CHAPTER ONE

UNITS AND MEASUREMENT

1.1 INTRODUCTION

Measurement of any physical quantity involves comparison with a certain basic, arbitrarily chosen, internationally accepted reference standard called **unit**. The result of a measurement of a physical quantity is expressed by a number (or numerical measure) accompanied by a unit. Although the number of physical quantities appears to be very large, we need only a limited number of units for expressing all the physical quantities, since they are inter-related with one another. The units for the fundamental or base quantities are called **fundamental** or **base units**. The units of all other physical quantities can be expressed as combinations of the base units. Such units obtained for the derived quantities are called **derived units**. A complete set of these units, both the base units and derived units, is known as the **system of units**.

1.2 THE INTERNATIONAL SYSTEM OF UNITS

In earlier time scientists of different countries were using different systems of units for measurement. Three such systems, the CGS, the FPS (or British) system and the MKS system were in use extensively till recently.

The base units for length, mass and time in these systems were as follows :

- In CGS system they were centimetre, gram and second respectively.
- In FPS system they were foot, pound and second respectively.
- In MKS system they were metre, kilogram and second respectively.

The system of units which is at present internationally accepted for measurement is the *Système Internationale d' Unites* (French for International System of Units), abbreviated as SI. The SI, with standard scheme of symbols, units and abbreviations, developed by the Bureau International des Poids et measures (The International Bureau of Weights and Measures, BIPM) in 1971 were recently revised by the General Conference on Weights and Measures in November 2018. The scheme is now for international usage in scientific, technical, industrial and commercial work. Because SI units used decimal system, conversions within the system are quite simple and convenient. We shall follow the SI units in this book.

In SI, there are seven base units as given in Table 1.1. Besides the seven base units, there are two more units that are defined for (a) plane angle $d\theta$ as the ratio of length of arc ds to the radius r and (b) solid angle $d\Omega$ as the ratio of the intercepted area dA of the spherical surface, described about the apex O as the centre, to the square of its radius r, as shown in Fig. 1.1(a) and (b) respectively. The unit for plane angle is radian with the symbol rad and the unit for the solid angle is steradian with the symbol sr. Both these are dimensionless quantities.

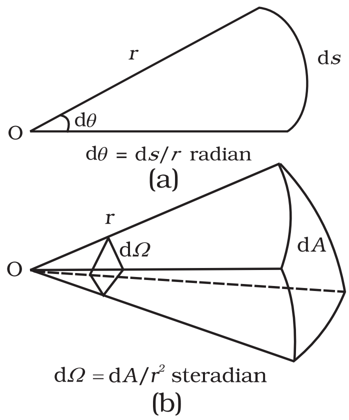


Fig. 1.1 *Description of (a) plane angle $d\theta$ and (b) solid angle $d\Omega$.*

Table 1.1 **SI Base Quantities and Units***

Base quantity	SI Units		
	Name	Symbol	Definition
Length	metre	m	The metre, symbol m, is the SI unit of length. It is defined by taking the fixed numerical value of the speed of light in vacuum c to be 299792458 when expressed in the unit m s^{-1}, where the second is defined in terms of the caesium frequency $\Delta \nu cs$.
Mass	kilogram	kg	The kilogram, symbol kg, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant h to be 6.62607015 10^{-34} when expressed in the unit J s, which is equal to kg $m^2 s^{-1}$, where the metre and the second are defined in terms of c and $\Delta \nu cs$.
Time	second	s	The second, symbol s, is the SI unit of time. It is defined by taking the fixed numerical value of the caesium frequency $\Delta \nu cs$, the unperturbed ground-state hyperfine transition frequency of the caesium-133 atom, to be 9192631770 when expressed in the unit Hz, which is equal to s^{-1}.
Electric	ampere	A	The ampere, symbol A, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge e to be 1.602176634 10^{-19} when expressed in the unit C, which is equal to A s, where the second is defined in terms of $\Delta \nu cs$.
Thermo dynamic Temperature	kelvin	K	The kelvin, symbol K, is the SI unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant k to be 1.380649 10^{-23} when expressed in the unit J K^{-1}, which is equal to kg $m^2 s^{-2} k^{-1}$, where the kilogram, metre and second are defined in terms of h, c and $\Delta \nu cs$.
Amount of substance	mole	mol	The mole, symbol mol, is the SI unit of amount of substance. One mole contains exactly 6.02214076 10^{23} elementary entities. This number is the fixed numerical value of the Avogadro constant, N_A, when expressed in the unit mol^{-1} and is called the Avogadro number. The amount of substance, symbol n, of a system is a measure of the number of specified elementary entities. An elementary entity may be an atom, a molecule, an ion, an electron, any other particle or specified group of particles.
Luminous intensity	candela	cd	The candela, symbol cd, is the SI unit of luminous intensity in given direction. It is defined by taking the fixed numerical value of the luminous efficacy of monochromatic radiation of frequency 540 10^{12} Hz, K_{cd}, to be 683 when expressed in the unit lm W^{-1}, which is equal to cd sr W^{-1}, or cd sr $kg^{-1} m^{-2} s^3$, where the kilogram, metre and second are defined in terms of h, c and $\Delta \nu cs$.

* *The values mentioned here need not be remembered or asked in a test. They are given here only to indicate the extent of accuracy to which they are measured. With progress in technology, the measuring techniques get improved leading to measurements with greater precision. The definitions of base units are revised to keep up with this progress.*

Table 1.2 **Some units retained for general use (Though outside SI)**

Name	Symbol	Value in SI Unit
minute	min	60 s
hour	h	60 min = 3600 s
day	d	24 h = 86400 s
year	y	365.25 d = 3.156×10^7 s
degree	°	$1° = (\pi/\ 180)$ rad
litre	L	I $dm^3 = 10^{-3}\ m^3$
tonne	t	10^3 kg
carat	c	200 mg
bar	bar	0.1 MPa = 10^5 Pa
curie	Ci	$3.7 \times 10^{10}\ s^{-1}$
roentgen	R	2.58×10^{-4} C/kg
quintal	q	100 kg
barn	b	100 $fm^2 = 10^{-28}\ m^2$
are	a	1 $dam^2 = 10^2\ m^2$
hectare	ha	1 $hm^2 = 10^4\ m^2$
standard atmospheric pressure	atm	101325 Pa = 1.013×10^5 Pa

Note that when mole is used, the elementary entities must be specified. These entities may be atoms, molecules, ions, electrons, other particles or specified groups of such particles.

We employ units for some physical quantities that can be derived from the seven base units (Appendix A 6). Some derived units in terms of the SI base units are given in (Appendix A 6.1). Some SI derived units are given special names (Appendix A 6.2) and some derived SI units make use of these units with special names and the seven base units (Appendix A 6.3). These are given in Appendix A 6.2 and A 6.3 for your ready reference. Other units retained for general use are given in Table 1.2.

Common SI prefixes and symbols for multiples and sub-multiples are given in Appendix A2. General guidelines for using symbols for physical quantities, chemical elements and nuclides are given in Appendix A7 and those for SI units and some other units are given in Appendix A8 for your guidance and ready reference.

1.3 SIGNIFICANT FIGURES

As discussed above, every measurement involves errors. Thus, the result of measurement should be reported in a way that indicates the precision of measurement. Normally, the reported result of measurement is a number that includes all digits in the number that are known reliably plus the first digit that is uncertain. The reliable digits plus the first uncertain digit are known as **significant digits** or **significant figures**. If we say the period of oscillation of a simple pendulum is 1.62 s, the digits 1 and 6 are reliable and certain, while the digit 2 is uncertain. Thus, the measured value has three significant figures. The length of an object reported after measurement to be 287.5 cm has four significant figures, the digits 2, 8, 7 are certain while the digit 5 is uncertain. Clearly, reporting the result of measurement that includes more digits than the significant digits is superfluous and also misleading since it would give a wrong idea about the precision of measurement.

The rules for determining the number of significant figures can be understood from the following examples. Significant figures indicate, as already mentioned, the precision of measurement which depends on the least count of the measuring instrument. **A choice of change of different units does not change the number of significant digits or figures in a measurement.** This important remark makes most of the following observations clear:

(1) For example, the length 2.308 cm has four significant figures. But in different units, the same value can be written as 0.02308 m or 23.08 mm or 23080 μm.

All these numbers have the same number of significant figures (digits 2, 3, 0, 8), namely four.

This shows that the location of decimal point is of no consequence in determining the number of significant figures.

The example gives the following rules :

- **All the non-zero digits are significant.**
- **All the zeros between two non-zero digits are significant, no matter where the decimal point is, if at all.**
- **If the number is less than 1, the zero(s) on the right of decimal point but to the left of the first non-zero digit are not significant.** [In $\underline{0}.\underline{00}$ 2308, the underlined zeroes are not significant].
- **The terminal or trailing zero(s) in a number without a decimal point are not significant.**

  [Thus 123 m = 12300 cm = 123000 mm has *three* significant figures, the trailing zero(s) being not significant.] However, you can also see the next observation.
- **The trailing zero(s) in a number with a decimal point are significant.**
  [The numbers 3.500 or 0.06900 have four significant figures each.]

(2) There can be some confusion regarding the trailing zero(s). Suppose a length is reported to be 4.700 m. It is evident that the zeroes here are meant to convey the precision of measurement and are, therefore, significant. [If these were not, it would be superfluous to write them explicitly, the reported measurement would have been simply 4.7 m]. Now suppose we change units, then

4.700 m = 470.0 cm = 4700 mm = 0.004700 km

Since the last number has trailing zero(s) in a number with no decimal, we would conclude erroneously from observation (1) above that the number has *two* significant figures, while in fact, it has four significant figures and a mere change of units cannot change the number of significant figures.

(3) **To remove such ambiguities in determining the number of significant figures, the best way is to report every measurement in scientific notation (in the power of 10).** In this notation, every number is expressed as $a \times 10^b$, where a is a number between 1 and 10, and b is any positive or negative exponent (or power) of 10. In order to get an approximate idea of the number, we may round off the number a to 1 (for $a \le 5$) and to 10 (for $5 < a \le 10$). Then the number can be expressed approximately as 10^b in which the exponent (or power) b of 10 is called **order of magnitude** of the physical quantity. When only an estimate is required, the quantity is of the order of 10^b. For example, the diameter of the earth (1.28×10^7m) is of the order of 10^7m with the order of magnitude 7. The diameter of hydrogen atom (1.06×10^{-10}m) is of the order of 10^{-10}m, with the order of magnitude –10. Thus, the diameter of the earth is 17 orders of magnitude larger than the hydrogen atom.

It is often customary to write the decimal after the first digit. Now the confusion mentioned in (a) above disappears :

$$4.700 \text{ m} = 4.700 \times 10^2 \text{ cm}$$
$$= 4.700 \times 10^3 \text{ mm} = 4.700 \times 10^{-3} \text{ km}$$

The power of 10 is irrelevant to the determination of significant figures. However, all zeroes appearing in the base number in the scientific notation are significant. Each number in this case has *four* significant figures.

Thus, in the scientific notation, no confusion arises about the trailing zero(s) in the base number a. They are always significant.

(4) The scientific notation is ideal for reporting measurement. But if this is not adopted, we use the rules adopted in the preceding example :

- **For a number greater than 1, without any decimal, the trailing zero(s) are not significant.**
- **For a number with a decimal, the trailing zero(s) are significant.**

(5) The digit 0 conventionally put on the left of a decimal for a number less than 1 (like 0.1250) is never significant. However, the zeroes at the end of such number are significant in a measurement.

(6) The multiplying or dividing factors which are neither rounded numbers nor numbers representing measured values are exact and have infinite number of significant digits. For example in $r = \dfrac{d}{2}$ or $s = 2\pi r$, the factor 2 is an exact number and it can be written as 2.0, 2.00

or 2.0000 as required. Similarly, in $T = \frac{t}{n}$, n is an exact number.

1.3.1 Rules for Arithmetic Operations with Significant Figures

The result of a calculation involving approximate measured values of quantities (i.e. values with limited number of significant figures) must reflect the uncertainties in the original measured values. It cannot be more accurate than the original measured values themselves on which the result is based. In general, the final result should not have more significant figures than the original data from which it was obtained. Thus, if mass of an object is measured to be, say, 4.237 g (four significant figures) and its volume is measured to be 2.51 cm^3, then its density, by mere arithmetic division, is 1.68804780876 g/cm^3 upto 11 decimal places. It would be clearly absurd and irrelevant to record the calculated value of density to such a precision when the measurements on which the value is based, have much less precision. The following rules for arithmetic operations with significant figures ensure that the final result of a calculation is shown with the precision that is consistent with the precision of the input measured values :

(1) **In multiplication or division, the final result should retain as many significant figures as are there in the original number with the least significant figures.**

Thus, in the example above, density should be reported to *three* significant figures.

$$\text{Density} = \frac{4.237\text{g}}{2.51\ \text{cm}^3} = 1.69\ \text{g cm}^{-3}$$

Similarly, if the speed of light is given as 3.00×10^8 m s^{-1} (three significant figure) and one year (1y = 365.25 d) has 3.1557×10^7 s (*five* significant figures), the light year is 9.47×10^{15} m (*three* significant figures).

(2) **In addition or subtraction, the final result should retain as many decimal places as are there in the number with the least decimal places.**

For example, the sum of the numbers 436.32 g, 227.2 g and 0.301 g by mere arithmetic addition, is 663.821 g. But the least precise measurement (227.2 g) is correct to only one decimal place. The final result should, therefore, be rounded off to 663.8 g.

Similarly, the difference in length can be expressed as :

$0.307\text{ m} - 0.304\text{ m} = 0.003\text{ m} = 3 \times 10^{-3}\text{ m}.$

Note that we should not use the *rule* (1) applicable for multiplication and division and write 664 g as the result in the example of **addition** and 3.00×10^{-3} m in the example of **subtraction**. They do not convey the precision of measurement properly. For addition and subtraction, the rule is in terms of decimal places.

1.3.2 Rounding off the Uncertain Digits

The result of computation with approximate numbers, which contain more than one uncertain digit, should be rounded off. The rules for rounding off numbers to the appropriate significant figures are obvious in most cases. A number $2.74\underline{6}$ rounded off to three significant figures is 2.75, while the number 1.743 would be 1.74. The *rule* by convention is that the **preceding digit is raised by 1 if the insignificant digit to be dropped (the underlined digit in this case) is more than 5, and is left unchanged if the latter is less than 5**. But what if the number is $2.74\underline{5}$ in which the insignificant digit is 5. Here, the convention is that **if the preceding digit is even, the insignificant digit is simply dropped and, if it is odd, the preceding digit is raised by** 1. Then, the number $2.74\underline{5}$ rounded off to three significant figures becomes 1.74. On the other hand, the number $2.73\underline{5}$ rounded off to three significant figures becomes 1.74 since the preceding digit is odd.

In any involved or complex multi-step calculation, you should retain, in intermediate steps, one digit more than the significant digits and round off to proper significant figures at the end of the calculation. Similarly, a number known to be within many significant figures, such as in 2.99792458×10^8 m/s for the speed of light in vacuum, is rounded off to an approximate value 3×10^8 m/s , which is often employed in computations. Finally, remember that exact numbers that appear in formulae like 2π in $T = 2\pi\sqrt{\frac{L}{g}}$, have a large (infinite) number of significant figures. The value of π = 3.1415926.... is known to a large number of significant figures. You may take the value as 3.142 or 3.14 for π, with limited number of significant figures as required in specific cases.

> ***Example 1.1*** Each side of a cube is measured to be 7.203 m. What are the total surface area and the volume of the cube to appropriate significant figures?

Answer The number of significant figures in the measured length is 4. The calculated area and the volume should therefore be rounded off to 4 significant figures.

Surface area of the cube $= 6(7.203)^2 \text{ m}^2$
$= 311.299254 \text{ m}^2$
$= 311.3 \text{ m}^2$

Volume of the cube $= (7.203)^3 \text{ m}^3$
$= 373.714754 \text{ m}^3$
$= 373.7 \text{ m}^3$

> ***Example 1.2*** 5.74 g of a substance occupies 1.2 cm^3. Express its density by keeping the significant figures in view.

Answer There are 3 significant figures in the measured mass whereas there are only 2 significant figures in the measured volume. Hence the density should be expressed to only 2 significant figures.

$$\text{Density} = \frac{5.74}{1.2} \text{ g cm}^{-3}$$
$$= 4.8 \text{ g cm}^{-3}.$$

1.3.3 Rules for Determining the Uncertainty in the Results of Arithmetic Calculations

The rules for determining the uncertainty or error in the number/measured quantity in arithmetic operations can be understood from the following examples.

(1) If the length and breadth of a thin rectangular sheet are measured, using a metre scale as 16.2 cm and, 10.1 cm respectively, there are three significant figures in each measurement. It means that the length l may be written as

$$l = 16.2 \pm 0.1 \text{ cm}$$
$$= 16.2 \text{ cm} \pm 0.6 \text{ \%}.$$

Similarly, the breadth b may be written as

$$b = 10.1 \pm 0.1 \text{ cm}$$
$$= 10.1 \text{ cm} \pm 1 \text{ \%}$$

Then, the error of the product of two (or more) experimental values, using the combination of errors rule, will be

$$l\,b = 163.62 \text{ cm}^2 \pm 1.6\%$$
$$= 163.62 \pm 2.6 \text{ cm}^2$$

This leads us to quote the final result as

$$l\,b = 164 \pm 3 \text{ cm}^2$$

Here 3 cm^2 is the uncertainty or error in the estimation of area of rectangular sheet.

(2) **If a set of experimental data is specified to n significant figures, a result obtained by combining the data will also be valid to n significant figures.**

However, if data are subtracted, the number of significant figures can be reduced.

For example, 12.9 g – 7.06 g, both specified to three significant figures, cannot properly be evaluated as 5.84 g but only as 5.8 g, as uncertainties in subtraction or addition combine in a different fashion (smallest number of decimal places rather than the number of significant figures in any of the number added or subtracted).

(3) **The relative error of a value of number specified to significant figures depends not only on n but also on the number itself.**

For example, the accuracy in measurement of mass 1.02 g is ± 0.01 g whereas another measurement 9.89 g is also accurate to ± 0.01 g. The relative error in 1.02 g is

$$= (\pm 0.01/1.02) \times 100 \text{ \%}$$
$$= \pm 1\%$$

Similarly, the relative error in 9.89 g is

$$= (\pm 0.01/9.89) \times 100 \text{ \%}$$
$$= \pm 0.1 \text{ \%}$$

Finally, remember that **intermediate results in a multi-step computation should be calculated to one more significant figure in every measurement than the number of digits in the least precise measurement.** These should be justified by the data and then the arithmetic operations may be carried out;

otherwise rounding errors can build up. For example, the reciprocal of 9.58, calculated (after rounding off) to the same number of significant figures (three) is 0.104, but the reciprocal of 0.104 calculated to three significant figures is 9.62. However, if we had written 1/9.58 = 0.1044 and then taken the reciprocal to three significant figures, we would have retrieved the original value of 9.58.

This example justifies the idea to retain one more extra digit (than the number of digits in the least precise measurement) in intermediate steps of the complex multi-step calculations in order to avoid additional errors in the process of rounding off the numbers.

1.4 DIMENSIONS OF PHYSICAL QUANTITIES

The nature of a physical quantity is described by its dimensions. All the physical quantities represented by derived units can be expressed in terms of some combination of seven fundamental or base quantities. We shall call these base quantities as the seven dimensions of the physical world, which are denoted with square brackets [ ]. Thus, length has the dimension [L], mass [M], time [T], electric current [A], thermodynamic temperature [K], luminous intensity [cd], and amount of substance [mol]. **The dimensions of a physical quantity are the powers (or exponents) to which the base quantities are raised to represent that quantity**. Note that using the square brackets [ ] round a quantity means that we are dealing with '**the dimensions of**' the quantity.

In mechanics, all the physical quantities can be written in terms of the dimensions [L], [M] and [T]. For example, the volume occupied by an object is expressed as the product of length, breadth and height, or three lengths. Hence the dimensions of volume are $[L] \times [L] \times [L] = [L]^3 = [L^3]$. As the volume is independent of mass and time, it is said to possess zero dimension in mass [M°], zero dimension in time [T°] and three dimensions in length.

Similarly, force, as the product of mass and acceleration, can be expressed as

Force = mass × acceleration

= mass × (length)/(time)^2

The dimensions of force are $[M]\ [L]/[T]^2 = [M\ L\ T^{-2}]$. Thus, the force has one dimension in mass, one dimension in length, and –2 dimensions in time. The dimensions in all other base quantities are zero.

Note that in this type of representation, the magnitudes are not considered. It is the quality of the type of the physical quantity that enters. Thus, a change in velocity, initial velocity, average velocity, final velocity, and speed are all equivalent in this context. Since all these quantities can be expressed as length/time, their dimensions are [L]/[T] or $[L\ T^{-1}]$.

1.5 DIMENSIONAL FORMULAE AND DIMENSIONAL EQUATIONS

The expression which shows how and which of the base quantities represent the dimensions of a physical quantity is called the *dimensional formula* of the given physical quantity. For example, the dimensional formula of the volume is $[M^\circ\ L^3\ T^\circ]$, and that of speed or velocity is $[M^\circ\ L\ T^{-1}]$. Similarly, $[M^\circ\ L\ T^{-2}]$ is the dimensional formula of acceleration and $[M\ L^{-3}\ T^\circ]$ that of mass density.

An equation obtained by equating a physical quantity with its dimensional formula is called the **dimensional equation** of the physical quantity. Thus, the dimensional equations are the equations, which represent the dimensions of a physical quantity in terms of the base quantities. For example, the dimensional equations of volume $[V]$, speed $[v]$, force $[F]$ and mass density $[\rho]$ may be expressed as

$$[V] = [M^0\ L^3\ T^0]$$
$$[v] = [M^0\ L\ T^{-1}]$$
$$[F] = [M\ L\ T^{-2}]$$
$$[\rho] = [M\ L^{-3}\ T^0]$$

The dimensional equation can be obtained from the equation representing the relations between the physical quantities. The dimensional formulae of a large number and wide variety of physical quantities, derived from the equations representing the relationships among other physical quantities and expressed in terms of base quantities are given in Appendix 9 for your guidance and ready reference.

1.6 DIMENSIONAL ANALYSIS AND ITS APPLICATIONS

The recognition of concepts of dimensions, which guide the description of physical behaviour is of basic importance as only those physical quantities can be added or subtracted which have the same dimensions. A thorough understanding of dimensional analysis helps us in deducing certain relations among different physical quantities and checking the derivation, accuracy and dimensional consistency or homogeneity of various mathematical expressions. When magnitudes of two or more physical quantities are multiplied, their units should be treated in the same manner as ordinary algebraic symbols. We can cancel identical units in the numerator and denominator. The same is true for dimensions of a physical quantity. Similarly, physical quantities represented by symbols on both sides of a mathematical equation must have the same dimensions.

1.6.1 Checking the Dimensional Consistency of Equations

The magnitudes of physical quantities may be added together or subtracted from one another only if they have the same dimensions. In other words, we can add or subtract similar physical quantities. Thus, velocity cannot be added to force, or an electric current cannot be subtracted from the thermodynamic temperature. This simple principle called **the principle of homogeneity of dimensions** in an equation is extremely useful in checking the correctness of an equation. If the dimensions of all the terms are not same, the equation is wrong. Hence, if we derive an expression for the length (or distance) of an object, regardless of the symbols appearing in the original mathematical relation, when all the individual dimensions are simplified, the remaining dimension must be that of length. Similarly, if we derive an equation of speed, the dimensions on both the sides of equation, when simplified, must be of length/time, or [L T^{-1}].

Dimensions are customarily used as a preliminary test of the consistency of an equation, when there is some doubt about the correctness of the equation. However, the dimensional consistency does not guarantee correct equations. It is uncertain to the extent of dimensionless quantities or functions. The arguments of special functions, such as the trigonometric, logarithmic and exponential functions must be dimensionless. A pure number, ratio of similar physical quantities, such as angle as the ratio (length/length), refractive index as the ratio (speed of light in vacuum/speed of light in medium) etc., has no dimensions.

Now we can test the dimensional consistency or homogeneity of the equation

$$x = x_0 + v_0\, t + (1/2)\, a\, t^2$$

for the distance x travelled by a particle or body in time t which starts from the position x_0 with an initial velocity v_0 at time $t = 0$ and has uniform acceleration a along the direction of motion.

The dimensions of each term may be written as

$$[x] = [L]$$
$$[x_0] = [L]$$
$$[v_0\, t] = [L\, T^{-1}]\ [T] = [L]$$
$$[(1/2)\, a\, t^2] = [L\, T^{-2}]\ [T^2] = [L]$$

As each term on the right hand side of this equation has the same dimension, namely that of length, which is same as the dimension of left hand side of the equation, hence this equation is a dimensionally correct equation.

It may be noted that a test of consistency of dimensions tells us no more and no less than a test of consistency of units, but has the advantage that we need not commit ourselves to a particular choice of units, and we need not worry about conversions among multiples and sub-multiples of the units. It may be borne in mind that **if an equation fails this consistency test, it is proved wrong, but if it passes, it is not proved right. Thus, a dimensionally correct equation need not be actually an exact (correct) equation, but a dimensionally wrong (incorrect) or inconsistent equation must be wrong**.

▶ ***Example 1.3*** Let us consider an equation

$$\frac{1}{2} m\, v^2 = m\, g\, h$$

where m is the mass of the body, v its velocity, g is the acceleration due to gravity and h is the height. Check whether this equation is dimensionally correct.

Answer The dimensions of LHS are

$$[M]\ [L\, T^{-1}]^2 = [M]\ [L^2 T^{-2}] = [M\, L^2\, T^{-2}]$$

The dimensions of RHS are

$$[M][L\,T^{-2}]\ \ [L] = [M][L^2\,T^{-2}]$$
$$= [M\,L^2\,T^{-2}]$$

The dimensions of LHS and RHS are the same and hence the equation is dimensionally correct. ◀

Example 1.4 The SI unit of energy is J = kg m^2 s^{-2}; that of speed v is m s^{-1} and of acceleration a is m s^{-2}. Which of the formulae for kinetic energy (K) given below can you rule out on the basis of dimensional arguments (m stands for the mass of the body) :

(a) $K = m^2 v^3$
(b) $K = (1/2)mv^2$
(c) $K = ma$
(d) $K = (3/16)mv^2$
(e) $K = (1/2)mv^2 + ma$

Answer Every correct formula or equation must have the same dimensions on both sides of the equation. Also, only quantities with the same physical dimensions can be added or subtracted. The dimensions of the quantity on the right side are [M^2 L^3 T^{-3}] for (a); [M L^2 T^{-2}] for (b) and (d); [M L T^{-2}] for (c). The quantity on the right side of (e) has no proper dimensions since two quantities of different dimensions have been added. Since the kinetic energy K has the dimensions of [M L^2 T^{-2}], formulas (a), (c) and (e) are ruled out. Note that dimensional arguments cannot tell which of the two, (b) or (d), is the correct formula. For this, one must turn to the actual definition of kinetic energy (see Chapter 5). The correct formula for kinetic energy is given by (b). ◀

1.6.2 Deducing Relation among the Physical Quantities

The method of dimensions can sometimes be used to deduce relation among the physical quantities. For this we should know the dependence of the physical quantity on other quantities (upto three physical quantities or linearly independent variables) and consider it as a product type of the dependence. Let us take an example.

