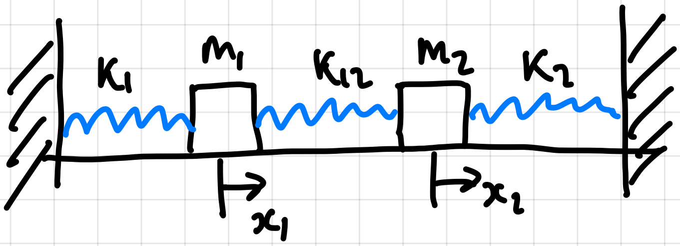
Mechanics and Oscillations

Week 2

Problem Sheet answers

2.1)



$m_1 = m_2 = m$

$K_1 = K_{12} = K$

$K_2 = 2K$

a) Two normal modes "±" frequencies are:

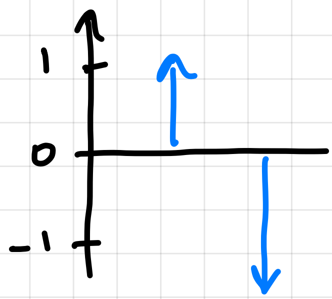
$$\omega_+ = \sqrt{\frac{5+\sqrt{5}}{2}}\sqrt{\frac{K}{m}} \qquad \omega_- = \sqrt{\frac{5-\sqrt{5}}{2}}\sqrt{\frac{K}{m}}$$

Normal mode amplitude vectors:

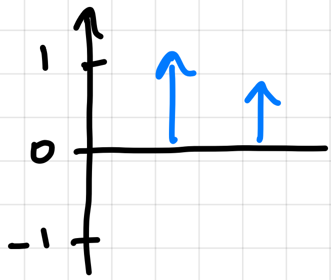
$$\underline{x}_+ = \begin{pmatrix} 1 \\ -\frac{(1+\sqrt{5})}{2} \end{pmatrix} \qquad \underline{x}_- = \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$$

-1.62 $\qquad$ 0.62

"+"



"−"



b) $m = 1\ \text{kg} \quad K = 1\ \text{N/m} \quad \Rightarrow \sqrt{\frac{K}{m}} = 1\ s^{-1}$

$$\underline{x}(t) = a_+ \cos\left(\sqrt{\frac{5+\sqrt{5}}{2}}\,t + \varphi_+\right)\begin{pmatrix} 1 \\ -\frac{(1+\sqrt{5})}{2} \end{pmatrix}$$

$$+\ a_- \cos\left(\sqrt{\frac{5-\sqrt{5}}{2}}\,t + \varphi_-\right)\begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$$

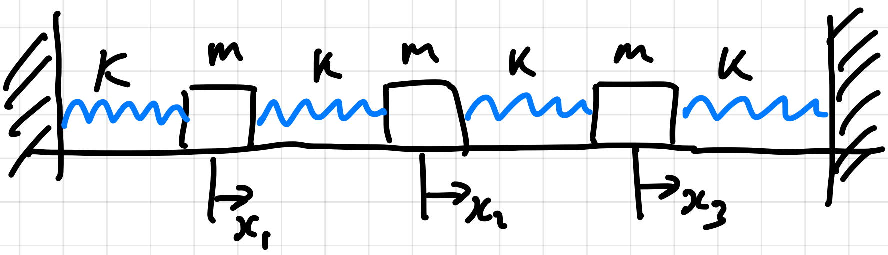
(c) Using given initial conditions:

$$\underline{x}(t) = \frac{\sqrt{5}+1}{2\sqrt{5}} \cos\left(\sqrt{\frac{5+\sqrt{5}}{2}}\, t\right) \begin{pmatrix} 1 \\ -\frac{(1+\sqrt{5})}{2} \end{pmatrix}$$

$$+ \frac{\sqrt{5}-1}{2\sqrt{5}} \cos\left(\sqrt{\frac{5-\sqrt{5}}{2}}\, t\right) \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix} [m]$$

with time t [s].

