Mathematical Physics 202 Part B
Course Notes

Joel Goldstein

October 6, 2022

Contents

1 Introduction

Much of physics consists of finding a differential equation and solving it with appropriate boundary conditions. In this course we will study differential equations in one and more dimensions, and introduce several tools for solving them. These tools will prove invaluable in finding solutions for realistic systems, and examples of common physics problems will be used throughout.

1.1 Learning Outcomes

- **The Dirac Delta Function** (Section 4): Understand the use of the Dirac δ function to model impulses and be able to apply it in simple cases.
- **Fourier Series** (Section 5): Able to calculate real and complex Fourier coefficients for simple periodic functions; find the solution to differential equations given periodic boundary conditions in terms of Fourier series.

- **Fourier Transforms** (Section 6): Understand the concept of a Fourier transform, and able to calculate the transform for functions with infinite domain. Understand the relation between the width of a function and its Fourier transform, the convolution of functions and the use of Fourier transforms in problems of infinite extent and in diffraction theory.
- **Ordinary and Partial Differential Equations** (Section 8): Apply the method of separation of variables to two- and three-dimensional problems in physics and understand the significance of boundary conditions in determining the solution. Understand the role of Legendre and Bessel functions in finding solutions to differential equations with spherical and cylindrical symmetry.

It would be possible to achieve the formal learning outcomes by blindly learning the tools and methods above, and how to apply them to commonly encountered physics problems. However, It will be much easier and more productive to first introduce the concepts of linear algebra and use them to understand the structure behind the methods being taught. This approach will give much deeper mathematical insight, and also be of great benefit for future courses especially in quantum mechanics.

1.2 Previous Knowledge

This course builds directly on all previous core maths and physics knowledge. You will be expected to be proficient in all of the topics listed below before this course starts. Some of these are directly addressed in the revision questions on the Problem Sheet.

- Common functions: polynomial, trigonometric, exponential, logarithmic, hyperbolic
- Differential calculus: chain rule, product rule, quotient rule, differentials of common functions
- Taylor approximations, Taylor series, convergence of the series, ratio test for power series applications of Taylor series, maxima and minima, l'Hopital's rule
- Complex numbers: Argand diagram, polar form, complex exponential, complex roots
- Integration: integrals of common functions, integration by parts, infinite integrands, infinite ranges of integration
- Differential equations: first order separable and first order linear differential equations, boundary conditions
- Second order linear differential equations with constant coefficients, homogenous including simple harmonic motion, inhomogeneous including resonance
- Multivariate calculus: partial differentiation, vector calculus, non-Cartesian coordinate systems
- Fourier series: full range, half range, use as solution to partial differential equation
- Matrices and vectors: matrix algebra, eigenvectors and eigenvalues

2 Revision of Differential Equations

A familiar physics problem is the motion of a mass m on a spring with restoring force kx and possibly a damping force $b\dot{x}$. Newton's second law leads to a second order Ordinary Differential Equation (ODE) which can be solved to give $x(t)$. The solutions to the homogeneous equation (i.e. $m\ddot{x} + b\dot{x} + kx = 0$) are exponential or oscillatory, depending on the size of the damping force.

A general solution to a second order ODE will contain two arbitrary constants of integration. These can be determined by the boundary conditions which can be:

- Dirichlet: the value of x is specified at two different times
- Neumann: the value of x' is specified at two different times
- Cauchy: the values of x and x' are specified at a given time

If there is an additional driving force, the equation becomes inhomogeneous ($m\ddot{x} + b\dot{x} + kx = F(t)$). Such equations can be solved in specific cases by finding a particular solution. For example, if $F(t) = \sin\omega t$ it can be guessed that $x(t)$ will also be $\sim \sin\omega t$ or $\cos\omega t$.

Exercise: Consider the particle on a spring moving in one dimension, displaced by a distance x from equilibrium. The restoring force acts opposite to the displacement and the damping force works opposite to motion, so Newton's second law gives

$$F = -kx - b\dot{x} + F(t) = m\ddot{x}$$

or

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

This differential equation is second order (because the highest differential is d^2/dx^2) and happens to have constant coefficients.

To solve this, we first find the *complementary function*, i.e. the solution to the homogeneous equation

$$m\ddot{x}_c + b\dot{x}_c + kx_c = 0$$

Second order homogeneous equations with constant coefficients can be solved by considering the auxiliary equation

$$my^2 + by + k = 0$$

which has roots

$$y = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

If $b^2 > 4mk$ then there are two real roots, say y_1 and y_2. The general solution for x_c is then

$$x_c = Ae^{y_1 t} + Be^{y_2 t}$$

where A and B are arbitrary (possibly complex) constants that will be determined by the boundary conditions. These (normally negative) exponential solutions result from a large b and are correspondingly called *over-damped*.

If $b^2 < 4mk$ then the roots of the auxiliary equation are complex, say $\alpha \pm i\omega$. In this case the solution will be

$$x_c = e^{\alpha t}(Ae^{i\omega t} + Be^{-i\omega t})$$

If it is known that the solution is real (as in this case), this can be written as

$$x_c = e^{\alpha t}\left[A' \cos(\omega t) + B' \sin(\omega t)\right]$$

with A' and B' now some real arbitrary coefficients, or equivalently

$$x_c = Ke^{\alpha t} \cos(\omega t + \phi)$$

where K and ϕ are arbitrary real constants, or

$$x_c = \mathrm{Re}\{Ce^{\alpha t} e^{i\omega t}\}$$

where C is complex. This gives an oscillating solution, decaying slowly in amplitude if $\alpha < 0$, and is called *under-damped*.

If $b^2 = 4mk$ then the solution will be $x_c = (At + B)e^{\alpha t}$ and is called *critically-damped*.

The second step is to find a particular solution x_p to the full differential equation when $F(t) \neq 0$. The full general solution will then be $x = x_c + x_p$.

Let us first take the example $F(t) = F_0 e^{pt}$. We can guess the solution will be of the form of an exponential, i.e. $x_p = Ce^{qt}$, and substitute this in to the differential equation.

$$\begin{aligned} m\ddot{x_p} + b\dot{x_p} + kx_p &= F_0 e^{pt} \\ \Rightarrow mCq^2 e^{qt} + bCqe^{qt} + kCe^{qt} &= F_0 e^{pt} \end{aligned}$$

For this to be true for all values of t, p must be equal to q and then

$$C = \frac{F_0}{mp^2 + bp + k}$$

The general solution is then

$$x = x_c + x_p = Ae^{y_1 t} + Be^{y_2 t} + \frac{F_0 e^{pt}}{mp^2 + bp + k}$$

For completeness we note that if p is equal to one or both of the roots of the auxiliary equation, the particular solutions will be of the form $x_p \sim te^{pt}$ or $t^2 e^{pt}$.

Next let us take the example of an oscillating driving function, $F(t) = |F_0| \cos(\omega_0 t + \phi)$. We can represent this as the real part of a complex exponential, i.e.

$$F(t) = \mathrm{Re}\left\{F_0 e^{i\omega_0 t}\right\}$$

The phase ϕ is now incorporated into a complex amplitude F_0, and following normal convention from now on we will not explicitly write "Re". It should be obvious from context when we are using a complex mathematical function to represent a real physical quantity, and taking the real part is then implicit.

We will guess that the particular solution in this case will be a complex exponential, $x_p = Ce^{i\omega_1 t}$ and substitute this into the differential equation to give

$$-mC\omega_1^2 e^{i\omega_1 t} + biC\omega_1 e^{i\omega_1 t} + kCe^{i\omega_1 t} = F_0 e^{i\omega_0 t}$$

As above, this means that $\omega_1 = \omega_0$ and $C = F_0/(k - m\omega_0^2 + i\omega_0 b)$ and the particular solution is

$$x_p = \frac{F_0 e^{i\omega_0 t}}{k - m\omega_0^2 + i\omega_0 b}$$

The physical, real part of this can be written as

$$x_p = K \cos(\omega_0 t + \delta)$$

where the (real) amplitude is

$$K = \frac{|F_0|}{\sqrt{(k - m\omega_0^2)^2 + \omega_0^2 b^2}}$$

and the relative phase

$$\delta = \phi - \tan^{-1}\left(\frac{\omega_0 b}{k - m\omega_0^2}\right)$$

This shows that the response has a resonance when $k = m\omega_0^2$ (in the limit of small b), i.e. at $\omega_0 = \sqrt{k/m}$, the natural undamped spring frequency.

Physical systems will rarely have a pure sine wave input with such an obvious particular solution. They also frequently have more than one degree of freedom. This course will discuss methods and techniques to solve such real-world problems: ordinary differential equations with arbitrary driving functions and multidimensional partial differential equations.

3 Linear Algebra

Many of the concepts you already have encountered with vectors and matrices can be generalised. Table 1 is a list of the most important properties of a general "vector space" and how they can be applied to functions.

Note that the representation of a 3D vector as the coefficients of an expansion in the orthonormal basis $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$ is so natural that we use the word "vector" to refer to both the vector itself and its representation as a single-column matrix

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

interchangeably. It is also straightforward to expand this into n-dimensional vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix}$$

and a function can be thought of as an infinite-dimensional vector with the values of the components being the values of the function at successive points (see Section 4 below).

This general behaviour of linear operators on vector spaces is known as linear algebra, and has a very general application. For example, we can regard an ODE as an operator equation e.g.

$$\hat{D}x(t) = f(t); \quad \hat{D} = m\frac{d^2}{dt^2} + b\frac{d}{dx} + k \tag{1}$$

If $\hat{D}$ is linear (and in this course we only consider such cases) we know that

$$\hat{D}[k_1 f + k_2 g] = k_1 \hat{D} f + k_2 \hat{D} g \tag{2}$$

Note that this explains how we were able to find general solutions for inhomogeneous equations:

$$\left.\begin{array}{l} \hat{D}x_c = 0 \\ \hat{D}x_p = f(t) \end{array}\right\} + \Rightarrow \hat{D}(x_c + x_p) = f(t) \tag{3}$$

Table 1: Correspondence between vectors and functions as a vector space

General	**Vectors**	**Functions**
Set of objects	$\mathbf{a}, \mathbf{b}, ...$	$f(x), g(x)...$
Closed under addition and scalar multiplication	$\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b}$	$h(x) = k_1 f(x) + k_2 g(x)$
Operators	Matrices: $M\mathbf{a} = \mathbf{b}$	$\hat{O}f(x) = g(x)$
Linear	$M[k_1\mathbf{a} + k_2\mathbf{b}] = k_1 M\mathbf{a} + k_2 M\mathbf{b}$	$\hat{O}[k_1 f + k_2 g] = k_1\hat{O}f + k_2\hat{O}g$
Scalar Product	$\mathbf{a}.\mathbf{b}$	$\langle f, g\rangle = \int_{x_0}^{x_1} f(x)^* g(x)\, dx$
Norm (magnitude)	$\mathbf{a}.\mathbf{a} = \lvert\mathbf{a}\rvert^2$	$\langle f(x), f(x)\rangle = [norm(f)]^2$
Linearly Dependent	$\mathbf{a} = k\mathbf{b}$	$f(x) = kg(x)$
Orthogonal	$\mathbf{a}.\mathbf{b} = 0$	$\langle f, g\rangle = 0$
Orthonormal Set	$\mathbf{a}_i.\mathbf{a}_j = \delta_{ij}$	$\langle f_i, f_j\rangle = \delta_{ij}$
Expansion in basis	$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + ...$	$f(x) = a_0 e_0(x) + a_1 e_1(x) + ...$
NB basis set must be complete i.e. ***span*** *the space, orthonormal sets are most convenient*		
Eigenvalues/vectors/ functions	$M\mathbf{e}_i = \lambda_i\mathbf{e}_i$	$\hat{O}f_i = \lambda_i f_i$
Adjoint	$(M^T\mathbf{a}).\mathbf{b} = \mathbf{a}.(M\mathbf{b})$	$\left\langle \hat{O}^\dagger f, g\right\rangle = \left\langle f, \hat{O}g\right\rangle$
Self-adjoint	Symmetric $M^T = M$	Hermitian $\hat{O}^\dagger = \hat{O}$

4 The Dirac Delta Function

Exercise:

As a first "real-world" example consider a stationary mass on the spring given a sudden impulse $\Delta p = J$ applied at $t = 0$. We expect the mass to have a momentum $mv = J$ immediately after the impulse, which we should check with our formalism.

