Wavepackets

8.1. In this question we will consider a wavepacket of a free particle. Assume the wavepacket has the form (4.14) from the notes, namely

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(k) e^{-i\hbar k^2 t/2M} e^{ikx} dk,$$

where $c(k)$ is given by

$$c(k) = \begin{cases} a & \text{if } |k| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

and where a is a positive constant.

(a) Sketch the function $c(k)$.

(b) Find a such that $c(k)$ is normalised.

(c) By evaluating the above integral expression, show that the wavefunction at $t = 0$ is

$$\Psi(x,0) = \frac{1}{\sqrt{\pi}} \frac{\sin x}{x}.$$

(d) Sketch the wavefunction $\Psi(x,0)$ and the probability density $P(x,0)$.

8.2. In this question we will derive the result (4.23) from Example 4.1 in the notes (Gaussian wavepackets). Assume again a wavepacket of the form (4.14) (given above), with $c(k)$ given by the Gaussian function

$$c(k) = \left(\frac{2a^2}{\pi}\right)^{1/4} e^{-a^2(k-k_0)^2},$$

where a and k_0 are arbitrary real constants.

(a) By evaluating the above integral expression for $\Psi(x,t)$ when $t = 0$, show that

$$\Psi(x,0) = \left(\frac{1}{a\sqrt{2\pi}}\right)^{1/2} e^{-x^2/4a^2} e^{ik_0 x}.$$

Hint: You may use the following result for Gaussian integrals in order to carry out this calculation:

$$\int_{-\infty}^{\infty} e^{-\alpha y^2 + \beta y} dy = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha}$$

where α and β are complex numbers, and the real part of α is positive, $\mathrm{Re}(\alpha) > 0$.

(b) Write down the probability density $P(x,0)$ for this wavefunction.

(c) Make a sketch of the probability density and use it (without calculation) to find the expectation value of the position of the particle $\langle x \rangle$.

(d) Calculate the standard deviation of the probability density $P(x,0)$. That is, calculate $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$, where

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x,0) dx.$$

Hint: You may use the following result for Gaussian integrals in order to carry out this calculation:

$$\int_{-\infty}^{\infty} y^2 e^{-by^2} dy = \frac{1}{2}\sqrt{\frac{\pi}{b^3}}$$

where b is real and positive.

Momentum

8.3. Consider a particle with the same wavefunction at time t_0 as Problem 7.1 of Problem Sheet – Week 7,

$$\Psi(x,t_0) = \begin{cases} \frac{\sqrt{15}}{4}(1-x^2) & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the momentum wavefunction of the particle at time t_0 is

$$\tilde{\Psi}(p,t_0) = \sqrt{\frac{15\hbar^3}{2\pi}} \left(\frac{\hbar \sin(p/\hbar)}{p^3} - \frac{\cos(p/\hbar)}{p^2} \right).$$

***Hint**: You will need to integrate by parts twice in order to obtain this result.*

(b) Sketch the wavefunction $\tilde{\Psi}(p,t_0)$ and the probability density $P(p,t_0)$.
You may want to use a computer to help in this (i.e. Python, Matlab, Mathematica, etc.)

(c) What is the probability amplitude and probability density for the particle to have momentum $p = \pi\hbar$?

8.4. Consider two particles, the first of which has wavefunction $\Psi(x,t_0)$, and the second of which has wavefunction $\Psi'(x,t_0)$, related to $\Psi(x,t_0)$ via

$$\Psi'(x,t_0) = \Psi(x,t_0)e^{ik_0x},$$

where k_0 is a real constant. This is the same situation considered in Problem 7.5.

The momentum wavefunction of the first particle is

$$\tilde{\Psi}(p,t_0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,t_0)e^{-ipx/\hbar}dx.$$

(a) Write down the momentum wavefunction of the second particle, i.e. the momentum wavefunction associated to $\Psi'(x,t_0)$.

(b) Show that the momentum wavefunctions of the two particles are related via

$$\tilde{\Psi}'(p,t_0) = \tilde{\Psi}(p - \hbar k_0, t_0)$$

***Hint**: It may be useful to introduce the new variable $p' = p - \hbar k_0$ in your answer to part (a).*

(c) In a single plot, make representative sketches of $|\tilde{\Psi}(p,t_0)|^2$ and $|\tilde{\Psi}'(p,t_0)|^2$.

(d) Use your answers to part (b) and (c) to explain how the state of a particle changes when we multiply the spatial wavefunction by e^{ik_0x}. How does this relate to your answer to Problem 7.5 (c).

8.5. Consider a particle with the following momentum wavefunction at time t_0,

$$\tilde{\Psi}(p,t_0) = \begin{cases} \frac{1}{\sqrt{p_b - p_a}} & \text{if } p_a \leq p \leq p_b, \\ 0 & \text{otherwise.} \end{cases}$$

where $p_a < p_b$. This wavefunction has the form of a "box", of width $\Delta = p_b - p_a$ and centre at $p_c = (p_a + p_b)/2$.

(a) Show that the spatial wavefunction $\Psi(x,t_0)$ of the particle at time t_0 is

$$\Psi(x,t_0) = \sqrt{\frac{\hbar}{2\pi(p_b - p_a)}} \frac{(e^{ip_bx/\hbar} - e^{ip_ax/\hbar})}{ix}$$

(b) By expressing p_a and p_b in terms of the centre p_c and width Δ, show that the wavefunction can alternatively be written as

$$\Psi(x,t_0) = \sqrt{\frac{\Delta}{2\pi\hbar}} e^{ip_c x/\hbar} \operatorname{sinc}(x\Delta/2\hbar),$$

where $\operatorname{sinc}(y) = \sin(y)/y$.

(c) ***(Tricky)*** Consider now a particle with a spatial wavefunction

$$\Psi'(x,t_0) = \sqrt{\frac{\Delta}{2\pi\hbar}} \operatorname{sinc}(x\Delta/2\hbar).$$

Use Problem 8.4 to write down (i.e. without calculating explicitly) the momentum wavefunction $\tilde{\Psi}'(p,t_0)$ of the particle at t_0. What is the centre and width of this wavefunction?