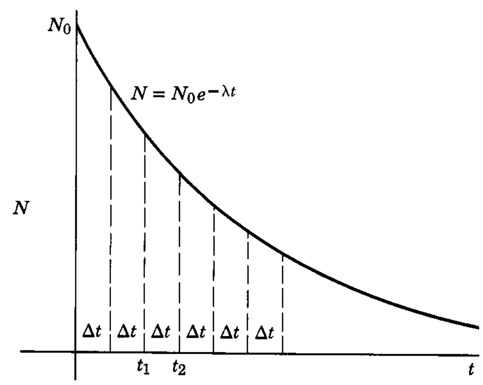


Figure 15.1

$$-\lambda N_0 e^{-\lambda t} = \frac{N_0 e^{-\lambda t_2} - N_0 e^{-\lambda t_1}}{\Delta t}. \tag{15.2}$$

Multiplying this equation by $(\Delta t/N_0)e^{\lambda(t_1+t_2)/2}$, we get

$$-\lambda\,\Delta t\, e^{-\lambda[t-(t_1+t_2)/2]} = e^{-\lambda(t_2-t_1)/2} - e^{\lambda(t_2-t_1)/2} = e^{-\lambda\,\Delta t/2} - e^{\lambda\,\Delta t/2} \tag{15.3}$$

since $t_2 - t_1 = \Delta t$. Since we assumed $\lambda\,\Delta t$ to be small, we can expand the exponentials on the right-hand side of (15.3) in power series; this gives

$$-\lambda\,\Delta t\, e^{-\lambda[t-(t_1+t_2)/2]} = -\lambda\,\Delta t - \frac{1}{3}\left(\frac{\lambda\,\Delta t}{2}\right)^3 \cdots \tag{15.4}$$

or, canceling $(-\lambda\,\Delta t)$,

$$e^{-\lambda[t-(t_1+t_2)/2]} = 1 + \frac{1}{24}(\lambda\,\Delta t)^2 \cdots. \tag{15.5}$$

Suppose $\lambda\,\Delta t$ is small enough so that we can neglect the term $\frac{1}{24}(\lambda\,\Delta t)^2$. Then

(15.5) reduces to

$$
\begin{aligned}
e^{-\lambda[t-(t_1+t_2)/2]} &= 1, \\
-\lambda\left(t - \frac{t_1+t_2}{2}\right) &= 0, \\
t &= \frac{t_1+t_2}{2}.
\end{aligned}
$$

Thus we have justified the usual practice of plotting $\Delta N/\Delta t$ at the midpoint of the interval Δt.

Next consider a more accurate approximation. From (15.5) we get

$$-\lambda\left(t - \frac{t_1+t_2}{2}\right) = \ln\left(1 + \frac{1}{24}(\lambda\,\Delta t)^2\cdots\right).$$

Since $\frac{1}{24}(\lambda\,\Delta t)^2 \ll 1$, we can expand the logarithm by (13.4) to get

$$-\lambda\left(t - \frac{t_1+t_2}{2}\right) = \frac{1}{24}(\lambda\,\Delta t)^2\cdots.$$

Then we have

$$t = \frac{t_1+t_2}{2} - \frac{1}{24\lambda}(\lambda\,\Delta t)^2\cdots.$$

Thus the measured $\Delta N/\Delta t$ should be plotted a little to the left of the midpoint of Δt, as we claimed.

PROBLEMS, SECTION 15

In Problems 1 to 4, use power series to evaluate the function at the given point. Compare with computer results, using the computer to find the series, and also to do the problem without series. Resolve any disagreement in results (see Example 1).

1. $e^{\arcsin x} + \ln\left(\dfrac{1-x}{e}\right)$ at $x = 0.0003$

2. $\dfrac{1}{\sqrt{1+x^4}} - \cos x^2$ at $x = 0.012$

3. $\ln\left(x + \sqrt{1+x^2}\right) - \sin x$ at $x = 0.001$

4. $e^{\sin x} - (1/x^3)\ln(1 + x^3 e^x)$ at $x = 0.00035$

Use Maclaurin series to evaluate each of the following. Although you could do them by computer, you can probably do them in your head faster than you can type them into the computer. So use these to practice quick and skillful use of basic series to make simple calculations.

