Mechanics and Oscillations **Week 1**

<u>Problem Sheet answers</u>

1.1) a)

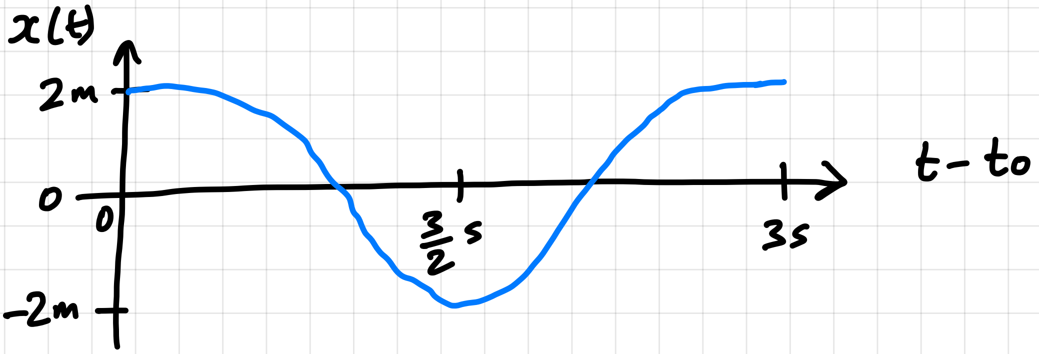
$$\Rightarrow \quad T = \frac{2\pi}{\omega}$$

here $\omega = \frac{2\pi}{3}\ s^{-1}$ $\therefore$ $T = \frac{2\pi}{2\pi} \cdot 3\ s = 3s$

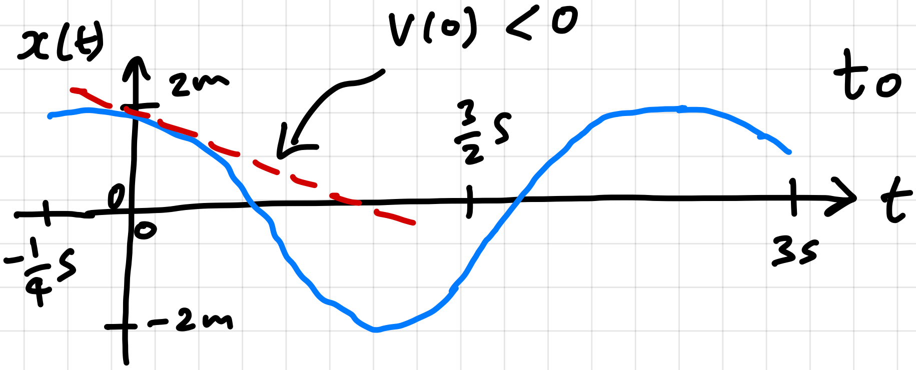
b) A = the amplitude of oscillation.

c) $t_0 = -\frac{\varphi}{\omega}$ $\therefore$ $x(t) = A\cos\left(\omega\left(t + \frac{\varphi}{\omega}\right)\right)$

d)



e)



$t_0 = -\frac{1}{4}\ s$

f) $x(0) = A\cos(\omega t_0)$

$V(0) = A\omega\sin(\omega t_0)$

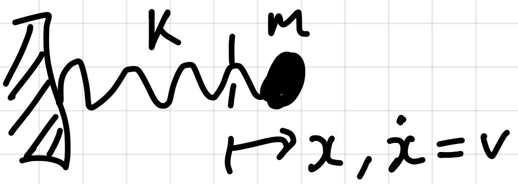
g) $x(0) = B \quad V(0) = C\omega$

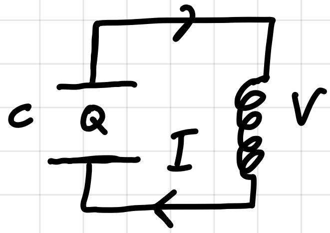
$$x(t) = x(0)\cos(\omega t) + \frac{V(0)}{\omega}\sin(\omega t)$$

h) $$x(t) = 2\cos(3t) + 2\sin(3t) \ [\mathrm{m}]$$
$$x(t) = \sqrt{8}\cos\left(3t - \frac{\pi}{4}\right) \ [\mathrm{m}]$$ with $t\ [\mathrm{s}]$

1.2) a) $$\frac{d^2 I}{dt^2} = -\frac{1}{LC} I \quad \text{or} \quad \frac{d^2 Q}{dt^2} = \frac{dI}{dt} = \frac{-1}{LC} Q$$

b) $$\omega = \frac{1}{\sqrt{LC}}$$

c) 



