Equation of Motion: The Schrödinger Equation

How does the state of a particle evolve in time?

$$\boxed{i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)}$$

Schrödinger's Equation (SE)

$\frac{h}{2\pi}$

potential energy (conservative force)

The evolution is determined by the SE. This is the equation of motion in QM.

- Specify initial condition $\psi(x, t_0) \rightarrow$ Solve for $\psi(x,t)$

- linear partial differential equation

The superposition Principle

Because the SE is linear it satisfies the superposition principle:

If $\psi_1(x,t)$ & $\psi_2(x,t)$ are both solutions of SE

then $\psi(x,t) = \alpha\,\psi_1(x,t) + \beta\,\psi_2(x,t)$ is also a solution of SE

α, β complex numbers

- Solutions to SE can be superposed.
- $\psi(x,t)$ is a superposition of $\psi_1(x,t)$ & $\psi_2(x,t)$

Proof:

$$\alpha\, i\hbar \frac{\partial \psi_1}{\partial t} = -\alpha \frac{\hbar^2}{2M}\frac{\partial^2 \psi_1}{\partial x^2} + \alpha V(x)\psi_1(x,t)$$

$$\beta\, i\hbar \frac{\partial \psi_2}{\partial t} = -\beta \frac{\hbar^2}{2M}\frac{\partial^2 \psi_2}{\partial x^2} + \beta V(x)\psi_2(x,t)$$

Recall: $\frac{\partial^2}{\partial x^2}\left(f(x) + g(x)\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2}$

$$\underbrace{\alpha\, i\hbar \frac{\partial \psi_1}{\partial t} + \beta\, i\hbar \frac{\partial \psi_2}{\partial t}}_{i\hbar \frac{\partial \psi'}{\partial t}} = \underbrace{-\alpha \frac{\hbar^2}{2M}\frac{\partial^2 \psi_1}{\partial x^2} - \beta \frac{\hbar^2}{2M}\frac{\partial^2 \psi_2}{\partial x^2}}_{-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2}\left[\underbrace{\alpha\psi_1(x,t) + \beta\psi_2(x,t)}_{\psi'(x,t)}\right]} + \underbrace{\alpha V(x)\psi_1(x,t) + \beta V(x)\psi_2(x,t)}_{V(x)\left[\underbrace{\alpha\psi_1(x,t) + \beta\psi_2(x,t)}_{\psi'(x,t)}\right]}$$

$$\hookrightarrow \quad i\hbar \frac{\partial \psi'}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2 \psi'}{\partial x^2} + V(x)\psi'(x,t)$$

i.e. $\psi'(x,t)$ satisfies SE.

* All properties of $\psi_1(x,t)$ & $\psi_2(x,t)$ are superposed.

<u>Shows:</u> Evolution of a superposition is the superposition of the evolutions

<u>See later</u>: this provides our general strategy for solving SE.

- Find wavefunctions whose evolution is SIMPLE
- Find a way of expressing an arbitrary wavefunction as a superposition of these special wavefunctions