5. $\dfrac{d^4}{dx^4}\ln(1+x^3)$ at $x = 0$

6. $\dfrac{d^3}{dx^3}\left(\dfrac{x^2 e^x}{1-x}\right)$ at $x = 0$

7. $\dfrac{d^{10}}{dx^{10}}(x^8 \tan^2 x)$ at $x = 0$

8. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ **9.** $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$ **10.** $\lim_{x\to 0} \dfrac{1-e^{x^3}}{x^3}$

11. $\lim_{x\to 0} \dfrac{\sin^2 2x}{x^2}$ **12.** $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$ **13.** $\lim_{x\to 0} \dfrac{\ln(1-x)}{x}$

Find a two term approximation for each of the following integrals and an error bound for the given t interval.

14. $\int_0^t e^{-x^2}\,dx, \quad 0<t<0.1$ **15.** $\int_0^t \sqrt{x}\,e^{-x}\,dx, \quad 0<t<0.01$

Find the sum of each of the following series by recognizing it as the Maclaurin series for a function evaluated at a point.

16. $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$ **17.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!}\left(\dfrac{\pi}{2}\right)^{2n}$

18. $\sum_{n=1}^{\infty} \dfrac{1}{n\,2^n}$ **19.** $\sum_{n=0}^{\infty} \dbinom{-1/2}{n}\left(-\dfrac{1}{2}\right)^n$

20. By computer or tables, find the exact sum of each of the following series.

(a) $\sum_{n=1}^{\infty} \dfrac{n}{(4n^2-1)^2}$ (b) $\sum_{n=1}^{\infty} \dfrac{n^3}{n!}$ (c) $\sum_{n=1}^{\infty} \dfrac{n(n+1)}{3^n}$

21. By computer, find a numerical approximation for the sum of each of the following series.

(a) $\sum_{n=1}^{\infty} \dfrac{n}{(n^2+1)^2}$ (b) $\sum_{n=2}^{\infty} \dfrac{\ln n}{n^2}$ (c) $\sum_{n=1}^{\infty} \dfrac{1}{n^n}$

22. The series $\sum_{n=1}^{\infty} 1/n^s$, $s>1$, is called the Riemann Zeta function, $\zeta(s)$. (In Problem 14.2(a) you found $\zeta(2)=\pi^2/6$. When n is an even integer, these series can be summed exactly in terms of π.) By computer or tables, find

(a) $\zeta(4)=\sum_{n=1}^{\infty} \dfrac{1}{n^4}$ (b) $\zeta(3)=\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ (c) $\zeta\left(\dfrac{3}{2}\right)=\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$

23. Find the following limits using Maclaurin series and check your results by computer. *Hint:* First combine the fractions. Then find the first term of the denominator series and the first term of the numerator series.

(a) $\lim_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$ (b) $\lim_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{\cos x}{\sin^2 x}\right)$

(c) $\lim_{x\to 0}\left(\csc^2 x-\dfrac{1}{x^2}\right)$ (d) $\lim_{x\to 0}\left(\dfrac{\ln(1+x)}{x^2}-\dfrac{1}{x}\right)$

24. Evaluate the following indeterminate forms by using L'Hôpital's rule and check your results by computer. (Note that Maclaurin series would not be useful here because x does not tend to zero, or because a function ($\ln x$, for example) is not expandable in a Maclaurin series.)

(a) $\lim_{x\to\pi} \dfrac{x\sin x}{x-\pi}$ (b) $\lim_{x\to\pi/2} \dfrac{\ln(2-\sin x)}{\ln(1+\cos x)}$

(c) $\lim_{x\to 1} \dfrac{\ln(2-x)}{x-1}$ (d) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

(e) $\lim_{x\to 0} x\ln 2x$

(f) $\lim_{x\to\infty} x^n e^{-x}$ (n not necessarily integral)

25. In general, we do not expect Maclaurin series to be useful in evaluating indeterminate forms except when x tends to zero (see Problem 24). Show, however, that Problem 24(f) can be done by writing $x^n e^{-x} = x^n/e^x$ and using the series (13.3) for e^x. *Hint:* Divide numerator and denominator by x^n before you take the limit. What is special about the e^x series which makes it possible to know what the limit of the infinite series is?