2.2)



a)

$$m\frac{d^2}{dt^2}x_1 = -Kx_1 - K(x_1 - x_2)$$

$$m\frac{d^2}{dt^2}x_2 = -K(x_2 - x_1) - K(x_2 - x_3)$$

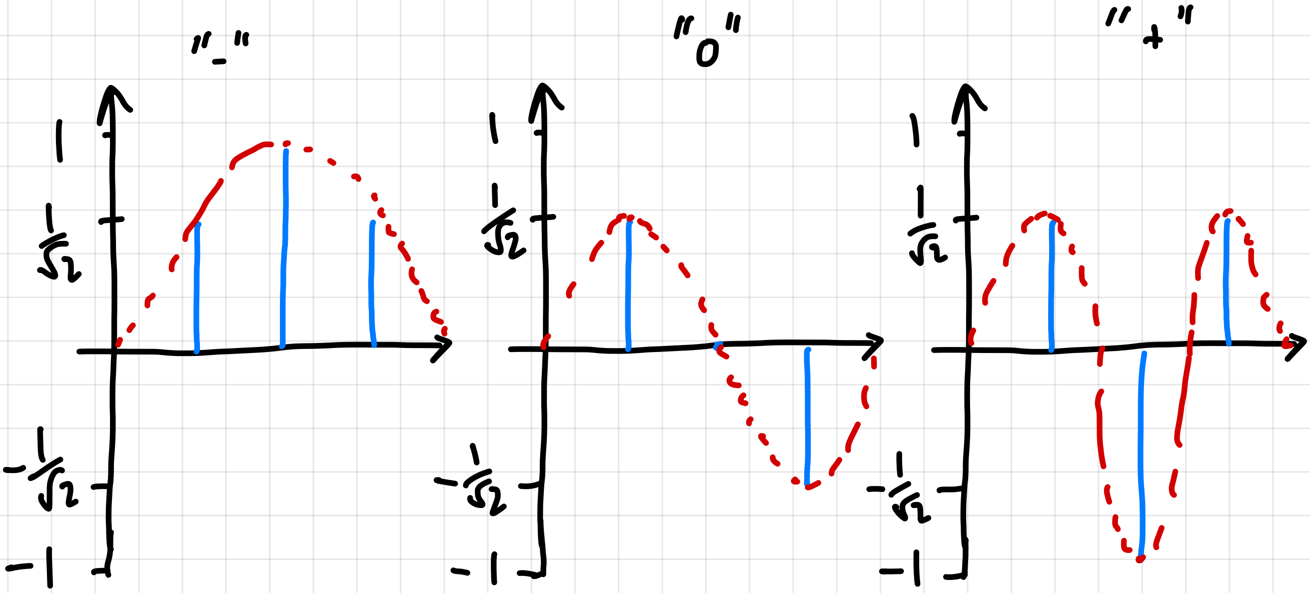
$$m\frac{d^2}{dt^2}x_3 = -Kx_3 - K(x_3 - x_2)$$

b) Three normal mode angular frequencies:

$$\omega_- = \sqrt{2-\sqrt{2}}\sqrt{\frac{K}{m}} \qquad \omega_0 = \sqrt{\frac{2K}{m}} \qquad \omega_+ = \sqrt{2+\sqrt{2}}\sqrt{\frac{K}{m}}$$

Associated amplitude vectors:

$$x_- = \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix} \qquad x_0 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \qquad x_+ = \begin{pmatrix} 1/\sqrt{2} \\ -1 \\ 1/\sqrt{2} \end{pmatrix}$$



$$\omega_- = \sqrt{2-\sqrt{2}}\,\sqrt{\frac{k}{m}} \qquad \omega_0 = \sqrt{2}\,\sqrt{\frac{k}{m}} \qquad \omega_+ = \sqrt{2+\sqrt{2}}\,\sqrt{\frac{k}{m}}$$

c)

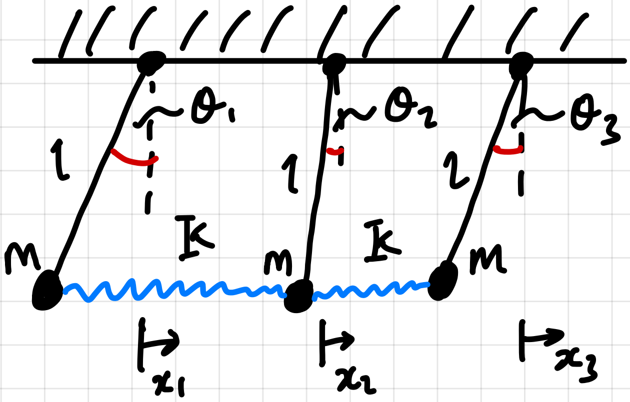
$$\underline{x}(t) = \frac{a_-}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2-\sqrt{2}}\sqrt{\frac{k}{m}}\,t + \varphi_-\right)$$

$$+ \frac{a_0}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos\left(\sqrt{2}\sqrt{\frac{k}{m}}\,t + \varphi_0\right)$$

$$+ \frac{a_+}{\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2+\sqrt{2}}\sqrt{\frac{k}{m}}\,t + \varphi_+\right)$$