Example 1.5 Consider a simple pendulum, having a bob attached to a string, that oscillates under the action of the force of gravity. Suppose that the period of oscillation of the simple pendulum depends on its length (l), mass of the bob (m) and acceleration due to gravity (g). Derive the expression for its time period using method of dimensions.

Answer The dependence of time period T on the quantities l, g and m as a product may be written as :

$T = k\, l^x\, g^y\, m^z$

where k is dimensionless constant and x, y and z are the exponents.

By considering dimensions on both sides, we have

$$[L^o M^o T^1] = [L^1]^x\,[L^1 T^{-2}]^y\,[M^1]^z$$
$$= L^{x+y}\,T^{-2y}\,M^z$$

On equating the dimensions on both sides, we have

$x + y = 0$; $-2y = 1$; and $z = 0$

So that $x = \dfrac{1}{2}, y = -\dfrac{1}{2}, z = 0$

Then, $T = k\, l^{½}\, g^{-½}$

or, $T = k\sqrt{\dfrac{l}{g}}$

Note that value of constant k can not be obtained by the method of dimensions. Here it does not matter if some number multiplies the right side of this formula, because that does not affect its dimensions.

Actually, $k = 2\pi$ so that $T = 2\pi\sqrt{\dfrac{l}{g}}$ ◀

Dimensional analysis is very useful in deducing relations among the interdependent physical quantities. However, dimensionless constants cannot be obtained by this method. The method of dimensions can only test the dimensional validity, but not the exact relationship between physical quantities in any equation. It does not distinguish between the physical quantities having same dimensions.

A number of exercises at the end of this chapter will help you develop skill in dimensional analysis.

SUMMARY

1. Physics is a quantitative science, based on measurement of physical quantities. Certain physical quantities have been chosen as fundamental or base quantities (such as length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity).
2. Each base quantity is defined in terms of a certain basic, arbitrarily chosen but properly standardised reference standard called unit (such as metre, kilogram, second, ampere, kelvin, mole and candela). The units for the fundamental or base quantities are called fundamental or base units.
3. Other physical quantities, derived from the base quantities, can be expressed as a combination of the base units and are called derived units. A complete set of units, both fundamental and derived, is called a system of units.
4. The International System of Units (SI) based on seven base units is at present internationally accepted unit system and is widely used throughout the world.
5. The SI units are used in all physical measurements, for both the base quantities and the derived quantities obtained from them. Certain derived units are expressed by means of SI units with special names (such as joule, newton, watt, etc).
6. The SI units have well defined and internationally accepted unit symbols (such as m for metre, kg for kilogram, s for second, A for ampere, N for newton etc.).
7. Physical measurements are usually expressed for small and large quantities in scientific notation, with powers of 10. Scientific notation and the prefixes are used to simplify measurement notation and numerical computation, giving indication to the precision of the numbers.
8. Certain general rules and guidelines must be followed for using notations for physical quantities and standard symbols for SI units, some other units and SI prefixes for expressing properly the physical quantities and measurements.
9. In computing any physical quantity, the units for derived quantities involved in the relationship(s) are treated as though they were algebraic quantities till the desired units are obtained.
10. In measured and computed quantities proper significant figures only should be retained. Rules for determining the number of significant figures, carrying out arithmetic operations with them, and 'rounding off ' the uncertain digits must be followed.
11. The dimensions of base quantities and combination of these dimensions describe the nature of physical quantities. Dimensional analysis can be used to check the dimensional consistency of equations, deducing relations among the physical quantities, etc. A dimensionally consistent equation need not be actually an exact (correct) equation, but a dimensionally wrong or inconsistent equation must be wrong.

EXERCISES

Note : In stating numerical answers, take care of significant figures.

1.1 Fill in the blanks

(a) The volume of a cube of side 1 cm is equal to .....m^3

(b) The surface area of a solid cylinder of radius 2.0 cm and height 10.0 cm is equal to ...$(mm)^2$

(c) A vehicle moving with a speed of 18 km h^{-1} covers....m in 1 s

(d) The relative density of lead is 11.3. Its density is ....g cm^{-3} or ....kg m^{-3}.

1.2 Fill in the blanks by suitable conversion of units

(a) 1 kg m^2 s^{-2} = ...g cm^2 s^{-2}

(b) 1 m = ... ly

(c) 3.0 m s^{-2} = ... km h^{-2}

(d) $G = 6.67 \times 10^{-11}$ N m^2 $(kg)^{-2}$ = ... $(cm)^3$ s^{-2} g^{-1}.

1.3 A calorie is a unit of heat (energy in transit) and it equals about 4.2 J where 1J = 1 kg m^2 s^{-2}. Suppose we employ a system of units in which the unit of mass equals α kg, the unit of length equals β m, the unit of time is γ s. Show that a calorie has a magnitude 4.2 α^{-1} β^{-2} γ^{2} in terms of the new units.

1.4 Explain this statement clearly :
"To call a dimensional quantity 'large' or 'small' is meaningless without specifying a standard for comparison". In view of this, reframe the following statements wherever necessary :
(a) atoms are very small objects
(b) a jet plane moves with great speed
(c) the mass of Jupiter is very large
(d) the air inside this room contains a large number of molecules
(e) a proton is much more massive than an electron
(f) the speed of sound is much smaller than the speed of light.

1.5 A new unit of length is chosen such that the speed of light in vacuum is unity. What is the distance between the Sun and the Earth in terms of the new unit if light takes 8 min and 20 s to cover this distance ?

1.6 Which of the following is the most precise device for measuring length :
(a) a vernier callipers with 20 divisions on the sliding scale
(b) a screw gauge of pitch 1 mm and 100 divisions on the circular scale
(c) an optical instrument that can measure length to within a wavelength of light ?

1.7 A student measures the thickness of a human hair by looking at it through a microscope of magnification 100. He makes 20 observations and finds that the average width of the hair in the field of view of the microscope is 3.5 mm. What is the estimate on the thickness of hair ?

1.8 Answer the following :
(a) You are given a thread and a metre scale. How will you estimate the diameter of the thread ?
(b) A screw gauge has a pitch of 1.0 mm and 200 divisions on the circular scale. Do you think it is possible to increase the accuracy of the screw gauge arbitrarily by increasing the number of divisions on the circular scale ?
(c) The mean diameter of a thin brass rod is to be measured by vernier callipers. Why is a set of 100 measurements of the diameter expected to yield a more reliable estimate than a set of 5 measurements only ?

1.9 The photograph of a house occupies an area of 1.75 cm^2 on a 35 mm slide. The slide is projected on to a screen, and the area of the house on the screen is 1.55 m^2. What is the linear magnification of the projector-screen arrangement.

1.10 State the number of significant figures in the following :
(a) 0.007 m^2
(b) 2.64×10^{24} kg
(c) 0.2370 g cm^{-3}
(d) 6.320 J
(e) 6.032 N m^{-2}
(f) 0.0006032 m^2

1.11 The length, breadth and thickness of a rectangular sheet of metal are 4.234 m, 1.005 m, and 2.01 cm respectively. Give the area and volume of the sheet to correct significant figures.

1.12 The mass of a box measured by a grocer's balance is 2.30 kg. Two gold pieces of masses 20.15 g and 20.17 g are added to the box. What is (a) the total mass of the box, (b) the difference in the masses of the pieces to correct significant figures ?

1.13 A famous relation in physics relates 'moving mass' m to the 'rest mass' m_o of a particle in terms of its speed v and the speed of light, c. (This relation first arose as a consequence of special relativity due to Albert Einstein). A boy recalls the relation almost correctly but forgets where to put the constant c. He writes :

$$m = \frac{m_0}{\left(1 - v^2\right)^{1/2}} .$$

Guess where to put the missing c.

1.14 The unit of length convenient on the atomic scale is known as an angstrom and is denoted by Å: 1 Å = 10^{-10} m. The size of a hydrogen atom is about 0.5 Å. What is the total atomic volume in m^3 of a mole of hydrogen atoms ?

1.15 One mole of an ideal gas at standard temperature and pressure occupies 22.4 L (molar volume). What is the ratio of molar volume to the atomic volume of a mole of hydrogen ? (Take the size of hydrogen molecule to be about 1 Å). Why is this ratio so large ?

1.16 Explain this common observation clearly : If you look out of the window of a fast moving train, the nearby trees, houses etc. seem to move rapidly in a direction opposite to the train's motion, but the distant objects (hill tops, the Moon, the stars etc.) seem to be stationary. (In fact, since you are aware that you are moving, these distant objects seem to move with you).

1.17 The Sun is a hot plasma (ionized matter) with its inner core at a temperature exceeding 10^7 K, and its outer surface at a temperature of about 6000 K. At these high temperatures, no substance remains in a solid or liquid phase. In what range do you expect the mass density of the Sun to be, in the range of densities of solids and liquids or gases ? Check if your guess is correct from the following data : mass of the Sun = 2.0×10^{30} kg, radius of the Sun = 7.0×10^8 m.

CHAPTER TWO

MOTION IN A STRAIGHT LINE

2.1 INTRODUCTION

Motion is common to everything in the universe. We walk, run and ride a bicycle. Even when we are sleeping, air moves into and out of our lungs and blood flows in arteries and veins. We see leaves falling from trees and water flowing down a dam. Automobiles and planes carry people from one place to the other. The earth rotates once every twenty-four hours and revolves round the sun once in a year. The sun itself is in motion in the Milky Way, which is again moving within its local group of galaxies.

Motion is change in position of an object with time. How does the position change with time ? In this chapter, we shall learn how to describe motion. For this, we develop the concepts of velocity and acceleration. We shall confine ourselves to the study of motion of objects along a straight line, also known as **rectilinear motion**. For the case of rectilinear motion with uniform acceleration, a set of simple equations can be obtained. Finally, to understand the relative nature of motion, we introduce the concept of relative velocity.

In our discussions, we shall treat the objects in motion as point objects. This approximation is valid so far as the size of the object is much smaller than the distance it moves in a reasonable duration of time. In a good number of situations in real-life, the size of objects can be neglected and they can be considered as point-like objects without much error.

In **Kinematics**, we study ways to describe motion without going into the causes of motion. What causes motion described in this chapter and the next chapter forms the subject matter of Chapter 4.

2.2 INSTANTANEOUS VELOCITY AND SPEED

The average velocity tells us how fast an object has been moving over a given time interval but does not tell us how fast it moves at different instants of time during that interval. For this, we define **instantaneous velocity** or simply velocity v at an instant t.

The velocity at an instant is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small. In other words,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{2.1a}$$

$$= \frac{dx}{dt} \tag{2.1b}$$

where the symbol $\lim_{\Delta t \to 0}$ stands for the operation of taking limit as $\Delta t \to 0$ of the quantity on its right. In the language of calculus, the quantity on the right hand side of Eq. (2.1a) is the differential coefficient of x with respect to t and is denoted by $\frac{dx}{dt}$ (see Appendix 2.1). It is the rate of change of position with respect to time, at that instant.

We can use Eq. (2.1a) for obtaining the value of velocity at an instant either **graphically** or **numerically**. Suppose that we want to obtain graphically the value of velocity at time $t = 4$ s (point P) for the motion of the car represented in Fig.2.1 calculation. Let us take $\Delta t = 2$ s centred at $t = 4$ s. Then, by the definition of the average velocity, the slope of line P_1P_2 (Fig. 2.1) gives the value of average velocity over the interval 3 s to 5 s.

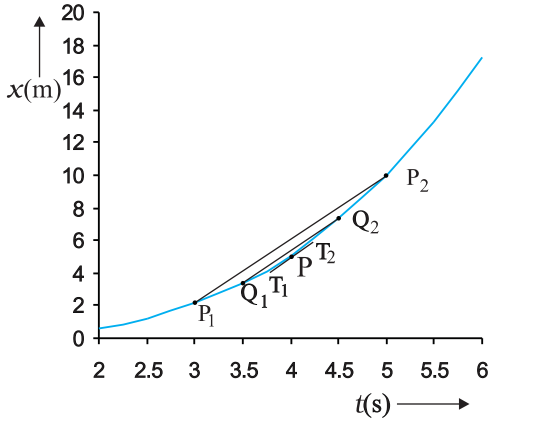


Fig. 2.1 Determining velocity from position-time graph. Velocity at t = 4 s is the slope of the tangent to the graph at that instant.

Now, we decrease the value of Δt from 2 s to 1 s. Then line P_1P_2 becomes Q_1Q_2 and its slope gives the value of the average velocity over the interval 3.5 s to 4.5 s. In the limit $\Delta t \to 0$, the line P_1P_2 becomes tangent to the position-time curve at the point P and the velocity at t = 4 s is given by the slope of the tangent at that point. **It is difficult to show this process graphically. But if we use numerical method to obtain the value of the velocity, the meaning of the limiting process becomes clear**. For the graph shown in Fig. 2.1, $x = 0.08\ t^3$. Table 2.1 gives the value of $\Delta x/\Delta t$ calculated for Δt equal to 2.0 s, 1.0 s, 0.5 s, 0.1 s and 0.01 s centred at t = 4.0 s. The second and third columns give the value of $t_1 = \left(t - \frac{\Delta t}{2}\right)$ and $t_2 = \left(t + \frac{\Delta t}{2}\right)$ and the fourth and the fifth columns give the

Table 2.1 Limiting value of $\frac{\Delta x}{\Delta t}$ at $t = 4$ s

Δt (s)	t_1 (s)	t_2 (s)	$x(t_1)$ (m)	$x(t_2)$ (m)	Δx (m)	$\Delta x / \Delta t$ (m s^{-1})
2.0	3.0	5.0	2.16	10.0	7.84	3.92
1.0	3.5	4.5	3.43	7.29	3.86	3.86
0.5	3.75	4.25	4.21875	6.14125	1.9225	3.845
0.1	3.95	4.05	4.93039	5.31441	0.38402	3.8402
0.01	3.995	4.005	5.100824	5.139224	0.0384	3.8400

corresponding values of x, i.e. $x(t_1) = 0.08\, t_1^3$ and $x(t_2) = 0.08\, t_2^3$. The sixth column lists the difference $\Delta x = x(t_2) - x(t_1)$ and the last column gives the ratio of Δx and Δt, i.e. the average velocity corresponding to the value of Δt listed in the first column.

We see from Table 2.1 that as we decrease the value of Δt from 2.0 s to 0.010 s, the value of the average velocity approaches the limiting value 3.84 m s^{-1} which is the value of velocity at $t = 4.0$ s, i.e. the value of $\dfrac{dx}{dt}$ at $t = 4.0$ s. In this manner, we can calculate velocity at each instant for motion of the car.

The graphical method for the determination of the instantaneous velocity is always not a convenient method. For this, we must carefully plot the position–time graph and calculate the value of average velocity as Δt becomes smaller and smaller. It is easier to calculate the value of velocity at different instants if we have data of positions at different instants or exact expression for the position as a function of time. Then, we calculate $\Delta x/\Delta t$ from the data for decreasing the value of Δt and find the limiting value as we have done in Table 2.1 or use differential calculus for the given expression and calculate $\dfrac{dx}{dt}$ at different instants as done in the following example.

> ***Example 2.1*** *The position of an object moving along x-axis is given by $x = a + bt^2$ where $a = 8.5$ m, $b = 2.5$ m s^{-2} and t is measured in seconds. What is its velocity at $t = 0$ s and $t = 2.0$ s. What is the average velocity between $t = 2.0$ s and $t = 4.0$ s ?*

Answer In notation of differential calculus, the velocity is

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(a + bt^2\right) = 2b\,t = 5.0\ t\ \text{m s}^{-1}$$

At $t = 0$ s, $v = 0$ m s^{-1} and at $t = 2.0$ s, $v = 10$ m s^{-1} .

$$Average\ velocity = \frac{x(4.0) - x(2.0)}{4.0 - 2.0}$$

$$= \frac{a + 16b - a - 4b}{2.0} = 6.0 \times b$$

$$= 6.0 \times 2.5 = 15\ \text{m s}^{-1}$$ ◀

Note that for uniform motion, velocity is the same as the average velocity at all instants.

Instantaneous speed or simply speed is the magnitude of velocity. For example, a velocity of + 24.0 m s^{-1} and a velocity of – 24.0 m s^{-1} — both have an associated speed of 24.0 m s^{-1}. It should be noted that though average speed over a finite interval of time is greater or equal to the magnitude of the average velocity, instantaneous speed at an instant is equal to the magnitude of the instantaneous velocity at that instant. Why so ?

2.3 ACCELERATION

The velocity of an object, in general, changes during its course of motion. How to describe this change? Should it be described as the rate of change in velocity **with distance** or **with time** ? This was a problem even in Galileo's time. It was first thought that this change could be described by the rate of change of velocity with distance. But, through his studies of motion of freely falling objects and motion of objects on an inclined plane, Galileo concluded that the rate of change of velocity with time is a constant of motion for all objects in free fall. On the other hand, the change in velocity with distance is not constant – it decreases with the increasing distance of fall. This led to the concept of acceleration as the rate of change of velocity with time.

The average acceleration $\bar{a}$ over a time interval is defined as the change of velocity divided by the time interval :

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \tag{2.2}$$

where v_2 and v_1 are the instantaneous velocities or simply velocities at time t_2 and t_1 . It is the average change of velocity per unit time. The SI unit of acceleration is m s^{-2} .

On a plot of velocity versus time, the average acceleration is the slope of the straight line connecting the points corresponding to (v_2, t_2) and (v_1, t_1).

Instantaneous acceleration is defined in the same way as the instantaneous velocity :

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \quad (2.3)$$

The acceleration at an instant is the slope of the tangent to the v–t curve at that instant.

Since velocity is a quantity having both magnitude and direction, a change in velocity may involve either or both of these factors. Acceleration, therefore, may result from a change in speed (magnitude), a change in direction or changes in both. Like velocity, acceleration can also be positive, negative or zero. Position-time graphs for motion with positive, negative and zero acceleration are shown in Figs. 2.4 (a), (b) and (c), respectively. Note that the graph curves upward for positive acceleration; downward for negative acceleration and it is a straight line for zero acceleration.

Although acceleration can vary with time, our study in this chapter will be restricted to motion with constant acceleration. In this case, the average acceleration equals the constant value of acceleration during the interval. If the velocity of an object is v_o at t = 0 and v at time t, we have

$$\bar{a} = \frac{v - v_0}{t - 0} \quad \text{or,} \quad v = v_0 + at \quad (2.4)$$

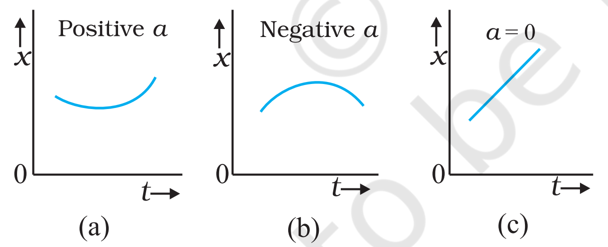


Fig. 2.2 *Position-time graph for motion with (a) positive acceleration; (b) negative acceleration, and (c) zero acceleration.*

Let us see how velocity-time graph looks like for some simple cases. Fig. 2.3 shows velocity-time graph for motion with constant acceleration for the following cases :

(a) An object is moving in a positive direction with a positive acceleration.
(b) An object is moving in positive direction with a negative acceleration.
(c) An object is moving in negative direction with a negative acceleration.
(d) An object is moving in positive direction till time t_1, and then turns back with the same negative acceleration.

An interesting feature of a velocity-time graph for any moving object is that **the area under the curve represents the displacement over a given time interval**. A general proof of this statement requires use of calculus. We can, however, see that it is true for the simple case of an object moving with constant velocity u. Its velocity-time graph is as shown in Fig. 2.4.

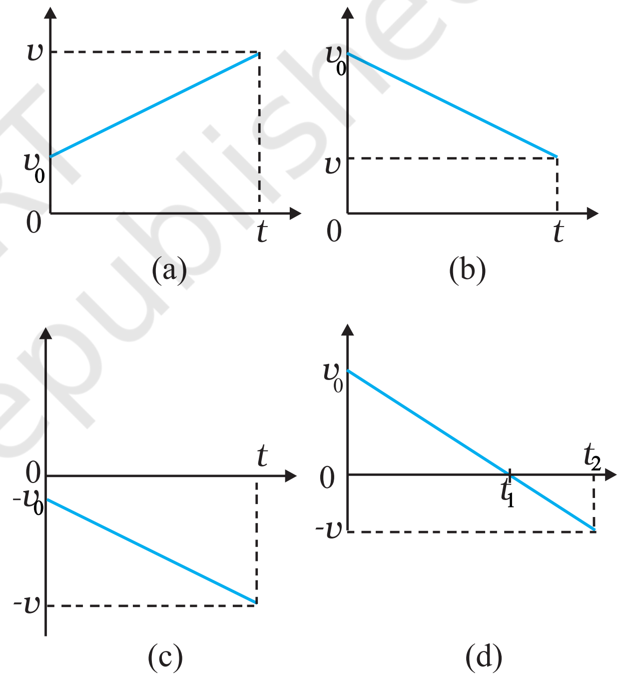


Fig. 2.3 *Velocity–time graph for motions with constant acceleration. (a) Motion in positive direction with positive acceleration, (b) Motion in positive direction with negative acceleration, (c) Motion in negative direction with negative acceleration, (d) Motion of an object with negative acceleration that changes direction at time t_1. Between times 0 to t_1, it moves in positive x - direction and between t_1 and t_2 it moves in the opposite direction.*

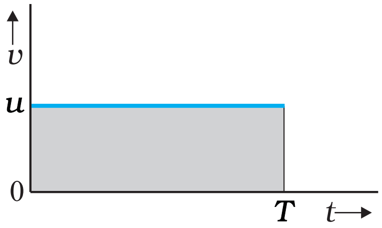


Fig. 2.4 *Area under v–t curve equals displacement of the object over a given time interval.*

The v-t curve is a straight line parallel to the time axis and the area under it between $t = 0$ and $t = T$ is the area of the rectangle of height u and base T. Therefore, area = $u \times T = uT$ which is the displacement in this time interval. How come in this case an area is equal to a distance? Think! Note the dimensions of quantities on the two coordinate axes, and you will arrive at the answer.

Note that the *x-t, v-t,* and *a-t* graphs shown in several figures in this chapter have sharp kinks at some points implying that the functions are not differentiable at these points. In any realistic situation, the functions will be differentiable at all points and the graphs will be smooth.

What this means physically is that acceleration and velocity cannot change values abruptly at an instant. Changes are always continuous.

2.4 KINEMATIC EQUATIONS FOR UNIFORMLY ACCELERATED MOTION

For uniformly accelerated motion, we can derive some simple equations that relate displacement (x), time taken (t), initial velocity (v_0), final velocity (v) and acceleration (a). Equation (2.4) already obtained gives a relation between final and initial velocities v and v_0 of an object moving with uniform acceleration a :

$$v = v_0 + at \tag{2.4}$$

This relation is graphically represented in Fig. 2.5. The area under this curve is :

Area between instants 0 and t = Area of triangle ABC + Area of rectangle OACD

$$= \frac{1}{2}(v - v_0)t + v_0 t$$

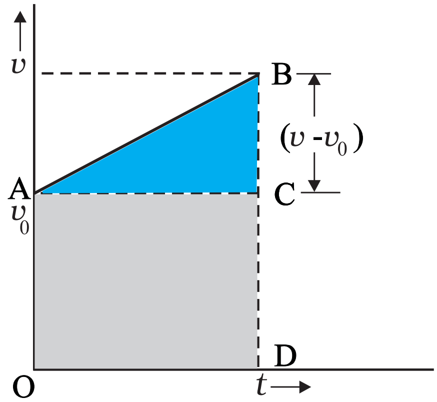


Fig. 2.5 *Area under v-t curve for an object with uniform acceleration.*

As explained in the previous section, the area under v-t curve represents the displacement. Therefore, the displacement x of the object is :

$$x = \frac{1}{2}(v - v_0)t + v_0 t \tag{2.5}$$

But $\quad v - v_0 = a\,t$

Therefore, $x = \frac{1}{2}a\,t^2 + v_0 t$

or, $$x = v_0 t + \frac{1}{2}at^2 \tag{2.6}$$

Equation (2.5) can also be written as

$$x = \frac{v + v_0}{2}t = \bar{v}\,t \tag{2.7a}$$

where,

$$\bar{v} = \frac{v + v_0}{2} \quad \textbf{(constant acceleration only)} \tag{2.7b}$$

Equations (2.7a) and (2.7b) mean that the object has undergone displacement x with an average velocity equal to the arithmetic average of the initial and final velocities.

From Eq. (2.4), $t = (v - v_0)/a$. Substituting this in Eq. (2.7a), we get

$$x = \bar{v}\,t = \left(\frac{v + v_0}{2}\right)\left(\frac{v - v_0}{a}\right) = \frac{v^2 - v_0^2}{2a}$$

$$v^2 = v_0^2 + 2ax \tag{2.8}$$

This equation can also be obtained by substituting the value of t from Eq. (2.4) into Eq. (2.6). Thus, we have obtained three important equations :

$$v = v_0 + at$$

$$x = v_0 t + \frac{1}{2} a t^2$$

$$v^2 = v_0^2 + 2ax \qquad (2.9a)$$

connecting five quantities v_0, v, a, t and x. These are kinematic equations of rectilinear motion for constant acceleration.

The set of Eq. (2.9a) were obtained by assuming that at $t = 0$, the position of the particle, x is 0. We can obtain a more general equation if we take the position coordinate at $t = 0$ as non-zero, say x_0. Then Eqs. (2.9a) are modified (replacing x by $x - x_0$) to :

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \qquad (2.9b)$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad (2.9c)$$

▶ ***Example 2.2*** Obtain equations of motion for constant acceleration using method of calculus.

Answer By definition

$$a = \frac{dv}{dt}$$

$$dv = a\,dt$$

Integrating both sides

$$\int_{v_0}^{v} dv = \int_0^t a \; dt$$

$$= a \int_0^t dt \qquad (a \text{ is constant})$$

$$v - v_0 = at$$

$$v = v_0 + at$$

Further, $$v = \frac{dx}{dt}$$

$$dx = v\,dt$$

Integrating both sides

$$\int_{x_0}^{x} dx = \int_0^t v \; dt$$

$$= \int_0^t (v_0 + at)\; dt$$

$$x - x_0 = v_0 \; t + \frac{1}{2} a \; t^2$$

$$x \; = x_0 + v_0 \; t + \frac{1}{2} \; a \; t^2$$

We can write

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

or, $v\,dv = a\,dx$

Integrating both sides,

$$\int_{v_0}^{v} v\,dv \; = \; \int_{x_0}^{x} a\,dx$$

$$\frac{v^2 - v_0^2}{2} \; = \; a\,(x - x_0)$$

$$v^2 = v_0^2 + 2a\,(x - x_0)$$

The advantage of this method is that it can be used for motion with non-uniform acceleration also.

Now, we shall use these equations to some important cases. ◀

▶ ***Example 2.3*** A ball is thrown vertically upwards with a velocity of 20 m s^{-1} from the top of a multistorey building. The height of the point from where the ball is thrown is 25.0 m from the ground. (a) How high will the ball rise ? and (b) how long will it be before the ball hits the ground? Take g = 10 m s^{-2}.

Answer (a) Let us take the y-axis in the vertically upward direction with zero at the ground, as shown in Fig. 2.6.

Now v_o = + 20 m s^{-1},

$a \; = \; -g = -10$ m s^{-2},

$v \; = \; 0$ m s^{-1}

If the ball rises to height y from the point of launch, then using the equation

$$v^2 \; = \; v_0^2 + 2 \; a \; \left(y - y_0\right)$$

we get

$$0 = (20)^2 + 2(-10)(y - y_0)$$

Solving, we get, $(y - y_0)$ = 20 m.