26. Find the values of several derivatives of e^{-1/t^2} at $t = 0$. *Hint:* Calculate a few derivatives (as functions of t); then make the substitution $x = 1/t^2$, and use the result of Problem 24(f) or 25.

27. The velocity v of electrons from a high energy accelerator is very near the velocity c of light. Given the voltage V of the accelerator, we often want to calculate the ratio v/c. The relativistic formula for this calculation is (approximately, for $V \gg 1$)

$$\frac{v}{c} = \sqrt{1 - \left(\frac{0.511}{V}\right)^2}, \quad V = \text{number of million volts.}$$

Use two terms of the binomial series (13.5) to find $1 - v/c$ in terms of V. Use your result to find $1 - v/c$ for the following values of V. *Caution:* $V =$ the number of *million* volts.

(a) $V = 100$ million volts

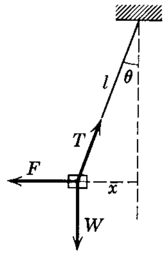
(b) $V = 500$ million volts

(c) $V = 25,000$ million volts

(d) $V = 100$ gigavolts (100×10^9 volts $= 10^5$ million volts)

28. The energy of an electron at speed v in special relativity theory is $mc^2(1-v^2/c^2)^{-1/2}$, where m is the electron mass, and c is the speed of light. The factor mc^2 is called the rest mass energy (energy when $v = 0$). Find two terms of the series expansion of $(1 - v^2/c^2)^{-1/2}$, and multiply by mc^2 to get the energy at speed v. What is the second term in the energy series? (If v/c is very small, the rest of the series can be neglected; this is true for everyday speeds.)

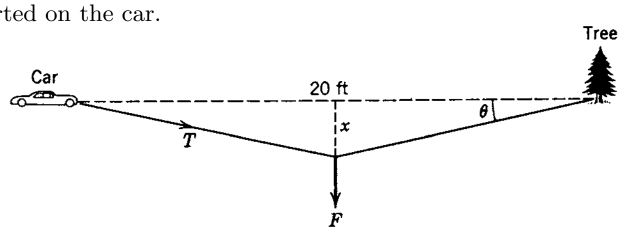
29. The figure shows a heavy weight suspended by a cable and pulled to one side by a force F. We want to know how much force F is required to hold the weight in equilibrium at a given distance x to one side (say to place a cornerstone correctly). From elementary physics, $T\cos\theta = W$, and $T\sin\theta = F$.



(a) Find F/W as a series of powers of θ.

(b) Usually in a problem like this, what we know is not θ, but x and l in the diagram. Find F/W as a series of powers of x/l.

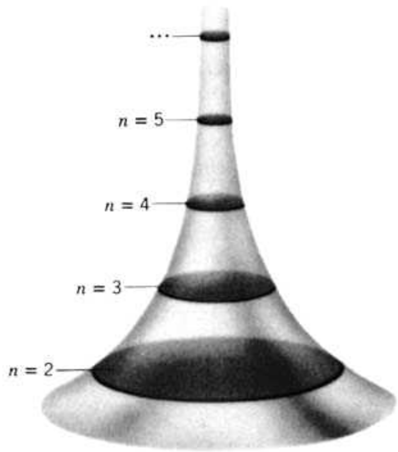
30. Given a strong chain and a convenient tree, could you pull your car out of a ditch in the following way? Fasten the chain to the car and to the tree. Pull with a force F at the center of the chain as shown in the figure. From mechanics, we have $F = 2T\sin\theta$, or $T = F/(2\sin\theta)$, where T is the tension in the chain, that is, the force exerted on the car.

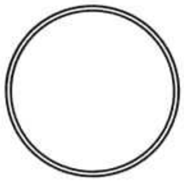


(a) Find T as x^{-1} times a series of powers of x.

