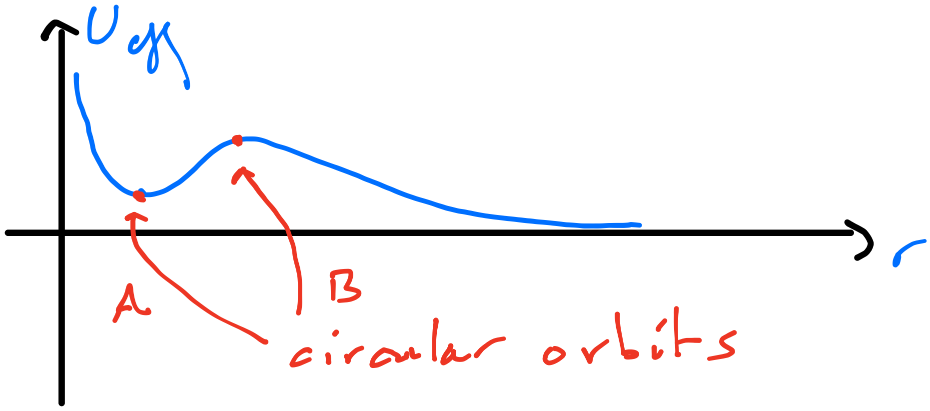
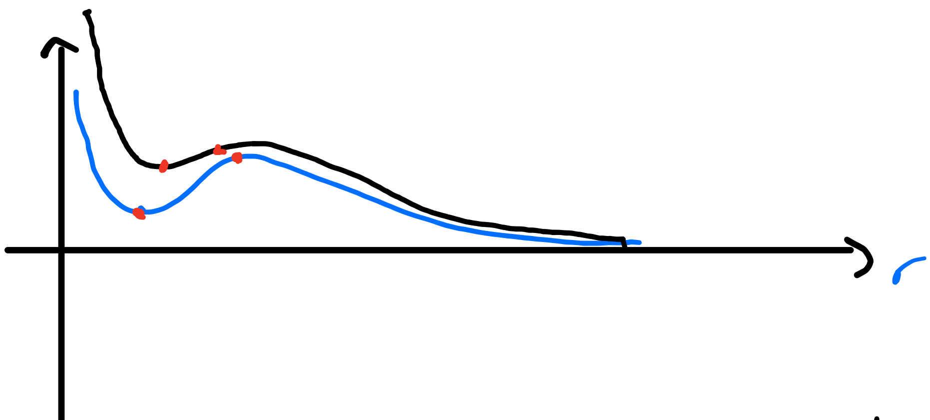
4.1

a)



b) A: stable (local minimum)
B: unstable (local maximum)

c)