We can solve this by integrating the differential equation over the duration of the pulse, taking the limit of small a. If the width of the pulse is a the force is J/a during the pulse and zero otherwise, so

$$\int_{-a/2}^{a/2} m\ddot{x} + b\dot{x} + kx \, dt = \int_{-a/2}^{a/2} F(t) \, dt$$

The integral on the right-hand side gives us back the impulse J, so we have

$$[m\dot{x} + bx]_{-a/2}^{a/2} + k \int_{-a/2}^{a/2} x \, dt = J$$

At $t = -a/2$ we know that $\ddot{x} = \dot{x} = x = 0$. If a is small, then at $t = +a$ the mass has not moved much so $x \approx 0$. This means that only the first term on the left hand side contributes hence immediately after the kick

$$m\dot{x} = J$$

as we expected.

4.1 Definition

As stated above, a function can be represented as a vector with the components giving the values of the function at successive points. We can visualise this by starting with a discrete approximation where the basis functions would be something like rectangles (or "top-hat" functions) with width a:

$$\Pi_a(x) = \begin{cases} 1, & |x| < a/2 \\ 0, & |x| > a/2 \end{cases} \tag{4}$$

For a set of points x_i separated by a, the sum

$$f(x) \approx \sum_i f(x_i) \Pi_a(x - x_i) \tag{5}$$

would then be a sequence of rectangles of height $f(x_i)$ as required.

What is the limit of this as $a \to 0$, i.e. as the rectangles become infinitesimally narrow? The approximation becomes exact, the sum becomes an integral, and the set of discrete x_is becomes a continuous variable, say x'. Equation 5 then becomes

$$\boxed{f(x) = \int_{-\infty}^{+\infty} f(x')\delta(x - x')\,\mathrm{d}x'} \tag{6}$$

for some appropriate function $\delta(x-x')$. This is known as the Dirac delta function, and we take this equation as its formal definition. One way of looking at this equation is that integrating with a delta function has the effect of "picking out" the particular value of the function $f(x')$ when the argument of the delta function is zero, i.e. when $x' = x$.

What properties must the delta function have in order for $\delta(x - x')\,\mathrm{d}x'$ to act as a basis for $f(x)$ as we require?

Firstly, it is only non-zero at the exact point under consideration i.e.

$$\delta(x - x') = 0, \;\; x \neq x' \tag{7}$$

Secondly, from considering the simple case that $f(x) = 1$, we see that

$$\int_{-\infty}^{+\infty} \delta(x - x')\,\mathrm{d}x' = 1 \tag{8}$$

In other words, it is a function with infinitesimal width but finite, non-zero area (and hence infinite value when $x = x'$). This is a mathematical representation of the point masses and instantaneous events we often discuss. A sudden "kick" at $t = 0$ to the mass on a spring can be represented as an instantaneous impulse $J\delta(t)$, and a point mass at the origin has a mass distribution $m\delta(x)$.

Note that throughout this course we will treat $\delta(x)$ as a normal function, even though it should only really exist within an integral.

Exercise: A simple example of integrating a delta function:

$$\int_{-\infty}^{+\infty} x^2\delta(3 - x)\,\mathrm{d}x = x^2|_{x=3} = 9$$

4.2 Representations of the Delta Function

There are many representations of the delta function as the limit of functions that approach a infinitely tall, infinitely narrow spike. In this section, for simplicity, we will set $x' = 0$ and just look at delta functions placed at the origin.

The most obvious representation is probably to make a top hat function with unit area (width a, height $1/a$) and take the limit as $a \to 0$, i.e.

$$\delta(x) = \lim_{a \to 0} \frac{1}{a} \times \Pi_a(x) \tag{9}$$

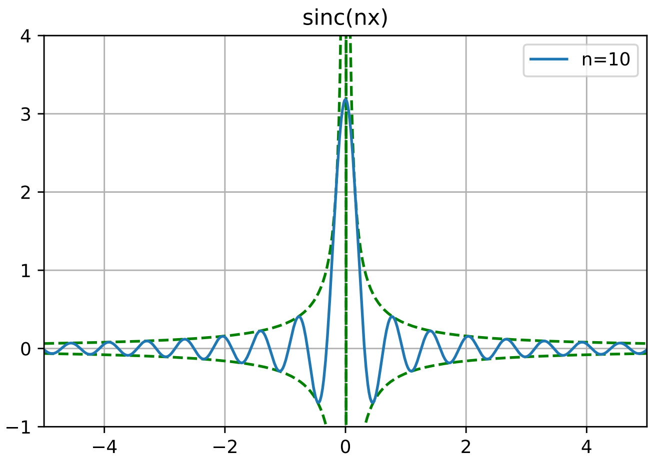
A slightly more complicated, but very useful representation is:

$$\delta(x) = \lim_{n \to \infty} \frac{\sin nx}{\pi x} \tag{10}$$

Not only is this continuous, but it can be expressed in an integral form:

$$\delta(x) = \lim_{n \to \infty} \frac{\sin nx}{\pi x} = \lim_{n \to \infty} \frac{1}{2\pi} \int_{-n}^{n} e^{ikx} dk \tag{11}$$

Exercise: This function is often called sinc(x) and can be seen in the figure with $n = 10$. It is an oscillating sine term multiplied by $1/x$ (shown as the dashed lines), giving a spike at $x = 0$. In the limit as $n \to \infty$ the oscillations become infinitely fast and so will have zero contribution to any integral except at the infinite spike at $x = 0$.



Exercise: Consider the integral on the right hand side of Equation 11.

$$\begin{aligned}\frac{1}{2\pi}\int_{-n}^{n} e^{ikx}dk &= \frac{1}{2\pi}\left[\frac{e^{ikx}}{ix}\right]_{-n}^{n} \\ &= \frac{1}{2\pi}\frac{2i\sin nx}{ix} \\ &= \frac{\sin nx}{\pi x}\end{aligned}$$

4.3 Generalisations of the Delta Function

It is normally straightforward to see what happens if the integration limits are finite, e.g.

$$\int_a^b \delta(x_0 - x)f(x) = \begin{cases} f(x_0) & a < x_0 < b \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

The effect of other changes to the argument can be seen by a change of variables, e.g.

$$\begin{aligned}\delta(-x) &= \delta(x) \\ \delta(ax) &= \frac{1}{|a|}\delta(x)\end{aligned}$$

Similarly, if $f(x)$ is a continuous, differentiable function with zeros at x_i

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|df/dx|} \tag{13}$$

Exercise: To see the effect of $\delta(-x)$ use the substitution $y = -x \Leftrightarrow dy = -dx$:

$$\int_{x=-\infty}^{x=\infty} \delta(-x)g(x)dx \to \int_{y=\infty}^{y=-\infty} \delta(y)g(-y)(-dy)$$

The negative sign swaps the limits, so we have

$$\int_{y=-\infty}^{y=\infty} \delta(y)g(-y)(dy) = g(0)$$

so the result is the same as for $\delta(+x)$.

Similarly, for $\delta(ax)$ we substitute $y = ax$ (and $dy = adx$):

$$\int_{x=-\infty}^{x=\infty} \delta(ax)g(x)dx \to \int_{y=-\infty}^{y=\infty} \delta(y)g(y/a)\frac{dy}{a}$$

Notice that the limits do not change as $a \times \infty = \infty$ as long as $a > 0$. This gives

$$\int_{-\infty}^{+\infty} \frac{\delta(y)}{a} g(y/a) dy = \frac{g(0)}{a}$$

for $a > 0$. If $a < 0$ then the limits would swap and we would obtain $g(0)/(-a)$.

For $\delta(f(x))$ there will be a delta-function "spike" wherever the argument of the delta function is zero, i.e. for each x_i with $f(x_i) = 0$. The contribution of each of these can be calculated by expanding $f(x)$ around x_i as a Taylor series.

$$\int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\, \delta[f(x)] dx = \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\, \delta[f(x_i) + (x - x_i) f'(x_i) + ...] dx$$

The value of $f(x)$ at x_i is zero, so we have

$$\int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\, \delta[0 + (x - x_i) f'(x_i) + ...] dx = \int_{x_i-\epsilon}^{x_i+\epsilon} g(x)\, \delta[(x - x_i) f'(x_i) + ...] dx$$

Taking the limit of small $|x - x_0|$ we then have just a constant scaling as previously with $a = f'(x_i)$.

It is possible to define a delta function in more than one dimension, e.g.

$$\int_{-\infty}^{+\infty} \delta(\vec{r} - \vec{r}_0)\, g(\vec{r})\, d^3 r = g(\vec{r}_0) \tag{14}$$

(Note that the n-dimensional delta function is sometimes written as δ^n - this should never be interpreted as $(\delta(x))^n$ which is ill-defined).

Exercise: We know that the moment of inertia around the z axis is given by the volume integral

$$I = \int_V \rho(\vec{r})\, [x^2 + y^2] dV$$

where $\rho(\vec{r})$ is the mass distribution or density.

If we are looking at a system of point particles, we can write the density as a sum of delta functions:

$$\rho = m_1 \delta^3(\vec{r} - \vec{r_1}) + m_2 \delta^3(\vec{r} - \vec{r_2}) + ...$$

where m_i is the mass of particle i and $\vec{r_i}$ is its position. Note that the three-dimensional delta function is NOT the cube of a delta function (which is not defined).

We can now find the moment of inertia as

$$\int_V \sum_i m_i \delta^3(\vec{r} - \vec{r}_i)[x^2 + y^2] dV = \sum_i m_i \int\int\int \delta(x - x_i)\delta(y - y_i)\delta(z - z_i)[x^2 + y^2] dx dy dz$$

$$= \sum_i m_i [x_i^2 + y_i^2]$$

as expected.

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5 Fourier Series

5.1 Representation of Function over a Finite Interval

You have previously seen that it is possible to represent a function $f(x)$ within a finite domain $x_0 < x < x_1 = x_0 + L$ as a sum of sine and cosine terms:

$$\boxed{f(x) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{L}} \tag{15}$$

for appropriate coefficients a_n and b_n.

You have also seen how this can be used to solve differential equations: if the solutions of a homogeneous linear differential equation are sines and cosines, an arbitrary sum of them such as a Fourier series must also be solution. The coefficients can be chosen to match the boundary conditions.

5.2 Representation of Periodic Functions

The series in Equation 15 can be seen to be periodic with period L. It can therefore be used to represent not only a function limited to the domain $x_0 \to x_1$, but also an infinitely repeating function of period L.

We first note that appropriate sines and cosines form a complete, orthogonal set of basis functions. The full orthogonality relations are:

$$\boxed{\begin{aligned}
\frac{2}{L}\int_{x_0}^{x_1} \cos\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx &= \frac{2}{L}\left\langle \cos\frac{2n\pi x}{L}, \cos\frac{2m\pi x}{L}\right\rangle = \delta_{nm} \\
\frac{2}{L}\int_{x_0}^{x_1} \sin\frac{2n\pi x}{L}\sin\frac{2m\pi x}{L}dx &= \frac{2}{L}\left\langle \sin\frac{2n\pi x}{L}, \sin\frac{2m\pi x}{L}\right\rangle = \delta_{nm} \\
\frac{2}{L}\int_{x_0}^{x_1} \sin\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx &= \frac{2}{L}\left\langle \sin\frac{2n\pi x}{L}, \cos\frac{2m\pi x}{L}\right\rangle = 0
\end{aligned}}$$

for non-zero m and n. Remember the definition of the Kronecker delta function:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{16}$$

Exercise: Let us demonstrate the orthogonality relationships for cosines by considering

$$\left\langle \cos\left(\frac{2\pi nx}{L}\right), \cos\left(\frac{2\pi mx}{L}\right)\right\rangle = \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi nx}{L}\right)\cos\left(\frac{2\pi x}{L}\right) dx$$

If n and m are both zero, this becomes

$$\int_{x_0}^{x_0+L} dx = L$$

For non-zero n and m it is best to use the formulae for addition and subtraction of angles.