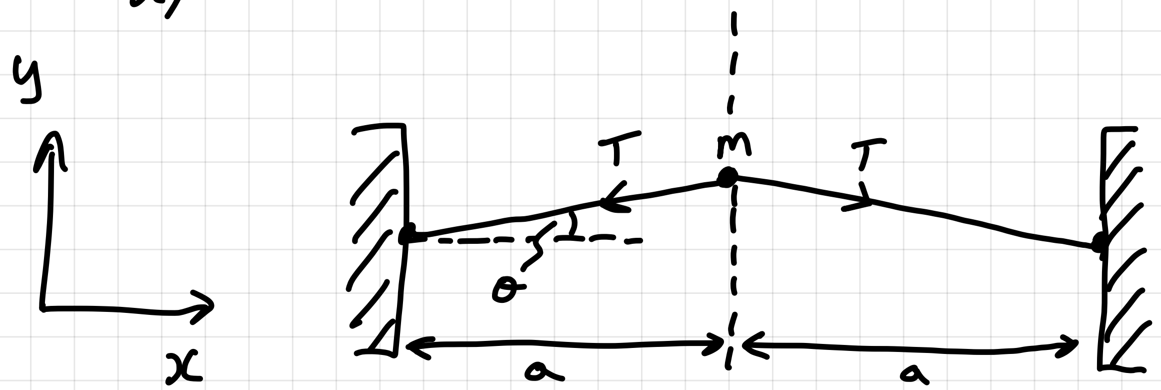
$\frac{dx}{dt} = v$	$\frac{dQ}{dt} = I$
$\frac{dv}{dt} = -\frac{K}{m} x$	$\frac{dI}{dt} = \frac{-1}{LC} Q$
$U = \frac{1}{2} K x^2$	$E_C = \frac{Q^2}{2C}$
$K = \frac{1}{2} m v^2$	$E_L = \frac{1}{2} L I^2$

So $Q \mapsto x$ $\quad I \mapsto \dot{x}$

$\frac{1}{C} \mapsto K$ $\quad L \mapsto m$

$E_C \mapsto U$ $\quad E_L \mapsto K$

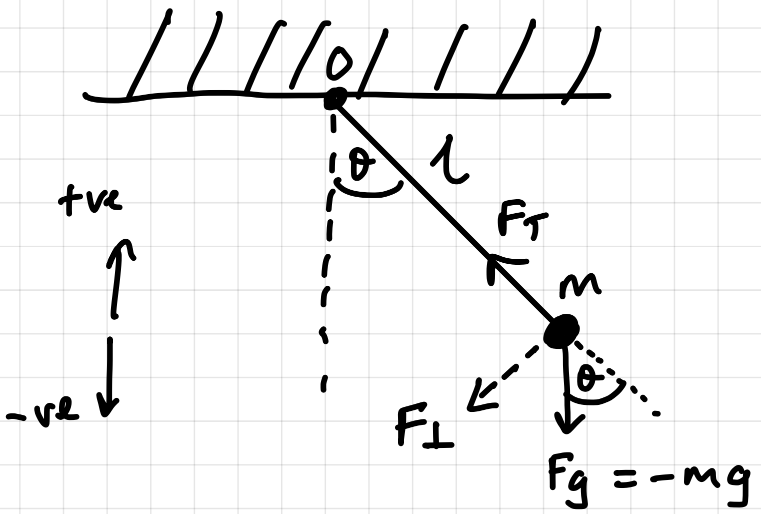
1.3) a)



No gravity!

b) $\frac{d^2y}{dt^2} = -\frac{2T}{ma}y \quad \therefore \quad \omega = \sqrt{2}\sqrt{\frac{T}{ma}}$

1.4)