(b) Find T as θ^{-1} times a series of powers of θ.

31. A tall tower of circular cross section is reinforced by horizontal circular disks (like large coins), one meter apart and of negligible thickness. The radius of the disk at height n is $1/(n \ln n)$ $(n \geq 2)$.





Assuming that the tower is of infinite height:

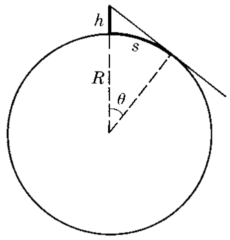
(a) Will the total area of the disks be finite or not? *Hint:* Can you compare the series with a simpler one?

(b) If the disks are strengthened by wires going around their circumferences like tires, will the total length of wire required be finite or not?

(c) Explain why there is not a contradiction between your answers in (a) and (b). That is, how is it possible to start with a set of disks of finite area, remove a little strip around the circumference of each, and get an infinite total length of these strips? *Hint:* Think about units—you can't compare area and length. Consider two cases: (1) Make the width of each strip equal to one percent of the radius of the disk from which you cut it. Now the total length is infinite but what about the total area? (2) Try to make the strips all the same width; what happens? Also see Chapter 5, Problem 3.31(b).

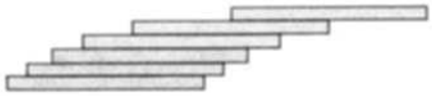
32. Show that the "doubling time" (time for your money to double) is n periods at interest rate $i\%$ per period with $ni = 69$, approximately. Show that the error in the approximation is less than 10% if $i\% \leq 20\%$. (Note that n does not have to be the number of years; it can be the number of months with $i =$ interest rate per month, etc.) *Hint:* You want $(1 + i/100)^n = 2$; take ln of both sides of this equation and use equation (13.4). Also see theorem (14.3).

33. If you are at the top of a tower of height h above the surface of the earth, show that the distance you can see along the surface of the earth is approximately $s = \sqrt{2Rh}$, where R is the radius of the earth. *Hints:* See figure. Show that $h/R = \sec\theta - 1$; find two terms of the series for $\sec\theta = 1/\cos\theta$, and use $s = R\theta$. Thus show that the distance in miles is approximately $\sqrt{3h/2}$ with h in feet.



16. MISCELLANEOUS PROBLEMS

1. (a) Show that it is possible to stack a pile of identical books so that the top book is as far as you like to the right of the bottom book. Start at the top and each time place the pile already completed on top of another book so that the pile is just at the point of tipping. (In practice, of course, you can't let them overhang quite this much without having the stack topple. Try it with

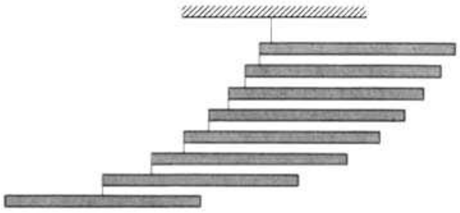


a deck of cards.) Find the distance from the right-hand end of each book to the right-hand end of the one beneath it. To find a general formula for this distance, consider the three forces acting on book n, and write the equation for the torque about its right-hand end. Show that the sum of these setbacks is a divergent series (proportional to the harmonic series). [See "Leaning Tower of *The Physical Reviews,"* Am. J. Phys. **27**, 121–122 (1959).]

(b) By computer, find the sum of N terms of the harmonic series with $N = 25$, 100, 200, 1000, 10^6, 10^{100}.

(c) From the diagram in (a), you can see that with 5 books (count down from the top) the top book is completely to the right of the bottom book, that is, the overhang is slightly over one book. Use your series in (a) to verify this. Then using parts (a) and (b) and a computer as needed, find the number of books needed for an overhang of 2 books, 3 books, 10 books, 100 books.

2. The picture is a mobile constructed of dowels (or soda straws) connected by thin threads. Each thread goes from the left-hand end of a rod to a point on the rod below. Number the rods from the bottom and find, for rod n, the distance from its left end to the thread so that all rods of the mobile will be horizontal. *Hint:* Can you see the relation between this problem and Problem 1?



