Forced Oscillators

3.1. Consider a harmonic oscillator of mass M connected to a fixed wall and to a moving wall by springs k and K as illustrated in Fig. 1.

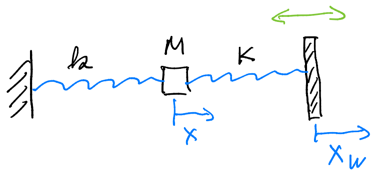

Figure 1: Oscillator coupled to a moving wall.

The moving wall is very heavy, so its movement is not affected by the mass M. The position of the moving wall is given by $x_w = D\cos(\omega t)$. The mass is also subjected to friction with damping constant b.
Find the movement of the mass in the long time limit.

3.2. A piece of very sensitive equipment of mass M needs to be fixed to the floor of the space station. (i.e. no gravity). The floor vibrates with the frequency ω and an amplitude A. Vibrations at this frequency or higher may damage the equipment; vibrations of a lower frequency are not problematic. A way to fix it to the floor needs to be found to reduce the amplitude of vibration at this frequency and higher, i.e. to "filter out high frequencies". There are various ways of fixing the equipment to the floor. The direct one is to use screws and fix it directly to the floor, in which case it vibrates with the same amplitude A as the floor. Another possibility that is considered, is to fix it via a spring of spring constant K as illustrated in the figure.

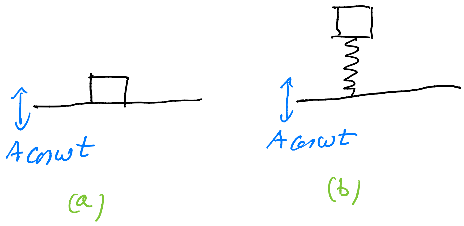

Figure 2: (a) Equipment fixed to the vibrating floor, or (b) attached to an oscillator that's fixed to the floor.

Explain why this setup is useful. Taking damping to be negligible, what should be the spring constant K in order to reduce the amplitude of vibration at frequency ω and higher by at least $1/2$?

Angular Momentum

3.3. Consider two vectors $\vec{v}$ and $\vec{w}$ given by

$$\begin{aligned} \vec{v} &= 2\hat{i} + \hat{j} + 3\hat{k} \\ \vec{w} &= 2\hat{i} - 2\hat{j} + \hat{k} \end{aligned}$$

with $\hat{i}$, $\hat{j}$, and $\hat{k}$ being unit vectors along the x, y and z axis.

(a) Compute the cross product

$$\vec{u} = \vec{v} \times \vec{w}\,.$$

(b) Show that $\vec{u}$ is orthogonal on $\vec{v}$ and on $\vec{w}$.

3.4. Consider an object of mass $M = 1kg$, situated at a point of coordinates $\vec{r}_1 = (0, 1, 4)$ (in meters) and moving with velocity $1m/s$ in the negative x direction.

(a) Make a drawing of the situation.
(b) Calculate the angular momentum relative to the origin and mark it on the drawing.
(c) Find the unit vectors in the direction of the position, velocity and angular momentum of the particle.
(d) Consider 3 more particles, situated at $\vec{r}_2 = (5, 1, 3)$, $\vec{r}_3 = (5, 2, 8)$, and $\vec{r}_4 = (3, -1, 4)$ (again in meters). Tell which of them, if any, are on the plane defined by the position and velocity of the first particle.

3.5. Consider a particle of mass m located at position $\vec{r}$ in a frame of reference with origin O. The particle moves with velocity $\vec{v}$ and is acted upon by a total force $\vec{F}$.

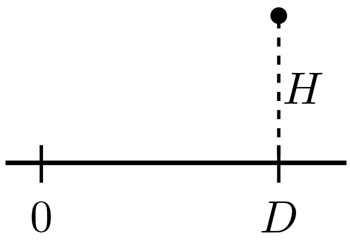

Figure 3: Sketch of the starting point of the particle

(a) Write the formulas for the angular momentum $\vec{L}$ and the torque $\vec{\tau}$ relative to the point O. Give the formulas both in vectorial notation as well as explicitly for the x, y and z components.
(b) Prove that

$$\frac{d\vec{L}}{dt} = \vec{\tau}\,.$$

(c) Consider now that O denotes a point on the ground and the particle is at horizontal distance D and height H from O, as illustrated in Figure 3. For concreteness, let x and y be in the horizontal plane and z in the vertical direction. The particle is located initially at $\vec{r}_0 = (D, 0, H)$. The particle is in the gravitational field of the Earth. The initial velocity of the particle is zero and then it falls to the ground. Calculate the time T that it takes the particle to hit the ground and the velocity $\vec{v}_{final}$ with which it hits the ground.
(d) Let $\vec{L}_{in}$ denote the initial angular momentum relative to O and and $\vec{L}_{final}$ the angular momentum relative to O when the particle reaches the ground. Calculate $\vec{L}_{in}$ and $\vec{L}_{final}$.
(e) Calculate the torque $\vec{\tau}(t)$ at an arbitrary time t during the fall.
(f) A consequence of the formula

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

is that

$$\vec{L}_{final} - \vec{L}_{in} = \int_0^T \vec{\tau}(t)dt\,.$$

Show that this formula is valid in our case by separately computing the LHS and RHS of this equation.