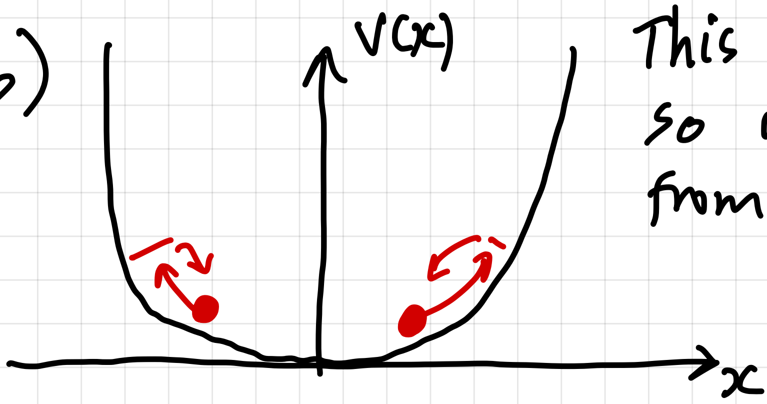


$\frac{d^2\theta}{dt^2} \simeq -\frac{g}{l}\theta$

and $\omega = \sqrt{\frac{g}{l}}$

1.5) a) $V(x) = \frac{1}{4}Kx^4 + C$

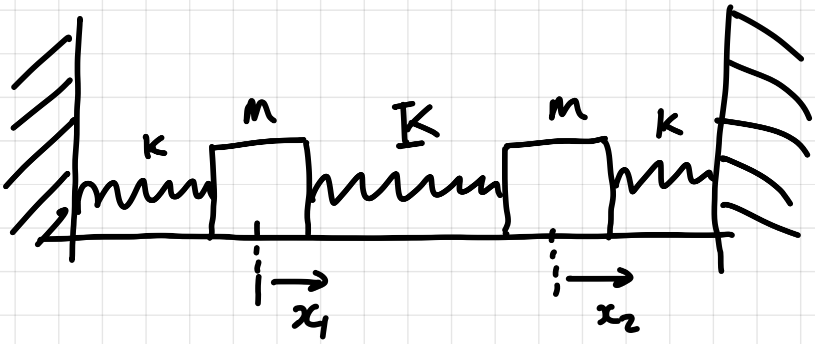
b)



This is a restoring force so we expect oscillations from KE ⟷ PE exchange

c) $\omega \propto A\sqrt{\frac{K}{m}}$

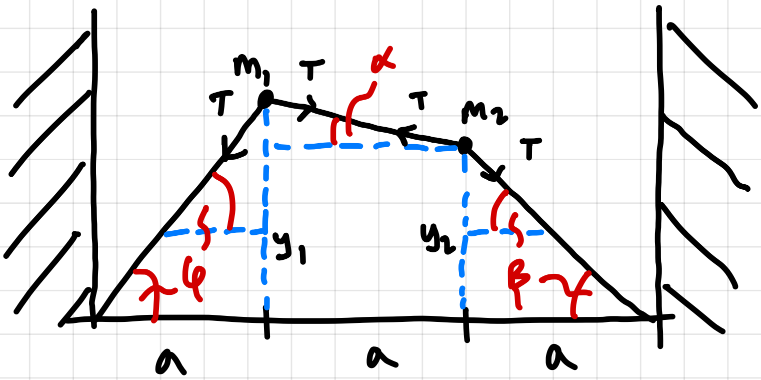
1.6)



$$x_1(t) = \frac{x_1(0)}{2}\left(\cos(\omega_1 t) + \cos(\omega_2 t)\right) + \frac{x_2(0)}{2}\left(\cos(\omega_1 t) - \cos(\omega_2 t)\right)$$
$$+ \frac{\dot{x}_1(0)}{2}\left(\sin(\omega_1 t) + \sin(\omega_2 t)\right) + \frac{\dot{x}_2(0)}{2}\left(\sin(\omega_1 t) - \sin(\omega_2 t)\right)$$

$$x_2(t) = \frac{x_1(0)}{2}\left(\cos(\omega_1 t) - \cos(\omega_2 t)\right) + \frac{x_2(0)}{2}\left(\cos(\omega_1 t) + \cos(\omega_2 t)\right)$$
$$+ \frac{\dot{x}_1(0)}{2}\left(\sin(\omega_1 t) - \sin(\omega_2 t)\right) + \frac{\dot{x}_2(0)}{2}\left(\sin(\omega_1 t) + \sin(\omega_2 t)\right)$$

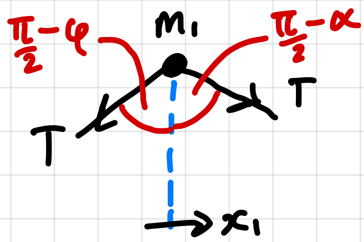
1.7) a)



$$m_1 \frac{d^2 y_1}{dt^2} = -\frac{T}{a} y_1 - \frac{T}{a}(y_1 - y_2) \qquad (1)$$

$$m_2 \frac{d^2 y_2}{dt^2} = \frac{T}{a}(y_1 - y_2) - \frac{T}{a} y_2 \qquad (2)$$

b)