2.3)

a)

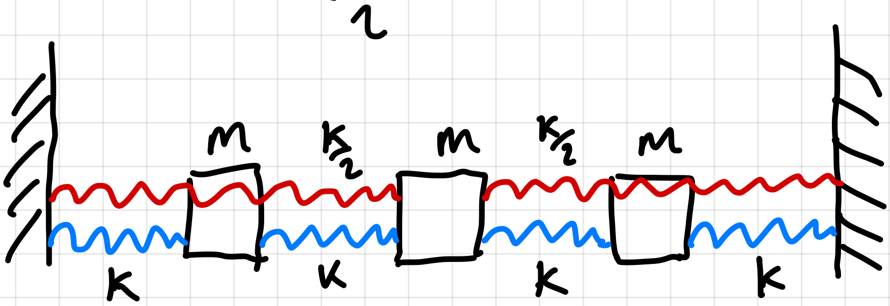


(1) $$m\frac{d^2x_1}{dt^2} = -\frac{mg}{l}x_1 - K(x_1 - x_2)$$

(2) $$m\frac{d^2x_2}{dt^2} = -\frac{mg}{l}x_2 - K(x_2 - x_1) - K(x_2 - x_3)$$

(3) $$m\frac{d^2x_3}{dt^2} = -\frac{mg}{l}x_3 - K(x_3 - x_2)$$

b) Equivalent to $\frac{mg}{l} = K$



c) (1) $$m\frac{d^2x_1}{dt^2} = -Kx_1 - K(x_1 - x_2)$$

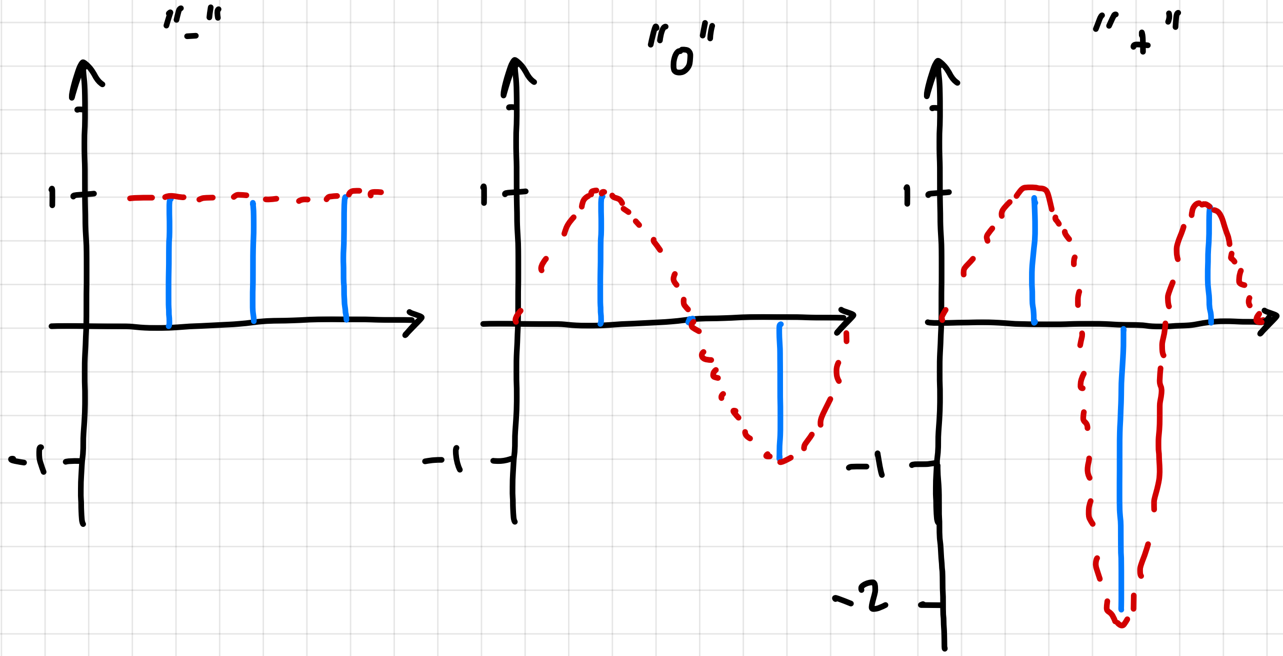
(2) $$m\frac{d^2x_2}{dt^2} = -Kx_2 - K(x_2 - x_1) - K(x_2 - x_3)$$