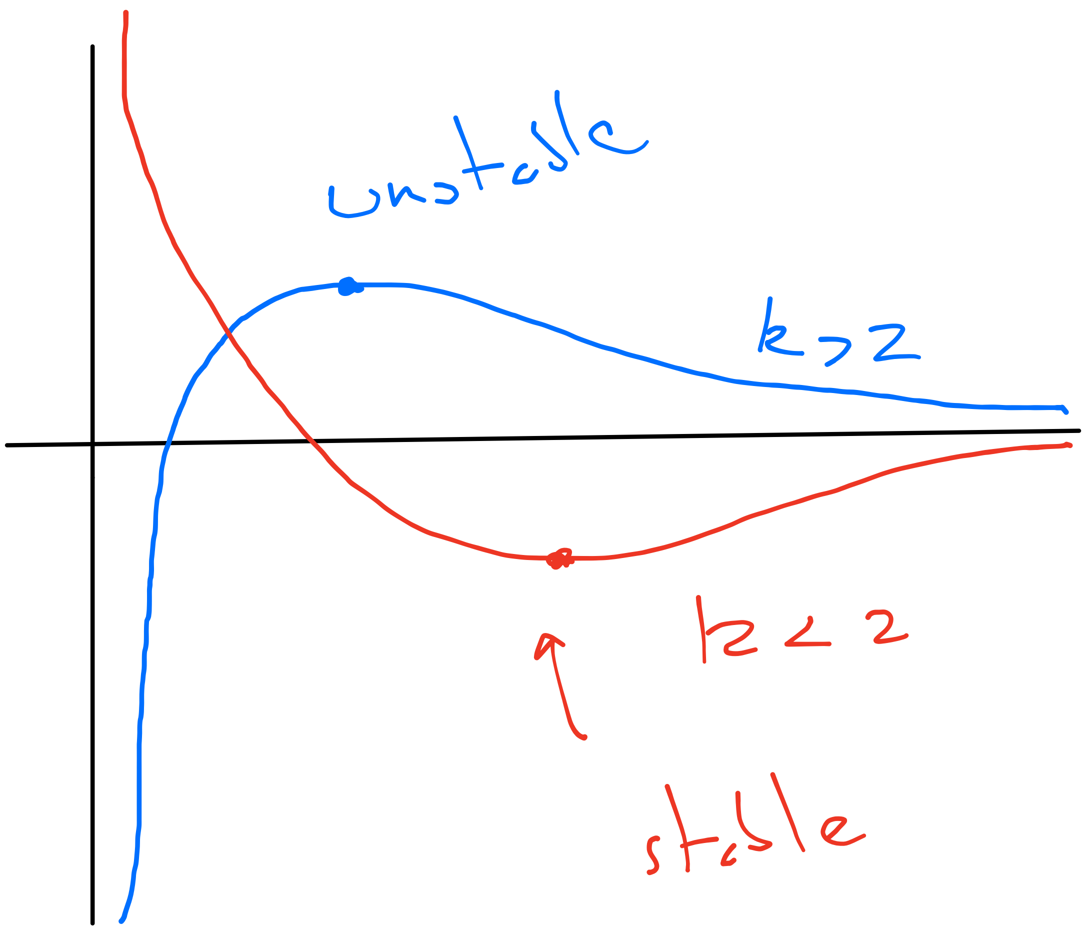


d) Stable goes right (higher r)
unstable goes left (lower r)

4.2

$$U(r) = -\frac{A}{r^k} \qquad A > 0 \quad (k > 0)$$

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{A}{r^k}$$



We answer this question by qualitatively plotting the effective potentials. You can do it by calculation, which is done below. The calculation is long and fiddly however.

4.2 In detail

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{A}{r^k}$$

circular orbit $\Rightarrow \frac{\partial U}{\partial r} = 0$

$$\frac{\partial U}{\partial r} = -\frac{L^2}{mr^3} + \frac{kA}{r^{k+1}}$$

$$= 0 \Rightarrow r = r^* = \left(\frac{mkA}{L^2}\right)^{\frac{1}{k-2}}$$

long, fiddly

stable if $\left.\frac{\partial^2 U}{\partial r^2}\right|_{r=r^*} > 0$

$$\left.\frac{\partial^2 U}{\partial r^2}\right|_{r=r^*} = -A(k-2)k\left(\frac{Akm}{L^2}\right)^{\frac{2+k}{2-k}}$$

Now $A > 0$, $L^2 > 0$, $k > 0$, $m > 0$

Assume $k \neq 2$ then

$$C = Ak\left(\frac{Akm}{L^2}\right)^{\frac{2+k}{2-k}} > 0$$

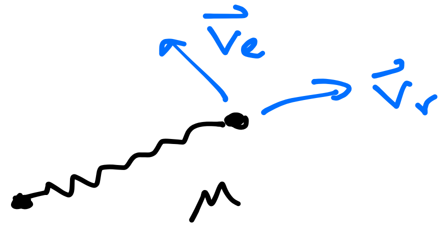
so $$\left.\frac{\partial^2 U}{\partial^2 r}\right|_{r=r^*} = -(k-2)C$$