(b) We can solve this part of the problem in two ways. **Note carefully the methods used.**

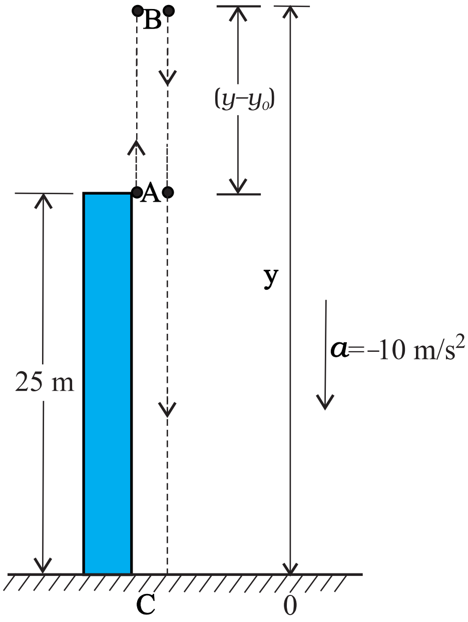


Fig. 2.6

FIRST METHOD : In the first method, we split the path in two parts : the upward motion (A to B) and the downward motion (B to C) and calculate the corresponding time taken t_1 and t_2. Since the velocity at B is zero, we have :

$$v = v_o + at$$
$$0 = 20 - 10t_1$$
Or, $$t_1 = 2 \text{ s}$$

This is the time in going from A to B. From B, or the point of the maximum height, the ball falls freely under the acceleration due to gravity. The ball is moving in negative y direction. We use equation

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

We have, $y_0 = 45$ m, $y = 0$, $v_0 = 0$, $a = -g = -10$ m s^{-2}

$$0 = 45 + (½)(-10)\, t_2^2$$

Solving, we get $t_2 = 3$ s

Therefore, the total time taken by the ball before it hits the ground = $t_1 + t_2$ = 2 s + 3 s = 5 s.

SECOND METHOD : The total time taken can also be calculated by noting the coordinates of initial and final positions of the ball with respect to the origin chosen and using equation

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

Now $y_0 = 25$ m $\quad y = 0$ m
$v_o = 20$ m s^{-1}, $a = -10$m s^{-2}, t = ?

$$0 = 25 + 20\,t + (½)\,(-10)\,t^2$$
Or, $$5t^2 - 20t - 25 = 0$$

Solving this quadratic equation for t, we get

$$t = 5\text{s}$$

Note that the second method is better since we do not have to worry about the path of the motion as the motion is under constant acceleration.

Example 2.4 Free-fall : Discuss the motion of an object under free fall. Neglect air resistance.

Answer An object released near the surface of the Earth is accelerated downward under the influence of the force of gravity. The magnitude of acceleration due to gravity is represented by $\boldsymbol{g}$. If air resistance is neglected, the object is said to be in **free fall**. If the height through which the object falls is small compared to the earth's radius, g can be taken to be constant, equal to 9.8 m s^{-2}. Free fall is thus a case of motion with uniform acceleration.

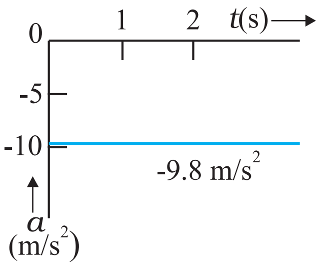
We assume that the motion is in y-direction, more correctly in $-y$-direction because we choose upward direction as positive. Since the acceleration due to gravity is always downward, it is in the negative direction and we have

$$a = -g = -9.8 \text{ m s}^{-2}$$

The object is released from rest at $y = 0$. Therefore, $v_0 = 0$ and the equations of motion become:

$$v = 0 - g\,t \quad = -9.8\,t \quad \text{m s}^{-1}$$
$$y = 0 - ½\,g\,t^2 \quad = -4.9\,t^2 \quad \text{m}$$
$$v^2 = 0 - 2\,g\,y \quad = -19.6\,y \quad \text{m}^2\text{ s}^{-2}$$

These equations give the velocity and the distance travelled as a function of time and also the variation of velocity with distance. The variation of acceleration, velocity, and distance, with time have been plotted in Fig. 2.7(a), (b) and (c).



(a)

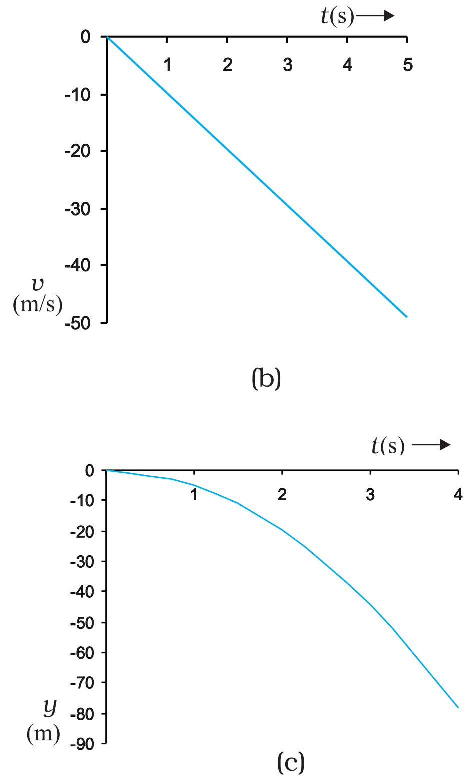


Fig. 2.7 *Motion of an object under free fall.*
(a) Variation of acceleration with time.
(b) Variation of velocity with time.
(c) Variation of distance with time ◀

> ▶ ***Example 2.5*** **Galileo's law of odd numbers** : *"The distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity [namely, 1: 3: 5: 7......]." Prove it.*

Answer Let us divide the time interval of motion of an object under free fall into many equal intervals τ and find out the distances traversed during successive intervals of time. Since initial velocity is zero, we have

$$y = -\frac{1}{2}gt^2$$

Using this equation, we can calculate the position of the object after different time intervals, 0, τ, 2τ, 3τ... which are given in second column of Table 2.2. If we take $(-1/2)\, g\tau^2$ as y_0 — the position coordinate after first time interval τ, then third column gives the positions in the unit of y_o. The fourth column gives the distances traversed in successive τs. We find that the distances are in the simple ratio 1: 3: 5: 7: 9: 11... as shown in the last column. This law was established by Galileo Galilei (1564-1642) who was the first to make quantitative studies of free fall. ◀

> ▶ ***Example 2.6*** **Stopping distance of vehicles :** When brakes are applied to a moving vehicle, the distance it travels before stopping is called stopping distance. It is an important factor for road safety and depends on the initial velocity (υ_0) and the braking capacity, or deceleration, $-a$ that is caused by the braking. Derive an expression for stopping distance of a vehicle in terms of υ_o and a.

Answer Let the distance travelled by the vehicle before it stops be d_s. Then, using equation of motion $\upsilon^2 = \upsilon_o^{\,2} + 2\,ax$, and noting that $\upsilon = 0$, we have the stopping distance

$$d_s = \frac{-\upsilon_0^2}{2a}$$

Thus, the stopping distance is proportional to the square of the initial velocity. Doubling the

Table 2.2

t	***y***	***y* in terms of y_o [=(- ½) g τ^2]**	**Distance traversed in successive intervals**	**Ratio of distances traversed**
0	0	0		
τ	$-(1/2)\,g\,\tau^2$	y_o	y_o	1
$2\,\tau$	$-4(1/2)\,g\,\tau^2$	$4\,y_o$	$3\,y_o$	3
$3\,\tau$	$-9(1/2)\,g\,\tau^2$	$9\,y_o$	$5\,y_o$	5
$4\,\tau$	$-16(1/2)\,g\,\tau^2$	$16\,y_o$	$7\,y_o$	7
$5\,\tau$	$-25(1/2)\,g\,\tau^2$	$25\,y_o$	$9\,y_o$	9
$6\,\tau$	$-36(1/2)\,g\,\tau^2$	$36\,y_o$	$11\,y_o$	11

initial velocity increases the stopping distance by a factor of 4 (for the same deceleration).

For the car of a particular make, the braking distance was found to be 10 m, 20 m, 34 m and 50 m corresponding to velocities of 11, 15, 20 and 25 m/s which are nearly consistent with the above formula.

Stopping distance is an important factor considered in setting speed limits, for example, in school zones. ◀

▶ ***Example 2.7*** **Reaction time :** When a situation demands our immediate action, it takes some time before we really respond. Reaction time is the time a person takes to observe, think and act. For example, if a person is driving and suddenly a boy appears on the road, then the time elapsed before he slams the brakes of the car is the reaction time. Reaction time depends on complexity of the situation and on an individual.

You can measure your reaction time by a simple experiment. Take a ruler and ask your friend to drop it vertically through the gap between your thumb and forefinger (Fig. 2.8). After you catch it, find the distance d travelled by the ruler. In a particular case, d was found to be 21.0 cm. Estimate reaction time.

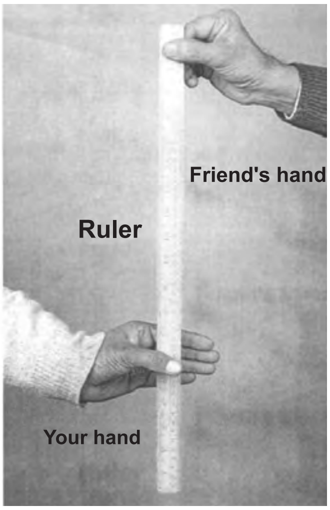


Fig. 2.8 *Measuring the reaction time.*

Answer The ruler drops under free fall. Therefore, $v_o = 0$, and $a = -g = -9.8$ m s^{-2}. The distance travelled d and the reaction time t_r are related by

$$d = -\frac{1}{2} g t_r^2$$

Or, $$t_r = \sqrt{\frac{2d}{g}} \text{ s}$$

Given $d = 21.0$ cm and $g = 9.8$ m s^{-2} the reaction time is

$$t_r = \sqrt{\frac{2 \times 0.21}{9.8}} \text{ s} \cong 0.2 \text{ s}.$$ ◀

SUMMARY

1. An object is said to be in *motion* if its position changes with time. The position of the object can be specified with reference to a conveniently chosen origin. For motion in a straight line, position to the right of the origin is taken as positive and to the left as negative.

   The average speed of an object is greater or equal to the magnitude of the average velocity over a given time interval.

2. *Instantaneous velocity* or simply velocity is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small :

$$v = \lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

   The velocity at a particular instant is equal to the slope of the tangent drawn on position-time graph at that instant.

3. *Average acceleration* is the change in velocity divided by the time interval during which the change occurs :

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

4. *Instantaneous acceleration* is defined as the limit of the average acceleration as the time interval Δt goes to zero :

$$a = \lim_{\Delta t \to 0} \bar{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

The acceleration of an object at a particular time is the slope of the velocity-time graph at that instant of time. For uniform motion, acceleration is zero and the x-t graph is a straight line inclined to the time axis and the v-t graph is a straight line parallel to the time axis. For motion with uniform acceleration, x-t graph is a parabola while the v-t graph is a straight line inclined to the time axis.

5. The area under the velocity-time curve between times t_1 and t_2 is equal to the displacement of the object during that interval of time.

6. For objects in uniformly accelerated rectilinear motion, the five quantities, displacement x, time taken t, initial velocity v_0, final velocity v and acceleration a are related by a set of simple equations called *kinematic equations of motion :*

$$v = v_0 + at$$

$$x = v_0 t + \frac{1}{2} a t^2$$

$$v^2 = v_0^2 + 2ax$$

if the position of the object at time $t = 0$ is 0. If the particle starts at $x = x_0$, x in above equations is replaced by $(x - x_0)$.

Physical quantity	Symbol	Dimensions	Unit	Remarks
Path length		[L]	m	
Displacement	Δx	[L]	m	$= x_2 - x_1$ In one dimension, its sign indicates the direction.
Velocity		$[LT^{-1}]$	m s^{-1}	
(a) Average	$\bar{v}$			$= \frac{\Delta x}{\Delta t}$
(b) Instantaneous	v			$= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$ In one dimension, its sign indicates the direction.

Speed (a) Average (b) Instantaneous		$[LT^{-1}]$	$m\ s^{-1}$	$= \dfrac{\textit{Path length}}{\textit{Time interval}}$ $= \dfrac{dx}{dt}$
Acceleration (a) Average (b) Instantaneous	 $\bar{a}$ a	$[LT^{-2}]$	$m\ s^{-2}$	$= \dfrac{\Delta v}{\Delta t}$ $= \lim_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t} = \dfrac{dv}{dt}$ In one dimension, its sign indicates the direction.

POINTS TO PONDER

1. The origin and the positive direction of an axis are a matter of choice. You should first specify this choice before you assign signs to quantities like displacement, velocity and acceleration.
2. If a particle is speeding up, acceleration is in the direction of velocity; if its speed is decreasing, acceleration is in the direction opposite to that of the velocity. This statement is independent of the choice of the origin and the axis.
3. The sign of acceleration does not tell us whether the particle's speed is increasing or decreasing. The sign of acceleration (as mentioned in point 3) depends on the choice of the positive direction of the axis. For example, if the vertically upward direction is chosen to be the positive direction of the axis, the acceleration due to gravity is negative. If a particle is falling under gravity, this acceleration, though negative, results in increase in speed. For a particle thrown upward, the same negative acceleration (of gravity) results in decrease in speed.
4. The zero velocity of a particle at any instant does not necessarily imply zero acceleration at that instant. A particle may be momentarily at rest and yet have non-zero acceleration. For example, a particle thrown up has zero velocity at its uppermost point but the acceleration at that instant continues to be the acceleration due to gravity.
5. In the kinematic equations of motion [Eq. (2.9)], the various quantities are algebraic, i.e. they may be positive or negative. The equations are applicable in all situations (for one dimensional motion with constant acceleration) provided the values of different quantities are substituted in the equations with proper signs.
6. The definitions of instantaneous velocity and acceleration (Eqs. (2.1) and (2.3)) are exact and are always correct while the kinematic equations (Eq. (2.9)) are true only for motion in which the magnitude and the direction of acceleration are constant during the course of motion.

EXERCISES

2.1 In which of the following examples of motion, can the body be considered approximately a point object:
(a) a railway carriage moving without jerks between two stations.
(b) a monkey sitting on top of a man cycling smoothly on a circular track.
(c) a spinning cricket ball that turns sharply on hitting the ground.
(d) a tumbling beaker that has slipped off the edge of a table.

2.2 The position-time (x-t) graphs for two children A and B returning from their school O to their homes P and Q respectively are shown in Fig. 2.9. Choose the correct entries in the brackets below ;
(a) (A/B) lives closer to the school than (B/A)
(b) (A/B) starts from the school earlier than (B/A)
(c) (A/B) walks faster than (B/A)
(d) A and B reach home at the (same/different) time
(e) (A/B) overtakes (B/A) on the road (once/twice).

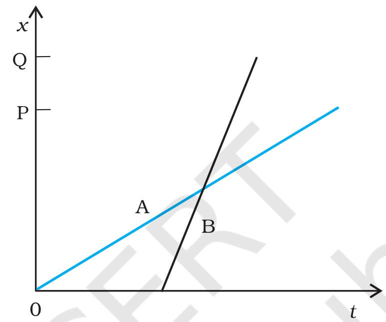


Fig. 2.9

2.3 A woman starts from her home at 9.00 am, walks with a speed of 5 km h^{-1} on a straight road up to her office 2.5 km away, stays at the office up to 5.00 pm, and returns home by an auto with a speed of 25 km h^{-1}. Choose suitable scales and plot the x-t graph of her motion.

2.4 A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward and 3 steps backward, and so on. Each step is 1 m long and requires 1 s. Plot the x-t graph of his motion. Determine graphically and otherwise how long the drunkard takes to fall in a pit 13 m away from the start.

2.5 A car moving along a straight highway with speed of 126 km h^{-1} is brought to a stop within a distance of 200 m. What is the retardation of the car (assumed uniform), and how long does it take for the car to stop ?

2.6 A player throws a ball upwards with an initial speed of 29.4 m s^{-1}.
(a) What is the direction of acceleration during the upward motion of the ball ?
(b) What are the velocity and acceleration of the ball at the highest point of its motion ?
(c) Choose the x = 0 m and t = 0 s to be the location and time of the ball at its highest point, vertically downward direction to be the positive direction of x-axis, and give the signs of position, velocity and acceleration of the ball during its upward, and downward motion.
(d) To what height does the ball rise and after how long does the ball return to the player's hands ? (Take g = 9.8 m s^{-2} and neglect air resistance).

2.7 Read each statement below carefully and state with reasons and examples, if it is true or false ;
A particle in one-dimensional motion
(a) with zero speed at an instant may have non-zero acceleration at that instant
(b) with zero speed may have non-zero velocity,
(c) with constant speed must have zero acceleration,
(d) with positive value of acceleration *must* be speeding up.

2.8 A ball is dropped from a height of 90 m on a floor. At each collision with the floor, the ball loses one tenth of its speed. Plot the speed-time graph of its motion between t = 0 to 12 s.

2.9 Explain clearly, with examples, the distinction between :

(a) magnitude of displacement (sometimes called distance) over an interval of time, and the total length of path covered by a particle over the same interval;

(b) magnitude of average velocity over an interval of time, and the average speed over the same interval. [Average speed of a particle over an interval of time is defined as the total path length divided by the time interval]. Show in both (a) and (b) that the second quantity is either greater than or equal to the first. When is the equality sign true ? [For simplicity, consider one-dimensional motion only].

2.10 A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km h^{-1}. Finding the market closed, he instantly turns and walks back home with a speed of 7.5 km h^{-1}. What is the

(a) magnitude of average velocity, and

(b) average speed of the man over the interval of time (i) 0 to 30 min, (ii) 0 to 50 min, (iii) 0 to 40 min ? [Note: You will appreciate from this exercise why it is better to define average speed as total path length divided by time, and not as magnitude of average velocity. You would not like to tell the tired man on his return home that his average speed was zero !]

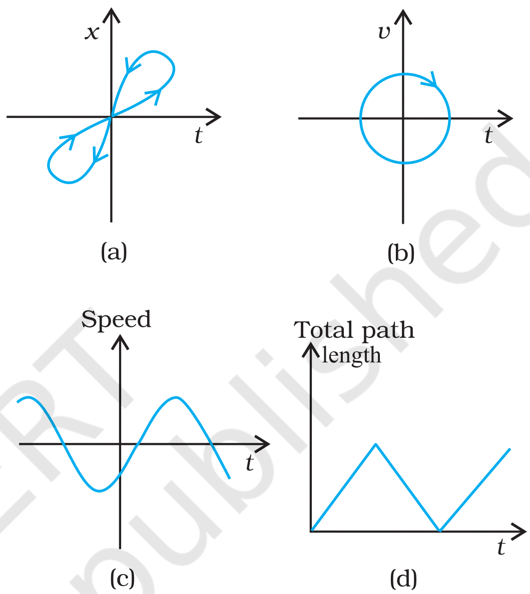


Fig. 2.10

2.11 In Exercises 2.9 and 2.10, we have carefully distinguished between *average* speed and magnitude of *average* velocity. No such distinction is necessary when we consider instantaneous speed and magnitude of velocity. The instantaneous speed is always equal to the magnitude of instantaneous velocity. Why?

2.12 Look at the graphs (a) to (d) (Fig. 2.10) carefully and state, with reasons, which of these *cannot* possibly represent one-dimensional motion of a particle.

2.13 Figure 2.11shows the x-t plot of one-dimensional motion of a particle. Is it correct to say from the graph that the particle moves in a straight line for $t < 0$ and on a parabolic path for $t > 0$? If not, suggest a suitable physical context for this graph.

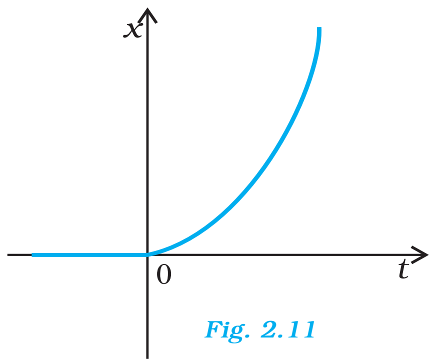


Fig. 2.11

2.14 A police van moving on a highway with a speed of 30 km h^{-1} fires a bullet at a thief's car speeding away in the same direction with a speed of 192 km h^{-1}. If the muzzle speed of the bullet is 150 m s^{-1}, with what speed does the bullet hit the thief's car ? (Note: Obtain that speed which is relevant for damaging the thief's car).

2.15 Suggest a suitable physical situation for each of the following graphs (Fig 2.12):

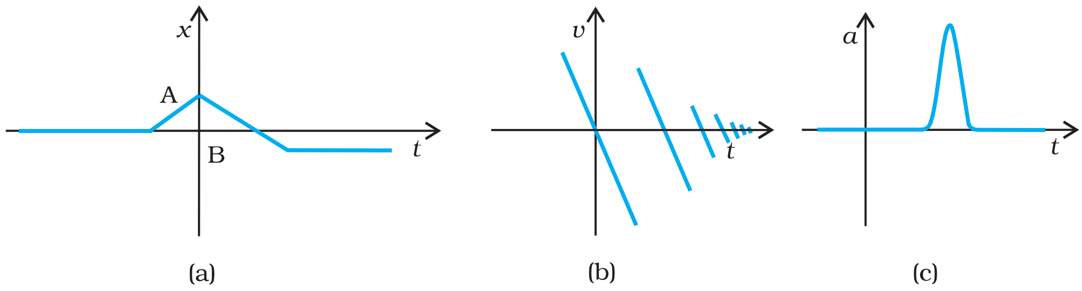


Fig. 2.12

2.16 Figure 2.13 gives the x-t plot of a particle executing one-dimensional simple harmonic motion. (You will learn about this motion in more detail in Chapter13). Give the signs of position, velocity and acceleration variables of the particle at t = 0.3 s, 1.2 s, – 1.2 s.

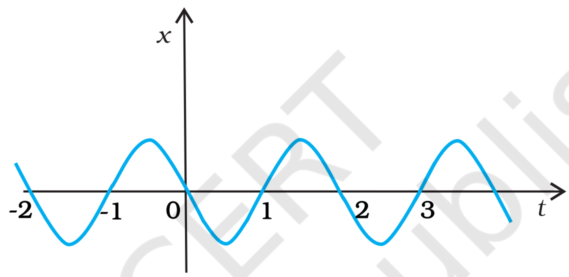


Fig. 2.13

2.17 Figure 2.14 gives the x-t plot of a particle in one-dimensional motion. Three different equal intervals of time are shown. In which interval is the average speed greatest, and in which is it the least ? Give the sign of average velocity for each interval.

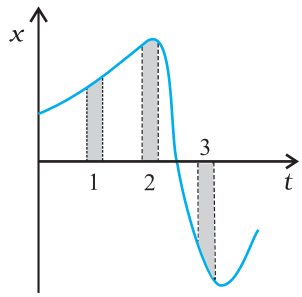


Fig. 2.14

2.18 Figure 2.15 gives a speed-time graph of a particle in motion along a constant direction. Three equal intervals of time are shown. In which interval is the average acceleration greatest in magnitude? In which interval is the average speed greatest ? Choosing the positive direction as the constant direction of motion, give the signs of v and a in the three intervals. What are the accelerations at the points A, B, C and D ?

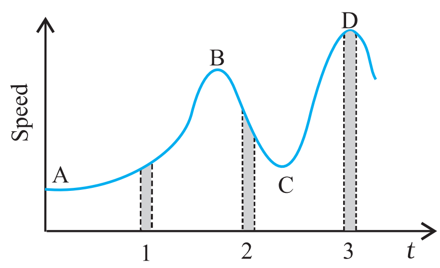


Fig. 2.15

Chapter Three

Motion in a Plane

3.1 INTRODUCTION

In the last chapter we developed the concepts of position, displacement, velocity and acceleration that are needed to describe the motion of an object along a straight line. We found that the directional aspect of these quantities can be taken care of by + and – signs, as in one dimension only two directions are possible. But in order to describe motion of an object in two dimensions (a plane) or three dimensions (space), we need to use vectors to describe the above-mentioned physical quantities. Therefore, it is first necessary to learn the language of vectors. What is a vector? How to add, subtract and multiply vectors ? What is the result of multiplying a vector by a real number ? We shall learn this to enable us to use vectors for defining velocity and acceleration in a plane. We then discuss motion of an object in a plane. As a simple case of motion in a plane, we shall discuss motion with constant acceleration and treat in detail the projectile motion. Circular motion is a familiar class of motion that has a special significance in daily-life situations. We shall discuss uniform circular motion in some detail.

The equations developed in this chapter for motion in a plane can be easily extended to the case of three dimensions.

3.2 SCALARS AND VECTORS

In physics, we can classify quantities as scalars or vectors. Basically, the difference is that a **direction** is associated with a vector but not with a scalar. A scalar quantity is a quantity with magnitude only. It is specified completely by a single number, along with the proper unit. Examples are : the distance between two points, mass of an object, the temperature of a body and the time at which a certain event happened. The rules for combining scalars are the rules of ordinary algebra. Scalars can be added, subtracted, multiplied and divided just as the ordinary numbers*. For example, if the length and breadth of a rectangle are 1.0 m and 0.5 m respectively, then its perimeter is the sum of the lengths of the four sides, 1.0 m + 0.5 m +1.0 m + 0.5 m = 3.0 m. The length of each side is a scalar and the perimeter is also a scalar. Take another example: the maximum and minimum temperatures on a particular day are 35.6 C and 24.2 C respectively. Then, the difference between the two temperatures is 11.4 C. Similarly, if a uniform solid cube of aluminium of side 10 cm has a mass of 2.7 kg, then its volume is 10^{-3} m^3 (a scalar) and its density is 2.7×10^3 kg m^{-3} (a scalar).

A **vector** quantity is a quantity that has both a magnitude and a direction and obeys the **triangle law of addition** or equivalently the **parallelogram law of addition**. So, a vector is specified by giving its magnitude by a number and its direction. Some physical quantities that are represented by vectors are displacement, velocity, acceleration and force.

To represent a vector, we use a bold face type in this book. Thus, a velocity vector can be represented by a symbol **v**. Since bold face is difficult to produce, when written by hand, a vector is often represented by an arrow placed over a letter, say $\vec{v}$. Thus, both **v** and $\vec{v}$ represent the velocity vector. The magnitude of a vector is often called its absolute value, indicated by $|\mathbf{v}| = v$. Thus, a vector is represented by a bold face, e.g. by **A, a, p, q, r, ... x, y**, with respective magnitudes denoted by light face $A, a, p, q, r, ... x, y$.

3.2.1 Position and Displacement Vectors

To describe the position of an object moving in a plane, we need to choose a convenient point, say O as origin. Let P and P′ be the positions of the object at time t and t', respectively [Fig. 3.1(a)]. We join O and P by a straight line. Then, **OP** is the position vector of the object at time t. An arrow is marked at the head of this line. It is represented by a symbol **r**, i.e. **OP** = **r**. Point P′ is represented by another position vector, **OP′** denoted by **r′**. The length of the vector **r** represents the magnitude of the vector and its direction is the direction in which P lies as seen from O. If the object moves from P to P′, the vector **PP′** (with tail at P and tip at P′) is called the **displacement vector** corresponding to motion from point P (at time t) to point P′ (at time t').