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B\end{aligned}$$

can be combined to give

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

which changes the integral above to

$$\int_{x_0}^{x_0+L} \frac{1}{2}\left[\cos\left(\frac{2\pi(n+m)x}{L}\right) + \cos\left(\frac{2\pi(n-m)x}{L}\right)\right] \mathrm{d}x$$

When $n \neq m$ both terms are integrations over some number of complete periods of a cosine, and are therefore zero.

If $n = m$ then the first term will still integrate to zero, but the second term is now

$$\frac{1}{2}\int_{x_0}^{x_0+L} dx = \frac{L}{2}$$

The orthogonality of two sines or a sine and cosine can be demonstrated in a similar fashion.

We can therefore "pick out" a specific term by multiplying the series by the appropriate sine or cosine and integrating over one period, resulting in:

$$\boxed{\begin{aligned}a_n &= \frac{2}{L}\int_{x_0}^{x_1} \cos\frac{2n\pi x}{L} f(x)dx\\ b_n &= \frac{2}{L}\int_{x_0}^{x_1} \sin\frac{2n\pi x}{L} f(x)dx\end{aligned}}$$

Note that the a_n equation gives the $n = 0$ constant coefficient correctly, as long as the factor of 1/2 in Equation 15 is used. This term gives the average value of the function.

Exercise: To obtain the coefficients for the cosine terms in a Fourier series, multiply both sides of Equation 15 by $\cos(2\pi mx/L)$ and integrate over a period:

$$\int_{x_0}^{x_0+L} \cos(2\pi mx/L) f(x) dx$$

$$= \int_{x_0}^{x_0+L} \cos(2\pi mx/L) \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi x}{L} \right) dx$$

The orthogonality relationships can then be applied to show that the terms involving a_n only contribute when $n = m$, and the sine terms are all zero, i.e.

$$\int_{x_0}^{x_0+L} \cos(2\pi mx/L) f(x) dx = \frac{a_0}{2} L \delta_{m0} + \sum_{n=1}^{\infty} a_n \frac{L}{2} \delta_{mn} = \frac{L}{2} a_m$$

i.e.

$$a_m = \frac{2}{L} \int_{x_0}^{x_0+L} \cos(2\pi mx/L) f(x) dx$$

It is often useful to consider the form of $f(x)$ to save calculation. If $f(x)$ is even (i.e. $f(-x) = f(x)$) then all of the sine terms must be zero. Similarly, for an odd $f(x)$ (i.e. $f(-x) = -f(x)$) all a_n coefficients must be zero.

Exercise: As an example, consider a square wave function with amplitude A and period L:

$$f(x) = \begin{cases} A & 0 \leq x \leq L/2 \\ -A & -L/2 < x < 0 \\ f(x+L) & \end{cases}$$

This is an odd function, so all of the a_n coefficients must be zero. The coefficients of the sine terms are given by

$$b_n = \frac{2}{L} \int_0^L \sin \frac{2n\pi x}{L} f(x) dx$$

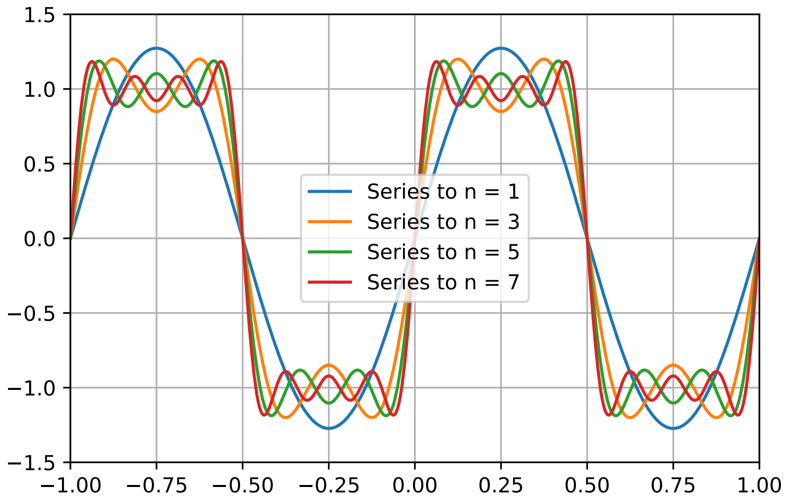
To evaluate this, we can split it into two integrals:

$$\begin{aligned}
b_n &= \frac{2}{L} \int_0^{L/2} \sin \frac{2n\pi x}{L} (A) dx + \frac{2}{L} \int_{L/2}^{L} \sin \frac{2n\pi x}{L} (-A) dx \\
&= \frac{2A}{L} \left[\frac{-\cos(2n\pi x/L)}{2n\pi/L} \right]_0^{L/2} + \frac{2A}{L} \left[\frac{\cos(2n\pi x/L)}{2n\pi/L} \right]_{L/2}^{L} \\
&= \frac{A}{n\pi} \left[-\cos(n\pi) + \cos(0) + \cos(2n\pi) - \cos(n\pi) \right] \\
&= \frac{2A}{n\pi} \left[1 - \cos(n\pi) \right]
\end{aligned}$$

This will be zero for all even n, and $4A/n\pi$ for odd n, so the Fourier series is

$$f(x) = \frac{4A}{\pi}\left[\sin(2\pi x/L) + \frac{\sin(6\pi x/L)}{3} + \frac{\sin(10\pi x/L)}{5} + ...\right]$$

This series will converge point-by-point on the original function as successive terms are added as shown in the figure.



At the discontinuities the series converges to the mid-point of the jump, and the Gibbs phenomenon of an overshoot that gradually approaches the discontinuity can be seen.

5.3 Dirichlet Conditions

The normal tests of convergence (e.g. the ratio test) are not very useful in determining whether a Fourier series converges to $f(x)$ at a particular x. However, it is known that a Fourier representation of $f(x)$ will converge as long as within one period $f(x)$:

- is single valued
- does not have an infinite number of discontinuities or any infinite discontinuities
- does not have an infinite number of maxima and minima

- the integral of its magnitude $\int |f(x)|$ is finite.

These are known as **Dirichlet conditions** which are true for most common functions, even those with a discontinuity (where the Fourier series will converge to the midpoint).

It should also be noted that these conditions are sufficient, but not necessary. So if $f(x)$ satisfies these conditions its Fourier series is guaranteed to converge, but the series may still converge even if one or more condition is violated.

5.4 Gibb's Phenomenon

At a discontinuous jump in a function, the Fourier representation will normally overshoot and then oscillate. Although the Fourier series for any $f(x)$ satisfying the Dirichlet conditions converges at any given x, this strangely enough does not mean that the size of the overshoot and oscillations decreases. In fact as higher-order terms are added the size of the overshoot approaches a constant, but its location moves closer and closer to the discontinuity.

This effect plays an important role in signal processing. For example, high frequency components may be lost from a square wave travelling down a cable. This will result in overshoots and "ringing" at the edges of the pulses.

5.5 Parseval's Theorem

There is a relationship between the squares of the coefficients and the square of the function:

$$\boxed{\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx = \left|\frac{a_0}{2}\right|^2 + \frac{1}{2}\sum_{i=1}^{\infty}(|a_i|^2 + |b_i|^2)} \tag{17}$$

This is **Parseval's Theorem**, and is a statement that the sine and cosine functions form a **complete** (and orthogonal) basis set of functions. Mathematically, this is equivalent to Pythagoras' theorem, i.e. Parseval's theorem tells us that sine and cosine are **basis functions** for representing general periodic functions in the same way that $\hat{\imath}$, $\hat{\jmath}$ etc. are **basis vectors** for representing general vectors. Note that they are orthogonal, but not orthonormal (i.e. normalised to one) - that is why there is an extra factor of 1/2.

There are also physical implications. For example, the power carried by all of the different Fourier components adds up to the total power carried by a wave.

Exercise: Parseval's theorem can be derived by considering the mean of the square of a function

$$\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx = \frac{1}{L}\int_{x_0}^{x_0+L} f^*(x)f(x)dx$$

Representing $f(x)$ as its Fourier series (and writing $k_n = 2\pi n/L$ for short), $f^*(x)f(x)$ becomes

$$\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n)\right]^* \times \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x)\right]$$

but using the orthogonality relations after multiplying out and integrating, all of the cross terms will disappear. We therefore have

$$\begin{aligned}\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx &= \frac{1}{L}\int_{x_0}^{x_0+L} \left[\left|\frac{a_0}{2}\right|^2 + \sum_{n=1}^{\infty} |a_n|^2 \cos^2(k_n x) + \sum_{n=1}^{\infty} |b_n|^2 \sin^2(k_n x)\right] dx \\ &= \left|\frac{a_0}{2}\right|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |b_n|^2\end{aligned}$$

as the average of $\cos^2$ and $\sin^2$ is 1/2.

5.6 Complex Fourier Series

A Fourier series can equivalently be written as a sum of terms involving complex exponentials:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x}} \tag{18}$$

and using the orthogonality relationship

$$\boxed{\frac{1}{L}\int_{x_0}^{x_1} e^{i\frac{2\pi n}{L}x} e^{-i\frac{2\pi m}{L}x} dx = \delta_{nm}} \tag{19}$$

the coefficients can be shown to be

$$\boxed{c_n = \frac{1}{L}\left\langle e^{i2\pi nx/L}, f(x)\right\rangle = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-i\frac{2\pi n}{L}x}dx} \tag{20}$$

The complex representation is often more convenient (normally unless $f(x)$ is even or odd). Parseval's theorem, for example, takes the form:

$$\boxed{\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2}$$

Exercise: The orthogonality relationship for complex exponentials can be easily demonstrated.

$$\frac{1}{L}\int_{x_0}^{x_1} e^{i\frac{2\pi n}{L}x}e^{-i\frac{2\pi m}{L}x}dx = \frac{1}{L}\int_{x_0}^{x_1} e^{i\frac{2\pi(n-m)x}{L}}dx$$

For $n = m$ the right hand side is

$$\frac{1}{L}\int_{x_0}^{x_1} dx = 1$$

If $n \neq m$ then the right hand side is an integral over some number of complete periods (remember that a complex exponential is equivalent to a sine/cosine oscillation) and must therefore be zero.

The Fourier coefficients and Parseval's theorem can be derived following the methods used for sine and cosine series.

Exercise: Consider a pulsed frequency source, amplitude A, period T and pulse width τ, i.e.

$$f(t) = \begin{cases} A & |t| < \tau/2 \\ 0 & \tau/2/ < |t| < T2 \\ f(t+T) & \end{cases}$$

This can be represented as a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{T}t}$$

with the coefficients given by Eqn 20 as

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i\frac{2\pi n}{T}t}\,\mathrm{d}t = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-i\frac{2\pi n}{T}t}\,\mathrm{d}t$$

For $n \neq 0$ The integral is evaluated in the normal fashion using Euler's formula to give

$$c_n = \frac{A}{T}\left[\frac{e^{-i\frac{2\pi n}{T}t}}{-2\pi i n/T}\right]_{-\tau/2}^{\tau/2} = \frac{A}{n\pi}\sin\left(\frac{n\pi\tau}{T}\right)$$

Note that for negative n, the sine also gains a minus sign so the coefficient is the same as that for the corresponding positive frequency.

For $n = 0$ we have

$$c_0 = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\,\mathrm{d}t = \frac{A\tau}{T}$$

Writing the first frequency as $\omega_1 = 2\pi/T$ the Fourier series is therefore

$$f(t) = \frac{A\tau}{T} + \frac{A}{\pi}\left\{ ... + \frac{1}{2}\sin\left(\frac{2\omega_1\tau}{2}\right)e^{-2i\omega_1 t} + \sin\left(\frac{\omega_1\tau}{2}\right)e^{-i\omega_1 t} \right.$$
$$\left. + \sin\left(\frac{\omega_1\tau}{2}\right)e^{i\omega_1 t} + \frac{1}{2}\sin\left(\frac{2\omega_1\tau}{2}\right)e^{2i\omega_1 t} + ... \right\}$$

Exercise:

We consider a series LCR circuit, which is described by the equation

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)$$

The auxiliary equation has solutions

$$y = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

We know that the complimentary function in the lightly damped case will therefore be

$$Q = e^{-\alpha t}\left(A\cos\omega t + B\sin\omega t\right)$$

with $\alpha = R/2L$ and $\omega \approx \sqrt{1/LC}$ for small R. The constants A and B are determined by the boundary conditions.