if $k < 2$ this is +ve ← stable

if $k > 2$ this is −ve ← unstable

4.3

a)



$$K = \frac{1}{2} m |\vec{v}|^2$$

$$= \frac{1}{2} m |\vec{v}_e|^2$$

$$= \frac{1}{2} m r^2 \dot{\phi}^2$$

$$K = \frac{L^2}{2mr^2} \quad \text{A}$$

$$\vec{v}_r = 0$$

$$\vec{v} = \vec{v}_e$$

$$\vec{v}_e = |\vec{v}_e| \vec{e}$$

$$|\vec{v}_e| = \sqrt{\frac{2K}{m}}$$

Velocity

Now we use the fact that the orbit is circular, so $\frac{\partial U_{eff}}{\partial r} = 0$

$$U_{eff} = \frac{L^2}{2mr^2} + \frac{k}{2} r^2 \qquad \frac{\partial U_{eff}}{\partial r} = 0$$

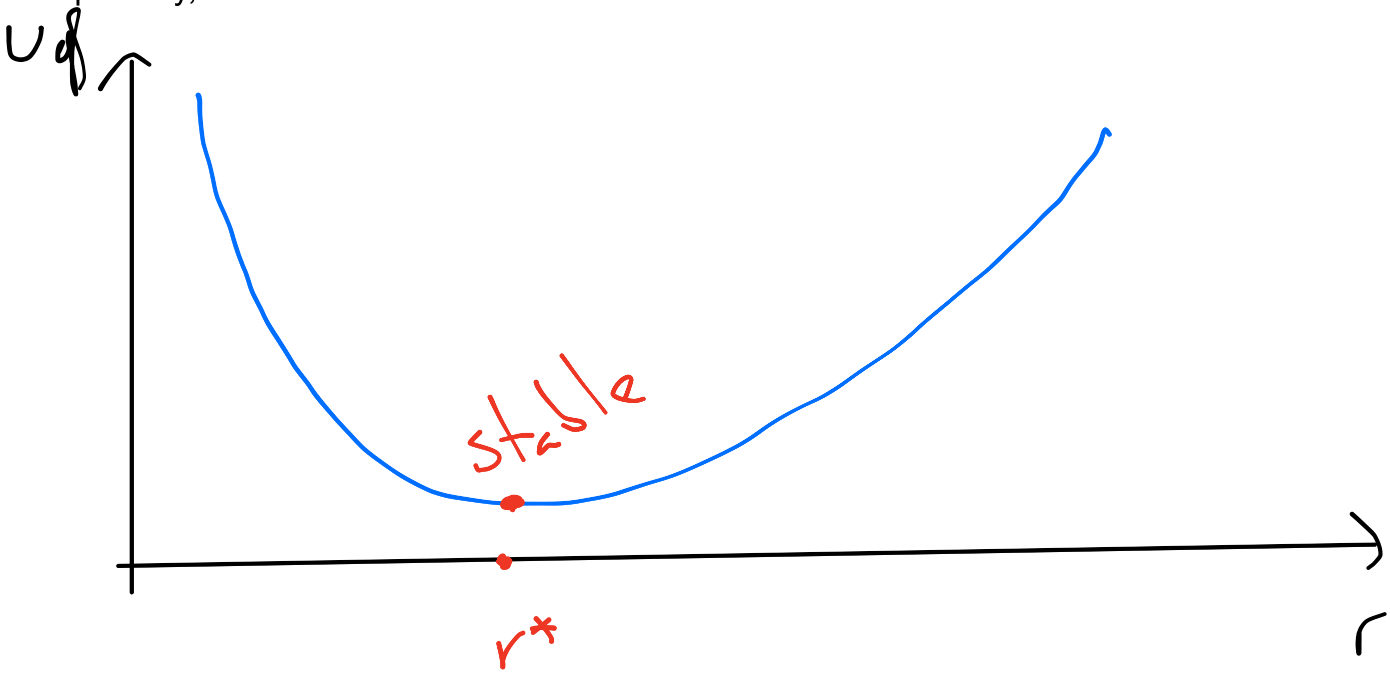
$$\Rightarrow r = r^* = \sqrt{\frac{L}{k^{1/2} m^{1/2}}} \quad \text{B}$$