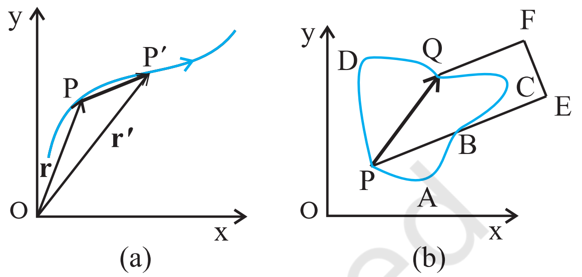


Fig. 3.1 (a) *Position and displacement vectors.* (b) *Displacement vector* **PQ** *and different courses of motion.*

It is important to note that displacement vector is the straight line joining the initial and final positions and does not depend on the actual path undertaken by the object between the two positions. For example, in Fig. 3.1(b), given the initial and final positions as P and Q, the displacement vector is the same **PQ** for different paths of journey, say PABCQ, PDQ, and PBEFQ. Therefore, the **magnitude of displacement is either less or equal to the path length of an object between two points**. This fact was emphasised in the previous chapter also while discussing motion along a straight line.

3.2.2 Equality of Vectors

Two vectors **A** and **B** are said to be equal if, and only if, they have the same magnitude and the same direction.**

Figure 3.2(a) shows two equal vectors **A** and **B**. We can easily check their equality. Shift **B** parallel to itself until its tail Q coincides with that of A, i.e. Q coincides with O. Then, since their tips S and P also coincide, the two vectors are said to be equal. In general, equality is indicated

* *Addition and subtraction of scalars make sense only for quantities with same units. However, you can multiply and divide scalars of different units.*

** *In our study, vectors do not have fixed locations. So displacing a vector parallel to itself leaves the vector unchanged. Such vectors are called* free vectors. *However, in some physical applications, location or line of application of a vector is important. Such vectors are called* localised vectors.

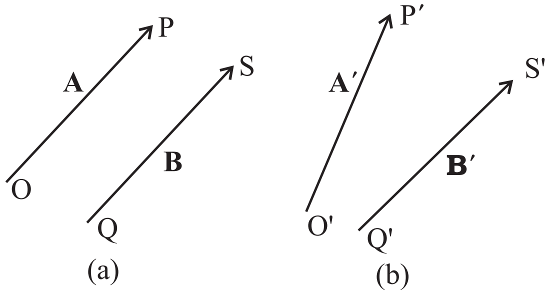


Fig. 3.2 (a) *Two equal vectors* **A** *and* **B**. (b) *Two vectors* **A′** *and* **B′** *are unequal though they are of the same length.*

as **A** = **B**. Note that in Fig. 3.2(b), vectors **A′** and **B′** have the same magnitude but they are not equal because they have different directions. Even if we shift **B′** parallel to itself so that its tail Q′ coincides with the tail O′ of **A′**, the tip S′ of **B′** does not coincide with the tip P′ of **A′**.

3.3 MULTIPLICATION OF VECTORS BY REAL NUMBERS

Multiplying a vector **A** with a positive number λ gives a vector whose magnitude is changed by the factor λ but the direction is the same as that of **A** :

$$|\lambda \mathbf{A}| = \lambda\, |\mathbf{A}| \text{ if } \lambda > 0.$$

For example, if **A** is multiplied by 2, the resultant vector 2**A** is in the same direction as **A** and has a magnitude twice of |**A**| as shown in Fig. 3.3(a).

Multiplying a vector **A** by a negative number –λ gives another vector whose direction is opposite to the direction of **A** and whose magnitude is λ times |**A**|.

Multiplying a given vector **A** by negative numbers, say –1 and –1.5, gives vectors as shown in Fig 3.3(b).

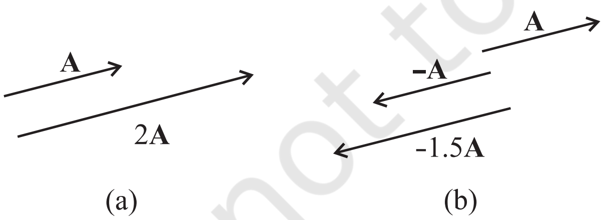


Fig. 3.3 (a) *Vector* **A** *and the resultant vector after multiplying* **A** *by a positive number 2.* (b) *Vector* **A** *and resultant vectors after multiplying it by a negative number –1 and –1.5.*

The factor λ by which a vector **A** is multiplied could be a scalar having its own physical dimension. Then, the dimension of λ **A** is the product of the dimensions of λ and **A**. For example, if we multiply a constant velocity vector by duration (of time), we get a displacement vector.

3.4 ADDITION AND SUBTRACTION OF VECTORS — GRAPHICAL METHOD

As mentioned in section 4.2, vectors, by definition, obey the triangle law or equivalently, the parallelogram law of addition. We shall now describe this law of addition using the graphical method. Let us consider two vectors **A** and **B** that lie in a plane as shown in Fig. 3.4(a). The lengths of the line segments representing these vectors are proportional to the magnitude of the vectors. To find the sum **A** + **B**, we place vector **B** so that its tail is at the head of the vector **A**, as in Fig. 3.4(b). Then, we join the tail of **A** to the head of **B**. This line OQ represents a vector **R,** that is, the sum of the vectors **A** and **B**. Since, in this procedure of vector addition, vectors are

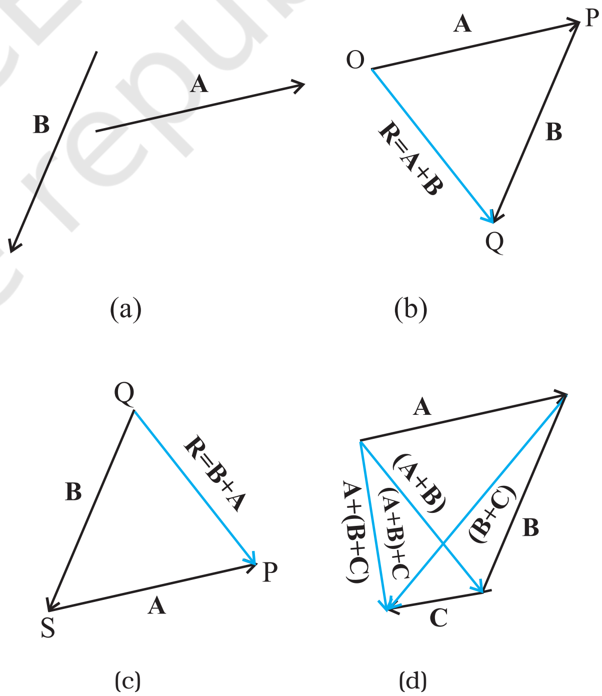


Fig. 3.4 (a) *Vectors* **A** *and* **B**. (b) *Vectors* **A** *and* **B** *added graphically.* (c) *Vectors* **B** *and* **A** *added graphically.* (d) *Illustrating the associative law of vector addition.*

arranged head to tail, this graphical method is called the **head-to-tail method**. The two vectors and their resultant form three sides of a triangle, so this method is also known as **triangle method of vector addition**. If we find the resultant of **B** + **A** as in Fig. 3.4(c), the same vector **R** is obtained. Thus, vector addition is **commutative**:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad (3.1)$$

The addition of vectors also obeys the associative law as illustrated in Fig. 3.4(d). The result of adding vectors **A** and **B** first and then adding vector **C** is the same as the result of adding **B** and **C** first and then adding vector **A** :

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \qquad (3.2)$$

What is the result of adding two equal and opposite vectors ? Consider two vectors **A** and –**A** shown in Fig. 3.3(b). Their sum is **A** + (–**A**). Since the magnitudes of the two vectors are the same, but the directions are opposite, the resultant vector has zero magnitude and is represented by **0** called a **null vector** or a **zero vector** :

$$\mathbf{A} - \mathbf{A} = \mathbf{0} \qquad |\mathbf{0}| = 0 \qquad (3.3)$$

Since the magnitude of a null vector is zero, its direction cannot be specified.

The null vector also results when we multiply a vector **A** by the number zero. The main properties of **0** are :

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$
$$\lambda\, \mathbf{0} = \mathbf{0}$$
$$0\, \mathbf{A} = \mathbf{0} \qquad (3.4)$$

What is the physical meaning of a zero vector? Consider the position and displacement vectors in a plane as shown in Fig. 3.1(a). Now suppose that an object which is at P at time *t*, moves to P′ and then comes back to P. Then, what is its displacement? Since the initial and final positions coincide, the displacement is a "null vector".

Subtraction of vectors can be defined in terms of addition of vectors. We define the difference of two vectors **A** and **B** as the sum of two vectors **A** and –**B** :

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \qquad (3.5)$$

It is shown in Fig 3.5. The vector –**B** is added to vector **A** to get $\mathbf{R}_2 = (\mathbf{A} - \mathbf{B})$. The vector $\mathbf{R}_1 = \mathbf{A} + \mathbf{B}$ is also shown in the same figure for comparison. We can also use the **parallelogram method** to find the sum of two vectors. Suppose we have two vectors **A** and **B**. To add these vectors, we bring their tails to a common origin O as shown in Fig. 3.6(a). Then we draw a line from the head of **A** parallel to **B** and another line from the head of **B** parallel to **A** to complete a parallelogram OQSP. Now we join the point of the intersection of these two lines to the origin O. The resultant vector **R** is directed from the common origin O along the diagonal (OS) of the parallelogram [Fig. 3.6(b)]. In Fig.3.6(c), the triangle law is used to obtain the resultant of **A** and **B** and we see that the two methods yield the same result. Thus, the two methods are equivalent.

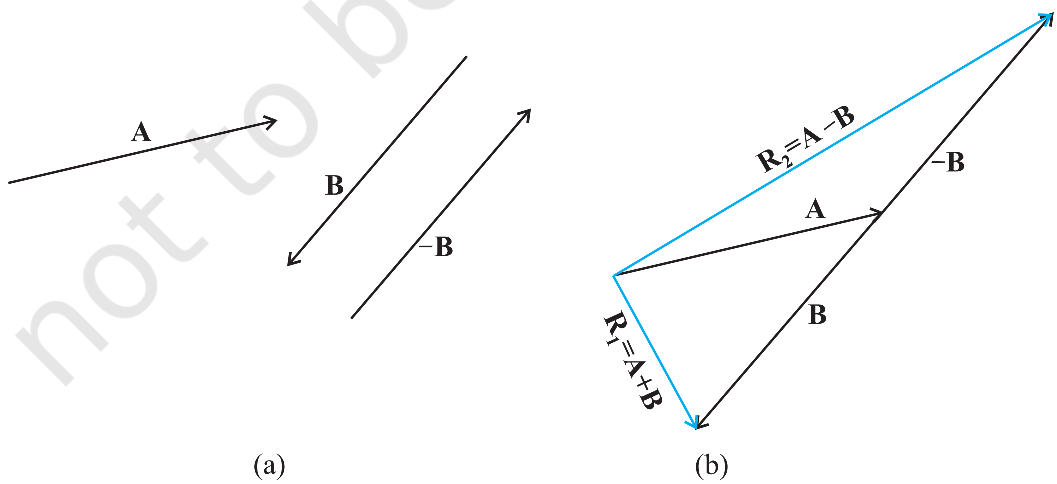


Fig. 3.5 (a) *Two vectors* **A** *and* **B**, – **B** *is also shown.* (b) *Subtracting vector* **B** *from vector* **A** *– the result is* $\mathbf{R}_2$. *For comparison, addition of vectors* **A** *and* **B**, *i.e.* $\mathbf{R}_1$ *is also shown.*

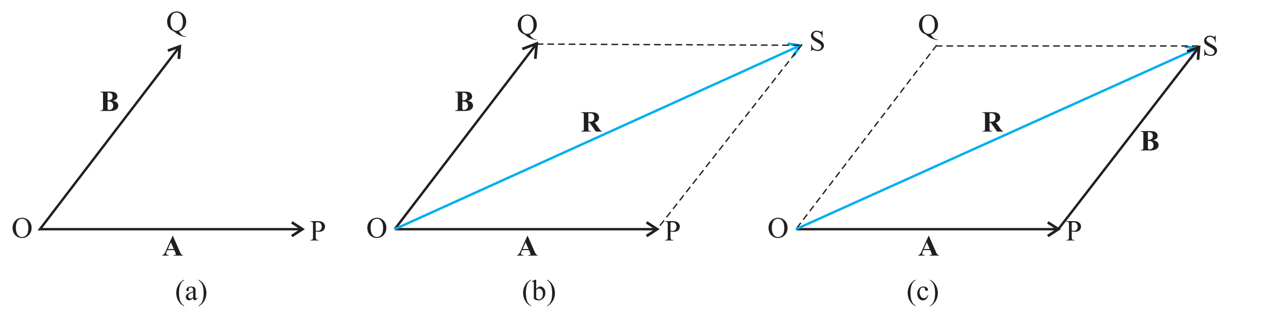


Fig. 3.6 (a) *Two vectors* **A** *and* **B** *with their tails brought to a common origin.* (b) *The sum* **A** + **B** *obtained using the parallelogram method.* (c) *The parallelogram method of vector addition is equivalent to the triangle method.*

Example 3.1 Rain is falling vertically with a speed of 35 m s^{-1}. Winds starts blowing after sometime with a speed of 12 m s^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?

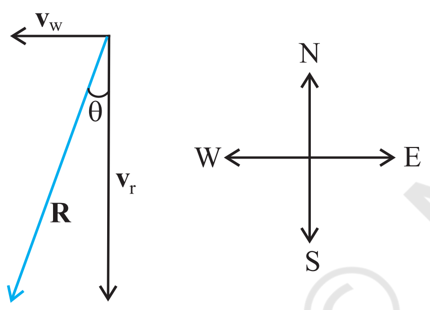


Fig. 3.7

Answer The velocity of the rain and the wind are represented by the vectors $\mathbf{v}_r$ and $\mathbf{v}_w$ in Fig. 3.7 and are in the direction specified by the problem. Using the rule of vector addition, we see that the resultant of $\mathbf{v}_r$ and $\mathbf{v}_w$ is **R** as shown in the figure. The magnitude of **R** is

$$R = \sqrt{v_r^2 + v_w^2} = \sqrt{35^2 + 12^2}\ \text{m s}^{-1} = 37\ \text{m s}^{-1}$$

The direction θ that R makes with the vertical is given by

$$\tan\theta = \frac{v_w}{v_r} = \frac{12}{35} = 0.343$$

Or, $\theta = \tan^{-1}(0.343) = 19°$

Therefore, the boy should hold his umbrella in the vertical plane at an angle of about 19° with the vertical towards the east.

3.5 RESOLUTION OF VECTORS

Let **a** and **b** be any two non-zero vectors in a plane with different directions and let **A** be another vector in the same plane (Fig. 3.8). **A** can be expressed as a sum of two vectors — one obtained by multiplying **a** by a real number and the other obtained by multiplying **b** by another real number. To see this, let O and P be the tail and head of the vector **A**. Then, through O, draw a straight line parallel to **a**, and through P, a straight line parallel to **b**. Let them intersect at Q. Then, we have

$$\mathbf{A} = \mathbf{OP} = \mathbf{OQ} + \mathbf{QP} \qquad (3.6)$$

But since **OQ** is parallel to **a**, and **QP** is parallel to **b**, we can write :

$$\mathbf{OQ} = \lambda\,\mathbf{a}, \text{ and } \mathbf{QP} = \mu\,\mathbf{b} \qquad (3.7)$$

where λ and μ are real numbers.

Therefore, $\mathbf{A} = \lambda\,\mathbf{a} + \mu\,\mathbf{b}$ (3.8)

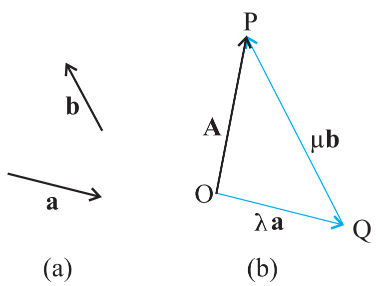


Fig. 3.8 (a) *Two non-colinear vectors* **a** *and* **b**. (b) *Resolving a vector* **A** *in terms of vectors* **a** *and* **b**.

We say that **A** has been resolved into two component vectors $\lambda\,\mathbf{a}$ and $\mu\,\mathbf{b}$ along **a** and **b** respectively. Using this method one can resolve a given vector into two component vectors along a set of two vectors – all the three lie in the same plane. It is convenient to resolve a general vector along the axes of a rectangular coordinate system using vectors of unit magnitude. These are called unit vectors that we discuss now. A unit vector is a vector of unit magnitude and points in a particular direction. It has no dimension and unit. It is used to specify a direction only. Unit vectors along the *x*-, *y*- and *z*-axes of a rectangular coordinate system are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively, as shown in Fig. 3.9(a).

Since these are unit vectors, we have

$$|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1 \qquad (3.9)$$

These unit vectors are perpendicular to each other. In this text, they are printed in bold face with a cap (^) to distinguish them from other vectors. Since we are dealing with motion in two dimensions in this chapter, we require use of only two unit vectors. If we multiply a unit vector, say $\hat{\mathbf{n}}$ by a scalar, the result is a vector $\boldsymbol{\lambda} = \lambda\hat{\mathbf{n}}$. In general, a vector **A** can be written as

$$\mathbf{A} = |\mathbf{A}|\,\hat{\mathbf{n}} \qquad (3.10)$$

where $\hat{\mathbf{n}}$ is a unit vector along **A**.

We can now resolve a vector **A** in terms of component vectors that lie along unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Consider a vector **A** that lies in *x-y* plane as shown in Fig. 3.9(b). We draw lines from the head of **A** perpendicular to the coordinate axes as in Fig. 3.9(b), and get vectors $\mathbf{A_1}$ and $\mathbf{A_2}$ such that $\mathbf{A_1} + \mathbf{A_2} = \mathbf{A}$. Since $\mathbf{A_1}$ is parallel to $\hat{\mathbf{i}}$ and $\mathbf{A_2}$ is parallel to $\hat{\mathbf{j}}$, we have :

$$\mathbf{A_1} = A_x\,\hat{\mathbf{i}}, \quad \mathbf{A_2} = A_y\,\hat{\mathbf{j}} \qquad (3.11)$$

where A_x and A_y are real numbers.

Thus, $$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} \qquad (3.12)$$

This is represented in Fig. 3.9(c). The quantities A_x and A_y are called *x*-, and *y*- components of the vector **A**. Note that A_x is itself not a vector, but $A_x\,\hat{\mathbf{i}}$ is a vector, and so is $A_y\,\hat{\mathbf{j}}$. Using simple trigonometry, we can express A_x and A_y in terms of the magnitude of **A** and the angle θ it makes with the *x*-axis :

$$A_x = A\cos\theta$$
$$A_y = A\sin\theta \qquad (3.13)$$

As is clear from Eq. (3.13), a component of a vector can be positive, negative or zero depending on the value of θ.

Now, we have two ways to specify a vector **A** in a plane. It can be specified by :

(i) its magnitude *A* and the direction θ it makes with the *x*-axis; or

(ii) its components A_x and A_y

If A and θ are given, A_x and A_y can be obtained using Eq. (3.13). If A_x and A_y are given, *A* and θ can be obtained as follows :

$$A_x^2 + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta$$
$$= A^2$$

Or, $$A = \sqrt{A_x^2 + A_y^2} \qquad (3.14)$$

And $$\tan\theta = \frac{A_y}{A_x}, \quad \theta = \tan^{-1}\frac{A_y}{A_x} \qquad (3.15)$$

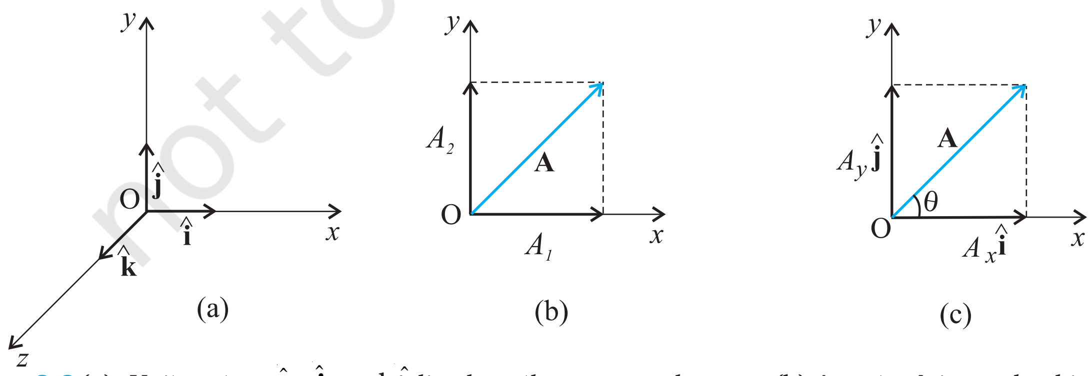


Fig. 3.9 (a) *Unit vectors* $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ *and* $\hat{\mathbf{k}}$ *lie along the x-, y-, and z-axes.* (b) *A vector* **A** *is resolved into its components* A_x *and* A_y *along x-, and y- axes.* (c) $\mathbf{A_1}$ *and* $\mathbf{A_2}$ *expressed in terms of* $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$.

So far we have considered a vector lying in an x-y plane. The same procedure can be used to resolve a general vector **A** into three components along x-, y-, and z-axes in three dimensions. If α, β, and γ are the angles* between **A** and the x-, y-, and z-axes, respectively [Fig. 3.9(d)], we have

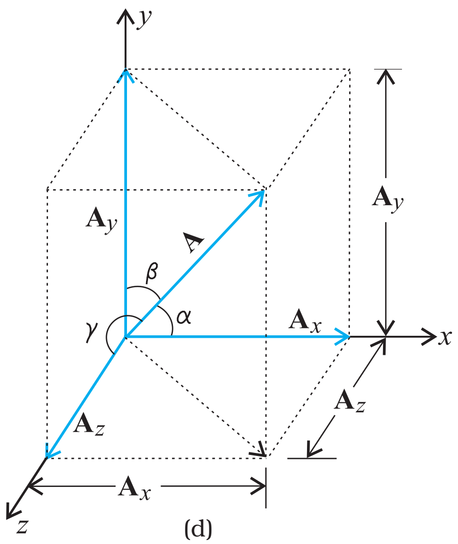


Fig. 3.9 (d) *A vector* **A** *resolved into components along x-, y-, and z-axes*

$$A_x = A\cos\alpha,\ A_y = A\cos\beta,\ A_z = A\cos\gamma \quad (3.16a)$$

In general, we have

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \quad (3.16b)$$

The magnitude of vector **A** is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (3.16c)$$

A position vector **r** can be expressed as

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \quad (3.17)$$

where x, y, and z are the components of **r** along x-, y-, z-axes, respectively.

3.6 VECTOR ADDITION – ANALYTICAL METHOD

Although the graphical method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components. Consider two vectors **A** and **B** in x-y plane with components A_x, A_y and B_x, B_y :

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \quad (3.18)$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}$$

Let **R** be their sum. We have

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

$$= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right) \quad (3.19a)$$

Since vectors obey the commutative and associative laws, we can arrange and regroup the vectors in Eq. (3.19a) as convenient to us :

$$\mathbf{R} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} \quad (3.19b)$$

$$\text{Since}\ \mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} \quad (3.20)$$

$$\text{we have,}\ R_x = A_x + B_x,\ R_y = A_y + B_y \quad (3.21)$$

Thus, each component of the resultant vector **R** is the sum of the corresponding components of **A** and **B**.

In three dimensions, we have

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$$

with

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

$$R_z = A_z + B_z \quad (3.22)$$

This method can be extended to addition and subtraction of any number of vectors. For example, if vectors **a**, **b** and **c** are given as

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

$$\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$$

$$\mathbf{c} = c_x\hat{\mathbf{i}} + c_y\hat{\mathbf{j}} + c_z\hat{\mathbf{k}} \quad (3.23a)$$

then, a vector **T** = **a** + **b** – **c** has components :

$$T_x = a_x + b_x - c_x$$

$$T_y = a_y + b_y - c_y \quad (3.23b)$$

$$T_z = a_z + b_z - c_z.$$

▶ ***Example 3.2*** Find the magnitude and direction of the resultant of two vectors **A** and **B** in terms of their magnitudes and angle θ between them.

* *Note that angles α, β, and γ are angles in space. They are between pairs of lines, which are not coplanar.*

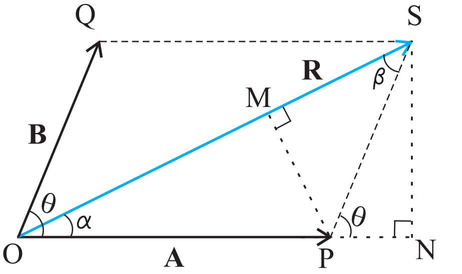


Fig. 3.10

Answer Let **OP** and **OQ** represent the two vectors **A** and **B** making an angle θ (Fig. 3.10). Then, using the parallelogram method of vector addition, **OS** represents the resultant vector **R** :

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

SN is normal to OP and PM is normal to OS.

From the geometry of the figure,

$$OS^2 = ON^2 + SN^2$$

but $\quad ON = OP + PN = A + B\cos\theta$

$\quad SN = B\sin\theta$

$\quad OS^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$

or, $\quad R^2 = A^2 + B^2 + 2AB\cos\theta$

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \tag{3.24a}$$

In Δ OSN, $\quad SN = OS\sin\alpha = R\sin\alpha$, and

in Δ PSN, $\quad SN = PS\sin\theta = B\sin\theta$

Therefore, $\quad R\sin\alpha = B\sin\theta$

$$\text{or,}\quad \frac{R}{\sin\theta} = \frac{B}{\sin\alpha} \tag{3.24b}$$

Similarly,

$$\text{PM} = A\sin\alpha = B\sin\beta$$

$$\text{or,}\quad \frac{A}{\sin\beta} = \frac{B}{\sin\alpha} \tag{3.24c}$$

Combining Eqs. (3.24b) and (3.24c), we get

$$\frac{R}{\sin\theta} = \frac{A}{\sin\beta} = \frac{\text{B}}{\sin\alpha} \tag{3.24d}$$

Using Eq. (3.24d), we get:

$$\sin\alpha = \frac{B}{R}\sin\theta \tag{3.24e}$$

where R is given by Eq. (3.24a).

$$\text{or,}\quad \tan\alpha = \frac{SN}{OP + PN} = \frac{B\sin\theta}{A + B\cos\theta} \tag{3.24f}$$

Equation (3.24a) gives the magnitude of the resultant and Eqs. (3.24e) and (3.24f) its direction. Equation (3.24a) is known as the **Law of cosines** and Eq. (3.24d) as the **Law of sines**. ◀

▶ ***Example 3.3*** A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

Answer The vector $\mathbf{v}_b$ representing the velocity of the motorboat and the vector $\mathbf{v}_c$ representing the water current are shown in Fig. 3.11 in directions specified by the problem. Using the parallelogram method of addition, the resultant **R** is obtained in the direction shown in the figure.