(3) $$m\frac{d^2x_3}{dt^2} = -Kx_3 - K(x_3 - x_2)$$

d) Normal modes:

$$\omega_- = \sqrt{\frac{K}{m}} \qquad \omega_0 = \sqrt{\frac{2K}{m}} \qquad \omega_+ = 2\sqrt{\frac{K}{m}}$$

$$\underline{x}_- = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \underline{x}_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \underline{x}_+ = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$



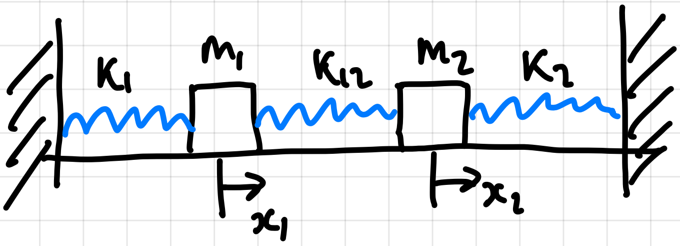
$$\omega_- = \sqrt{\frac{K}{m}} \qquad \omega_0 = \sqrt{2}\sqrt{\frac{K}{m}} \qquad \omega_+ = 2\sqrt{\frac{K}{m}}$$

e) $$\underline{x}(t) = a_- \cos\left(\sqrt{\frac{K}{m}}\, t + \varphi_-\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+ \; a_0 \cos\left(\sqrt{\frac{2K}{m}}\, t + \varphi_0\right)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$+ \; a_+ \cos\left(\sqrt{\frac{4K}{m}}\, t + \varphi_+\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

2.4)



$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\frac{d^2}{dt^2} \underline{x}(t) = - \underline{\underline{D}}\, \underline{x}(t) \qquad \underline{\underline{D}} = \begin{bmatrix} \frac{K_1 + K_{12}}{m_1} & -\frac{K_{12}}{m_1} \\ -\frac{K_{12}}{m_2} & \frac{K_2 + K_{12}}{m_2} \end{bmatrix}$$

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underline{\underline{R}}\, \underline{x} \qquad \underline{\underline{R}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

hence

$$\underline{x} = \underline{\underline{R}}^{-1} \underline{y}$$

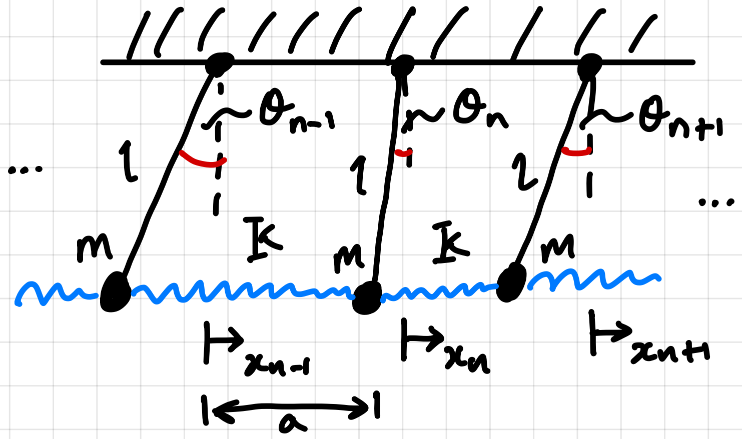
$$\underline{\underline{R}}^{-1} \frac{d^2}{dt^2} \underline{y} = - \underline{\underline{D}}\, \underline{\underline{R}}^{-1} \underline{y}$$

$$\Rightarrow \quad \ddot{\underline{y}} = -\left(\underline{\underline{R}}\, \underline{\underline{D}}\, \underline{\underline{R}}^{-1} \right) \underline{y} \overset{!}{=} - \underline{\underline{P}}\, \underline{y}$$

with $\underline{\underline{P}} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ diagonal $\equiv$ decoupled

$$\therefore \quad \underline{\underline{P}} = \underline{\underline{R}}\, \underline{\underline{D}}\, \underline{\underline{R}}^{-1}$$ diagnalisation of $\underline{\underline{D}}$