Putting in some typical values say $L = 1$ mH and $C = 1$ nF we get an undamped frequency of

$$\sqrt{\frac{1}{10^{-3}\times 10^{-9}}} = 10^6 \text{ rad s}^{-1}$$

so the oscillations will have a period of about 6 μs. For a resistance of 100 Ω this will decay with a time constant

$$\frac{1}{\alpha} = \frac{2\times 10^{-3}}{100} = 20\ \mu\text{s}$$

Now consider the inhomogeneous case with a driving function $V(t) = c_n e^{i\omega_n t}$ i.e. a pure sine oscillation with frequency ω_n and amplitude and phase contained in the (complex) coefficient c_n. We know that the particular solutions will be of the form

$$Q = d_n e^{i\omega_n t}$$

and putting this into the equation gives

$$d_n = \frac{Cc_n}{1 - w_n^2 LC + iRC\omega_n}$$

This is a typical resonant response: the amplitude is

$$|d_n| = \frac{|c_n|}{\sqrt{(1/C - \omega_n^2 L)^2 + \omega_n^2 R^2}}$$

with a maximum at $\omega_n^2 = 1/LC$ for small R. There is also a phase shift of

$$\tan^{-1} \frac{RC\omega_n}{1 - \omega_n^2 LC}$$

relative to the input.

The general solution is

$$Q = e^{-\alpha t} \left(A \cos \omega t + B \sin \omega t\right) + d_n e^{i\omega_n t}$$

Note that since the complementary function normally contains exponential damping, in the steady state (i.e. at large times) the first two terms will tend to zero, leaving only the particular solution. The complementary function terms are therefore often called the "transients".

Consider the case of a general periodic driving function with period T. If this can be represented by a Fourier series

$$V(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \quad \omega_n = \frac{2\pi n}{T}$$

we now understand that the particular solutions for the different frequency components can be added to give the general solution

$$Q = CF + \sum_{n=-\infty}^{\infty} d_n e^{i\omega_n t}$$

The d_n coefficients will depend on both the input Fourier coefficients (c_n) and the response of the circuit to that particular frequency - frequencies close to resonance will have a large response and therefore dominate the output.

Taking the particular case of a periodic pulse (as calculated previously) with width set to half of the period (i.e. $\tau = T/2$) we have

$$f(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{\sin(n\pi/2)}{n} e^{i\omega_n t}$$

The terms are zero for even n (except for $n = 0$), and for odd terms the amplitude is proportional to $1/n$:

$$|c_n| = \frac{A}{n\pi}$$

The magnitude of the corresponding component in the response is then

$$|d_n| = \frac{|c_n|}{\sqrt{(1/C - \omega_n^2 L)^2 + \omega_n^2 R^2}} = \frac{A}{Ln\pi\sqrt{(\Omega^2 - \omega_n^2)^2 + \omega_n^2 R^2 \Omega^4}}$$

where we have written the undamped oscillation frequency $\sqrt{1/LC}$ as Ω and rearranged to emphasise the resonance at frequencies near Ω.

- For $|\omega_n| \ll \Omega$, $|d_n| \approx AC^2/n\pi$, so this follows the $1/n$ behaviour of the driving waveform.
- For $|\omega_n|$ close to Ω the first term in the square root is small, so components close to the resonant frequency will be enhanced.
- For $|\omega_n| \gg \Omega$ the square root $\sim \omega_n^2$ so the magnitude overall goes as $1/n^3$

As a final note on Fourier analysis of circuits, we can consider the current due to one frequency component $V_n = V_0 e^{i\omega_n t}$ say:

$$I = \frac{dQ}{dt} = C\frac{i\omega_n V_n}{1 - \omega_n^2 LC + i\omega_n RC}$$

We can make this look like Ohm's law as

$$I = V/Z$$

by defining the impedance

$$Z = R + i\omega_n L + \frac{1}{i\omega_n C} = Z_R + Z_L + Z_C$$

In other words, we can define (frequency dependent) impedance as the AC analogue of resistance. The impedance of pure resistors is just their resistance, and for inductors and capacitors it is given as above. The impedance of individual components can be combined in series or parallel to form a complete circuit.

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6 Fourier Transforms

6.1 Definition

The formalism of Fourier series representing periodic functions can be extended to arbitrary non-periodic functions by considering the limit as the period becomes infinite. Defining $k_n = 2n\pi/L$, it is clear that $\delta k = k_{n+1} - k_n \to 0$ as $L \to \infty$. The discrete frequency spectrum of the Fourier series thus becomes continuous, the coefficients become a continuous function of k and the sum becomes an integral:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{ik_n x} \to f(x) = \int_{-\infty}^{+\infty} c(k) e^{ikx} dk \tag{21}$$

We can use Equation 11 to derive the orthogonality relation:

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik'x} e^{ikx} dx = \delta(k - k')} \tag{22}$$

We can then use this to show that

$$\boxed{c(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx} \tag{23}$$

Exercise: Consider the scalar product

$$\left\langle e^{ik'x}, e^{ikx} \right\rangle = \int_{-\infty}^{+\infty} e^{-ik'x} e^{ikx} dx = \int_{-\infty}^{+\infty} e^{i(k-k')x} dx$$

We can compare this to the integral representation of the delta function in Equation 11:

$$\delta(x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} e^{ikx} dk$$

to see that it is $2\pi\delta(k - k')$.

We want to find the function $c(k)$ such that

$$f(x) = \int_{-\infty}^{+\infty} c(k) e^{ikx} dk$$

As we did to find the (discrete) coefficients of a Fourier series, we will take the scalar product of a single component e^{ikx} with the Fourier representation:

$$\left\langle e^{ikx}, f(x) \right\rangle = \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-ikx} c(k') e^{ik'x} dk'\, dx$$

Integrating the right-hand side with respect to x and using the above result gives

$$\int_{-\infty}^{+\infty} c(k') 2\pi\delta(k' - k) dk' = c(k)$$

as required.

The function $c(k)$ is known as the Fourier transform of the function $f(x)$, and is often written as $\tilde{f}(k)$. The normalisation of the Fourier transform is arbitrary, and different texts therefore put the factor of 2π in different places. Boas uses the above convention, but in this course we will use a more symmetric convention to define the Fourier transform:

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ikx} dk \\ \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \qquad (24) \end{aligned}}$$

Note that the choice of sign in the exponents is also arbitrary (as long as they are opposite) but this is the normal convention.

The first equation is sometimes known as the **inverse transform**, but it is normal to regard the functions $f(x)$ and $\tilde{f}(k)$ as Fourier transforms of each other, or as a **Fourier pair**. They contain the same information and the symmetry implies that neither is more fundamental than the other, even though we normally choose f as the function in "real" space while $\tilde{f}$ is in "reciprocal" or "frequency" space.

Exercise: To find the Fourier transform of a top-hat function, i.e.

$$f(x) = \begin{cases} A & |x| < a/2 \\ 0 & \text{otherwise} \end{cases}$$

apply the transform in Equation 24

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} A e^{-ikx} dx = \frac{A}{\sqrt{2\pi}} \left[\frac{e^{-ikx}}{-ik} \right]_{-a/2}^{a/2} = A\sqrt{\frac{2}{\pi}} \frac{\sin(ka/2)}{k}$$

which is the "sinc" function we have seen several times before. Note that it has the first zeros on either side of the peak at $k = 2\pi/a$ so as the top hat gets wider in x its transform gets narrower in k.

Exercise: The Fourier transform of a delta function at $x = x_0$ is straightforward to calculate.

$$FT(\delta(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x - x_0) e^{-ikx} dx = \frac{e^{-ix_0 k}}{\sqrt{2\pi}}$$

i.e. it is a complex exponential.

Exercise: Now let us find the Fourier transform of a Gaussian. First we need to remind ourselves of how to integrate a Gaussian. We start from the standard Gaussian integral result:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

and a change of variables to $x' = \sqrt{a}\, x$ shows that

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Since the limits are infinite, a translation in x has no effect:

$$\int_{-\infty}^{+\infty} e^{-a(x-x_0)^2} dx = \sqrt{\frac{\pi}{a}}$$

Now let us find the Fourier transform of a Gaussian of width σ, mean zero and area normalised to one:

$$f(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

The transform is given by

$$\tilde{f}(k) = \frac{1}{2\pi\sigma} \int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2} e^{-ikx} dx = \frac{1}{2\pi\sigma} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2} - ikx\right) dx$$

We know the integral when the exponent is of the form $-a(x-x_0)^2$, so we write it in the form

$$-\frac{x^2}{2\sigma^2} - ikx = -\frac{1}{2\sigma^2}\left(x + ik\sigma^2\right)^2$$

This looks like the standard integral above – the first term is correct and the second one would match if we set $x_0 = -ik\sigma^2$. To "complete the square" we need an additional term in the exponent

$$-\frac{x_0^2}{2\sigma^2} = \frac{k^2\sigma^2}{2}$$

which we can "borrow", i.e. we write the integral as

$$\frac{1}{2\pi\sigma} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2} - ikx + \frac{k^2\sigma^2}{2} - \frac{k^2\sigma^2}{2}\right) dx$$

The first three terms in the exponential now form the complete square and the last term is a constant with respect to x, so

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi\sigma} \int_{-\infty}^{+\infty} e^{-(x+ik\sigma^2)^2/2\sigma^2} e^{-k^2\sigma^2/2} dx \\
&= \frac{1}{2\pi\sigma} . \sqrt{2\pi}\sigma . e^{-k^2\sigma^2/2} \\
&= \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma^2/2}
\end{aligned}$$

The Fourier transform of a Gaussian of width σ is therefore also a Gaussian, but with a width equal to $1/\sigma$, so again as the function gets wider its transform gets narrower.

6.2 More on Fourier Transforms

It is straightforward to show the following relationships:

Function	Transform
$f(x)$	$\tilde{f}(k)$
$f(x-a)$	$e^{-ika}\tilde{f}(k)$
$f(xa)$	$1/\lvert a\rvert\,\tilde{f}(k/a)$
$f'(x)$	$ik\tilde{f}(k)$
$f''(x)$	$-k^2\tilde{f}(k)$
$\delta(x-x_0)$	$\frac{1}{\sqrt{2\pi}}e^{-ikx_0}$
e^{ik_0x}	$\sqrt{2\pi}\,\delta(k-k_0)$

Exercise: The Fourier transform of $f(x-a)$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-a)e^{-ikx}dx$$

Change variables to $x' = x - a \Leftrightarrow dx' = dx$ and this becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x')e^{-ik(x'+a)}dx' = e^{-ika}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x')e^{-ikx'}dx' = e^{-ika}\tilde{f}(k)$$

since x' is a dummy variable.

Exercise: The Fourier transform of the differential of a function is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f'(x)e^{-ikx}dx$$

which we can integrate by parts:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f'(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\left\{\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{+\infty} f(x)(-ik)e^{-ikx}dx\right\}$$

As long as $f(x)$ goes to zero at infinity (one of the requirements for being able to take the Fourier transform) the first term is guaranteed to vanish, so we have

$$ik\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx = ik\tilde{f}(k)$$

Parseval's theorem applied to Fourier transforms states that

$$\boxed{\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\tilde{f}(k)|^2 dk} \tag{25}$$

Note that there may be a proportionality constant for different normalisations of Equation 24 – the lack of one here is one reason why the symmetric form was chosen.

6.3 Convolution

If you look through a telescope at the night sky and see a fuzzy circle, there are several possible explanations. You could be looking at a fuzzy, circular nebula, but perhaps the telescope is just out of focus and you are really looking at a point-like star. Perhaps the truth is somewhere inbetween, with a circular nebula made more fuzzy by a slightly out-of-focus telescope.

Similarly, you could repeatedly measure the x position of a particle in a box and plot the distribution of your results. Does the spread in your results come from errors in your measurement, genuine variation in the position of the particle or some combination of the two?