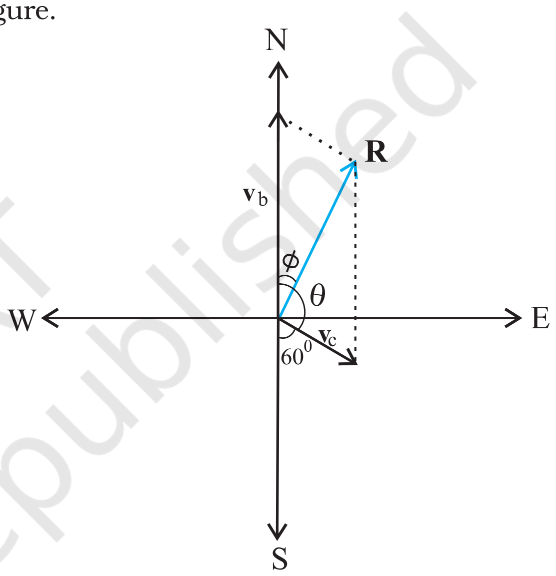


Fig. 3.11

We can obtain the magnitude of **R** using the Law of cosine :

$$R = \sqrt{v_b^2 + v_c^2 + 2v_b v_c \cos 120^\circ}$$

$$= \sqrt{25^2 + 10^2 + 2\times 25\times 10(-1/2)} \cong 22 \text{ km/h}$$

To obtain the direction, we apply the Law of sines

$$\frac{R}{\sin\theta} = \frac{v_c}{\sin\phi} \quad \text{or,}\ \sin\phi = \frac{v_c}{R}\sin\theta$$

$$= \frac{10\times\sin 120^\circ}{21.8} = \frac{10\sqrt{3}}{2\times 21.8} \cong 0.397$$

$$\phi \cong 23.4^\circ$$ ◀

3.7 MOTION IN A PLANE

In this section we shall see how to describe motion in two dimensions using vectors.

3.7.1 Position Vector and Displacement

The position vector $\mathbf{r}$ of a particle P located in a plane with reference to the origin of an x-y reference frame (Fig. 3.12) is given by

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$$

where x and y are components of $\mathbf{r}$ along x-, and y- axes or simply they are the coordinates of the object.

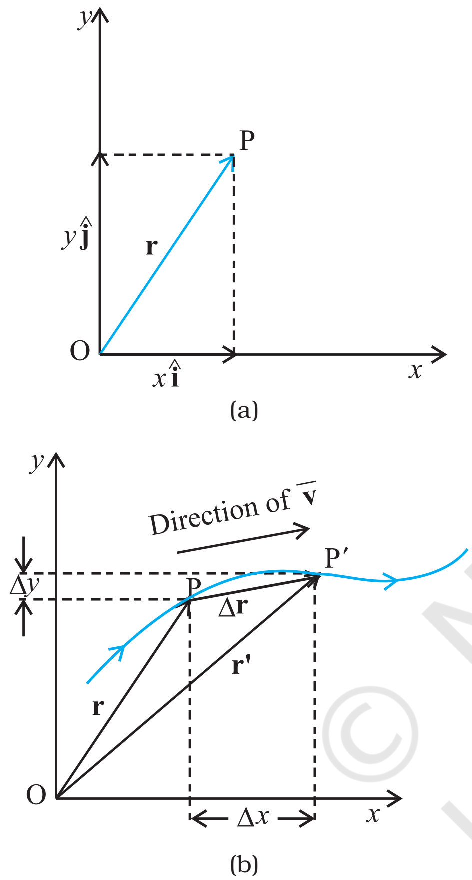


Fig. 3.12 (a) *Position vector* $\mathbf{r}$. (b) *Displacement* $\Delta\mathbf{r}$ *and average velocity* $\bar{\mathbf{v}}$ *of a particle.*

Suppose a particle moves along the curve shown by the thick line and is at P at time t and P′ at time t' [Fig. 3.12(b)]. Then, the displacement is :

$$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r} \qquad (3.25)$$

and is directed from P to P′.

We can write Eq. (3.25) in a component form:

$$\Delta\mathbf{r} = \left(x'\,\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}}\right) - \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}\right)$$

$$= \hat{\mathbf{i}}\Delta x + \hat{\mathbf{j}}\Delta y$$

where $\Delta x = x' - x,\ \Delta y = y' - y$ (3.26)

Velocity

The average velocity $(\bar{\mathbf{v}})$ of an object is the ratio of the displacement and the corresponding time interval :

$$\bar{\mathbf{v}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}}{\Delta t} = \hat{\mathbf{i}}\frac{\Delta x}{\Delta t} + \hat{\mathbf{j}}\frac{\Delta y}{\Delta t} \qquad (3.27)$$

Or, $\bar{\mathbf{v}} = \bar{v}_x\,\hat{\mathbf{i}} + \bar{v}_y\,\hat{\mathbf{j}}$

Since $\bar{\mathbf{v}} = \dfrac{\Delta\mathbf{r}}{\Delta t}$, the direction of the average velocity is the same as that of $\Delta\mathbf{r}$ (Fig. 3.12). The **velocity** (instantaneous velocity) is given by the limiting value of the average velocity as the time interval approaches zero :

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \qquad (3.28)$$

The meaning of the limiting process can be easily understood with the help of Fig 3.13(a) to (d). In these figures, the thick line represents the path of an object, which is at P at time t. P_1, P_2 and P_3 represent the positions of the object after times Δt_1, Δt_2, and Δt_3. $\Delta\mathbf{r}_1$, $\Delta\mathbf{r}_2$, and $\Delta\mathbf{r}_3$ are the displacements of the object in times Δt_1, Δt_2, and

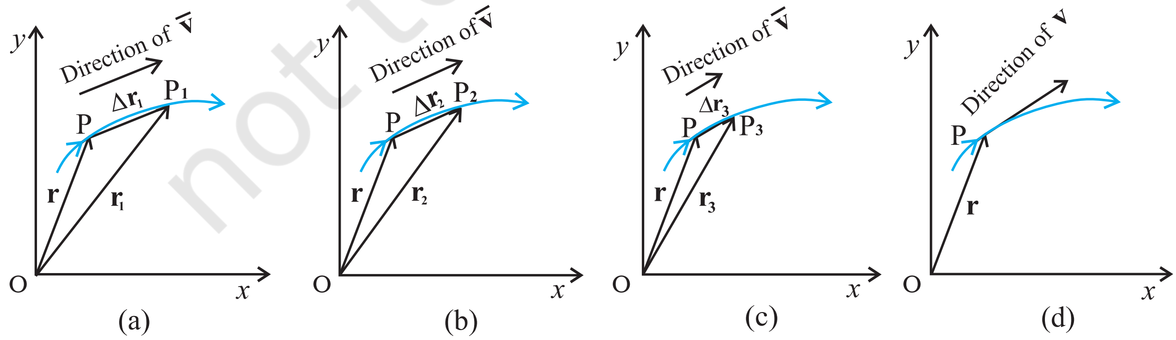


Fig. 3.13 *As the time interval* Δt *approaches zero, the average velocity approaches the velocity* $\mathbf{v}$. *The direction of* $\bar{\mathbf{v}}$ *is parallel to the line tangent to the path.*

Δt_3, respectively. The direction of the average velocity $\bar{\mathbf{v}}$ is shown in figures (a), (b) and (c) for three decreasing values of Δt, i.e. Δt_1,Δt_2, and Δt_3, ($\Delta t_1 > \Delta t_2 > \Delta t_3$). As $\Delta t \to 0$, $\Delta \mathbf{r} \to 0$ and is along the tangent to the path [Fig. 3.13(d)]. Therefore, **the direction of velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion**.

We can express **v** in a component form :

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

$$= \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta y}{\Delta t}\hat{\mathbf{j}}\right) \tag{3.29}$$

$$= \hat{\mathbf{i}} \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t}$$

Or, $$\mathbf{v} = \hat{\mathbf{i}}\frac{\mathrm{d}x}{\mathrm{d}t} + \hat{\mathbf{j}}\frac{\mathrm{d}y}{\mathrm{d}t} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}.$$

where $$v_x = \frac{\mathrm{d}x}{\mathrm{d}t}, v_y = \frac{\mathrm{d}y}{\mathrm{d}t} \tag{3.30a}$$

So, if the expressions for the coordinates x and y are known as functions of time, we can use these equations to find v_x and v_y.

The magnitude of **v** is then

$$v = \sqrt{v_x^2 + v_y^2} \tag{3.30b}$$

and the direction of **v** is given by the angle θ :

$$\tan\theta = \frac{v_y}{v_x}, \quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \tag{3.30c}$$

v_x, v_y and angle θ are shown in Fig. 3.14 for a velocity vector **v** at point **p**.

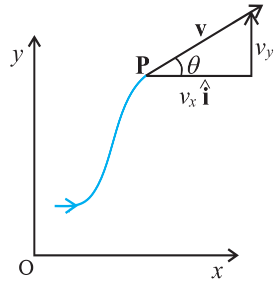


*Fig. 3.14 The components v_x and v_y of velocity **v** and the angle θ it makes with x-axis. Note that $v_x = v\cos\theta$, $v_y = v\sin\theta$.*

Acceleration

The **average acceleration a** of an object for a time interval Δt moving in x-y plane is the change in velocity divided by the time interval :

$$\bar{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta\left(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}\right)}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} + \frac{\Delta v_y}{\Delta t}\hat{\mathbf{j}} \tag{3.31a}$$

Or, $$\bar{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}. \tag{3.31b}$$

The **acceleration** (instantaneous acceleration) is the limiting value of the average acceleration as the time interval approaches zero :

$$\mathbf{a} = \lim_{\Delta t \to 0}\frac{\Delta\mathbf{v}}{\Delta t} \tag{3.32a}$$

Since $\Delta\boldsymbol{v} = \Delta v_x\hat{\mathbf{i}} + \Delta v_y\hat{\mathbf{j}}$, we have

$$\mathbf{a} = \hat{\mathbf{i}} \lim_{\Delta t \to 0}\frac{\Delta v_x}{\Delta t} + \hat{\mathbf{j}} \lim_{\Delta t \to 0}\frac{\Delta v_y}{\Delta t}$$

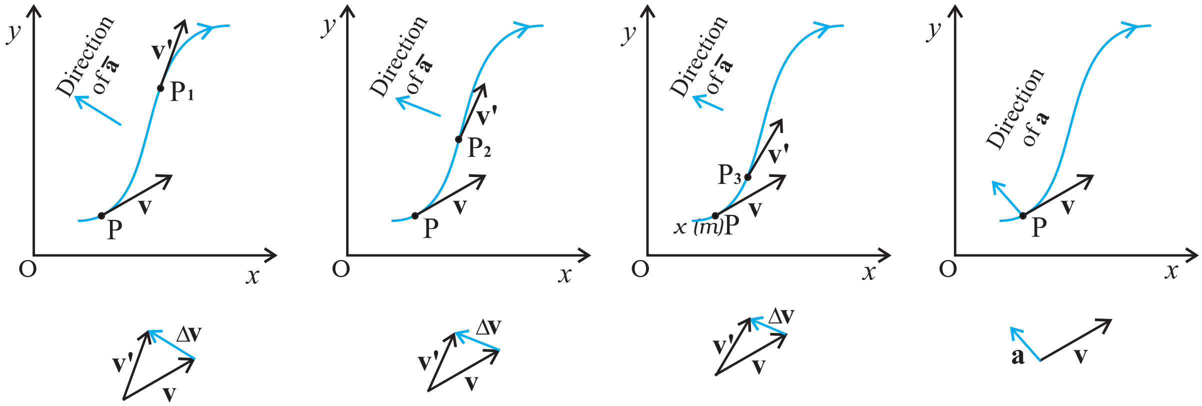
Or, $$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} \tag{3.32b}$$

where, $$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t}, \; a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} \tag{3.32c}$$*

As in the case of velocity, we can understand graphically the limiting process used in defining acceleration on a graph showing the path of the object's motion. This is shown in Figs. 3.15(a) to (d). P represents the position of the object at time t and P_1, P_2, P_3 positions after time Δt_1, Δt_2, Δt_3, respectively ($\Delta t_1 > \Delta t_2 > \Delta t_3$). The velocity vectors at points P, P_1, P_2, P_3 are also shown in Figs. 3.15 (a), (b) and (c). In each case of Δt, $\Delta\mathbf{v}$ is obtained using the triangle law of vector addition. By definition, the direction of average acceleration is the same as that of $\Delta\mathbf{v}$. We see that as Δt decreases, the direction of $\Delta\mathbf{v}$ changes and consequently, the direction of the acceleration changes. Finally, in the limit $\Delta t \to 0$ [Fig. 3.15(d)], the average acceleration becomes the instantaneous acceleration and has the direction as shown.

* *In terms of x and y, a_x and a_y can be expressed as*

$$a_x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \; a_y = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$



(a) (b) (c) (d)

Fig. 3.15 *The average acceleration for three time intervals* (a) Δt_1, (b) Δt_2, *and* (c) Δt_3, *(*$\Delta t_1 > \Delta t_2 > \Delta t_3$*).* (d) *In the limit* $\Delta t \to 0$*, the average acceleration becomes the acceleration.*

Note that in one dimension, the velocity and the acceleration of an object are always along the same straight line (either in the same direction or in the opposite direction). However, for motion in two or three dimensions, velocity and acceleration vectors may have any angle between 0° and 180° between them.

> ***Example 3.4*** The position of a particle is given by
>
> $$\mathbf{r} = 3.0t\,\hat{\mathbf{i}} + 2.0t^2\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}$$
>
> where t is in seconds and the coefficients have the proper units for $\mathbf{r}$ to be in metres. (a) Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ of the particle. (b) Find the magnitude and direction of $\mathbf{v}(t)$ at $t = 1.0$ s.

Answer

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(3.0\;t\,\hat{\mathbf{i}} + 2.0t^2\,\hat{\mathbf{j}} + 5.0\,\hat{\mathbf{k}}\right)$$

$$= 3.0\hat{\mathbf{i}} + 4.0t\,\hat{\mathbf{j}}$$

$$\mathbf{a}\,(t) = \frac{d\mathbf{v}}{dt} = +4.0\hat{\mathbf{j}}$$

$a = 4.0$ m s^{-2} along y- direction

At $t = 1.0$ s, $\mathbf{v} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$

It's magnitude is $v = \sqrt{3^2 + 4^2} = 5.0$ m s^{-1}

and direction is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) \cong 53^\circ \text{ with } x\text{-axis.}$$

◀

3.8 MOTION IN A PLANE WITH CONSTANT ACCELERATION

Suppose that an object is moving in x-y plane and its acceleration $\mathbf{a}$ is constant. Over an interval of time, the average acceleration will equal this constant value. Now, let the velocity of the object be $\mathbf{v}_0$ at time $t = 0$ and $\mathbf{v}$ at time t. Then, by definition

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v_0}}{t - 0} = \frac{\mathbf{v} - \mathbf{v_0}}{t}$$

Or, $\mathbf{v} = \mathbf{v_0} + \mathbf{a}t$ (3.33a)

In terms of components :

$$v_x = v_{ox} + a_x t$$

$$v_y = v_{oy} + a_y t \qquad (3.33b)$$

Let us now find how the position $\mathbf{r}$ changes with time. We follow the method used in the one-dimensional case. Let $\mathbf{r_o}$ and $\mathbf{r}$ be the position vectors of the particle at time 0 and t and let the velocities at these instants be $\mathbf{v_o}$ and $\mathbf{v}$. Then, over this time interval t, the average velocity is $(\mathbf{v_o} + \mathbf{v})/2$. The displacement is the average velocity multiplied by the time interval :

$$\mathbf{r}-\mathbf{r_0}=\left(\frac{\mathbf{v}+\mathbf{v_0}}{2}\right)t=\left(\frac{(\mathbf{v_0}+\mathbf{a}t)+\mathbf{v_0}}{2}\right)t$$

$$=\mathbf{v_0}t+\frac{1}{2}\mathbf{a}t^2$$

Or, $$\mathbf{r}=\mathbf{r_0}+\mathbf{v_0}t+\frac{1}{2}\mathbf{a}t^2 \qquad (3.34a)$$

It can be easily verified that the derivative of Eq. (3.34a), i.e. $\frac{d\mathbf{r}}{dt}$ gives Eq.(3.33a) and it also satisfies the condition that at t=0, $\mathbf{r} = \mathbf{r}_o$. Equation (3.34a) can be written in component form as

$$x = x_0 + v_{ox}t + \frac{1}{2}a_x t^2$$

$$y = y_0 + v_{oy}t + \frac{1}{2}a_y t^2 \qquad (3.34b)$$

One immediate interpretation of Eq.(3.34b) is that the motions in x- and y-directions can be treated independently of each other. That is, **motion in a plane (two-dimensions) can be treated as two separate simultaneous one-dimensional motions with constant acceleration along two perpendicular directions**. This is an important result and is useful in analysing motion of objects in two dimensions. A similar result holds for three dimensions. The choice of perpendicular directions is convenient in many physical situations, as we shall see in section 3.9 for projectile motion.

Example 3.5 A particle starts from origin at t = 0 with a velocity 5.0 $\hat{\mathbf{i}}$ m/s and moves in x-y plane under action of a force which produces a constant acceleration of $(3.0\hat{\mathbf{i}}+2.0\hat{\mathbf{j}})$ m/s². (a) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m ? (b) What is the speed of the particle at this time ?

Answer From Eq. (3.34a) for $\mathbf{r_0}$ = 0, the position of the particle is given by

$$\mathbf{r}(t)=\mathbf{v_0}t+\frac{1}{2}\mathbf{a}t^2$$
$$=5.0\hat{\mathbf{i}}t+(1/2)(3.0\hat{\mathbf{i}}+2.0\hat{\mathbf{j}})\,t^2$$
$$=(5.0t+1.5t^2)\hat{\mathbf{i}}+1.0t^2\hat{\mathbf{j}}$$

Therefore, $x(t)=5.0t+1.5t^2$

$y(t)=+1.0t^2$

Given $x(t)$ = 84 m, t = ?

$5.0\,t + 1.5\,t^2 = 84 \Rightarrow t = 6$ s

At t = 6 s, y = 1.0 (6)² = 36.0 m

Now, the velocity $\mathbf{v}=\frac{d\mathbf{r}}{dt}=(5.0+3.0t)\hat{\mathbf{i}}+2.0t\,\hat{\mathbf{j}}$

At t = 6 s, $\mathbf{v}=23.0\hat{\mathbf{i}}+12.0\hat{\mathbf{j}}$

speed $=|\mathbf{v}|=\sqrt{23^2+12^2}\cong 26\text{ m s}^{-1}$.

3.9 PROJECTILE MOTION

As an application of the ideas developed in the previous sections, we consider the motion of a projectile. An object that is in flight after being thrown or projected is called a **projectile**. Such a projectile might be a football, a cricket ball, a baseball or any other object. The motion of a projectile may be thought of as the result of two separate, simultaneously occurring components of motions. One component is along a horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to the force of gravity. It was Galileo who first stated this independency of the horizontal and the vertical components of projectile motion in his **Dialogue on the great world systems** (1632).

In our discussion, we shall assume that the air resistance has negligible effect on the motion of the projectile. Suppose that the projectile is launched with velocity $\mathbf{v}_o$ that makes an angle θ_o with the x-axis as shown in Fig. 3.16.

After the object has been projected, the acceleration acting on it is that due to gravity which is directed vertically downward:

$$\mathbf{a}=-g\hat{\mathbf{j}}$$

Or, $$a_x = 0,\ a_y = -g \qquad (3.35)$$

The components of initial velocity $\mathbf{v}_o$ are :

$$v_{ox} = v_o\cos\theta_o$$
$$v_{oy} = v_o\sin\theta_o \qquad (3.36)$$

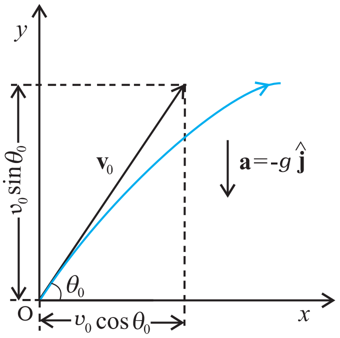


Fig 3.16 Motion of an object projected with velocity $\mathbf{v}_o$ *at angle* θ_0.

If we take the initial position to be the origin of the reference frame as shown in Fig. 3.16, we have :

$$x_o = 0,\ y_o = 0$$

Then, Eq.(3.34b) becomes :

$$x = v_{ox}\, t = (v_o \cos\theta_o)\, t$$

and $$y = (v_o \sin\theta_o)\, t - (\tfrac{1}{2})g\, t^2 \qquad (3.37)$$

The components of velocity at time t can be obtained using Eq.(3.33b) :

$$v_x = v_{ox} = v_o \cos\theta_o$$

$$v_y = v_o \sin\theta_o - g\, t \qquad (3.38)$$

Equation (3.37) gives the x-, and y-coordinates of the position of a projectile at time t in terms of two parameters — initial speed v_0 and projection angle θ_0. Notice that the choice of mutually perpendicular x-, and y-directions for the analysis of the projectile motion has resulted in a simplification. One of the components of velocity, i.e. x-component remains constant throughout the motion and only the y- component changes, like an object in free fall in vertical direction. This is shown graphically at few instants in Fig. 3.17. Note that at the point of maximum height, $v_y = 0$ and therefore,

$$\theta = \tan^{-1}\frac{v_y}{v_x} = 0$$

Equation of path of a projectile

What is the shape of the path followed by the projectile? This can be seen by eliminating the time between the expressions for x and y as given in Eq. (3.37). We obtain:

$$y = (\tan\theta_o)\, x - \frac{g}{2\,(v_o\cos\theta_o)^2}\, x^2 \qquad (3.39)$$

Now, since g, θ_o and v_o are constants, Eq. (3.39) is of the form $y = a\,x + b\,x^2$, in which a and b are constants. This is the equation of a parabola, i.e. the path of the projectile is a parabola (Fig. 3.17).

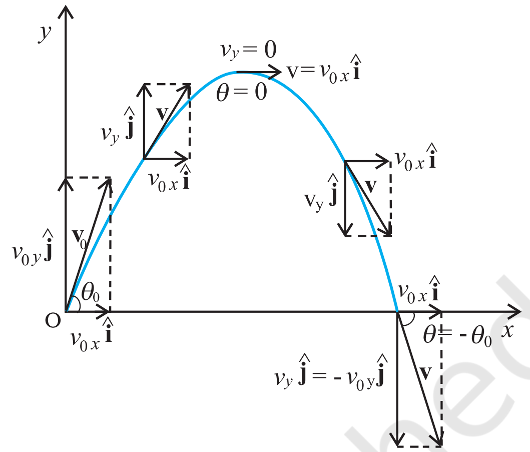


Fig. 3.17 The path of a projectile is a parabola.

Time of maximum height

How much time does the projectile take to reach the maximum height ? Let this time be denoted by t_m. Since at this point, $v_y = 0$, we have from Eq. (3.38):

$$v_y = v_o \sin\theta_o - g\, t_m = 0$$

Or, $$t_m = v_o \sin\theta_o / g \qquad (3.40a)$$

The total time T_f during which the projectile is in flight can be obtained by putting $y = 0$ in Eq. (3.37). We get :

$$T_f = 2\,(v_o \sin\theta_o)/g \qquad (3.40b)$$

T_f is known as the **time of flight** of the projectile. We note that $T_f = 2\,t_m$, which is expected because of the symmetry of the parabolic path.

Maximum height of a projectile

The maximum height h_m reached by the projectile can be calculated by substituting $t = t_m$ in Eq. (3.37) :

$$y = h_m = (v_0\sin\theta_0)\left(\frac{v_0\sin\theta_0}{g}\right) - \frac{g}{2}\left(\frac{v_0\sin\theta_0}{g}\right)^2$$

Or, $$h_m = \frac{(v_0\sin\theta_0)^2}{2g} \qquad (3.41)$$

Horizontal range of a projectile

The horizontal distance travelled by a projectile from its initial position ($x = y = 0$) to the position where it passes $y = 0$ during its fall is called the **horizontal**

range, R. It is the distance travelled during the time of flight T_f . Therefore, the range R is

$$R = (v_o \cos \theta_o)\ (T_f)$$

$$=(v_o \cos \theta_o)\ (2\ v_o \sin \theta_o)/g$$

Or, $$R = \frac{v_0^2 \sin 2\theta_0}{g} \qquad (3.42a)$$

Equation (3.42a) shows that for a given projection velocity v_o , R is maximum when sin $2\theta_0$ is maximum, i.e., when $\theta_0 = 45^0$.

The maximum horizontal range is, therefore,

$$R_m = \frac{v_0^2}{g} \qquad (3.42b)$$

Example 3.6 Galileo, in his book **Two new sciences,** stated that "for elevations which exceed or fall short of 45 by equal amounts, the ranges are equal". Prove this statement.

Answer For a projectile launched with velocity $\mathbf{v}_o$ at an angle θ_o , the range is given by

$$R = \frac{v_0^2 \sin 2\theta_0}{g}$$

Now, for angles, (45 + α) and (45 – α), $2\theta_o$ is (90 + 2α) and (90 – 2α) , respectively. The values of sin (90 + 2α) and sin (90 – 2α) are the same, equal to that of cos 2α. Therefore, ranges are equal for elevations which exceed or fall short of 45 by equal amounts α. ◀

Example 3.7 A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 m s^{-1}. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 m s^{-2}).

Answer We choose the origin of the x-,and y-axis at the edge of the cliff and t = 0 s at the instant the stone is thrown. Choose the positive direction of x-axis to be along the initial velocity and the positive direction of y-axis to be the vertically upward direction. The x-, and y-components of the motion can be treated independently. The equations of motion are :

$$x\,(t) = x_o + v_{ox}\, t$$

$$y\,(t) = y_o + v_{oy}\, t + (1/2)\, a_y\, t^2$$

Here, $x_o = y_o = 0,\ v_{oy} = 0,\ a_y = -g = -9.8$ m s^{-2}, v_{ox} = 15 m s^{-1}.

The stone hits the ground when $y(t)$ = – 490 m.

$$-\,490\text{ m} = -(1/2)(9.8)\ t^2.$$

This gives t =10 s.

The velocity components are $v_x = v_{ox}$ and

$$v_y = v_{oy} - g\,t$$

so that when the stone hits the ground :

$$v_{ox} = 15 \text{ m s}^{-1}$$

$$v_{oy} = 0 - 9.8 \quad 10 = -\,98 \text{ m s}^{-1}$$

Therefore, the speed of the stone is

$$\sqrt{v_x^2 + v_y^2} = \sqrt{15^2 + 98^2} = 99 \text{ m s}^{-1}$$ ◀

Example 3.8 A cricket ball is thrown at a speed of 28 m s^{-1} in a direction 30 above the horizontal. Calculate (a) the maximum height, (b) the time taken by the ball to return to the same level, and (c) the distance from the thrower to the point where the ball returns to the same level.

Answer (a) The maximum height is given by

$$h_m = \frac{(v_0 \sin\theta_o)^2}{2g} = \frac{(28\sin 30°)^2}{2\ (9.8)} \text{ m}$$

$$= \frac{14 \times 14}{2 \times 9.8} = 10.0 \text{ m}$$

(b) The time taken to return to the same level is

$$T_f = (2\ v_o \sin \theta_o)/g = (2 \quad 28 \quad \sin 30\ )/9.8$$

$$= 28/9.8 \text{ s} = 2.9 \text{ s}$$

(c) The distance from the thrower to the point where the ball returns to the same level is

$$R = \frac{(v_o^2 \sin 2\theta_o)}{g} = \frac{28 \times 28 \times \sin 60^o}{9.8} = 69 \text{ m}$$ ◀

3.10 UNIFORM CIRCULAR MOTION

When an object follows a circular path at a constant speed, the motion of the object is called **uniform circular motion**. The word "uniform" refers to the speed, which is uniform (constant) throughout the motion. Suppose an object is moving with uniform speed v in a circle of radius R as shown in Fig. 3.18. Since the velocity of the object is changing continuously in direction, the object undergoes acceleration. Let us find the magnitude and the direction of this acceleration.