3. Show that $\sum_{n=2}^{\infty} 1/n^{3/2}$ is convergent. What is wrong with the following "proof" that it diverges?

$$\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{27}} + \frac{1}{\sqrt{64}} + \frac{1}{\sqrt{125}} + \cdots > \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} + \frac{1}{\sqrt{81}} + \frac{1}{\sqrt{144}} + \cdots$$

which is

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \cdots = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right).$$

Since the harmonic series diverges, the original series diverges. *Hint:* Compare $3n$ and $n\sqrt{n}$.

Test for convergence:

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(n-1)^2}{1+n^2}$

6. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n-1}}{(n+1)^2-1}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{2n^3}{n^4-2}$

Find the interval of convergence, including end-point tests:

9. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$ **10.** $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$ **11.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2n-1}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n n^2}{5^n(n^2+1)}$ **13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{(-3)^n\sqrt{n}}$

Find the Maclaurin series for the following functions.

14. $\cos[\ln(1+x)]$ **15.** $\ln\left(\dfrac{\sin x}{x}\right)$ **16.** $\dfrac{1}{\sqrt{1+\sin x}}$

17. $e^{1-\sqrt{1-x^2}}$ **18.** $\arctan x = \displaystyle\int_0^x \frac{du}{1+u^2}$

Find the first few terms of the Taylor series for the following functions about the given points.

19. $\sin x,\ a=\pi$ **20.** $\sqrt[3]{x},\ a=8$ **21.** $e^x,\ a=1$

Use series you know to show that:

22. $1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots=\dfrac{\pi}{4}.$ *Hint:* See Problem 18.

23. $\dfrac{\pi^2}{3!}-\dfrac{\pi^4}{5!}+\dfrac{\pi^6}{7!}-\cdots=1$ **24.** $\ln 3+\dfrac{(\ln 3)^2}{2!}+\dfrac{(\ln 3)^3}{3!}+\cdots=2$

25. Evaluate the limit $\lim_{x\to 0} x^2/\ln\cos x$ by series (in your head), by L'Hôpital's rule, and by computer.

Use Maclaurin series to do Problems 26 to 29 and check your results by computer.

26. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{1-\cos^2 x}\right)$ **27.** $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$

28. $\displaystyle\lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\sin x}\right)$ **29.** $\displaystyle\left.\frac{d^6}{dx^6}(x^4 e^{x^2})\right|_{x=0}$

30. (a) It is clear that you (or your computer) can't find the sum of an infinite series just by adding up the terms one by one. For example, to get $\zeta(1.1)=\sum_{n=1}^{\infty} 1/n^{1.1}$ (see Problem 15.22) with error < 0.005 takes about 10^{33} terms. To see a simple alternative (for a series of positive decreasing terms) look at Figures 6.1 and 6.2. Show that when you have summed N terms, the sum R_N of the rest of the series is between $I_N=\int_N^\infty a_n\,dn$ and $I_{N+1}=\int_{N+1}^\infty a_n\,dn$.

(b) Find the integrals in (a) for the $\zeta(1.1)$ series and verify the claimed number of terms needed for error < 0.005. *Hint*: Find N such that $I_N = 0.005$. Also find upper and lower bounds for $\zeta(1.1)$ by computing $\sum_{n=1}^{N} 1/n^{1.1}+\int_N^\infty n^{-1.1}\,dn$ and $\sum_{n=1}^{N} 1/n^{1.1}+\int_{N+1}^\infty n^{-1.1}\,dn$ where N is far less than 10^{33}. *Hint*: You want the difference between the upper and lower limits to be about 0.005; find N so that term $a_N = 0.005$.

31. As in Problem 30, for each of the following series, find the number of terms required to find the sum with error < 0.005, and find upper and lower bounds for the sum using a much smaller number of terms.

(a) $\displaystyle\sum_{1}^{\infty}\frac{1}{n^{1.01}}$ (b) $\displaystyle\sum_{1}^{\infty}\frac{1}{n(1+\ln n)^2}$ (c) $\displaystyle\sum_{3}^{\infty}\frac{1}{n\ln n(\ln\ln n)^2}$