If we know the genuine distribution $f(x)$ and the error distribution $g(x)$, how can we combine them to predict the distribution of the results? The answer in mathematics is the **convolution** of the two functions written as $f * g$.

Mathematically, convolution is defined as

$$\boxed{f * g \equiv \int_{-\infty}^{+\infty} f(x')g(x-x')\, dx' = \int_{-\infty}^{+\infty} f(x-x')g(x')\, dx'} \tag{26}$$

The symmetry between f and g can be seen from the above arguments, or demonstrated by a simple change of variables.

If the genuine position actually has no variation and is always x_0 say, $f(x) = \delta(x - x_0)$. We then find

$$f * g = \int_{-\infty}^{+\infty} \delta(x' - x_0)g(x - x')\, dx' = g(x - x_0)$$

In other words, the convolution of $g(x)$ with a delta function at $x = x_0$ results in a "copy" of the distribution g being moved from the origin and placed at x_0. We can then understand general convolution as in some way adding up copies of g for each point in the function f.

Exercise: Convolution can be applied to experimental uncertainties. You can think of an experiment as having a "true" distribution which is then convoluted with some number of uncertainties that smear it out.

As an example, consider an experiment where for simplicity we can take the "true" distribution as a single value at $x = x_0$, i.e. a delta function $\delta(x - x_0)$. The apparatus with which we measure this introduces a Gaussian uncertainty, i.e. for a true value of 0 our measurement will fall in the probability distribution

$$G_1 = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-x^2/2\sigma_1^2}$$

The observed result will then be the convolution of the "true" signal with the uncertainty

$$\delta(x - x_0) * G_1 = \int_{-\infty}^{+\infty} \delta(x - x' - x_0) \frac{1}{\sqrt{2\pi}\sigma_1} e^{-x'^2/2\sigma_1^2} dx' = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(x-x_0)^2/2\sigma_1^2}$$

i.e. a Gaussian with the given uncertainty, centred at the value x_0.

For simplicity we will now take $x_0 = 0$ so the observed distribution is centred at zero and in fact is just G_1. Now consider the effect of a second source of Gaussian uncertainty, G_2 with width σ_2. The observed distribution will now be $G_1 * G_2$ which can be shown (see problem sheet) to also be a Gaussian, now with a width equal to the sum in quadrature of the widths of the originals (as you already know from error analysis).

The **convolution theorem** states that if $h = f * g$ then

$$\boxed{\tilde{h} = \sqrt{2\pi}\, \tilde{f} \times \tilde{g}} \tag{27}$$

i.e. the Fourier transform of the convolution of two functions is just the product of the individual Fourier transforms (to within a constant which depends on the normalisation chosen in Equation 24). This can be useful as a way to determine h without doing the actual integration, or for **deconvolution** (finding an unknown f given h and g).

Given the symmetry of Equation 24, it should not be surprising that the convolution theorem also works in reverse, i.e. the inverse transform of $\tilde{f} * \tilde{g}$ is just $\sqrt{2\pi} fg$.

Exercise: To prove the convolution theorem, we will consider two functions f and g, and their Fourier transforms, $\tilde{f}$ and $\tilde{g}$. We will then define $\sqrt{2\pi}$ times the product of the transforms to be $\tilde{h}$ i.e.

$$\begin{aligned}
\tilde{h}(k) &= \sqrt{2\pi}\tilde{f}(k).\tilde{g}(k) \\
&= \sqrt{2\pi}.\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x_1) e^{-ikx_1} dx_1 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x_2) e^{-ikx_2} dx_2
\end{aligned}$$

Note that the dummy x_1 and x_2 are different for the different integrals, but it is the same k throughout.

We can then take the inverse transform of this to find $h(x)$:

$$
\begin{aligned}
h(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{h}(k) e^{ikx} dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x_1) e^{-ikx_1} g(x_2) e^{-ikx_2} e^{ikx} dx_1 dx_2 dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x_1) g(x_2) e^{ik(x-x_1-x_2)} dx_1 dx_2 dk
\end{aligned}
$$

The only k dependence is in the exponential, so the integral over k can be done first giving a delta function.

$$
\begin{aligned}
h(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x_1) g(x_2) 2\pi\delta(x - x_1 - x_2) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} f(x_1) g(x - x_1) dx_1 \\
&= f * g
\end{aligned}
$$

so h is just the convolution of f and g as required.

A good example of the use of the convolution theorem is to consider the rather messy convolution of two Gaussians above. We use the standard Gaussian transforms:

$$\tilde{G}_1 = \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma_1^2/2}, \quad \tilde{G}_2 = \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma_2^2/2}$$

so the product is

$$\tilde{G}_1.\tilde{G}_2 = \frac{1}{2\pi} e^{-k^2(\sigma_1^2+\sigma_2^2)/2}$$

The convolution theorem states that the Fourier transform of $G_1 * G_2$ must be

$$\sqrt{2\pi}\tilde{G}_1.\tilde{G}_2 = \frac{1}{\sqrt{2\pi}} e^{-k^2(\sigma_1^2+\sigma_2^2)/2}$$

Again, the standard result can be used to show that the inverse transform of this is

$$\frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-x^2/2(\sigma_1^2+\sigma_2^2)}$$

as shown above.

6.4 Multidimensional Fourier Transforms

The transforms discussed so far have all been one dimensional, but the concept can be readily expanded into any number of dimensions. For example, a function in 3D space $f(\vec{r})$ will have a Fourier transform

$$\tilde{f}(\vec{k}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\vec{k}\cdot\vec{r}} f(\vec{r})\, dx\, dy\, dz \tag{28}$$

6.5 Application of Fourier Transforms to Diffraction

A common physical example that directly demonstrates Fourier transforms is that of **Fraunhofer (or far-field) diffraction**.

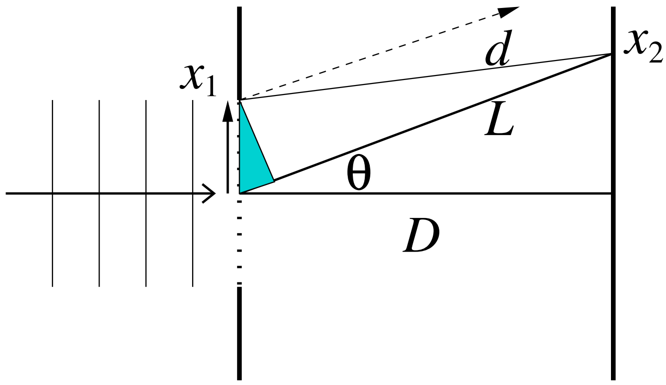


Figure 1: Fraunhofer diffraction.

As shown in Figure 1, coherent light of wavelength λ is incident on a screen which has a pattern on it. We can describe the light wave emerging from the screen as $\phi(x_1, t) = \phi_0(t)f(x_1)$, where the incident wave ϕ_0 has been modified by the screen function $f(x_1)$. In the simplest case, this function will be one where there is a hole in the screen and zero otherwise, but it can have any grey value between zero and one as well as a complex component to give a phase shift.

Huygen's construction can then be used to determine the wave ϕ' at any point x_2 on a distant projection screen. Incorporating overall phase and amplitude factors into an uninteresting (complex) constant $A(t)$ we find

$$\phi'(x_2, t) = A(t) \int_{-\infty}^{+\infty} f(x_1) e^{-i2\pi x_2\, x_1/D\lambda} dx_1 \sim A\tilde{f}(k) \tag{29}$$

where $k = 2\pi x_2/D\lambda$. The projected interference pattern is therefore proportional to the Fourier transform of the screen pattern, with an observed intensity proportional to $|\tilde{f}|^2$.

Exercise: Huygen's principle states that the wave ϕ' at x_2 can be constructed by adding the contribution from every point x_1. The contribution from a single source point x_1 to a single point on the screen at x_2 will be

$$d\phi'(x_2) = B\phi(x_1)e^{i2\pi d/\lambda}dx_1 = B\phi_0 f(x_1)e^{i2\pi d/\lambda}dx_1$$

where d is the straight-line distance from x_1 to x_2, and B is some amplitude factor that will depend on the geometry.

If we are far enough away ($D \gg x_1, x_2$), we can make several approximations that simplify this calculation:

- B is independent of x_1 and x_2
- $\sin\theta = \theta = x_2/D = x_2/L$
- the rays going to x_2 are "paraxial", i.e. they are parallel (as shown by the dashed arrow). This means that the difference in path length between d and a ray coming from $x_1 = 0$ (i.e. L) is given by the short side of the small blue triangle:

  $$d = L - x_1 \sin\theta$$

  and hence

  $$d\phi'(x_2) = B\phi_0 f(x_1)e^{i2\pi(L - x_1\sin\theta)/\lambda}dx_1$$

Incorporating all of the constant terms into a single term $A = B\phi_0 e^{i2\pi L/\lambda}$ we have

$$d\phi'(x_2) = Af(x_1)e^{-i2\pi x_1 x_2/D\lambda}$$

so

$$\phi'(x_2) = A\int_{-\infty}^{+\infty} e^{-i2\pi x_1 x_2/D\lambda} f(x_1)dx \sim \tilde{f}$$

Exercise: The diffraction pattern from a double slit can be most easily derived using the convolution theorem. First consider a single slit of width a centred at $x = 0$. This is represented by the function

$$f_1(x) = \begin{cases} 1 & |x| < a/2 \\ 0 & \text{otherwise} \end{cases}$$

and we know that the Fourier transform of this is the sinc function

$$\tilde{f}_1(k) = \sqrt{\frac{2}{\pi}}\frac{\sin(ka/2)}{k}$$

Two slits with separation b can then be represented as the convolution of the single slit pattern with two delta functions, i.e.

$$f_2 = f_1 * [\delta(x+b/2) + \delta(x-b/2)]$$

and hence the diffraction pattern will be the product of the single slit diffraction pattern with the Fourier transform of the double delta (multiplied by $\sqrt{2\pi}$ if the absolute value matters).

The Fourier transform of $[\delta(x+b/2) + \delta(x-b/2)]$ is

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} [\delta(x+b/2) + \delta(x-b/2)]\, e^{-ikx} dx = \frac{e^{ikb/2} + e^{-ikb/2}}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \cos kb/2$$

and so the double slit diffraction pattern will be proportional to

$$\frac{\sin(ka/2)\cos(kb/2)}{k}$$

Exercise: It is possible to put a piece of glass (or similar) in front of one of the slits to introduce a phase change in the light passing through. This will not change the magnitude of the function f but its phase, e.g. changing the phase by $\lambda/2$ will introduce a relative negative sign

$$\delta(x+b/2) + \delta(x-b/2) \rightarrow \delta(x+b/2) - \delta(x-b/2)$$

This has a Fourier transform of $e^{ikb/2} - e^{-ikb/2} = 2\sin kb/2$ and so the final diffraction pattern will be proportional to

$$\frac{\sin(ka/2)\sin(kb/2)}{k}$$

Exercise: We can extend this from a one-dimensional slit pattern to a two-dimensional one. Calling the slit coordinates x_1 and y_1 with the screen coordinates x_2 and y_2, we will obtain

$$\phi'(x_2, y_2) \sim \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x_1, y_1) e^{-ik_x x_1} e^{-ik_y y_1} dx_1 dy_1$$

Consider a rectangular hole of size $a \times b$. The diffraction pattern will be proportional to

$$\int_{x_1=-a/2}^{a/2} \int_{y_1=-b/2}^{b/2} e^{-ik_x x_1} e^{-ik_y y_1} dx_1 dy_1$$

In this case the integrals are independent, so it is straightforward to show that the pattern will be the product of two sinc functions:

$$\frac{\sin(k_x a/2)}{k_x} \frac{\sin(k_y b/2)}{k_y}$$

6.6 Discrete Fourier Transforms

In many instances, particularly those involving digital processing, we have limited information. Instead of a continuous function $f(x)$ that stretches from minus to plus infinity, we will typically have a sequence of N samples $f_0, ..., f_{N-1}$ taken over a period $L = N\,\delta(x)$ say. If it is assumed that the $f(x)$ is also periodic with period L then the Fourier transform $\tilde{f}(k)$ will also only be non-zero for discrete values f_m when $k = 2\pi m/L$ i.e. it returns to being a Fourier series. This transform between a discrete real-space function and a discrete frequency-space one is called a discrete Fourier transform (DFT) and can be written in a symmetric fashion as

$$
\begin{aligned}
f_n &= \frac{1}{\sqrt{N}} \sum_{m=0}^{m=N-1} \tilde{f}_m e^{-i2\pi nm/N} \\
\tilde{f}_m &= \frac{1}{\sqrt{N}} \sum_{n=0}^{n=N-1} f_n e^{i2\pi nm/N}
\end{aligned}
$$

This is now straightforward to compute, and particularly efficient DFT algorithms called fast Fourier transforms (FFTs) are extensively used in modern computing applications.