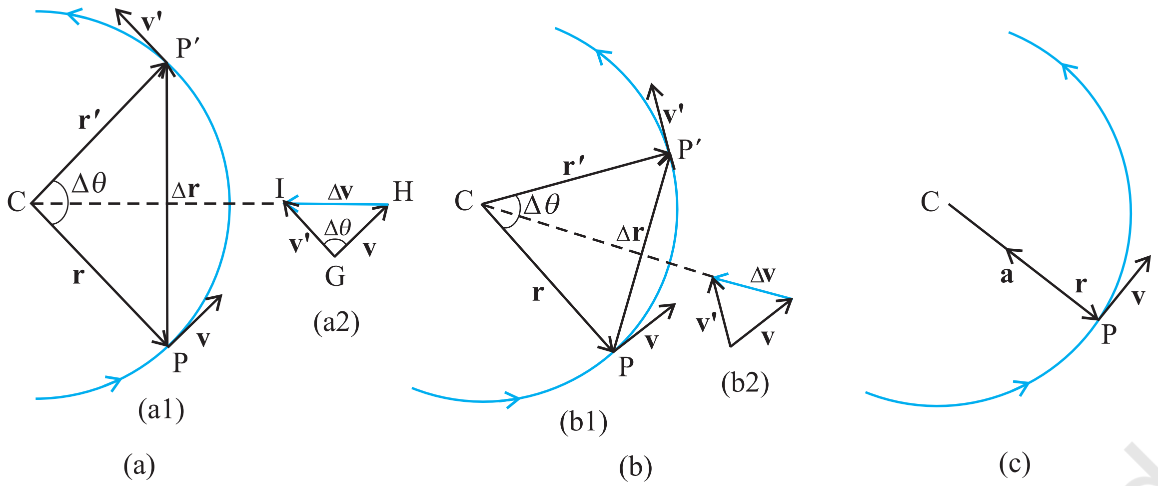


Fig. 3.18 *Velocity and acceleration of an object in uniform circular motion. The time interval Δt decreases from (a) to (c) where it is zero. The acceleration is directed, at each point of the path, towards the centre of the circle.*

Let **r** and **r′** be the position vectors and **v** and **v′** the velocities of the object when it is at point *P* and *P′* as shown in Fig. 3.18(a). By definition, velocity at a point is along the tangent at that point in the direction of motion. The velocity vectors **v** and **v′** are as shown in Fig. 3.18(a1). Δ**v** is obtained in Fig. 3.18 (a2) using the triangle law of vector addition. Since the path is circular, **v** is perpendicular to **r** and so is **v′** to **r′**. Therefore, Δ**v** is perpendicular to Δ**r**. Since average acceleration is along Δ**v** $\left(\bar{\mathbf{a}} = \frac{\Delta \mathbf{v}}{\Delta t}\right)$, the average acceleration $\bar{\mathbf{a}}$ is perpendicular to Δ**r.** If we place Δ**v** on the line that bisects the angle between **r** and **r′**, we see that it is directed towards the centre of the circle. Figure 3.18(b) shows the same quantities for smaller time interval. Δ**v** and hence $\bar{\mathbf{a}}$ is again directed towards the centre. In Fig. 3.18(c), $\Delta t \to 0$ and the average acceleration becomes the instantaneous acceleration. It is directed towards the centre*. Thus, we find that the acceleration of an object in uniform circular motion is always directed towards the centre of the circle. Let us now find the magnitude of the acceleration.

The magnitude of **a** is, by definition, given by

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{v}|}{\Delta t}$$

Let the angle between position vectors **r** and **r′** be $\Delta\theta$. Since the velocity vectors **v** and **v′** are always perpendicular to the position vectors, the angle between them is also $\Delta\theta$. Therefore, the triangle CPP′ formed by the position vectors and the triangle GHI formed by the velocity vectors **v**, **v′** and Δ**v** are similar (Fig. 3.18a). Therefore, the ratio of the base-length to side-length for one of the triangles is equal to that of the other triangle. That is :

$$\frac{|\Delta \mathbf{v}|}{v} = \frac{|\Delta \mathbf{r}|}{R}$$

Or, $$|\Delta \mathbf{v}| = v\frac{|\Delta \mathbf{r}|}{R}$$

Therefore,

$$|\mathbf{a}| = \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{v|\Delta \mathbf{r}|}{R\Delta t} = \frac{v}{R} \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{r}|}{\Delta t}$$

If Δt is small, Δθ will also be small and then arc *PP′* can be approximately taken to be |Δ**r**| :

$$|\Delta \mathbf{r}| \cong v\Delta t$$

$$\frac{|\Delta \mathbf{r}|}{\Delta t} \cong v$$

Or, $$\lim_{\Delta t \to 0} \frac{|\Delta \mathbf{r}|}{\Delta t} = v$$

Therefore, the centripetal acceleration a_c is :

* *In the limit Δt→0, Δ**r** becomes perpendicular to **r**. In this limit Δ**v**→**0** and is consequently also perpendicular to **V**. Therefore, the acceleration is directed towards the centre, at each point of the circular path.*

$$a_c = \left(\frac{v}{R}\right) v = v^2/R \qquad (3.43)$$

Thus, the acceleration of an object moving with speed v in a circle of radius R has a magnitude v^2/R and is always **directed towards the centre**. This is why this acceleration is called **centripetal acceleration** (a term proposed by Newton). A thorough analysis of centripetal acceleration was first published in 1673 by the Dutch scientist Christiaan Huygens (1629-1695) but it was probably known to Newton also some years earlier. "Centripetal" comes from a Greek term which means 'centre-seeking'. Since v and R are constant, the magnitude of the centripetal acceleration is also constant. However, the direction changes — pointing always towards the centre. Therefore, a centripetal acceleration is not a constant vector.

We have another way of describing the velocity and the acceleration of an object in uniform circular motion. As the object moves from P to P′ in time Δt (= $t' - t$), the line CP (Fig. 3.18) turns through an angle $\Delta\theta$ as shown in the figure. $\Delta\theta$ is called angular distance. We define the angular speed ω (Greek letter omega) as the time rate of change of angular displacement :

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad (3.44)$$

Now, if the distance travelled by the object during the time Δt is Δs, i.e. PP' is Δs, then :

$$v = \frac{\Delta s}{\Delta t}$$

but $\Delta s = R\,\Delta\theta$. Therefore :

$$v = R\frac{\Delta\theta}{\Delta t} = R\,\omega$$

$$v = R\,\omega \qquad (3.45)$$

We can express centripetal acceleration a_c in terms of angular speed :

$$a_c = \frac{v^2}{R} = \frac{\omega^2 R^2}{R} = \omega^2 R$$

$$a_c = \omega^2 R \qquad (3.46)$$

The time taken by an object to make one revolution is known as its time period T and the number of revolution made in one second is called its frequency ν (=$1/T$). However, during this time the distance moved by the object is $s = 2\pi R$.

Therefore, $v = 2\pi R/T = 2\pi R\nu$ (3.47)

In terms of frequency ν, we have

$$\omega = 2\pi\nu$$

$$v = 2\pi R\nu$$

$$a_c = 4\pi^2 \nu^2 R \qquad (3.48)$$

▶ ***Example 3.9*** An insect trapped in a circular groove of radius 12 cm moves along the groove steadily and completes 7 revolutions in 100 s. (a) What is the angular speed, and the linear speed of the motion? (b) Is the acceleration vector a constant vector ? What is its magnitude ?

Answer This is an example of uniform circular motion. Here R = 12 cm. The angular speed ω is given by

$$\omega = 2\pi/T = 2\pi \quad 7/100 = 0.44 \text{ rad/s}$$

The linear speed v is :

$$v = \omega\, R = 0.44 \text{ s}^{-1} \quad 12 \text{ cm} = 5.3 \text{ cm s}^{-1}$$

The direction of velocity $\mathbf{v}$ is along the tangent to the circle at every point. The acceleration is directed towards the centre of the circle. Since this direction changes continuously, acceleration here is *not* a constant vector. However, the magnitude of acceleration is constant:

$$a = \omega^2 R = (0.44 \text{ s}^{-1})^2 (12 \text{ cm})$$

$$= 2.3 \text{ cm s}^{-2}$$ ◀

SUMMARY

1. *Scalar quantities* are quantities with magnitudes only. Examples are distance, speed, mass and temperature.
2. *Vector quantities* are quantities with magnitude and direction both. Examples are displacement, velocity and acceleration. They obey special rules of vector algebra.
3. A vector **A** multiplied by a real number λ is also a vector, whose magnitude is λ times the magnitude of the vector **A** and whose direction is the same or opposite depending upon whether λ is positive or negative.
4. Two vectors **A** and **B** may be *added graphically* using *head-to-tail method* or *parallelogram method.*
5. Vector addition is *commutative* :
$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
It also obeys the *associative law* :
$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$
6. A *null* or *zero vector* is a vector with zero magnitude. Since the magnitude is zero, we don't have to specify its direction. It has the properties :
$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$
$$\lambda\mathbf{0} = \mathbf{0}$$
$$0\ \mathbf{A} = \mathbf{0}$$
7. The *subtraction* of vector **B** from **A** is defined as the sum of **A** and –**B** :
$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$
8. A vector **A** can be *resolved* into component along two given vectors **a** and **b** lying in the same plane :
$$\mathbf{A} = \lambda\,\mathbf{a} + \mu\,\mathbf{b}$$
where λ and μ are real numbers.
9. A *unit vector* associated with a vector **A** has magnitude 1 and is along the vector **A**:
$$\hat{\mathbf{n}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$
The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are vectors of unit magnitude and point in the direction of the *x*-, *y*-, and *z*-axes, respectively in a right-handed coordinate system.
10. A vector **A** can be expressed as
$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$
where A_x, A_y are its components along *x*-, and *y* -axes. If vector **A** makes an angle θ with the *x*-axis, then $A_x = A\cos\theta$, $A_y = A\sin\theta$ and $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$, $\tan\theta = \dfrac{A_y}{A_x}$.
11. Vectors can be conveniently added using *analytical method.* If sum of two vectors **A** and **B**, that lie in *x-y* plane, is **R**, then :
$\mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$, where, $R_x = A_x + B_x$, and $R_y = A_y + B_y$
12. The *position vector* of an object in *x-y* plane is given by $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ and the *displacement* from position **r** to position **r'** is given by
$$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$$
$$= (x' - x)\,\hat{\mathbf{i}} + (y' - y)\,\hat{\mathbf{j}}$$
$$= \Delta x\,\hat{\mathbf{i}} + \Delta y\,\hat{\mathbf{j}}$$
13. If an object undergoes a displacement $\Delta\mathbf{r}$ in time Δt, its *average velocity* is given by $\mathbf{v} = \dfrac{\Delta\mathbf{r}}{\Delta t}$. The *velocity* of an object at time *t* is the limiting value of the average velocity as Δt tends to zero :

$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$. It can be written in unit vector notation as :

$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ where $v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}, v_z = \frac{dz}{dt}$

When position of an object is plotted on a coordinate system, $\mathbf{v}$ is always tangent to the curve representing the path of the object.

14. If the velocity of an object changes from $\mathbf{v}$ to $\mathbf{v}'$ in time Δt, then its *average acceleration* is given by: $\bar{\mathbf{a}} = \frac{\mathbf{v} - \mathbf{v}'}{\Delta t} = \frac{\Delta \mathbf{v}}{\Delta t}$

The *acceleration* $\mathbf{a}$ at any time t is the limiting value of $\bar{\mathbf{a}}$ as $\Delta t \to 0$:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

In component form, we have : $\mathbf{a} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} + a_z \hat{\mathbf{k}}$

where, $a_x = \frac{dv_x}{dt}$, $a_y = \frac{dv_y}{dt}$, $a_z = \frac{dv_z}{dt}$

15. If an object is moving in a plane with constant acceleration $a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2}$ and its position vector at time $t = 0$ is $\mathbf{r}_o$, then at any other time t, it will be at a point given by:

$$\mathbf{r} = \mathbf{r_o} + \mathbf{v_o} t + \frac{1}{2} \mathbf{a} t^2$$

and its velocity is given by :

$$\mathbf{v} = \mathbf{v_o} + \mathbf{a}\, t$$

where $\mathbf{v_o}$ is the velocity at time $t = 0$

In component form :

$$x = x_o + v_{ox} t + \frac{1}{2} a_x t^2$$

$$y = y_o + v_{oy}\, t + \frac{1}{2} a_y t^2$$

$$v_x = v_{ox} + a_x t$$

$$v_y = v_{oy} + a_y t$$

Motion in a plane can be treated as superposition of two separate simultaneous one-dimensional motions along two perpendicular directions

16. An object that is in flight after being projected is called a *projectile*. If an object is projected with initial velocity $\mathbf{v_o}$ making an angle θ_o with x-axis and if we assume its initial position to coincide with the origin of the coordinate system, then the position and velocity of the projectile at time t are given by :

$$x = (v_o \cos \theta_o)\, t$$

$$y = (v_o \sin \theta_o)\, t - (1/2)\, g\, t^2$$

$$v_x = v_{ox} = v_o \cos \theta_o$$

$$v_y = v_o \sin \theta_o - g\, t$$

The path of a projectile is *parabolic* and is given by :

$$y = (\tan\theta_0)\, x - \frac{g x^2}{2\,(v_o \cos\theta_o)^2}$$

The *maximum height* that a projectile attains is :

$$h_m = \frac{(v_o \sin\theta_o)^2}{2g}$$

The *time* taken to reach this height is :

$$t_m = \frac{v_o \sin\theta_o}{g}$$

The horizontal distance travelled by a projectile from its initial position to the position it passes $y = 0$ during its fall is called the *range*, R of the projectile. It is :

$$R = \frac{v_o^2}{g} \sin 2\theta_o$$

17. When an object follows a circular path at constant speed, the motion of the object is called *uniform circular motion*. The magnitude of its acceleration is $a_c = v^2/R$. The direction of a_c is always towards the centre of the circle.
The angular speed ω, is the rate of change of angular distance. It is related to velocity v by $v = \omega R$. The acceleration is $a_c = \omega^2 R$.
If T is the time period of revolution of the object in circular motion and ν is its frequency, we have $\omega = 2\pi\nu$, $v = 2\pi\nu R$, $a_c = 4\pi^2\nu^2 R$

Physical Quantity	Symbol	Dimensions	Unit	Remark
Position vector	$\mathbf{r}$	[L]	m	Vector. It may be denoted by any other symbol as well.
Displacement	$\Delta\mathbf{r}$	[L]	m	- do -
Velocity		$[LT^{-1}]$	$m\ s^{-1}$	
(a) Average	$\bar{\mathbf{v}}$			$= \frac{\Delta\mathbf{r}}{\Delta t}$, vector
(b) Instantaneous	$\mathbf{v}$			$= \frac{d\mathbf{r}}{dt}$, vector
Acceleration		$[LT^{-2}]$	$m\ s^{-2}$	
(a) Average	$\bar{\mathbf{a}}$			$= \frac{\Delta\mathbf{v}}{\Delta t}$, vector
(b) Instantaneous	$\mathbf{a}$			$= \frac{d\mathbf{v}}{dt}$, vector
Projectile motion				
(a) Time of max. height	t_m	[T]	s	$= \frac{v_0 \sin\theta_0}{g}$
(b) Max. height	h_m	[L]	m	$= \frac{(v_0 \sin\theta_0)^2}{2g}$
(c) Horizontal range	R	[L]	m	$= \frac{v_0^2 \sin 2\theta_0}{g}$
Circular motion				
(a) Angular speed	ω	$[T^{-1}]$	rad/s	$= \frac{\Delta\theta}{\Delta t} = \frac{v}{r}$
(b) Centripetal acceleration	a_c	$[LT^{-2}]$	$m\ s^{-2}$	$= \frac{v^2}{r}$

POINTS TO PONDER

1. The path length traversed by an object between two points is, in general, not the same as the magnitude of displacement. The displacement depends only on the end points; the path length (as the name implies) depends on the actual path. The two quantities are equal only if the object does not change its direction during the course of motion. In all other cases, the path length is greater than the magnitude of displacement.
2. In view of point 1 above, the average speed of an object is greater than or equal to the magnitude of the average velocity over a given time interval. The two are equal only if the path length is equal to the magnitude of displacement.
3. The vector equations (3.33a) and (3.34a) do not involve any choice of axes. Of course, you can always resolve them along any two independent axes.
4. The kinematic equations for uniform acceleration do not apply to the case of uniform circular motion since in this case the magnitude of acceleration is constant but its direction is changing.
5. An object subjected to two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ has a resultant velocity $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Take care to distinguish it from velocity of object 1 relative to velocity of object 2 : $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$. Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are velocities with reference to some common reference frame.
6. The resultant acceleration of an object in circular motion is towards the centre only if the speed is constant.
7. The shape of the trajectory of the motion of an object is not determined by the acceleration alone but also depends on the initial conditions of motion (initial position and initial velocity). For example, the trajectory of an object moving under the same acceleration due to gravity can be a straight line or a parabola depending on the initial conditions.

EXERCISES

3.1 State, for each of the following physical quantities, if it is a scalar or a vector : volume, mass, speed, acceleration, density, number of moles, velocity, angular frequency, displacement, angular velocity.

3.2 Pick out the two scalar quantities in the following list :
force, angular momentum, work, current, linear momentum, electric field, average velocity, magnetic moment, relative velocity.

3.3 Pick out the only vector quantity in the following list :
Temperature, pressure, impulse, time, power, total path length, energy, gravitational potential, coefficient of friction, charge.

3.4 State with reasons, whether the following algebraic operations with scalar and vector physical quantities are meaningful :
(a) adding any two scalars, (b) adding a scalar to a vector of the same dimensions , (c) multiplying any vector by any scalar, (d) multiplying any two scalars, (e) adding any two vectors, (f) adding a component of a vector to the same vector.

3.5 Read each statement below carefully and state with reasons, if it is true or false :
(a) The magnitude of a vector is always a scalar, (b) each component of a vector is always a scalar, (c) the total path length is always equal to the magnitude of the displacement vector of a particle. (d) the average speed of a particle (defined as total path length divided by the time taken to cover the path) is either greater or equal to the magnitude of average velocity of the particle over the same interval of time, (e) Three vectors not lying in a plane can never add up to give a null vector.

3.6 Establish the following vector inequalities geometrically or otherwise :

(a) $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(b) $|\mathbf{a}+\mathbf{b}| \geq ||\mathbf{a}| - |\mathbf{b}||$

(c) $|\mathbf{a}-\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(d) $|\mathbf{a}-\mathbf{b}| \geq ||\mathbf{a}| - |\mathbf{b}||$

When does the equality sign above apply?

3.7 Given $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$, which of the following statements are correct :

(a) **a**, **b**, **c**, and **d** must each be a null vector,

(b) The magnitude of $(\mathbf{a} + \mathbf{c})$ equals the magnitude of $(\mathbf{b} + \mathbf{d})$,

(c) The magnitude of **a** can never be greater than the sum of the magnitudes of **b**, **c**, and **d**,

(d) $\mathbf{b} + \mathbf{c}$ must lie in the plane of **a** and **d** if **a** and **d** are not collinear, and in the line of **a** and **d**, if they are collinear ?

3.8 Three girls skating on a circular ice ground of radius 200 m start from a point *P* on the edge of the ground and reach a point Q diametrically opposite to *P* following different paths as shown in Fig. 3.19. What is the magnitude of the displacement vector for each ? For which girl is this equal to the actual length of path skate ?

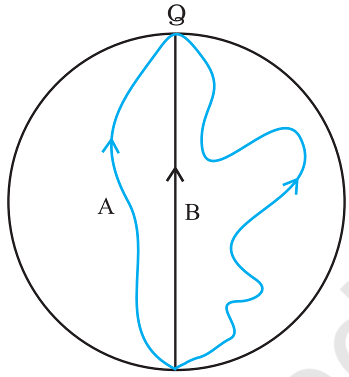


Fig. 3.19

3.9 A cyclist starts from the centre *O* of a circular park of radius 1 km, reaches the edge *P* of the park, then cycles along the circumference, and returns to the centre along QO as shown in Fig. 3.20. If the round trip takes 10 min, what is the (a) net displacement, (b) average velocity, and (c) average speed of the cyclist ?

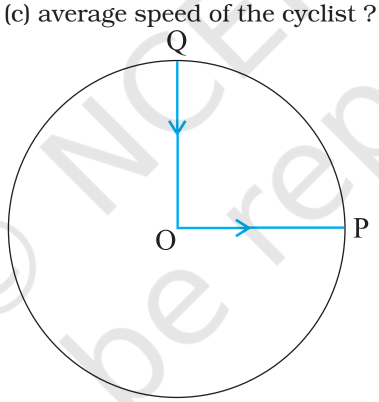


Fig. 3.20

3.10 On an open ground, a motorist follows a track that turns to his left by an angle of 60^0 after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of the displacement with the total path length covered by the motorist in each case.

3.11 A passenger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circuitous path 23 km long and reaches the hotel in 28 min. What is (a) the average speed of the taxi, (b) the magnitude of average velocity ? Are the two equal ?

3.12 The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 m s^{-1} can go without hitting the ceiling of the hall ?

3.13 A cricketer can throw a ball to a maximum horizontal distance of 100 m. How much high above the ground can the cricketer throw the same ball ?

3.14 A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, what is the magnitude and direction of acceleration of the stone ?

3.15 An aircraft executes a horizontal loop of radius 1.00 km with a steady speed of 900 km/h. Compare its centripetal acceleration with the acceleration due to gravity.

3.16 Read each statement below carefully and state, with reasons, if it is true or false :

(a) The net acceleration of a particle in circular motion is *always* along the radius of the circle towards the centre

(b) The velocity vector of a particle at a point is *always* along the tangent to the path of the particle at that point

(c) The acceleration vector of a particle in *uniform* circular motion averaged over one cycle is a null vector

3.17 The position of a particle is given by

$$\mathbf{r} = 3.0t\,\hat{\mathbf{i}} - 2.0t^2\,\hat{\mathbf{j}} + 4.0\,\hat{\mathbf{k}}\ \text{m}$$

where t is in seconds and the coefficients have the proper units for $\mathbf{r}$ to be in metres.

(a) Find the $\mathbf{v}$ and $\mathbf{a}$ of the particle? (b) What is the magnitude and direction of velocity of the particle at t = 2.0 s ?

3.18 A particle starts from the origin at t = 0 s with a velocity of 10.0 $\hat{\mathbf{j}}$ m/s and moves in the x-y plane with a constant acceleration of $\left(8.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}\right)$ m s^{-2}. (a) At what time is the x- coordinate of the particle 16 m? What is the y-coordinate of the particle at that time? (b) What is the speed of the particle at the time ?

3.19 $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along x- and y- axis respectively. What is the magnitude and direction of the vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, and $\hat{\mathbf{i}} - \hat{\mathbf{j}}$? What are the components of a vector $\mathbf{A}$= $2\,\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ along the directions of $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} - \hat{\mathbf{j}}$? [You may use graphical method]

3.20 For any arbitrary motion in space, which of the following relations are true :

(a) $\mathbf{v}_{\text{average}} = (1/2)\,(\mathbf{v}\,(t_1) + \mathbf{v}\,(t_2))$

(b) $\mathbf{v}_{\text{average}} = [\mathbf{r}(t_2) - \mathbf{r}(t_1)\,]\,/(t_2 - t_1)$

(c) $\mathbf{v}\,(t) = \mathbf{v}\,(0) + \mathbf{a}\,t$

(d) $\mathbf{r}\,(t) = \mathbf{r}\,(0) + \mathbf{v}\,(0)\,t + (1/2)\,\mathbf{a}\,t^2$

(e) $\mathbf{a}_{\text{average}} = [\,\mathbf{v}\,(t_2) - \mathbf{v}\,(t_1)\,]\,/(\,t_2 - t_1)$

(The 'average' stands for average of the quantity over the time interval t_1 to t_2)

3.21 Read each statement below carefully and state, with reasons and examples, if it is true or false :

A scalar quantity is one that

(a) is conserved in a process

(b) can never take negative values

(c) must be dimensionless

(d) does not vary from one point to another in space

(e) has the same value for observers with different orientations of axes.

3.22 An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10.0 s a part is 30°, wat is the speed of the aircraft ?

CHAPTER FOUR

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LAWS OF MOTION

4.1 INTRODUCTION

In the preceding Chapter, our concern was to describe the motion of a particle in space quantitatively. We saw that uniform motion needs the concept of velocity alone whereas non-uniform motion requires the concept of acceleration in addition. So far, we have not asked the question as to what governs the motion of bodies. In this chapter, we turn to this basic question.

Let us first guess the answer based on our common experience. To move a football at rest, someone must kick it. To throw a stone upwards, one has to give it an upward push. A breeze causes the branches of a tree to swing; a strong wind can even move heavy objects. A boat moves in a flowing river without anyone rowing it. Clearly, some external agency is needed to provide force to move a body from rest. Likewise, an external force is needed also to retard or stop motion. You can stop a ball rolling down an inclined plane by applying a force against the direction of its motion.

In these examples, the external agency of force (hands, wind, stream, etc) is in contact with the object. This is not always necessary. A stone released from the top of a building accelerates downward due to the gravitational pull of the earth. A bar magnet can attract an iron nail from a distance. **This shows that external agencies (e.g. gravitational and magnetic forces) can exert force on a body even from a distance.**

In short, a force is required to put a stationary body in motion or stop a moving body, and some external agency is needed to provide this force. The external agency may or may not be in contact with the body.

So far so good. But what if a body is moving uniformly (e.g. a skater moving straight with constant speed on a horizontal ice slab) ? **Is an external force required to keep a body in uniform motion?**

4.2 ARISTOTLE'S FALLACY

The question posed above appears to be simple. However, it took ages to answer it. Indeed, the correct answer to this question given by Galileo in the seventeenth century was the foundation of Newtonian mechanics, which signalled the birth of modern science.

The Greek thinker, Aristotle (384 B.C– 322 B.C.), held the view that if a body is moving, something external is required to keep it moving. According to this view, for example, an arrow shot from a bow keeps flying since the air behind the arrow keeps pushing it. The view was part of an elaborate framework of ideas developed by Aristotle on the motion of bodies in the universe. Most of the Aristotelian ideas on motion are now known to be wrong and need not concern us. For our purpose here, the Aristotelian law of motion may be phrased thus: **An external force is required to keep a body in motion**.

Aristotelian law of motion is flawed, as we shall see. However, it is a natural view that anyone would hold from common experience. Even a small child playing with a simple (non-electric) toy-car on a floor knows intuitively that it needs to constantly drag the string attached to the toy-car with some force to keep it going. If it releases the string, it comes to rest. This experience is common to most terrestrial motion. External forces seem to be needed to keep bodies in motion. Left to themselves, all bodies eventually come to rest.

What is the flaw in Aristotle's argument? The answer is: a moving toy car comes to rest because the external force of friction on the car by the floor opposes its motion. To counter this force, the child has to apply an external force on the car in the direction of motion. When the car is in uniform motion, there is no net external force acting on it: the force by the child cancels the force (friction) by the floor. The corollary is: if there were no friction, the child would not be required to apply any force to keep the toy car in uniform motion.

The opposing forces such as friction (solids) and viscous forces (for fluids) are always present in the natural world. This explains why forces by external agencies are necessary to overcome the frictional forces to keep bodies in uniform motion. Now we understand where Aristotle went wrong. He coded this practical experience in the form of a basic argument. To get at the true law of nature for forces and motion, one has to imagine a world in which uniform motion is possible with no frictional forces opposing. This is what Galileo did.