Resolve horizontally $-ve \xleftarrow{\;x\;}\!\!\rightarrow +ve$

Sign consistent with setup

$$-T\sin\left(\tfrac{\pi}{2} - \varphi\right) \longleftarrow \bullet \longrightarrow T\sin\left(\tfrac{\pi}{2} - \alpha\right)$$

$$\therefore \quad m_1 \frac{d^2 x_1}{dt^2} = -T\cos\varphi + T\cos\alpha$$

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but to linear order in y_1/a:

$$m_1 \frac{d^2 x_1}{dt^2} = T - T = 0$$

c) $\quad x_1 = y_1 - y_2 \qquad x_2 = y_1 + y_2$

$$m \frac{d^2 x_1}{dt^2} = -\frac{3T}{a} x_1 \qquad m \frac{d^2 x_2}{dt^2} = -\frac{T}{a} x_2$$

$$\Rightarrow \quad x_1(t) = A_1 \cos(\omega_1 t + \varphi_1) \qquad \omega_1 = \sqrt{\frac{3T}{am}}$$

$$x_2(t) = A_2 \cos(\omega_2 t + \varphi_2) \qquad \omega_2 = \sqrt{\frac{T}{am}}$$

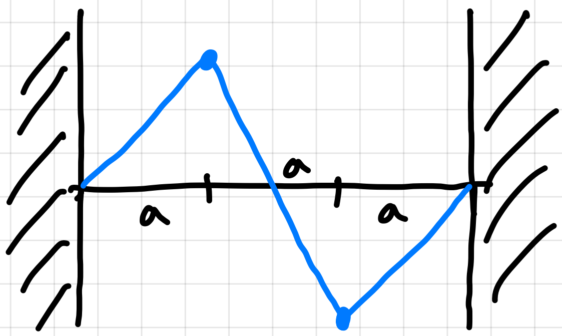
$$\therefore \quad y_1(t) = \frac{A_1}{2} \cos(\omega_1 t + \varphi_1) + \frac{A_2}{2} \cos(\omega_2 t + \varphi_2)$$

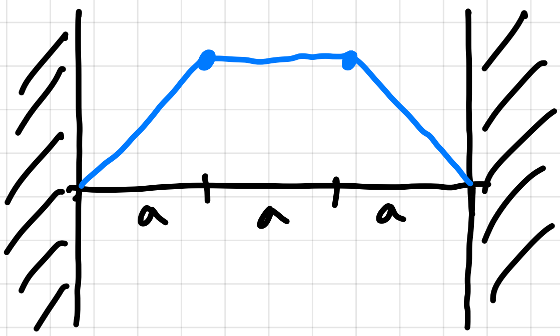
$$y_2(t) = -\frac{A_1}{2} \cos(\omega_1 t + \varphi_1) + \frac{A_2}{2} \cos(\omega_2 t + \varphi_2)$$

$A_2 = 0 \quad$ mode I $\quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{A_1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_1 t + \varphi_1)$

$A_1 = 0 \quad$ mode II $\quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{A_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_2 t + \varphi_2)$

d) mode I $\qquad$ mode II





e)

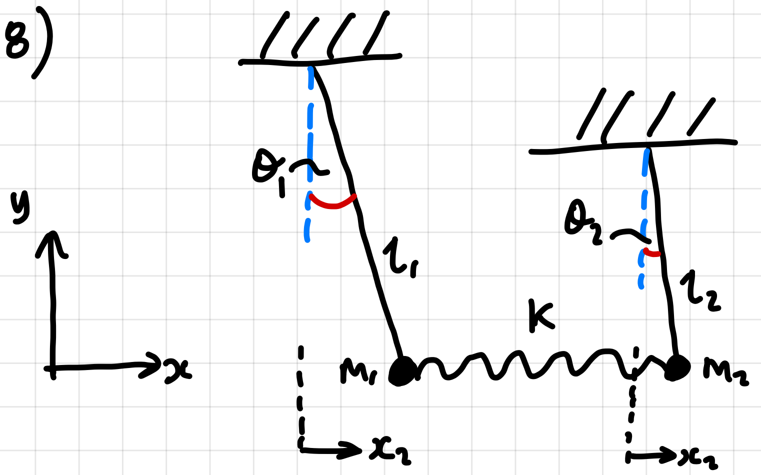
$$y_1(t) = A\cos(\omega_1 t) + B\sin(\omega_1 t) + C\cos(\omega_2 t) + D\sin(\omega_2 t)$$