7 Sturm-Liouville Theory

Symmetric matrices have some very useful properties you should remember:

- The eigenvalues of a symmetric matrix are real
- The eigenvectors of a symmetric matrix are orthogonal...
- ... and therefore form a complete basis set

i.e. in an n-dimensional vector space, any $n \times n$ symmetric matrix will have a set of n orthogonal eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ (which can be normalised to unity) and any vector in the space can be written as

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + ... + a_n\hat{\mathbf{e}}_n$$

Since this is an orthonormal basis, familiar techniques such as Pythagoras' theorem ($|\mathbf{a}|^2 = a_1^2 + a_2^2 + ... + a_n^2$) and taking the scalar product by multiplying components ($\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + ... + a_nb_n$) can be applied.

The equivalent in the vector space of functions is:

- Hermitian (i.e. self-adjoint) operators have real eigenvalues
- Their eigenfunctions form a complete, orthogonal set.
- *Any* function $g(x)$ can be expanded as a sum over eigenfunctions of a Hermitian operator

$$g(x) = a_0 f_0(x) + a_1 f_1(x) + a_2 f_2(x) + ...$$

A general second-order differential Hermitian operator is of the "Sturm-Liouville" form:

$$\frac{d}{dx}\left[A(x)\frac{d}{dx}\right] + B(x) \tag{30}$$

with appropriate boundary conditions (see problems), so it is useful to rearrange second order DEs to look like an eigenfunction equation:

$$\left(\frac{d}{dx}\left[A(x)\frac{d}{dx}\right] + B(x)\right) y_i = \lambda_i y_i \tag{31}$$

The solutions y_i are then orthogonal eigenfunctions forming a complete basis, and hence any other function can be written as a linear combination of these solutions.

Many common differential operators are of this form. Other second-order differential operators can be made Hermitian by multiplying them by an appropriate weight function, $w(x)$ (see problems). The eigenfunction equation then becomes the Sturm-Liouville equation:

$$\boxed{\frac{d}{dx}\left[A(x)y'\right] + B(x)y = \lambda w\, y} \tag{32}$$

The eigenfunctions are then orthogonal "under the weight function" w, that is with the modified definition of the scalar product:

$$\boxed{\langle f, g\rangle = \int_{x_0}^{x_1} f(x)^* g(x) w(x)\, dx} \tag{33}$$

8 Ordinary and Partial Differential Equations

8.1 Series Solutions

So far we have mainly encountered first and second order differential equations with constant coefficients. We will soon encounter equations where this is not the case, so sinusoidal functions will not be solutions.

One technique to solve such equations is to assume a polynomial solution, i.e.

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{34}$$

Consider substituting y as above into an equation of the form

$$f_2(x)y'' + f_1(x)y' + f_0(x)y = 0 \tag{35}$$

where the functions f_2, f_1 and f_0 are (or can be written as) polynomials in x. This will give an equation of the form

$$K_0x^0 + K_1x^1 + K_2x^2 ... + K_mx^m + ... = 0 \tag{36}$$

where K_m will depend on the functions f_0, f_1 and f_2 and the coefficients a_n of the power series (including, general terms where $m \neq n$).

Note that this is not an equation for an unknown x, but rather a statement that must be true for all values of x (just as with the original differential equation). Since the powers of x are independent, this means that each and every coefficient *must* be identically zero.

To see this, consider that when $x = 0$ equation 36 becomes $K_0 = 0$ so the zeroth term vanishes. We can then differentiate equation 36 and look at $x = 0$ again to prove that $K_1 = 0$, and so on to prove that $K_m = 0$ for all m.

By calculating an expression for K_m and setting it to be zero, we normally find a recursive relationship (or relationships) between the power series coefficients a_n called a **recurrence relation**. If one or more terms can be determined, the recurrence relation(s) may then give the complete series.

Exercise: To demonstrate the principle of series solutions, let us use the technique to solve the differential equation for a mass on a spring $m\ddot{x} + kx = 0$, given the initial conditions of $x = 0, \dot{x} = V$ at $t = 0$.

We assume a solution of the form $x = \sum_{n=0}^{\infty} a_n t^n$ i.e.

$$\begin{aligned} x &= a_0 + a_1t + a_2t^2 + a_3t^3 + ... + a_nt^n + ... \\ \Rightarrow \dot{x} &= 0 + a_1 + 2a_2t + 3a_3t^2 ... + na_nt^{n-1} + ... \\ \Rightarrow \ddot{x} &= 0 + 0 + 2a_2 + 3.2a_3t ... n(n-1)a_nt^{n-2} + ... \end{aligned}$$

We can now put these expressions into the equation, and it can be helpful to tabulate the coefficients as follows:

	t^0	t^1	t^2	t^3	t^n
$m\ddot{x}$	$2ma_2$	$6ma_3$			$(n+2)(n+1)ma_{n+2}$
$+kx$	ka_0	ka_1	ka_2	ka_3	ka_n

Each row corresponds to one term on the left-hand side of the differential equation, and each column collects all of the terms containing a particular power of t. Note that the term in the original series that provides the t^n power in the second differential must be $a_{n+2}t^{n+2}$.

We can then add the terms in each column to give us the series representation of the equation

$$(2ma_2 + ka_0) + (6ma_3 + ka_1)t + ... + ([n+2][n+1]ma_{n+2} + ka_n)t^n + ... = 0$$

This is an equation that is required to be true for all values of t, and the only way for this to be satisfied is for the coefficient of each power of t to be zero. Taking the t^n term that means

$$[n+2][n+1]ma_{n+2} + ka_n = 0 \Leftrightarrow a_{n+2} = -\frac{k}{[n+2][n+1]m}a_n$$

Writing $\omega^2 = k/m$ we have a recurrence relation

$$a_{n+2} = -\frac{\omega^2}{(n+2)(n+1)}a_n$$

giving two independent power series, one containing the even terms and one the odd.

We now apply the boundary condition that $x = 0$ when $t = 0$. This means that a_0 must be zero, and so by the recurrence relation all even terms are zero.

The second boundary condition is that $\dot{x}(t = 0) = a_1 = V$. We can now use the recurrence relation to generate the entire odd series:

$$\begin{aligned} x &= Vt - \frac{\omega^2}{(3)(2)}Vt^3 + \frac{\omega^2}{(5)(4)}\frac{\omega^2}{(3)(2)}Vt^5 + ... \\ &= \frac{V}{\omega}\left[\omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} + ...\right] \\ &= \frac{V}{\omega}\sin(\omega t) \end{aligned}$$

which should come as no surprise.

A generalisation of Equation 34 to

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \tag{37}$$

where r may be non-integer allows this method to be used for functions which have no Taylor series expansion (e.g. those with a singularity).

8.2 Partial Differential Equations

You have already encountered PDEs such as Schrödinger's equation or diffusion involving functions of time and one spatial dimension. We will also need to study PDEs in more than one spatial dimension. Such PDEs frequently involve the **Laplacian**, the second-order differential operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{38}$$

Note that the solution of PDEs normally leads to one or more ODEs.

Some of the most common PDEs are listed in the following table.

1D	3D	Name	Example
$\frac{d^2\phi}{dx^2} = 0$	$\nabla^2\phi = 0$	Laplace	Electrostatics, fluid flow
$\frac{d^2\phi}{dx^2} = \rho$	$\nabla^2\phi = \rho$	Poisson	Electrostatics with charge
$\frac{\partial^2\phi}{\partial x^2} = a\frac{\partial\phi}{\partial t}$	$\nabla^2\phi = a\frac{\partial\phi}{\partial t}$	Diffusion	
$-\frac{\partial^2\phi}{\partial x^2} = ia\frac{\partial\phi}{\partial t}$	$-\nabla^2\phi = ia\frac{\partial\phi}{\partial t}$	Schrödinger	Quantum Mechanics
$\frac{\partial^2\phi}{\partial x^2} = a\frac{\partial^2\phi}{\partial t^2}$	$\nabla^2\phi = a\frac{\partial^2\phi}{\partial t^2}$	Wave	

8.3 Solution by Fourier Transform

Fourier transforms can also be used to solve ODEs, or reduce PDEs to ODEs. Consider the Fourier transform of a differential equation involving $f(x, ...)$. This can be evaluated by first writing the function as the inverse transform

$$f(x, ...) = \int_{-\infty}^{+\infty} dk \tilde{f}(k, ...)e^{ikx} \tag{39}$$

The x dependence is now entirely in the exponential, and since the partial derivative with respect to x commutes with the k integral its evaluation is straightforward:

$$\frac{\partial f(x, ...)}{dx} = \int_{-\infty}^{+\infty} dk\,(ik)\tilde{f}(k, ...)e^{ikx} \tag{40}$$

and so on. Taking the Fourier transform of the entire equation will then give an equation for the transformed functions in which every differentiation with respect to x has been replaced by multiplying by a factor (ik).

Exercise: We will demonstrate how to use Fourier transforms to solve the diffusion equation

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $T(\vec{r}, t)$ is the temperature at a specific point, and α is a constant that is related to the thermal conductivity and specific heat of the material. For simplicity, we will consider a one-dimensional metal rod with negligible contact with its surroundings, so the diffusion equation simplifies to

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Let us take the Fourier transform of this equation with respect to x, leaving the time dependence unaffected. We know from Section 6.2 that the transform of the second differential with respect to x will introduce a factor of $-k^2$ on the left hand side, while the transform commutes with the time differential. Thus we have

$$-k^2 \tilde{T}(k,t) = \frac{1}{\alpha}\frac{\partial \tilde{T}(k,t)}{\partial t}$$

This is a standard first-order differential equation which can be solved by separating the variables

$$\int \frac{d\tilde{T}}{\tilde{T}} = -\alpha k^2 \int dt$$

As with e.g. radioactive decay, this can be solved to give

$$\tilde{T}(k,t) = \tilde{T}(k,0)e^{-\alpha k^2 t}$$

so if the Fourier transform of the initial temperature distribution is known, the transform at any later time can be calculated.

For example, take a simplistic (if unrealistic) initial temperature distribution as $T(x,0) = \delta(x)$, i.e. the temperature is zero everywhere except for a finite amount of heat deposited at $x = 0$. The Fourier transform of this initial state is easily found:

$$\tilde{T}(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \delta(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}$$

and so at a later time

$$\tilde{T}(k,t) = \frac{1}{\sqrt{2\pi}}e^{-\alpha k^2 t}$$

Of course, we want a solution in real space rather than Fourier space, so we have to take the inverse transform:

$$T(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\alpha k^2 t} e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-\alpha t k^2 + ikx\right) dk$$

As normal we can solve this by completing the square to make the exponential look like $-a(k-b)^2$ i.e.

$$-\alpha t k^2 + ikx = -\alpha t \left(k - \frac{ix}{2\alpha t}\right)^2 - \frac{x^2}{4\alpha t}$$

with the final term, which is a constant with respect to k, being there to subtract off the last term obtained when squaring the brackets. The integral then evaluates as

$$\begin{aligned} T(x,t) &= \frac{1}{2\pi} e^{-x^2/4\alpha t} \int_{-\infty}^{+\infty} \exp -\alpha t \left(k - \frac{ix}{2\alpha t}\right)^2 dk \\ &= \frac{1}{2\pi} e^{-x^2/4\alpha t} \sqrt{\frac{\pi}{\alpha t}} \end{aligned}$$

i.e.

$$T(x,t) = \frac{e^{-x^2/4\alpha t}}{2\sqrt{\pi \alpha t}}$$

This is a properly-normalised Gaussian, with a width that grows as $\sqrt{t}$ as expected for a "random-walk" type problem.