Using (A) & (B):

$$r = \sqrt{\frac{2k}{k}} \qquad L = 2k\sqrt{\frac{m}{k}}$$

b)

Graphically, this is easier to see



by calculation:

$$\frac{\partial^2 U}{\partial r^2} = k + \frac{3L^2}{mr^4} \quad \text{at } r = r^*$$

$$= 4k > 0$$

$\Rightarrow$ stable

c)

$$m\ddot{r} = -\frac{\partial U_{eff}}{\partial r}$$

near $r = r^*$ say $r = r^* + \delta r$

then $\frac{\partial}{\partial r} = \frac{\partial}{\partial \delta r}$

and $U_{eff} \simeq \Big[\underbrace{U_{eff}(r^*) + \frac{\partial U_{eff}}{\partial r}\Big|_{r=r^*} \delta r}_{\text{Taylor series}}$

$+ \frac{1}{2}\frac{\partial^2 U_{eff}}{\partial r^2}\Big|_{r=r^*} \delta r^2 + \dots\Big]$

const.

$= 0$ as we are at circular orbit

hence

$$m\ddot{\delta r} = -\frac{\partial}{\partial \delta r}\left(\frac{1}{2}\frac{\partial^2 U_{eff}}{\partial r^2}\Big|_{r=r^*} \delta r^2\right)$$