$$y_2(t) = -A\cos(\omega_1 t) - B\sin(\omega_1 t) + C\cos(\omega_2 t) + D\sin(\omega_2 t)$$

$$A = \frac{y_1(0) - y_2(0)}{2} \qquad B = \frac{1}{2\omega_1}\left(V_1(0) - V_2(0)\right)$$

$$C = \frac{y_1(0) + y_2(0)}{2} \qquad D = \frac{1}{2\omega_2}\left(V_1(0) + V_2(0)\right)$$

1.8)



a)

$$m_1 \frac{d^2x_1}{dt^2} = -T_1 \sin\theta_1 - K(x_1 - x_2)$$

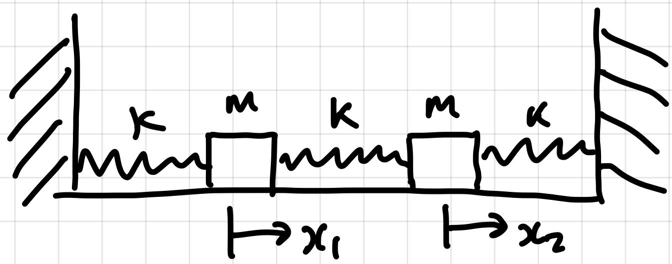
$$m_2 \frac{d^2x_2}{dt^2} = -T_2 \sin\theta_2 + K(x_1 - x_2)$$

b)

$$m_1 \frac{d^2x_1}{dt^2} = -\frac{m_1 g}{l} x_1 - K(x_1 - x_2)$$

$$m_2 \frac{d^2x_2}{dt^2} = -\frac{m_2 g}{l} x_2 - K(x_2 - x_1)$$

1.9)



a) $E = K + U$

$K = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$

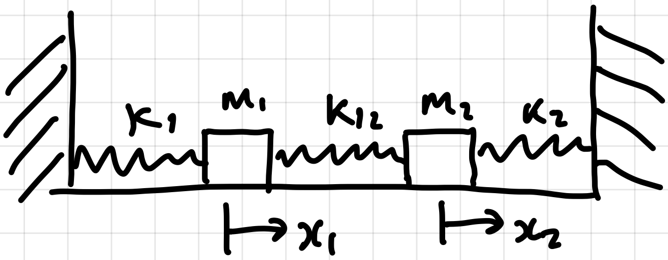
$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (x_1 - x_2)^2$

b) $m \ddot{y}_1 = -k y_1 \qquad m \ddot{y}_2 = -3k y_2$

c) $U' = \frac{1}{2} k y_1^2 + \frac{3}{2} k y_2^2 \overset{\text{show}}{=} U$

$K' = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 \overset{\text{show}}{=} K$

1.10)



a) (i) $m_1 \ddot{x}_1 = -k_1 x_1 - k_{12}(x_1 - x_2)$

(ii) $m_2 \ddot{x}_2 = -k_2 x_2 - k_{12}(x_2 - x_1)$

b) We see directly that it involves $x, \dot{x}, \ddot{x}, \dots$ linearly, they are coupled so form a system of 2 equations and there are no terms which are independent on $x, \dot{x}, \ddot{x}, \dots$

c) $\vec{\tilde{x}} = \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} = a \begin{pmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{pmatrix} + b \begin{pmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{pmatrix}$

$$\underline{\underline{M}} \frac{d^2}{dt^2} \vec{\tilde{x}}(t) \overset{\text{show}}{=} \underline{\underline{K}} \vec{\tilde{x}}(t)$$

1.11) (i) Damped harmonic oscillator:

$m\ddot{x} + b\dot{x} + Kx = 0$

$\Rightarrow$ linear homogeneous equation (b)

(ii) Damped harmonic oscillator with driving:

$m\ddot{x} + b\dot{x} + Kx = f(t)$

$\Rightarrow$ linear non-homogeneous (a)

1.12)

a) $m\ddot{x} = -Kx^3$

b) $m\ddot{x}' = -Kx'^3$ and $m\ddot{x}'' = -Kx''^3$

$x = \alpha x' + \beta x''$

$m\ddot{x} \neq -K(\alpha x' + \beta x'')^3 = -Kx^3$

hence x is <u>not</u> a solution.

c) $\Rightarrow$ non-linear equation