8.4 Separation of Variables

The technique of separating variables can be commonly used to solve first order differential equations, e.g. if the equation is of the form

$$A(x)\frac{dy}{dx} = B(x)C(y) \tag{41}$$

then it can be written as

$$\frac{1}{C(y)} dy = \frac{B(x)}{A(x)} dx \tag{42}$$

(apart from in the trivial cases that A or C are zero) and both sides integrated independently. This can sometimes be repeated to solve second or higher order equations.

A similar technique can be used to solve partial differential equations, by assuming that a solution can be found that is the product of separate functions in each variable. As an example, consider Laplace's equation $\nabla^2\phi = 0$. We will assume that a solution exists of the form

$$\phi = X(x)Y(y)Z(z) \tag{43}$$

Laplace's equation then becomes

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2}YZ + X\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}Z + XY\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0 \tag{44}$$

which can be rearranged to give

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0 \tag{45}$$

(for $X, Y, Z \neq 0$).

Now each term is a function of only one variable and therefore independent of the others. We can imagine changing x to any arbitrary value without changing y and z, but the the equation must still hold. The only possibility is that the x term itself is constant:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda \Leftrightarrow \frac{d^2X}{dx^2} = \lambda X \tag{46}$$

This is a familiar Sturm-Liouville type equation.

Of course, a similar argument can be applied to find equations for Y and Z.

Exercise: As an example of separating variables, consider the vibration of a rectangular membrane ("drum") of size $a \times b$. Calling the transverse displacement at a point $\phi(x, y)$, we will have boundary conditions $\phi = 0$ at $x = 0, a$ and $y = 0, b$, and the displacement will be described by the wave equation

$$\nabla^2\phi = \frac{1}{c^2}\ddot{\phi}$$

where c is the speed of transverse waves on the membrane. Since this is a two dimensional problem we can write explicitly in Cartesian coordinates

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

We now assume a solution of the form

$$\phi(x, y, t) = X(x)Y(y)T(t)$$

and substitute this into the wave equation to give either the trivial solution that $\phi = 0$ or

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = \frac{1}{Tc^2}\frac{\mathrm{d}^2T}{\mathrm{d}t^2}$$

This equation holds for any x, y and t which are all independent, and so we can say that since each term is only a function of a single variable they must each be equal to a constant. Taking the time dependence first,

$$\frac{1}{T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = \text{constant} = -\omega^2$$

say, or

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = -\omega^2 T$$

Similarly we have

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -k_x^2 X$$

and

$$\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = -k_y^2 Y$$

These all have the same form of solution e.g.

$$T(t) = Ae^{i\omega t} + Be^{-i\omega t} = \phi_0 \cos(\omega t + \delta)$$

where A and B or ϕ_0 and δ are constants of integration.

For X and Y, the solutions must be $X \sim \sin(k_x x)$ and $Y \sim \sin(k_y y)$ with $k_x = n\pi/a$ and $k_y = m\pi/b$ in order to satisfy the boundary conditions.

If we put these solutions into the equation

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0$$

we get

$$-k_x^2 - k_y^2 = -\frac{\omega^2}{c^2} \Rightarrow \omega^2 = \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2 c^2$$

so different spatial modes have different frequencies.

The full solution for a particular choice of n and m is then

$$\phi_{nm} = \Phi_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cos\left(\pi c\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, t + \delta_{nm}\right)$$

where Φ and δ are arbitrary constants. A general solution will be a linear combination of these eigenfunctions:

$$\phi = \sum_{n,m} \phi_{nm}$$

with the individual Φ_{nm} and δ_{nm} chosen to satisfy the initial conditions such as where and how hard the membrane was hit.

8.5 The Legendre Equation

The success of the separation of variables technique often relies on recognising the symmetry of the problem. Using Cartesian variables as above should give a solution for a rectangular box, but is unlikely to work for a cylinder or a sphere.

In spherical coordinates (r, θ, ϕ) Laplace's equation becomes

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \qquad (47)$$

Now assume a solution of the form $V = R(r)\Theta(\theta)\Phi(\phi)$ and substitute it into Equation 47.

Exercise: Substituting $V = R(r)\Theta(\theta)\Phi(\phi)$ into Equation 47 gives

$$\Theta\Phi\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + R\Phi\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Theta}{\partial \theta}\right) + R\Theta\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

The trivial solution is $V = 0$ otherwise we can multiply by $r^2\sin^2\theta/V$ to give

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Theta}{\partial \theta}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

The ϕ dependence will separate cleanly to give

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad (48)$$

where the sign of m^2 has been chosen to give solutions that are manifestly periodic in ϕ,

$$\Phi \sim e^{im\phi} \qquad (49)$$

(m is any integer) as may be expected for physical problems with rotational symmetry, i.e. $\Phi(2\pi) = \Phi(0)$.

The θ dependent part can be isolated and then simplified by substituting $x = \cos\theta$. The results in the **general Legendre equation**:

$$\boxed{(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left(K - \frac{m^2}{1-x^2}\right)\Theta = 0} \qquad (50)$$

where K is an arbitrary constant. Again, this is a Sturm-Liouville type equation (subject to boundary conditions etc.) as the first two terms are the differential of $(1-x^2)\Theta'$.

Exercise: To obtain the Legendre equation, we divide our original separated equation by V and substitute

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$$

to give

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{r^2\sin^2\theta} = 0$$

Once the r^2 has been cancelled we have separated R and Θ. The equation for R is

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \text{constant}$$

which turns out to have straightforward solutions (see later example).

The Θ equation is

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = \text{constant} = -K$$

say. It is conventional to make the substitution:

$$\begin{aligned} x &= \cos\theta \\ \Rightarrow dx &= -\sin\theta\, d\theta \\ 1-x^2 &= \sin^2\theta \\ \frac{\sin\theta}{d\theta} &= \frac{\sin^2\theta}{\sin\theta\, d\theta} = -\frac{1-x^2}{dx} \end{aligned}$$

The equation then becomes

$$\frac{1}{\Theta}\frac{-d}{dx}\left(-(1-x^2)\frac{d\Theta}{dx}\right) - \frac{m^2}{1-x^2} = -K$$

$$\Rightarrow \frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) + \left(K - \frac{m^2}{1-x^2}\right)\Theta = 0$$

and using the product rule on the first term gives the Legendre equation itself.

We will first examine the case when there is no ϕ dependence so $m = 0$ and Equation 50 becomes what is normally referred to simply as **Legendre's equation**:

$$\boxed{(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + K\Theta = 0} \tag{51}$$

Assuming a series solution of the form $\Theta = \sum_{n=0}^{\infty} a_n x^n$ and gathering the coefficients of x^n gives the recurrence relation

$$a_{n+2} = \frac{n(n+1)-K}{(n+1)(n+2)}a_n \tag{52}$$

which (for a given constant K) gives the solution as two independent series, one with odd terms and one with even terms, requiring two constants of integration.

Exercise: We have

$$\begin{aligned}\Theta(x) &= a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \\ \Rightarrow \Theta' &= a_1 + 2a_2 x + ... + n a_n x^{n-1} + ... \\ \Rightarrow \Theta'' &= 2a_2 + ... + n(n-1) a_n x^{n-2} + ...\end{aligned}$$

We then make the familiar table of coefficients with a row for each separate term in the equation:

	x^0	x^1	x^n
Θ''	$2a_2$		$(n+2)(n+1)a_{n+2}$
$-x^2\Theta''$	-		$-n(n-1)a_n$
$-2x\Theta'$	-	$-2a_1$	$-2na_n$
$K\Theta$	Ka_0	Ka_1	Ka_n

We can therefore say that the coefficient of x^n

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + Ka_n = 0$$

$$\Rightarrow a_{n+2} = \frac{n(n+1) - K}{(n+1)(n+2)} a_n$$

The ratio test will give

$$\rho = \lim_{n\to\infty} \frac{|a_{n+2}x^{n+2}|}{|a_n x^n|} = \lim_{n\to\infty} |x^2|$$

as the ratio a_{n+2}/a_n is given by the recurrence formula as $\sim n^2/n^2 \to 1$.

The ratio test applied to either the odd or even series shows that they converge for $|x| < 1$, but not in general when $|x| = 1$. However, from Equation 52 it can be seen that a_{n+2} will be zero if $K = n(n+1)$ for some n. And if a_{n+2} is zero then the recurrence relation says that $a_{n+4}, a_{n+6}....$ are also zero, i.e. a purely odd or even series will terminate at this point and hence will not diverge. Choosing

$$\boxed{K = l(l+1)} \tag{53}$$

where l is zero or a positive integer therefore ensures the solution is a finite series of order x^l, and in fact it the Legendre equations above are usually written using $l(l+1)$ rather than K.

Exercise: To ensure convergence at $|x| = 1$ we require the series to terminate at some $n = l$ say. This means that $a_{l+2} = 0$ but a_l is not, so from the recurrence relation

$$l(l+1) - K = 0$$

i.e.

$$K = l(l+1)$$

It turns out that setting $K = l(l+1)$ is the only way to get solutions bounded over the whole physical range $[-1, 1]$. In this case, the Legendre equation can be regarded as a Sturm-Liouville eigenvalue equation (equation 31) with eigenvalues $-l(l+1)$, so the corresponding eigenfunctions form a complete, orthogonal basis.

8.6 Legendre Polynomials

These solutions to Legendre's equation are called the **Legendre polynomials**, $P_l(x)$. They are orthogonal over the interval $-1 \leq x \leq 1$ with a uniform weight function, The first few (using the most common normalisation) are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2 - 1)/2 \\ P_3(x) &= (5x^3 - 3x)/2 \end{aligned}$$

The Legendre Polynomials can be expressed using **Rodrigues' formula**:

$$P_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{dx^l} (x^2 - 1)^l \tag{54}$$

They can also be obtained from a **generating function** i.e. as the coefficients in a Taylor series as follows:

$$\frac{1}{\sqrt{1 - 2hx + h^2}} = \sum_{l=0}^{\infty} P_l(x) h^l \tag{55}$$

This formula gives the "multipole" expansion of a potential due to a charge distribution.

Exercise: Consider a point charge q lying on the z axis some distance H from the origin. The potential due to this charge at an arbitrary point $\vec{r}$ will be

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 d}$$

where d is the direct distance to the charge.

The cosine rule gives d in terms of the polar angle of $\vec{r}$ as

$$d^2 = H^2 + r^2 - 2Hr\cos\theta$$

For convenience, we can define $h = H/r$ and $x = \cos\theta$ and write

$$d = r\sqrt{1 + h^2 - 2hx}$$

and so

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}\frac{1}{\sqrt{1 + h^2 - 2hx}}$$

This is normal potential due to a point charge at the origin multiplied by

$$\frac{1}{\sqrt{1 - 2hx + h^2}} = \sum_{l=0}^{\infty} P_l(x)h^l$$

so we should ask what the physical significance of the terms in the series are.

The $l = 0$ term is $P_0 h^0 = 1$, so this just represents the normal point charge potential. As there is a single charge, this is called a monopole.

The $l = 1$ term is $P_1(x)h = h\cos\theta$. This will give a potential that goes quickly to zero ($\sim 1/r^2$ as there is an r in both the h and in front). It will be positive for $z > 0$ and negative for $z < 0$. In fact,

$$\frac{qH\cos\theta}{4\pi\epsilon_0 r^2}$$

is the potential due to two equal and opposite charges $\pm q$ placed a distance H apart, an electric dipole of moment qH, centred at the origin. At large distances the effects of the charges cancel out (thus the $1/r^2$ dependence), but at smaller r the dipole structure can be seen.

Unsurprisingly, the higher l terms represent quadrupole and high-order multipole potentials. This formula thus relates the potential due to a charge displaced from the origin to a sum over multipoles placed at the origin. This can be a useful tool, especially as it allows the field due to arbitrary charge distributions to be calculated by summing the multipole formulations from each charge.

The general Legendre equation (equation 50) has solutions called **assocated Legendre functions**:

$$P_l^m(x) = (1 - x^2)^{m/2}\frac{d^m P_l(x)}{dx^m} \tag{56}$$

for $m \geq 0$ and P_l^{-m} proportional to P_l^m. As expected, these reduce to the Legendre polynomials when $m = 0$, and they are non-zero for $-l \leq m \leq l$.