2.5) a)



$$m\frac{d^2x_n}{dt^2} = -\frac{mg}{l}x_n - K(x_n - x_{n-1}) - K(x_n - x_{n+1})$$

b)

$$x_n(t) = A\cos(\omega_p(\lambda)t + \varphi)\sin\left(\frac{2\pi}{\lambda}an + \theta\right)$$

so long as λ and ω are related as:

$$\omega_p(\lambda) = \sqrt{\frac{4K}{m}\sin^2\left(\frac{\pi a}{\lambda}\right) + \frac{g}{l}} \qquad p = \text{pendulums}$$

c)

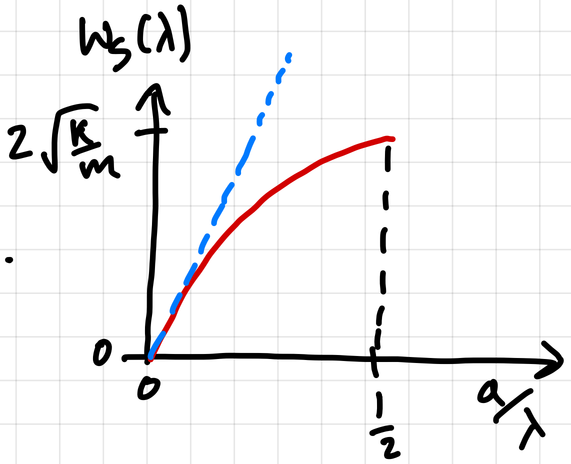
$$\omega_s = 2\sin\left(\frac{\pi a}{\lambda}\right)\sqrt{\frac{K}{m}} \qquad s = \text{springs}$$

$$\lim_{\lambda\to\infty} \omega_s(\lambda) = 0 \quad \text{but} \quad \lim_{\lambda\to\infty} \omega_p(\lambda) = \sqrt{\frac{g}{l}}$$

d)

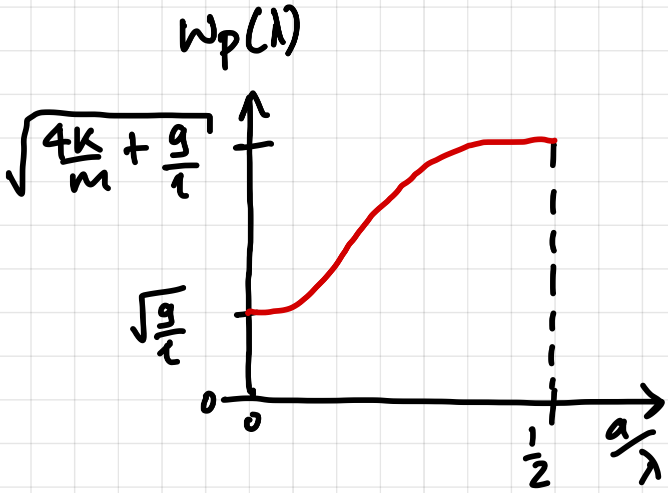
$$\omega_s(\lambda) = 2\sqrt{\frac{K}{m}}\left(\frac{\pi a}{\lambda}\right) - \dots$$

$\therefore$ non-dispersive

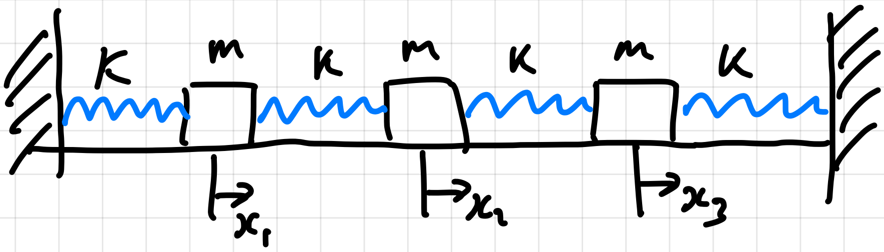


$$\omega_p(\lambda) \simeq \sqrt{\frac{g}{l}} + \frac{K}{m}\sqrt{\frac{l}{g}}\left(\frac{\pi a}{\lambda}\right)^2 - \dots$$

