Key features of probability density functions

Below, p(x) is the PDF that describes the random variable x.

- p(x) is not a probability. $\int_a^b p(x)dx$ is a probability. From this follows, that p(x) has units (the inverse units of x). If you remember the properties of probability functions for discrete values, most of those for PDFs follow if you take $P \longrightarrow dP = p(x)dx$ (which is an infinitesimally small probability) and substitute $\sum \longrightarrow \int$. So $p(x) \geq 0 \forall x$ (but in contrast to probabilities, $p(x) > 1$ is allowed, as long as any integral over p(x) is $\int_a^b p(x)dx \leq 1$).
- Normalisation: $\int_{\text{all x}} p(x)\, dx = 1$
- Expectation value (expected mean) $\langle x \rangle = \int x\, p(x)\, dx$
- Expectation value of any function of x: $\int f(x)\, p(x)\, dx$
- Standard deviation: $\sigma_x = \sqrt{\int (x - \langle x \rangle)^2\, p(x)\, dx} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$
- The weirdest thing is how PDFs behave under variable transformations. They change a lot: $p_2(y) = p_1(x) \left| \frac{dx}{dy} \right|$

All of the above generalises to multi-dimensional PDFs, as in

- Normalisation: $\int p(x, y)\, dx\, dy = 1$
- Expectation values (expected mean) $\langle x \rangle = \int x\, p(x, y)\, dx\, dy$, $\langle y \rangle = \int y\, p(x, y)\, dx\, dy$
- Expectation value of any function of x, y: $\int f(x, y)\, p(x, y)\, dx\, dy$
- Standard deviation: $\sigma_x = \sqrt{\int (x - \langle x \rangle)^2\, dx\, dy} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ (and analogously for y).
- Now we have more than one variable, we can define the covariance $\mathrm{cov}(x, y) = \int (x - \langle x \rangle)(y - \langle y \rangle)\, p(x, y)\, dx\, dy = \langle xy \rangle - \langle x \rangle \langle y \rangle$
- Coordinate transformation: $p_2(u, v) = p_1(x, y) \left\| \frac{\partial x, y}{\partial u, v} \right\|$, where $\left| \frac{\partial x, y}{\partial u, v} \right|$ is the Jacobian that we'll meet again in the vector calculus part of this course.