Since the angular part of this solution $\Theta\Phi$ occurs in many physical problems, it is normal to define **spherical harmonics** as

$$Y_{lm}(\theta,\phi) = AP_l^{|m|}(\cos\theta)e^{im\phi} \tag{57}$$

where the normalisation constant A is chosen such that

$$\int_{4\pi} d\Omega |Y_{lm}|^2 = 1 \tag{58}$$

Exercise: As an example of Laplace's equation in spherical polar coordinates let us consider fluid flow past a sphere.

We will describe the flow using the velocity $\vec{v}(\vec{r})$ at any point. If the fluid is incompressible and there are no sources (or sinks) the flow must be divergenceless, i.e.

$$\vec{\nabla}.\vec{v} = 0$$

We will also assume that there is no rotation (i.e. vortices) present, so

$$\vec{\nabla} \times \vec{v} = 0$$

If this condition is satisfied for some vector field, it can be represented as the gradient of a potential, so we will write

$$\vec{v} = \vec{\nabla}V$$

and since the flow is divergenceless, the potential will satisfy Laplace's equation:

$$\nabla^2 V = 0$$

The sphere will be placed at the origin so will have a surface at $r = r_0$ where r_0 is the radius. At large r the fluid flow will be undisturbed by the sphere, and we will set it as a constant and uniform flow in the z direction: $\vec{v}_0 = v_0\hat{k}$.

Since there is no rotation and perfect azimuthal symmetry, we can say that there will be no ϕ dependence in the flow or potential, and hence we will assume a solution of the form

$$V = R(r)\Theta(\theta)$$

Inserting this Equation 47 gives the trivial $V = 0$ solution or

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = 0$$

As normal, we state that the theta-dependent term must be equal to a constant i.e.

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

which can be rewritten with $x = \cos\theta$ as the Legendre equation 51. We therefore know that the solutions to the angular part will be the Legendre polynomials, $\Theta \sim P_l(\cos\theta)$.

Substituting this back into the previous equation gives

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - l(l+1) = 0$$

$$\Rightarrow r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$$

We can guess that r^n will be a solution and try it in the equation:

$$n(n-1)r^n + 2nr^n - l(l+1)r^n = 0$$

$$\Rightarrow n(n+1) = l(l+1)$$

This actually has two solutions - the obvious $n = l$ plus $n = -(l+1)$. The general solution is thus

$$V = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)P_l(\cos\theta)$$

[Note that if there had been an azimuthal dependence, we would have had to solve the general Legendre equation and so we would be summing over $R_{lm}Y_l^m$ terms.]

We now apply boundary conditions. First, to obtain the far-field flow $\vec{v}_0 = v_0\hat{k}$ we will need a potential $V_0 = v_0 z = v_0 r\cos\theta$ at large r (you can check that $\vec{\nabla}V = v_o\hat{k}$). By considering the $a_1 r^l P_1 = a_1 r\cos\theta$ term we can see that $a_1 = v_0$. We can also see a_l for $l > 1$ must be zero to ensure that this is the dominant term at $r \to \infty$.

The a_0 term will be a constant, and hence have no physical significance (we only care about the gradient of the potential, not its absolute value). We will therefore take it to be zero, and have now determined all a_l coefficients.

Now consider the boundary condition at the surface of the sphere which is that the flow must be parallel to this surface. The simplest way to impose this is to set the radial flow to be zero, i.e.

$$\left.\frac{\partial V}{\partial r}\right|_{r=r_0} = \sum\frac{\partial R}{\partial r}P_l = 0$$

In order to be satisfied at every θ the coefficient of each P_l must be identically zero, so

$$\left.\frac{\partial}{\partial r}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)\right|_{r=r_0} = 0$$

For $l = 0$ this can only be the case if $b_0 = 0$ (even if a_0 had not been set to zero).

For each $l > 0$,

$$l a_l r^{l-1} = \frac{(l+1)b_l}{r^{l+2}}$$

so as the only non-zero a_l is a_1 the only non-zero b_l will be

$$b_1 = \left.\frac{a_1 r^3}{2}\right|_{r=r_0} = \frac{v_0 r_0^3}{2}$$

The solution is therefore

$$V = \left(v_0 r + \frac{v_0 r_0^3}{2r^2}\right)\cos\theta$$

which gives a flow of

$$\begin{aligned}\vec{v} = \vec{\nabla} V &\equiv \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} \\ &= v_0\left(1 - \frac{r_0^3}{r^3}\right)\cos\theta\hat{r} - v_0\left(1 + \frac{r_0^3}{2r^3}\right)\sin\theta\hat{\theta}\end{aligned}$$

8.7 Bessel's Equation

Now consider solving Laplace's equation in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{59}$$

Assuming a solution of the form $V = P(\rho)\Phi(\phi)Z(z)$, as in the case of spherical coordinates in Section 8.5, the ϕ dependence separates to give solutions of the form $\Phi = e^{im\phi}$.

The ρ dependence is given by the equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right) - \frac{m^2}{\rho^2}P = -K \tag{60}$$

(with K an arbitrary constant) which can be rearranged and with the substitution $x = \sqrt{K}\rho$ written as

$$\boxed{x^2\frac{d^2 P}{dx^2} + x\frac{dP}{dx} + (x^2 - m^2)P = 0} \tag{61}$$

which is **Bessel's equation**.

Bessel's equation can be solved using the series solution

$$P = x^s \sum_{n=0}^{\infty} a_n x^n \tag{62}$$

and collecting coefficients of x^n gives the recurrence relation

$$a_n = \frac{a_{n-2}}{m^2 - (s+n)^2} \tag{63}$$

Looking at the coefficients of x^s reveals that $s = \pm m$.

The general solutions to Equation 61 are called **Bessel functions** of the mth order and are of the form

$$J_m(x) = x^m(a_0 + ...) \tag{64}$$

These are always infinite but converge, and are oscillatory functions (but without a regular periodicity).

Exercise: Let us go back to vibrating membranes, but now consider a circular membrane with radius ρ_0. The displacement ψ will still satisfy the wave equation, but we will use the more appropriate separation of variables in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

We are two dimensions, so we can ignore the z part. We therefore assume a solution of the form $\psi(\rho,\theta,t) = P(\rho)\Phi(\phi)T(t)$ which we substitute into the wave equation and for $\psi \neq 0$ obtain

$$\frac{1}{P\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right) + \frac{1}{\Phi\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = \frac{1}{Tc^2}\frac{\partial^2 T}{\partial t^2}$$

Applying the normal arguments gives

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2 \Rightarrow T \sim e^{i\omega t}$$

and

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2 \Rightarrow \Phi \sim e^{im\phi}$$

Inserting these into the full equation then gives

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) - \frac{m^2}{\rho^2} = -\frac{\omega^2}{c^2}$$

$$\Rightarrow \rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \left(\frac{\omega^2\rho^2}{c^2} - m^2\right)P = 0$$

We now substitute $x = \omega\rho/c \to dx = \omega d\rho/c$ and $x/dx = \rho/d\rho$ to get

$$x\frac{d}{dx}\left(x\frac{dP}{dx}\right) + \left(x^2 - m^2\right)P = x^2\frac{d^2P}{dx^2} + x\frac{dP}{dx} + \left(x^2 - m^2\right)P = 0$$

which is Bessel's equation and has Bessel functions as its solutions. We note that this is not a Sturm-Liouville equation, but we can make it so with an integrating factor

$$w = \frac{1}{x^2} e^{\int x/x^2 dx} = \frac{1}{x}$$

This means the solutions will be orthogonal under the weight $w = 1/x$.

Try a generalised series solution $P = J = x^s \sum a_n x^n$:

$$\begin{aligned}
J &= a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + a_3 x^{s+3} + ... + a_n x^{s+n} + ... \\
\Rightarrow J' &= s a_0 x^{s-1} + (s+1) a_1 x^s + (s+2) a_2 x^{s+1} + (s+3) a_3 x^{s+2} + ... + (s+n) a_n x^{s+n-1} + ... \\
\Rightarrow J'' &= s(s-1) a_0 x^{s-2} + (s+1)(s) a_1 x^{s-1} + (s+2)(s+1) a_2 x^s \\
&\quad + (s+3)(s+2) a_3 x^{s+1} + ... + (s+n)(s+n-1) a_n x^{s+n-2} + ...
\end{aligned}$$

We will make a table to help us collate the coefficients as before:

	x^s	x^{s+1}	x^{s+n}
$x^2 J''$	$s(s-1)a_0$	$(s+1)(s)a_1$	$(s+n)(s+n-1)a_n$
xJ'	sa_0	$(s+1)a_1$	$(s+n)a_n$
$x^2 J$	n/a	n/a	a_{n-2}
$-m^2 J$	$-m^2 a_0$	$-m^2 a_1$	$-m^2 a_n$

Requiring the coefficient of x^{s+n} to be zero means that

$$(s+n)(s+n-1)a_n + (s+n)a_n + a_{n-2} - m^2 a_n = 0$$

$$\Rightarrow a_n = \frac{a_{n-2}}{m^2 - (s+n)^2}$$

which gives us the recurrence relation for Bessel functions (of the first kind), but we still need to find s.

From the coefficients of x^s we can see

$$s(s-1)a_0 + sa_0 - m^2 a_0 = 0$$

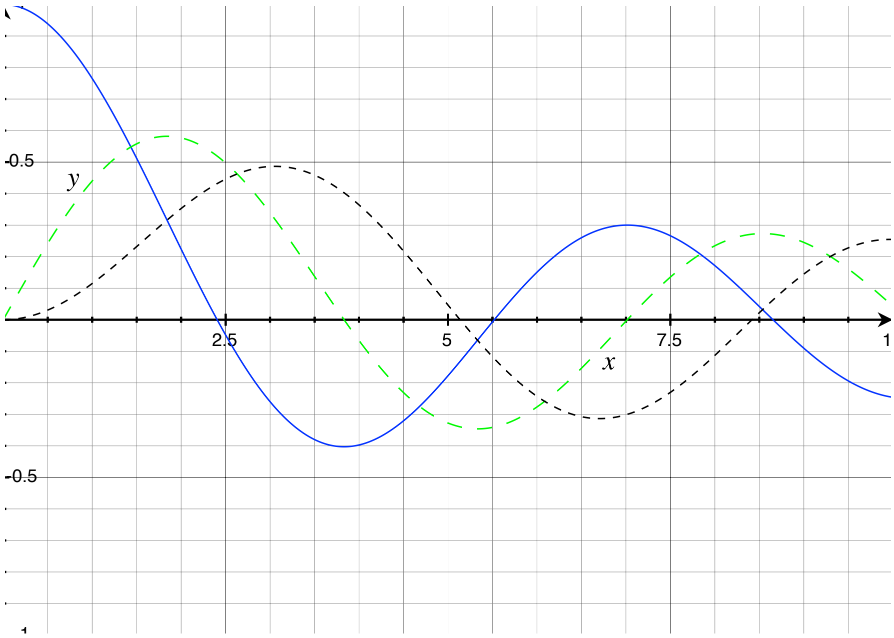
which for a non-zero a_0 means $s^2 = m^2$ or $s = \pm m$.

The ratio test gives

$$\rho = \lim_{n \to \infty} \left| \frac{a_n x^n}{a_{n-2} x^{n-2}} \right| = \lim_{n \to \infty} \frac{|x^2|}{n^2} = 0$$

demonstrating absolute convergence.

The first three Bessel functions J_0 (solid blue), J_1 (long dash green) and J_2 (short dash grey) are shown.



The general solution will therefore be

$$\psi = \sum_{n,m} A_{nm} J_m(k_{nm}\rho) e^{im\phi} e^{i\omega_{nm} t}$$

where m can be positive or negative and gives the number of azimuthal nodes, k_{nm} are all possible values that satisfy the boundary condition $J_m(k_{nm}\rho_0) = 0$ (so $n = 0, 1...$ can give the number of radial nodes), and $\omega_{nm} = k_{nm}c$. The A_{nm}s are constants that can be set to match initial or other boundary conditions.