$$= -4k\,\delta r$$

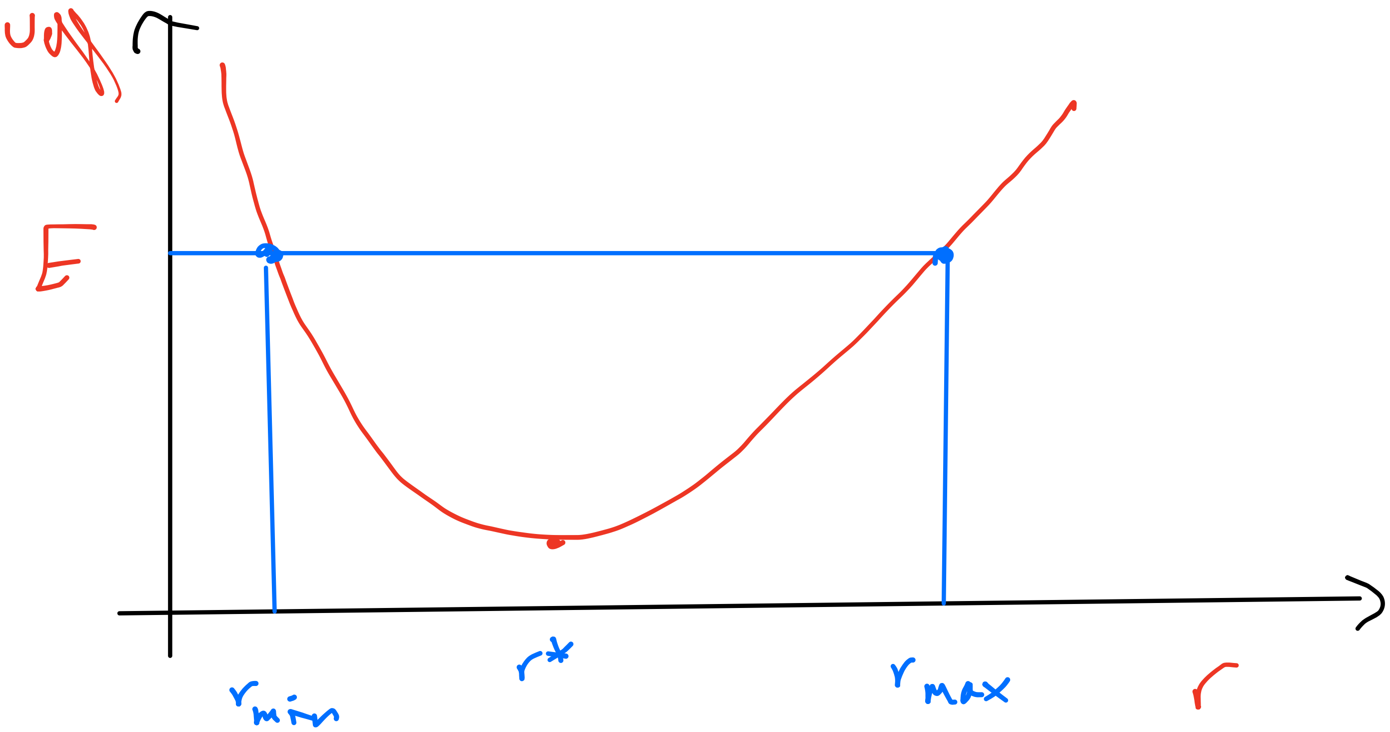
S.H.M $\ddot{x} = -\omega^2 x$

we have $\ddot{\delta r} = -\frac{4k}{m}\delta r$

$\Rightarrow \omega = 2\sqrt{\frac{k}{m}}$

d) No, potential is not quadratic

e) Do this graphically



$$E = \frac{L^2}{2mr_*^2} + \frac{1}{2}kr_*^2 + \frac{1}{2}m\omega^2$$

radial k.e.

$$= L\sqrt{\frac{k}{m}} + \frac{1}{2}m\omega^2$$

at r_{min}, r_{max} radial k.e. $= 0$

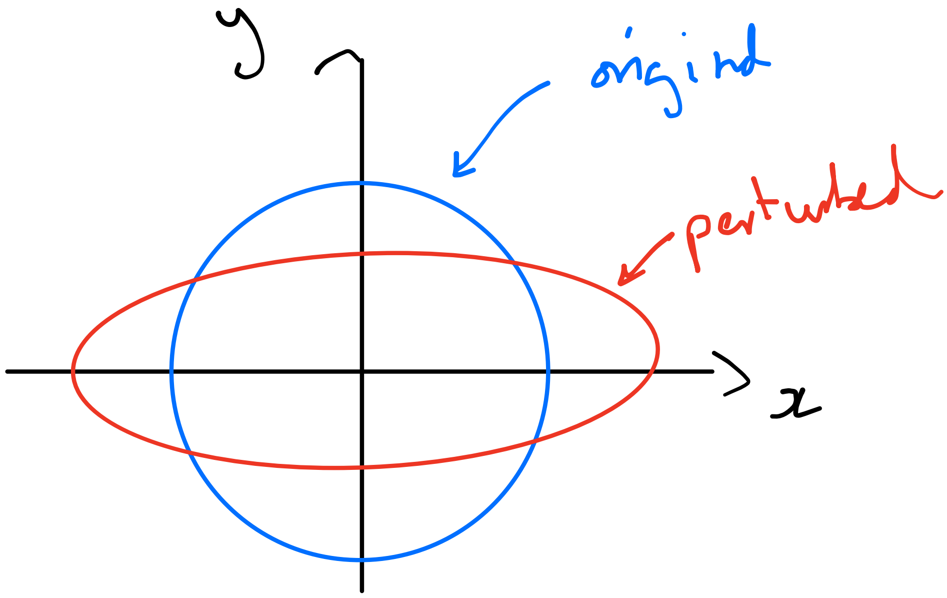
$\Rightarrow$

$$L\sqrt{\frac{k}{m}} + \frac{1}{2}m\omega^2 = \frac{L^2}{2mr_{min}^2} + \frac{1}{2}kr_{min}^2$$

also solved by r_{max}

Exact soln tricky

f)



radial freq $= 2\omega$ yes, it matches

(r oscillates twice as fast as x, y)

why?