3.6. Consider two point-like particles of mass M_1 and M_2 and velocities $\vec{v}_1$ and $\vec{v}_2$ respectively that undergo a collision. Calculate the angular momentum relative to some arbitrary point O just before the collision and just after it and show that it is conserved. Does your result depend on whether the collision is elastic or plastic?

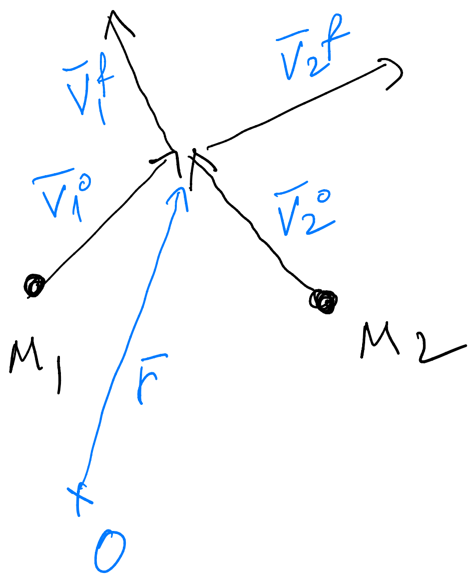

Figure 4: Two particle collision

3.7. Consider the situation illustrated in Fig. 5. On a table with a small hole in the middle, there is an object of mass m, connected to a string. The string goes through the hole and at the other end is connected to a mass M. Consider that the string has zero mass, and that there is no friction. Let the distance of the mass m from the hole be $\vec{r}_0$. Suppose that we start with the system at rest, by holding the mass M in place. We then give the mass m a velocity $\vec{v}_0$ orthogonal to the string. Calculate $\vec{r}_{min}$, the minimal distance from the hole the mass m can reach.

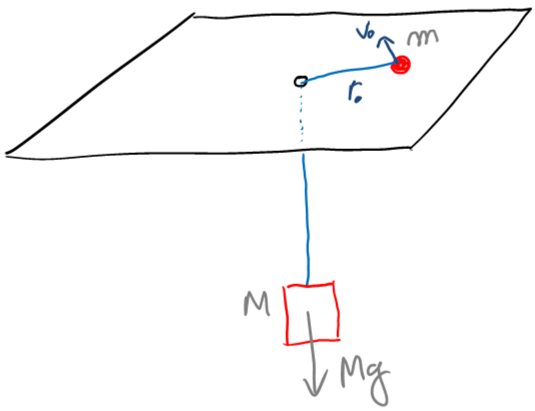

Figure 5: Particle on a table

3.8. Consider a system of two particles of masses m_1 and m_2 and described by positions $\vec{r}_1$ and $\vec{r}_2$. Consider furthermore two fictitious particles, one of mass $M = m_1 + m_2$ and position $\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$ (i.e. the centre of mass of the two masses), and one of the mass $\mu = \frac{m_1 m_2}{M}$, the "reduced mass" of the two particles and of position $\vec{r}_r = \vec{r}_1 - \vec{r}_2$, (i.e.the relative position of the two particles).

(a) Calculate the velocities $\vec{v}_{CM}$ and $\vec{v}_r$ of the two fictitious masses.
(b) Calculate the kinetic energy of the two fictitious masses and show that their sum equals to the kinetic energy of the two original masses.
(c) Show that $\vec{L}$, the total angular momentum of the two original particles versus the origin, is equal to the sum of the angular momenta of the two fictitious particles versus their origins, i.e.

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{r}_{CM} \times \vec{p}_{CM} + \vec{r}_r \times \vec{p}_r$$

where $\vec{p}_{CM} = M\vec{v}_{CM}$ and $\vec{p}_r = \mu\vec{v}_r$.

Polar Coordinates

3.9. Consider a particle of mass $m = 1kg$ located on the x, y plane at $\vec{r} = 2\hat{i} + 3\hat{j}$ with the distance given in meters and where $\hat{i}$ and $\hat{j}$ are the unit vectors along the x and y axes. The particle has velocity $\vec{v} = 5\hat{i} + \hat{j}$ in units of meters/second. Consider now the problem in polar coordinates.

(a) Calculate its the distance $\vec{r}$ to the origin.
(b) Calculate the unit vectors $\vec{r}$ and $\hat{e}$.
(c) Calculate $\tan(\varphi)$.
(d) Calculate the angular speed $\frac{d\varphi}{dt}$.
(e) Calculate the radial component of the velocity as well as the component orthogonal to $\vec{r}$.
(f) Calculate the angular momentum around the origin.