4.3 THE LAW OF INERTIA

Galileo studied motion of objects on an inclined plane. Objects (i) moving down an inclined plane accelerate, while those (ii) moving up retard. (iii) Motion on a horizontal plane is an intermediate situation. Galileo concluded that an object moving on a frictionless horizontal plane must neither have acceleration nor retardation, i.e. it should move with constant velocity (Fig. 4.1(a)).

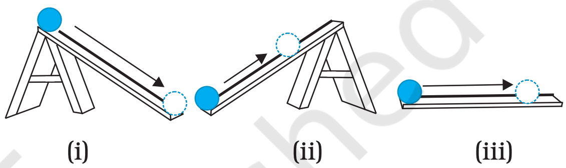


Fig. 4.1(a)

Another experiment by Galileo leading to the same conclusion involves a double inclined plane. A ball released from rest on one of the planes rolls down and climbs up the other. If the planes are smooth, the final height of the ball is nearly the same as the initial height (a little less but never greater). In the ideal situation, when friction is absent, the final height of the ball is the same as its initial height.

If the slope of the second plane is decreased and the experiment repeated, the ball will still reach the same height, but in doing so, it will travel a longer distance. In the limiting case, when the slope of the second plane is zero (i.e. is a horizontal) the ball travels an infinite distance. In other words, its motion never ceases. This is, of course, an idealised situation (Fig. 4.1(b)).

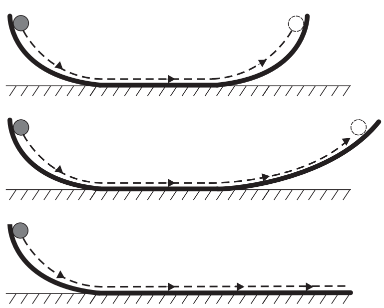


Fig. 4.1(b) The law of inertia was inferred by Galileo from observations of motion of a ball on a double inclined plane.

In practice, the ball does come to a stop after moving a finite distance on the horizontal plane, because of the opposing force of friction which can never be totally eliminated. However, if there were no friction, the ball would continue to move with a constant velocity on the horizontal plane.

Galileo thus, arrived at a new insight on motion that had eluded Aristotle and those who followed him. The state of rest and the state of uniform linear motion (motion with constant velocity) are equivalent. In both cases, there is no net force acting on the body. It is incorrect to assume that a net force is needed to keep a body in uniform motion. To maintain a body in uniform motion, we need to apply an external force to ecounter the frictional force, so that the two forces sum up to zero net external force.

To summarise, if the net external force is zero, a body at rest continues to remain at rest and a body in motion continues to move with a uniform velocity. This property of the body is called inertia. Inertia means **'resistance to change'**. A body does not change its state of rest or uniform motion, unless an external force compels it to change that state.

4.4 NEWTON'S FIRST LAW OF MOTION

Galileo's simple, but revolutionary ideas dethroned Aristotelian mechanics. A new mechanics had to be developed. This task was accomplished almost single-handedly by Isaac Newton, one of the greatest scientists of all times.

Newton built on Galileo's ideas and laid the foundation of mechanics in terms of three laws of motion that go by his name. Galileo's law of inertia was his starting point which he formulated as the **first law of motion**:

Every body continues to be in its state of rest or of uniform motion in a straight line unless compelled by some external force to act otherwise.

The state of rest or uniform linear motion both imply zero acceleration. The first law of motion can, therefore, be simply expressed as:

If the net external force on a body is zero, its acceleration is zero. Acceleration can be non zero only if there is a net external force on the body.

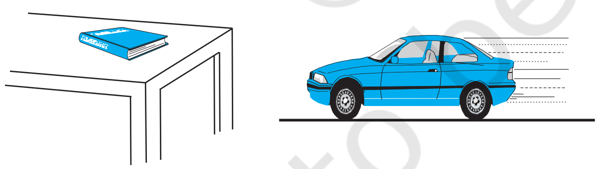
Two kinds of situations are encountered in the application of this law in practice. In some examples, we know that the net external force on the object is zero. In that case we can conclude that the acceleration of the object is zero. For example, a spaceship out in interstellar space, far from all other objects and with all its rockets turned off, has no net external force acting on it. Its acceleration, according to the first law, must be zero. If it is in motion, it must continue to move with a uniform velocity.

Ideas on Motion in Ancient Indian Science

Ancient Indian thinkers had arrived at an elaborate system of ideas on motion. Force, the cause of motion, was thought to be of different kinds : force due to continuous pressure (nodan), as the force of wind on a sailing vessel; impact (abhighat), as when a potter's rod strikes the wheel; persistent tendency (sanskara) to move in a straight line(vega) or restoration of shape in an elastic body; transmitted force by a string, rod, etc. The notion of (vega) in the Vaisesika theory of motion perhaps comes closest to the concept of inertia. Vega, the tendency to move in a straight line, was thought to be opposed by contact with objects including atmosphere, a parallel to the ideas of friction and air resistance. It was correctly summarised that the different kinds of motion (translational, rotational and vibrational) of an extended body arise from only the translational motion of its constituent particles. A falling leaf in the wind may have downward motion as a whole (patan) and also rotational and vibrational motion (bhraman, spandan), but each particle of the leaf at an instant only has a definite (small) displacement. There was considerable focus in Indian thought on measurement of motion and units of length and time. It was known that the position of a particle in space can be indicated by distance measured along three axes. Bhaskara (1150 A.D.) had introduced the concept of 'instantaneous motion' (*tatkaliki gati*), which anticipated the modern notion of instantaneous velocity using Differential Calculus. The difference between a wave and a current (of water) was clearly understood; a current is a motion of particles of water under gravity and fluidity while a wave results from the transmission of vibrations of water particles.

More often, however, we do not know all the forces to begin with. In that case, if we know that an object is unaccelerated (i.e. it is either at rest or in uniform linear motion), we can infer from the first law that the net external force on the object must be zero. Gravity is everywhere. For terrestrial phenomena, in particular, every object experiences gravitational force due to the earth. Also objects in motion generally experience friction, viscous drag, etc. If then, on earth, an object is at rest or in uniform linear motion, it is not because there are no forces acting on it, but because the various external forces cancel out i.e. add up to zero net external force.

Consider a book at rest on a horizontal surface Fig. (4.2(a)). It is subject to two external forces : the force due to gravity (i.e. its weight W) acting downward and the upward force on the book by the table, the normal force R . R is a self-adjusting force. This is an example of the kind of situation mentioned above. The forces are not quite known fully but the state of motion is known. We observe the book to be at rest. Therefore, we conclude from the first law that the magnitude of R equals that of W. A statement often encountered is : "Since $W = R$, forces cancel and, therefore, the book is at rest". This is incorrect reasoning. The correct statement is : "Since the book is observed to be at rest, the net external force on it must be zero, according to the first law. This implies that the normal force R must be equal and opposite to the weight W ".



(a) (b)

Fig. 4.2 (a) *a book at rest on the table, and* (b) *a car moving with uniform velocity. The net force is zero in each case.*

Consider the motion of a car starting from rest, picking up speed and then moving on a smooth straight road with uniform speed (Fig. (4.2(b)). When the car is stationary, there is no net force acting on it. During pick-up, it accelerates. This must happen due to a net external force. Note, it has to be an external force. The acceleration of the car cannot be accounted for by any internal force. This might sound surprising, but it is true. The only conceivable external force along the road is the force of friction. It is the frictional force that accelerates the car as a whole. (You will learn about friction in section 4.9). When the car moves with constant velocity, there is no net external force.

The property of inertia contained in the First law is evident in many situations. Suppose we are standing in a stationary bus and the driver starts the bus suddenly. We get thrown backward with a jerk. Why ? Our feet are in touch with the floor. If there were no friction, we would remain where we were, while the floor of the bus would simply slip forward under our feet and the back of the bus would hit us. However, fortunately, there is some friction between the feet and the floor. If the start is not too sudden, i.e. if the acceleration is moderate, the frictional force would be enough to accelerate our feet along with the bus. But our body is not strictly a rigid body. It is deformable, i.e. it allows some relative displacement between different parts. What this means is that while our feet go with the bus, the rest of the body remains where it is due to inertia. Relative to the bus, therefore, we are thrown backward. As soon as that happens, however, the muscular forces on the rest of the body (by the feet) come into play to move the body along with the bus. A similar thing happens when the bus suddenly stops. Our feet stop due to the friction which does not allow relative motion between the feet and the floor of the bus. But the rest of the body continues to move forward due to inertia. We are thrown forward. The restoring muscular forces again come into play and bring the body to rest.

Example 4.1 An astronaut accidentally gets separated out of his small spaceship accelerating in inter stellar space at a constant rate of 100 m s^{-2}. What is the acceleration of the astronaut the instant after he is outside the spaceship ? (Assume that there are no nearby stars to exert gravitational force on him.)

Answer Since there are no nearby stars to exert gravitational force on him and the small spaceship exerts negligible gravitational attraction on him, the net force acting on the astronaut, once he is out of the spaceship, is zero. By the first law of motion the acceleration of the astronaut is zero. ◀

4.5 NEWTON'S SECOND LAW OF MOTION

The first law refers to the simple case when the net external force on a body is zero. The second law of motion refers to the general situation when there is a net external force acting on the body. It relates the net external force to the acceleration of the body.

Momentum

Momentum of a body is defined to be the product of its mass m and velocity $\mathbf{v}$, and is denoted by $\mathbf{p}$:

$$\mathbf{p} = m\,\mathbf{v} \qquad (4.1)$$

Momentum is clearly a vector quantity. The following common experiences indicate the importance of this quantity for considering the effect of force on motion.

- Suppose a light-weight vehicle (say a small car) and a heavy weight vehicle (say a loaded truck) are parked on a horizontal road. We all know that a much greater force is needed to push the truck than the car to bring them to the same speed in same time. Similarly, a greater opposing force is needed to stop a heavy body than a light body in the same time, if they are moving with the same speed.
- If two stones, one light and the other heavy, are dropped from the top of a building, a person on the ground will find it easier to catch the light stone than the heavy stone. The mass of a body is thus an important parameter that determines the effect of force on its motion.
- Speed is another important parameter to consider. A bullet fired by a gun can easily pierce human tissue before it stops, resulting in casualty. The same bullet fired with moderate speed will not cause much damage. Thus for a given mass, the greater the speed, the greater is the opposing force needed to stop the body in a certain time. Taken together, the product of mass and velocity, that is momentum, is evidently a relevant variable of motion. The greater the change in the momentum in a given time, the greater is the force that needs to be applied.
- A seasoned cricketer catches a cricket ball coming in with great speed far more easily than a novice, who can hurt his hands in the act. One reason is that the cricketer allows a longer time for his hands to stop the ball. As you may have noticed, he draws in the hands backward in the act of catching the ball (Fig. 4.3). The novice, on the other hand, keeps his hands fixed and tries to catch the ball almost instantly. He needs to provide a much greater force to stop the ball instantly, and this hurts. The conclusion is clear: force not only depends on the change in momentum, but also on how fast the change is brought about. The same change in momentum brought about in a shorter time needs a greater applied force. In short, the greater the rate of change of momentum, the greater is the force.

Fig. 4.3 *Force not only depends on the change in momentum but also on how fast the change is brought about. A seasoned cricketer draws in his hands during a catch, allowing greater time for the ball to stop and hence requires a smaller force.*

- Observations confirm that the product of mass and velocity (i.e. momentum) is basic to the effect of force on motion. Suppose a fixed force is applied for a certain interval of time on two bodies of different masses, initially at rest, the lighter body picks up a greater speed than the heavier body. However, at the end of the time interval, observations show that each body acquires the same momentum. **Thus the same force for the same time causes the same change in momentum for different bodies**. This is a crucial clue to the second law of motion.
- In the preceding observations, the vector

character of momentum has not been evident. In the examples so far, momentum and change in momentum both have the same direction. But this is not always the case. Suppose a stone is rotated with uniform speed in a horizontal plane by means of a string, the magnitude of momentum is fixed, but its direction changes (Fig. 4.4). A force is needed to cause this change in momentum vector. This force is provided by our hand through the string. Experience suggests that our hand needs to exert a greater force if the stone is rotated at greater speed or in a circle of smaller radius, or both. This corresponds to greater acceleration or equivalently a greater rate of change in momentum vector. This suggests that the greater the rate of change in momentum vector the greater is the force applied.

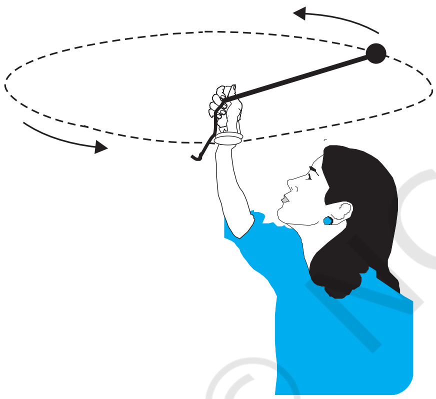


Fig. 4.4 *Force is necessary for changing the direction of momentum, even if its magnitude is constant. We can feel this while rotating a stone in a horizontal circle with uniform speed by means of a string.*

These qualitative observations lead to the **second law of motion** expressed by Newton as follows :

The rate of change of momentum of a body is directly proportional to the applied force and takes place in the direction in which the force acts.

Thus, if under the action of a force **F** for time interval Δt, the velocity of a body of mass m changes from $\mathbf{v}$ to $\mathbf{v} + \Delta\mathbf{v}$ i.e. its initial momentum $\mathbf{p} = m\mathbf{v}$ changes by $\Delta\mathbf{p} = m\Delta\mathbf{v}$. According to the Second Law,

$$\mathbf{F} \propto \frac{\Delta\mathbf{p}}{\Delta t} \quad \text{or} \quad \mathbf{F} = k\,\frac{\Delta\mathbf{p}}{\Delta t}$$

where k is a constant of proportionality. Taking the limit $\Delta t \to 0$, the term $\frac{\Delta\mathbf{p}}{\Delta t}$ becomes the derivative or differential co-efficient of **p** with respect to t, denoted by $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. Thus

$$\mathbf{F} = k\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{4.2}$$

For a body of fixed mass m,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\,\mathbf{v}) = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\,\mathbf{a} \tag{4.3}$$

i.e the Second Law can also be written as

$$\mathbf{F} = k\,m\,\mathbf{a} \tag{4.4}$$

which shows that force is proportional to the product of mass m and acceleration **a**.

The unit of force has not been defined so far. In fact, we use Eq. (4.4) to define the unit of force. We, therefore, have the liberty to choose any constant value for k. For simplicity, we choose $k = 1$. The second law then is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\,\mathbf{a} \tag{4.5}$$

In SI unit force is one that causes an acceleration of 1 m s^{-2} to a mass of 1 kg. This unit is known as **newton** : 1 N = 1 kg m s^{-2}.

Let us note at this stage some important points about the second law :

1. In the second law, $\mathbf{F} = 0$ implies $\mathbf{a} = 0$. The second law is obviously consistent with the first law.
2. The second law of motion is a vector law. It is equivalent to three equations, one for each component of the vectors :

$$F_x = \frac{\mathrm{d}p_x}{\mathrm{d}t} = ma_x$$

$$F_y = \frac{\mathrm{d}p_y}{\mathrm{d}t} = ma_y$$

$$F_z = \frac{\mathrm{d}p_z}{\mathrm{d}t} = m\,a_z \tag{4.6}$$

This means that if a force is not parallel to the velocity of the body, but makes some angle with it, it changes only the component of velocity along the direction of force. The component of velocity normal to the force remains unchanged. For example, in the motion of a projectile under the vertical gravitational force, the horizontal component of velocity remains unchanged (Fig. 4.5).

3. The second law of motion given by Eq. (4.5) is applicable to a single point particle. The force **F** in the law stands for the net external force on the particle and **a** stands for acceleration of the particle. It turns out, however, that the law in the same form applies to a rigid body or, even more generally, to a system of particles. In that case, **F** refers to the total external force on the system and **a** refers to the acceleration of the system as a whole. More precisely, **a** is the acceleration of the centre of mass of the system about which we shall study in detail in Chapter 6. **Any internal forces in the system are not to be included in F**.

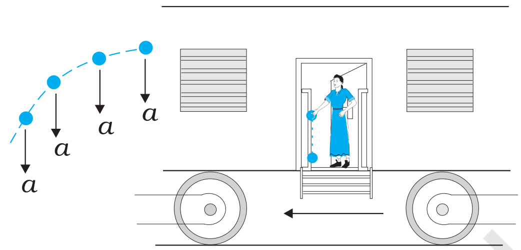


Fig. 4.5 *Acceleration at an instant is determined by the force at that instant. The moment after a stone is dropped out of an accelerated train, it has no horizontal acceleration or force, if air resistance is neglected. The stone carries no memory of its acceleration with the train a moment ago.*

4. The second law of motion is a local relation which means that force **F** at a point in space (location of the particle) at a certain instant of time is related to **a** at that point at that instant. Acceleration here and now is determined by the force here and now, **not by any history of the motion of the particle (See Fig. 4.5)**.

Example 4.2 A bullet of mass 0.04 kg moving with a speed of 90 m s^{-1} enters a heavy wooden block and is stopped after a distance of 60 cm. What is the average resistive force exerted by the block on the bullet?

Answer The retardation 'a' of the bullet (assumed constant) is given by

$$a = \frac{-u^2}{2s} = \frac{-90 \times 90}{2 \times 0.6} \text{ m s}^{-2} = -6750 \text{ m s}^{-2}$$

The retarding force, by the second law of motion, is

$$= 0.04 \text{ kg} \quad 6750 \text{ m s}^{-2} = 270 \text{ N}$$

The actual resistive force, and therefore, retardation of the bullet may not be uniform. The answer therefore, only indicates the average resistive force.

Example 4.3 The motion of a particle of mass m is described by $y = ut + \frac{1}{2}gt^2$. Find the force acting on the particle.

Answer We know

$$y = ut + \frac{1}{2}gt^2$$

Now,

$$v = \frac{dy}{dt} = u + gt$$

acceleration, $a = \frac{dv}{dt} = g$

Then the force is given by Eq. (4.5)

$$F = ma = mg$$

Thus the given equation describes the motion of a particle under acceleration due to gravity and y is the position coordinate in the direction of g.

Impulse

We sometimes encounter examples where a large force acts for a very short duration producing a finite change in momentum of the body. For example, when a ball hits a wall and bounces back, the force on the ball by the wall acts for a very short time when the two are in contact, yet the force is large enough to reverse the momentum of the ball. Often, in these situations, the force and the time duration are difficult to ascertain separately. However, the product of force and time, which is the change in momentum of the body remains a measurable quantity. This product is called impulse:

Impulse = Force time duration
= Change in momentum (4.7)

A large force acting for a short time to produce a finite change in momentum is called an *impulsive force*. In the history of science, impulsive forces were put in a conceptually different category from ordinary forces. Newtonian mechanics has no such distinction. Impulsive force is like any other force – except that it is large and acts for a short time.

▶ ***Example 4.4*** A batsman hits back a ball straight in the direction of the bowler without changing its initial speed of 12 m s^{-1}. If the mass of the ball is 0.15 kg, determine the impulse imparted to the ball. (Assume linear motion of the ball)

Answer Change in momentum

$= 0.15 \times 12 – (–0.15 \times 12)$

$= 3.6$ N s,

Impulse $= 3.6$ N s,

in the direction from the batsman to the bowler.

This is an example where the force on the ball by the batsman and the time of contact of the ball and the bat are difficult to know, but the impulse is readily calculated. ◀

4.6 NEWTON'S THIRD LAW OF MOTION

The second law relates the external force on a body to its acceleration. What is the origin of the external force on the body ? What agency provides the external force ? The simple answer in Newtonian mechanics is that the external force on a body always arises due to some other body. Consider a pair of bodies *A* and *B*. *B* gives rise to an external force on *A*. A natural question is: Does *A* in turn give rise to an external force on *B* ? In some examples, the answer seems clear. If you press a coiled spring, the spring is compressed by the force of your hand. The compressed spring in turn exerts a force on your hand and you can feel it. But what if the bodies are not in contact ? The earth pulls a stone downwards due to gravity. Does the stone exert a force on the earth ? The answer is not obvious since we hardly see the effect of the stone on the earth. The answer according to Newton is: Yes, the stone does exert an equal and opposite force on the earth. We do not notice it since the earth is very massive and the effect of a small force on its motion is negligible.

Thus, according to Newtonian mechanics, force never occurs singly in nature. Force is the mutual interaction between two bodies. Forces always occur in pairs. Further, the mutual forces between two bodies are always equal and opposite. This idea was expressed by Newton in the form of the **third law of motion**.

To every action, there is always an equal and opposite reaction.

Newton's wording of the third law is so crisp and beautiful that it has become a part of common language. For the same reason perhaps, misconceptions about the third law abound. Let us note some important points about the third law, particularly in regard to the usage of the terms : action and reaction.

1. The terms action and reaction in the third law mean nothing else but 'force'. Using different terms for the same physical concept can sometimes be confusing. A simple and clear way of stating the third law is as follows :

   Forces always occur in pairs. Force on a body A by B is equal and opposite to the force on the body B by A.

2. The terms action and reaction in the third law may give a wrong impression that action comes before reaction i.e action is the cause and reaction the effect. **There is no cause-effect relation implied in the third law. The force on *A* by *B* and the force on *B* by *A* act at the same instant.** By the same reasoning, any one of them may be called action and the other reaction.
3. Action and reaction forces act on different bodies, not on the same body. Consider a pair of bodies *A* and *B*. According to the third law,

   $$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \tag{4.8}$$

   (force on *A* by *B*) = – (force on *B* by *A*)

   Thus if we are considering the motion of any one body (*A* or *B*), only one of the two forces is relevant. It is an error to add up the two forces and claim that the net force is zero.

   However, if you are considering the system of two bodies as a whole, $\mathbf{F}_{AB}$ and $\mathbf{F}_{BA}$ are internal forces of the system ($A + B$). They add up to give a null force. Internal forces in a body or a system of particles thus cancel away in pairs. This is an important fact that enables the second law to be applicable to a body or a system of particles (See Chapter 6).

Example 4.5 Two identical billiard balls strike a rigid wall with the same speed but at different angles, and get reflected without any change in speed, as shown in Fig. 4.6. What is (i) the direction of the force on the wall due to each ball? (ii) the ratio of the magnitudes of impulses imparted to the balls by the wall ?

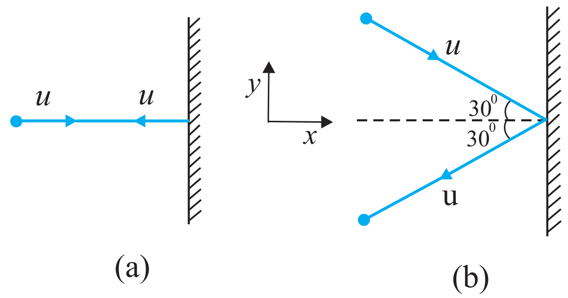


Fig. 4.6

Answer An instinctive answer to (i) might be that the force on the wall in case (a) is normal to the wall, while that in case (b) is inclined at 30 to the normal. This answer is wrong. The force on the wall is normal to the wall in both cases.

How to find the force on the wall? The trick is to consider the force (or impulse) on the ball due to the wall using the second law, and then use the third law to answer (i). Let u be the speed of each ball before and after collision with the wall, and m the mass of each ball. Choose the x and y axes as shown in the figure, and consider the change in momentum of the ball in each case :

Case (a)

$$(p_x)_{\text{initial}} = mu \qquad (p_y)_{\text{initial}} = 0$$

$$(p_x)_{\text{final}} = -mu \qquad (p_y)_{\text{final}} = 0$$

Impulse is the change in momentum vector. Therefore,

x-component of impulse = $-2\,m\,u$
y-component of impulse = 0

Impulse and force are in the same direction. Clearly, from above, the force on the ball due to the wall is normal to the wall, along the negative x-direction. Using Newton's third law of motion, the force on the wall due to the ball is normal to the wall along the positive x-direction. The magnitude of force cannot be ascertained since the small time taken for the collision has not been specified in the problem.

Case (b)

$$(p_x)_{\text{initial}} = m\,u\cos 30^\circ,\ (p_y)_{\text{initial}} = -\,m\,u\sin 30^\circ$$

$$(p_x)_{\text{final}} = -\,m\,u\cos 30^\circ,\ (p_y)_{\text{final}} = -\,m\,u\sin 30^\circ$$

Note, while p_x changes sign after collision, p_y does not. Therefore,

x-component of impulse = $-2\,m\,u\,\cos\,30$
y-component of impulse = 0

The direction of impulse (and force) is the same as in (a) and is normal to the wall along the negative x direction. As before, using Newton's third law, the force on the wall due to the ball is normal to the wall along the positive x direction.

The ratio of the magnitudes of the impulses imparted to the balls in (a) and (b) is

$$2\,m\,u/\left(2\,m\,u\cos 30^\circ\right) = \frac{2}{\sqrt{3}} \approx 1.2$$

4.7 CONSERVATION OF MOMENTUM

The second and third laws of motion lead to an important consequence: the law of conservation of momentum. Take a familiar example. A bullet is fired from a gun. If the force on the bullet by the gun is **F**, the force on the gun by the bullet is $-$**F**, according to the third law. The two forces act for a common interval of time Δt. According to the second law, $\mathbf{F}\,\Delta t$ is the change in momentum of the bullet and $-\mathbf{F}\ \Delta t$ is the change in momentum of the gun. Since initially, both are at rest, the change in momentum equals the final momentum for each. Thus if $\mathbf{p}_b$ is the momentum of the bullet after firing and $\mathbf{p}_g$ is the recoil momentum of the gun, $\mathbf{p}_g = -\mathbf{p}_b$ *i.e.* $\mathbf{p}_b + \mathbf{p}_g = 0$. That is, the total momentum of the (bullet + gun) system is conserved.

Thus in an isolated system (i.e. a system with no external force), mutual forces between pairs of particles in the system can cause momentum change in individual particles, but since the mutual forces for each pair are equal and opposite, the momentum changes cancel in pairs and the total momentum remains unchanged. This fact is known as the **law of conservation of momentum** :

The total momentum of an isolated system of interacting particles is conserved.

An important example of the application of the law of conservation of momentum is the collision of two bodies. Consider two bodies A and B, with initial momenta $\mathbf{p}_A$ and $\mathbf{p}_B$. The bodies collide, get apart, with final momenta $\mathbf{p}'_A$ and $\mathbf{p}'_B$ respectively. By the Second Law

$$\mathbf{F}_{AB}\Delta t = \mathbf{p}'_A - \mathbf{p}_A \quad \text{and}$$

$$\mathbf{F}_{BA}\Delta t = \mathbf{p}'_B - \mathbf{p}_B$$

(where we have taken a common interval of time for both forces i.e. the time for which the two bodies are in contact.)

Since $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$ by the third law,

$$\mathbf{p}'_A - \mathbf{p}_A = -(\mathbf{p}'_B - \mathbf{p}_B)$$

i.e. $$\mathbf{p}'_A + \mathbf{p}'_B = \mathbf{p}_A + \mathbf{p}_B \qquad (4.9)$$

which shows that the total final momentum of the isolated system equals its initial momentum. Notice that this is true whether the collision is elastic or inelastic. In elastic collisions, there is a second condition that the total initial kinetic energy of the system equals the total final kinetic energy (See Chapter 5).

4.8 EQUILIBRIUM OF A PARTICLE

Equilibrium of a particle in mechanics refers to the situation when the net external force on the particle is zero.* According to the first law, this means that, the particle is either at rest or in uniform motion.