$\therefore$ dispersive



2.6)



$N = 3$ oscillators

For $q = 1, 2, 3$ we get:

$$x_n^{(q)}(t) = A \sin\left(\frac{\pi}{4} q n\right) \cos\left(\omega^{(q)} t + \varphi\right)$$

with $\omega^{(q)} = 2 \sin\left(\frac{\pi}{8} q\right)\sqrt{\frac{K}{m}}$

Explicitly:

$$\omega^{(1)} = 2 \sin\left(\frac{\pi}{8}\right)\sqrt{\frac{K}{m}} = \sqrt{2-\sqrt{2}}\,\sqrt{\frac{K}{m}}$$

$$\underline{A}^{(1)} = \begin{pmatrix} \sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{3\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix}$$

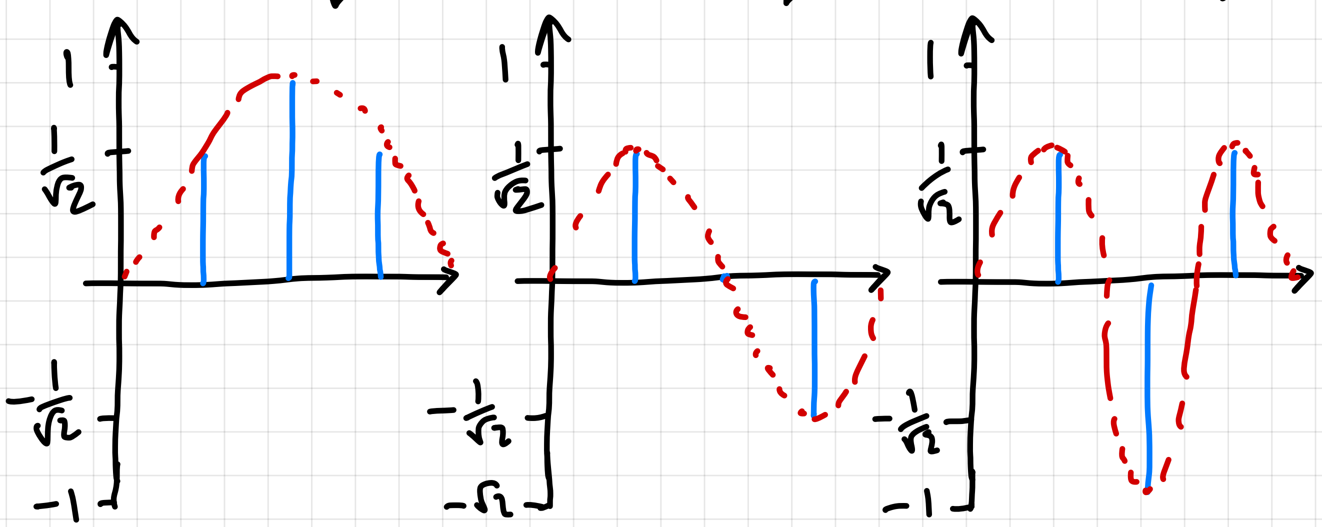
$$\omega^{(2)} = 2\sin\left(\frac{\pi}{4}\right)\sqrt{\frac{K}{m}} = \sqrt{2}\sqrt{\frac{K}{m}}$$

$$\underline{A}^{(2)} = \begin{pmatrix} \sin\left(\frac{\pi}{2}\right) \\ \sin(\pi) \\ \sin\left(\frac{3\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\omega^{(3)} = 2\sin\left(\frac{3\pi}{8}\right)\sqrt{\frac{K}{m}} = \sqrt{2+\sqrt{2}}\sqrt{\frac{K}{m}}$$

$$\underline{A}^{(3)} = \begin{pmatrix} \sin\left(\frac{3\pi}{4}\right) \\ \sin\left(\frac{3}{2}\pi\right) \\ \sin\left(\frac{9}{4}\pi\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

"-" $= q = 1$ "0" $= q = 2$ "+" $= q = 3$



$\omega_1 = \sqrt{2-\sqrt{2}}\sqrt{\frac{K}{m}}$ $\omega_2 = \sqrt{2}\sqrt{\frac{K}{m}}$ $\omega_3 = \sqrt{2+\sqrt{2}}\sqrt{\frac{K}{m}}$