If two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, act on a particle, equilibrium requires

$$\mathbf{F}_1 = -\mathbf{F}_2 \qquad (4.10)$$

i.e. the two forces on the particle must be equal and opposite. Equilibrium under three concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ requires that the vector sum of the three forces is zero.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0 \qquad (4.11)$$

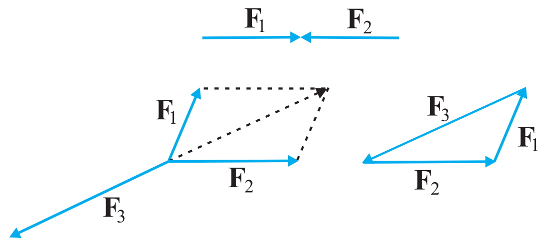


Fig. 4.7 *Equilibrium under concurrent forces.*

In other words, the resultant of any two forces say $\mathbf{F}_1$ and $\mathbf{F}_2$, obtained by the parallelogram law of forces must be equal and opposite to the third force, $\mathbf{F}_3$. As seen in Fig. 4.7, the three forces in equilibrium can be represented by the sides of a triangle with the vector arrows taken in the same sense. The result can be generalised to any number of forces. A particle is in equilibrium under the action of forces $\mathbf{F}_1$, $\mathbf{F}_2$,... $\mathbf{F}_n$ if they can be represented by the sides of a closed n-sided polygon with arrows directed in the same sense.

Equation (4.11) implies that

$$F_{1x} + F_{2x} + F_{3x} = 0$$
$$F_{1y} + F_{2y} + F_{3y} = 0$$
$$F_{1z} + F_{2z} + F_{3z} = 0 \qquad (4.12)$$

where F_{1x}, F_{1y} and F_{1z} are the components of F_1 along x, y and z directions respectively.

▶ ***Example 4.6*** See Fig. 4.8. A mass of 6 kg is suspended by a rope of length 2 m from the ceiling. A force of 50 N in the horizontal direction is applied at the mid-point P of the rope, as shown. What is the angle the rope makes with the vertical in equilibrium ? (Take $g = 10\ \text{m s}^{-2}$). Neglect the mass of the rope.

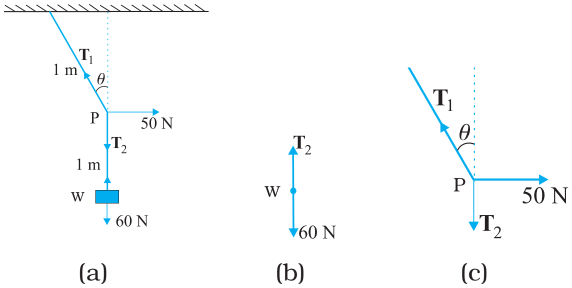


Fig. 4.8

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* *Equilibrium of a body requires not only translational equilibrium (zero net external force) but also rotational equilibrium (zero net external torque), as we shall see in Chapter 6.*

Answer Figures 4.8(b) and 4.8(c) are known as free-body diagrams. Figure 4.8(b) is the free-body diagram of W and Fig. 4.8(c) is the free-body diagram of point P.

Consider the equilibrium of the weight W. Clearly, $T_2 = 6 \times 10 = 60$ N.

Consider the equilibrium of the point P under the action of three forces - the tensions T_1 and T_2, and the horizontal force 50 N. The horizontal and vertical components of the resultant force must vanish separately :

$$T_1 \cos\theta = T_2 = 60 \text{ N}$$

$$T_1 \sin\theta = 50 \text{ N}$$

which gives that

$$\tan\theta = \frac{5}{6} \text{ or } \theta = \tan^{-1}\left(\frac{5}{6}\right) = 40^{\circ}$$

Note the answer does not depend on the length of the rope (assumed massless) nor on the point at which the horizontal force is applied. ◀

4.9 COMMON FORCES IN MECHANICS

In mechanics, we encounter several kinds of forces. The gravitational force is, of course, pervasive. Every object on the earth experiences the force of gravity due to the earth. Gravity also governs the motion of celestial bodies. The gravitational force can act at a distance without the need of any intervening medium.

All the other forces common in mechanics are contact forces.* As the name suggests, a contact force on an object arises due to contact with some other object: solid or fluid. When bodies are in contact (e.g. a book resting on a table, a system of rigid bodies connected by rods, hinges and other types of supports), there are mutual contact forces (for each pair of bodies) satisfying the third law. The component of contact force normal to the surfaces in contact is called normal reaction. The component parallel to the surfaces in contact is called friction. Contact forces arise also when solids are in contact with fluids. For example, for a solid immersed in a fluid, there is an upward bouyant force equal to the weight of the fluid displaced. The viscous force, air resistance, etc are also examples of contact forces (Fig. 4.9).

Two other common forces are tension in a string and the force due to spring. When a spring is compressed or extended by an external force, a restoring force is generated. This force is usually proportional to the compression or elongation (for small displacements). The spring force F is written as $F = -k\,x$ where x is the displacement and k is the force constant. The negative sign denotes that the force is opposite to the displacement from the unstretched state. For an inextensible string, the force constant is very high. The restoring force in a string is called tension. It is customary to use a constant tension T throughout the string. This assumption is true for a string of negligible mass.

We learnt that there are four fundamental forces in nature. Of these, the weak and strong forces appear in domains that do not concern us here. Only the gravitational and electrical forces are relevant in the context of mechanics. The different contact forces of mechanics mentioned above fundamentally arise from electrical forces. This may seem surprising

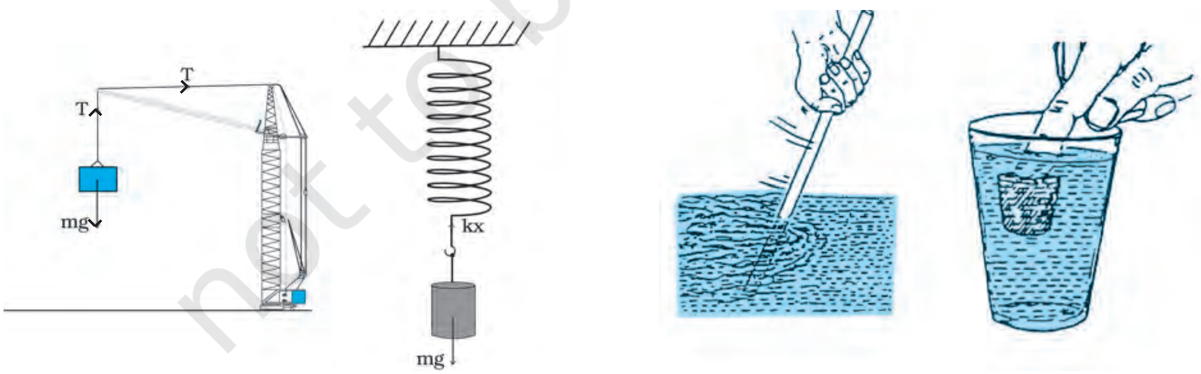


Fig. 4.9 *Some examples of contact forces in mechanics.*

* *We are not considering, for simplicity, charged and magnetic bodies. For these, besides gravity, there are electrical and magnetic non-contact forces.*

since we are talking of uncharged and non-magnetic bodies in mechanics. At the microscopic level, all bodies are made of charged constituents (nuclei and electrons) and the various contact forces arising due to elasticity of bodies, molecular collisions and impacts, etc. can ultimately be traced to the electrical forces between the charged constituents of different bodies. The detailed microscopic origin of these forces is, however, complex and not useful for handling problems in mechanics at the macroscopic scale. This is why they are treated as different types of forces with their characteristic properties determined empirically.

4.9.1 Friction

Let us return to the example of a body of mass m at rest on a horizontal table. The force of gravity (mg) is cancelled by the normal reaction force (N) of the table. Now suppose a force F is applied horizontally to the body. We know from experience that a small applied force may not be enough to move the body. But if the applied force F were the only external force on the body, it must move with acceleration F/m, however small. Clearly, the body remains at rest because some other force comes into play in the horizontal direction and opposes the applied force F, resulting in zero net force on the body. This force f_s parallel to the surface of the body in contact with the table is known as frictional force, or simply friction (Fig. 4.10(a)). The subscript stands for static friction to distinguish it from kinetic friction f_k that we consider later (Fig. 4.10(b)). Note that static friction does not

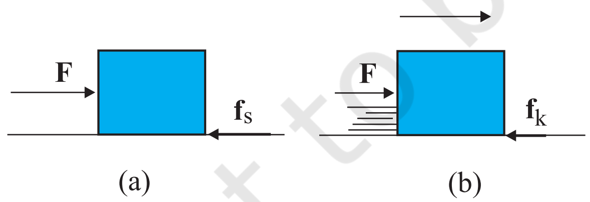


Fig. 4.10 *Static and sliding friction: (a) Impending motion of the body is opposed by static friction. When external force exceeds the maximum limit of static friction, the body begins to move. (b) Once the body is in motion, it is subject to sliding or kinetic friction which opposes relative motion between the two surfaces in contact. Kinetic friction is usually less than the maximum value of static friction.*

exist by itself. When there is no applied force, there is no static friction. It comes into play the moment there is an applied force. As the applied force F increases, f_s also increases, remaining equal and opposite to the applied force (up to a certain limit), keeping the body at rest. Hence, it is called **static friction**. Static friction opposes **impending motion**. The term impending motion means motion that would take place (but does not actually take place) under the applied force, if friction were absent.

We know from experience that as the applied force exceeds a certain limit, the body begins to move. It is found experimentally that the limiting value of static friction $\left(f_s\right)_{\max}$ is independent of the area of contact and varies with the normal force (N) approximately as :

$$\left(f_s\right)_{\max} = \mu_s N \tag{4.13}$$

where μ_s is a constant of proportionality depending only on the nature of the surfaces in contact. The constant μ_s is called the coefficient of static friction. The law of static friction may thus be written as

$$f_s \le \mu_s N \tag{4.14}$$

If the applied force F exceeds $\left(f_s\right)_{\max}$ the body begins to slide on the surface. It is found experimentally that when relative motion has started, the frictional force decreases from the static maximum value $\left(f_s\right)_{\max}$. Frictional force that opposes relative motion between surfaces in contact is called kinetic or sliding friction and is denoted by $\mathbf{f}_k$. Kinetic friction, like static friction, is found to be independent of the area of contact. Further, it is nearly independent of the velocity. It satisfies a law similar to that for static friction:

$$\mathbf{f}_k = \mu_k \mathbf{N} \tag{4.15}$$

where μ_k, the coefficient of kinetic friction, depends only on the surfaces in contact. As mentioned above, experiments show that μ_k is less than μ_s. When relative motion has begun, the acceleration of the body according to the second law is $(F - f_k)/m$. For a body moving with constant velocity, $F = f_k$. If the applied force on the body is removed, its acceleration is $-f_k/m$ and it eventually comes to a stop.

The laws of friction given above do not have the status of fundamental laws like those for gravitational, electric and magnetic forces. They are empirical relations that are only approximately true. Yet they are very useful in practical calculations in mechanics.

Thus, when two bodies are in contact, each experiences a contact force by the other. Friction, by definition, is the component of the contact force parallel to the surfaces in contact, which opposes impending or actual relative motion between the two surfaces. Note that it is not motion, but **relative motion** that the frictional force opposes. Consider a box lying in the compartment of a train that is accelerating. If the box is stationary relative to the train, it is in fact accelerating along with the train. What forces cause the acceleration of the box? Clearly, the only conceivable force in the horizontal direction is the force of friction. If there were no friction, the floor of the train would slip by and the box would remain at its initial position due to inertia (and hit the back side of the train). This impending relative motion is opposed by the static friction f_s. Static friction provides the same acceleration to the box as that of the train, keeping it stationary relative to the train.

Example 4.7 Determine the maximum acceleration of the train in which a box lying on its floor will remain stationary, given that the co-efficient of static friction between the box and the train's floor is 0.15.

Answer Since the acceleration of the box is due to the static friction,

$$ma = f_s \le \mu_s N = \mu_s \; m g$$

i.e. $a \le \mu_s g$

$$\therefore \; a_{max} = \mu_s g = 0.15 \text{ x } 10 \text{ m s}^{-2} = 1.5 \text{ m s}^{-2}$$

Example 4.8 See Fig. 4.11. A mass of 4 kg rests on a horizontal plane. The plane is gradually inclined until at an angle θ = 15° with the horizontal, the mass just begins to slide. What is the coefficient of static friction between the block and the surface ?

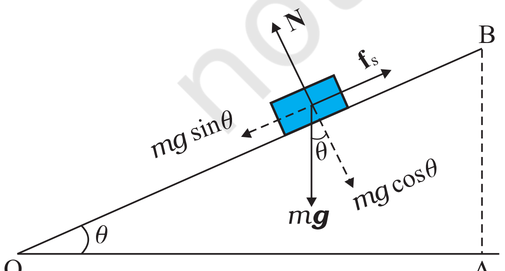


Fig. 4.11

Answer The forces acting on a block of mass *m* at rest on an inclined plane are (i) the weight mg acting vertically downwards (ii) the normal force *N* of the plane on the block, and (iii) the static frictional force f_s opposing the impending motion. In equilibrium, the resultant of these forces must be zero. Resolving the weight *mg* along the two directions shown, we have

$$m g \sin\theta = f_s \;, \quad m g \cos\theta = N$$

As θ increases, the self-adjusting frictional force f_s increases until at $\theta = \theta_{max}$, f_s achieves its maximum value, $(f_s)_{max} = \mu_s N$.

Therefore,

$$\tan\theta_{max} = \mu_s \text{ or } \theta_{max} = \tan^{-1}\mu_s$$

When θ becomes just a little more than θ_{max}, there is a small net force on the block and it begins to slide. Note that θ_{max} depends only on μ_s and is independent of the mass of the block.

For θ_{max} = 15°,

μ_s = tan 15°

= 0.27

Example 4.9 What is the acceleration of the block and trolley system shown in a Fig. 4.12(a), if the coefficient of kinetic friction between the trolley and the surface is 0.04? What is the tension in the string? (Take g = 10 m s^{-2}). Neglect the mass of the string.

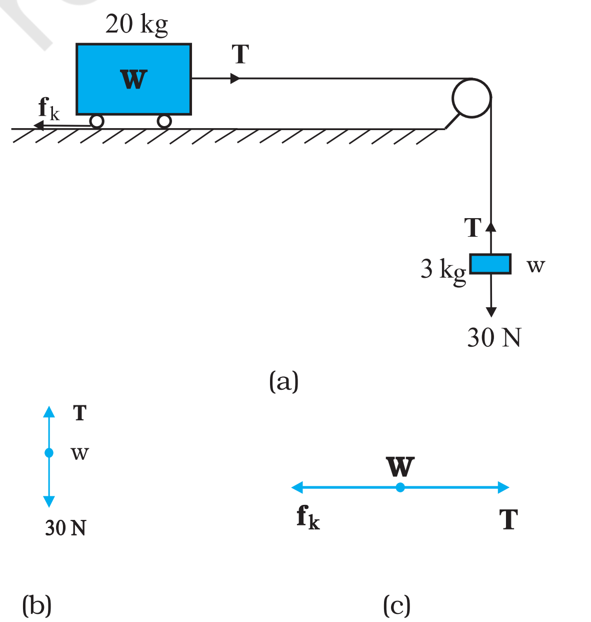


Fig. 4.12

Answer As the string is inextensible, and the pully is smooth, the 3 kg block and the 20 kg trolley both have same magnitude of acceleration. Applying second law to motion of the block (Fig. 4.12(b)),

$$30 - T = 3a$$

Apply the second law to motion of the trolley (Fig. 4.12(c)),

$$T - f_k = 20\ a.$$

Now $\quad f_k = \mu_k N,$

Here $\quad \mu_k = 0.04,$

$\quad N = 20 \times 10 = 200$ N.

Thus the equation for the motion of the trolley is

$$T - 0.04 \times 200 = 20\ a \qquad \text{Or} \quad T - 8 = 20a.$$

These equations give $a = \frac{22}{23}$ m s^{-2} = 0.96 m s^{-2} and T = 27.1 N. ◀

Rolling friction

A body like a ring or a sphere rolling without slipping over a horizontal plane will suffer no friction, in principle. At every instant, there is just one point of contact between the body and the plane and this point has no motion relative to the plane. In this ideal situation, kinetic or static friction is zero and the body should continue to roll with constant velocity. We know, in practice, this will not happen and some resistance to motion (rolling friction) does occur, i.e. to keep the body rolling, some applied force is needed. For the same weight, rolling friction is much smaller (even by 2 or 3 orders of magnitude) than static or sliding friction. This is the reason why discovery of the wheel has been a major milestone in human history.

Rolling friction again has a complex origin, though somewhat different from that of static and sliding friction. During rolling, the surfaces in contact get momentarily deformed a little, and this results in a finite area (not a point) of the body being in contact with the surface. The net effect is that the component of the contact force parallel to the surface opposes motion.

We often regard friction as something undesirable. In many situations, like in a machine with different moving parts, friction does have a negative role. It opposes relative motion and thereby dissipates power in the form of heat, etc. Lubricants are a way of reducing kinetic friction in a machine. Another way is to use ball bearings between two moving parts of a machine [Fig. 4.13(a)]. Since the rolling friction between ball bearings and the surfaces in contact is very small, power dissipation is reduced. A thin cushion of air maintained between solid surfaces in relative motion is another effective way of reducing friction (Fig. 4.13(a)).

In many practical situations, however, friction is critically needed. Kinetic friction that dissipates power is nevertheless important for quickly stopping relative motion. It is made use of by brakes in machines and automobiles. Similarly, static friction is important in daily life. We are able to walk because of friction. It is impossible for a car to move on a very slippery road. On an ordinary road, the friction between the tyres and the road provides the necessary external force to accelerate the car.

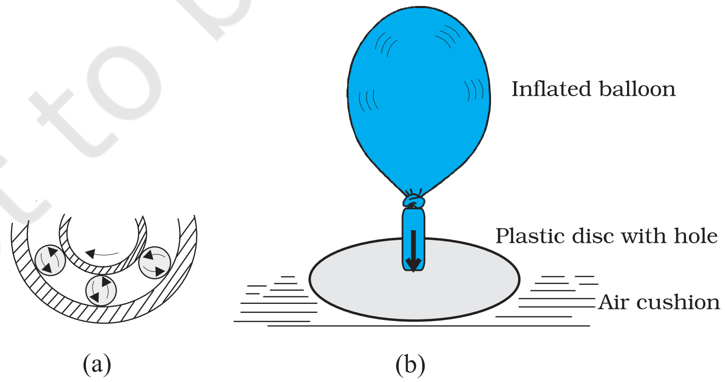


Fig. 4.13 *Some ways of reducing friction. (a) Ball bearings placed between moving parts of a machine. (b) Compressed cushion of air between surfaces in relative motion.*

4.10 CIRCULAR MOTION

We have seen in Chapter 4 that acceleration of a body moving in a circle of radius R with uniform speed v is v^2/R directed towards the centre. According to the second law, the force f_c providing this acceleration is :

$$f_c = \frac{mv^2}{R} \tag{4.16}$$

where m is the mass of the body. This force directed forwards the centre is called the centripetal force. For a stone rotated in a circle by a string, the centripetal force is provided by the tension in the string. The centripetal force for motion of a planet around the sun is the gravitational force on the planet due to the sun. For a car taking a circular turn on a horizontal road, the centripetal force is the force of friction.

The circular motion of a car on a flat and banked road give interesting application of the laws of motion.

Motion of a car on a level road

Three forces act on the car (Fig. 4.14(a):

(i) The weight of the car, mg
(ii) Normal reaction, N
(iii) Frictional force, f

As there is no acceleration in the vertical direction

$N - mg = 0$

$$N = mg \tag{4.17}$$

The centripetal force required for circular motion is along the surface of the road, and is provided by the component of the contact force between road and the car tyres along the surface. This by definition is the frictional force. Note that it is the static friction that provides the centripetal acceleration. Static friction opposes the impending motion of the car moving away from the circle. Using equation (4.14) & (4.16) we get the result

$$f = \frac{mv^2}{R} \le \mu_s N$$

$$v^2 \le \frac{\mu_s RN}{m} = \mu_s Rg \qquad [\because N = mg]$$

which is independent of the mass of the car. This shows that for a given value of μ_s and R, there is a maximum speed of circular motion of the car possible, namely

$$v_{max} = \sqrt{\mu_s Rg} \tag{4.18}$$

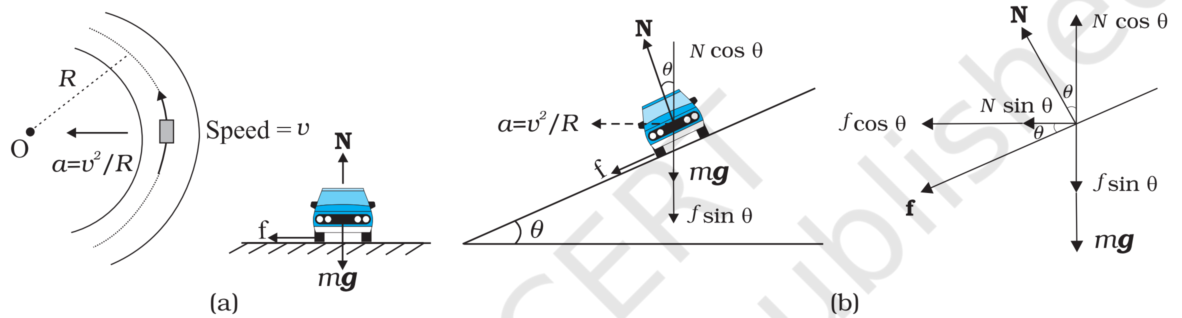


Fig. 4.14 Circular motion of a car on (a) a level road, (b) a banked road.

Motion of a car on a banked road

We can reduce the contribution of friction to the circular motion of the car if the road is banked (Fig. 4.14(b)). Since there is no acceleration along the vertical direction, the net force along this direction must be zero. Hence,

$$N \cos\theta = mg + f \sin\theta \tag{4.19a}$$

The centripetal force is provided by the horizontal components of N and f.

$$N \sin\theta + f \cos\theta = \frac{mv^2}{R} \tag{4.19b}$$

But $f \le \mu_s N$

Thus to obtain v_{max} we put

$f = \mu_s N$.

Then Eqs. (4.19a) and (4.19b) become

$$N \cos\theta = mg + \mu_s N \sin\theta \tag{4.20a}$$

$N\sin\theta + \mu_s N\cos\theta = mv^2/R$ (4.20b)

From Eq. (4.20a), we obtain

$$N = \frac{mg}{\cos\theta - \mu_s \sin\theta}$$

Substituting value of N in Eq. (4.20b), we get

$$\frac{mg(\sin\theta + \mu_s \cos\theta)}{\cos\theta - \mu_s \sin\theta} = \frac{mv_{max}^2}{R}$$

$$\text{or } v_{max} = \left(Rg\frac{\mu_s + \tan\theta}{1 - \mu_s \tan\theta}\right)^{\frac{1}{2}} \quad (4.21)$$

Comparing this with Eq. (4.18) we see that maximum possible speed of a car on a banked road is greater than that on a flat road.

For $\mu_s = 0$ in Eq. (4.21),

$$v_o = (Rg\tan\theta)^{1/2} \quad (4.22)$$

At this speed, frictional force is not needed at all to provide the necessary centripetal force. Driving at this speed on a banked road will cause little wear and tear of the tyres. The same equation also tells you that for $v < v_o$, frictional force will be up the slope and that a car can be parked only if $\tan\theta \le \mu_s$.

▶ ***Example 4.10*** A cyclist speeding at 18 km/h on a level road takes a sharp circular turn of radius 3 m without reducing the speed. The co-efficient of static friction between the tyres and the road is 0.1. Will the cyclist slip while taking the turn?

Answer On an unbanked road, frictional force alone can provide the centripetal force needed to keep the cyclist moving on a circular turn without slipping. If the speed is too large, or if the turn is too sharp (i.e. of too small a radius) or both, the frictional force is not sufficient to provide the necessary centripetal force, and the cyclist slips. The condition for the cyclist not to slip is given by Eq. (4.18) :

$$v^2 \le \mu_s R g$$

Now, $R = 3$ m, $g = 9.8$ m s^{-2}, $\mu_s = 0.1$. That is, $\mu_s R g = 2.94$ m^2 s^{-2}. $v = 18$ km/h $= 5$ m s^{-1}; i.e., $v^2 = 25$ m^2 s^{-2}. The condition is not obeyed. The cyclist will slip while taking the circular turn. ◀

▶ ***Example 4.11*** A circular racetrack of radius 300 m is banked at an angle of 15°. If the coefficient of friction between the wheels of a race-car and the road is 0.2, what is the (a) optimum speed of the race-car to avoid wear and tear on its tyres, and (b) maximum permissible speed to avoid slipping ?

Answer On a banked road, the horizontal component of the normal force and the frictional force contribute to provide centripetal force to keep the car moving on a circular turn without slipping. At the optimum speed, the normal reaction's component is enough to provide the needed centripetal force, and the frictional force is not needed. The optimum speed v_o is given by Eq. (4.22):

$$v_o = (R\,g\tan\theta)^{1/2}$$

Here $R = 300$ m, $\theta = 15°$, $g = 9.8$ m s^{-2}; we have

$$v_o = 28.1 \text{ m s}^{-1}.$$

The maximum permissible speed v_{max} is given by Eq. (4.21):

$$v_{max} = \left(R\,g\frac{\mu_s + \tan\theta}{1 - \mu_s\tan\theta}\right)^{1/2} = 38.1 \text{ m s}^{-1}$$ ◀

4.11 SOLVING PROBLEMS IN MECHANICS

The three laws of motion that you have learnt in this chapter are the foundation of mechanics. You should now be able to handle a large variety of problems in mechanics. A typical problem in mechanics usually does not merely involve a single body under the action of given forces. More often, we will need to consider an assembly of different bodies exerting forces on each other. Besides, each body in the assembly experiences the force of gravity. When trying to solve a problem of this type, it is useful to remember the fact that we can choose any part of the assembly and apply the laws of motion to that part provided we include all forces on the chosen part due to the remaining parts of the assembly. We may call the chosen part of the assembly as the system and the remaining part of the assembly (plus any other agencies of forces) as the environment. We have followed the same method in solved examples. To handle a typical problem in mechanics systematically, one should use the following steps :

(i) Draw a diagram showing schematically the various parts of the assembly of bodies, the links, supports, etc.

(ii) Choose a convenient part of the assembly as one system.

(iii) Draw a separate diagram which shows this system and all the forces on the system by the remaining part of the assembly. Include also the forces on the system by other agencies. **Do not include the forces on the environment by the system**. A diagram of this type is known as 'a free-body diagram'. (Note this does not imply that the system under consideration is without a net force).

(iv) In a free-body diagram, include information about forces (their magnitudes and directions) that are either given or you are sure of (e.g., the direction of tension in a string along its length). The rest should be treated as unknowns to be determined using laws of motion.

(v) If necessary, follow the same procedure for another choice of the system. In doing so, employ Newton's third law. That is, if in the free-body diagram of A, the force on A due to B is shown as **F**, then in the free-body diagram of B, the force on B due to A should be shown as –**F.**

The following example illustrates the above procedure :

▶ ***Example 4.12*** See Fig. 4.15. A wooden block of mass 2 kg rests on a soft horizontal floor. When an iron cylinder of mass 25 kg is placed on top of the block, the floor yields steadily and the block and the cylinder together go down with an acceleration of 0.1 m s^{-2}. What is the action of the block on the floor (a) before and (b) after the floor yields ? Take g = 10 m s^{-2}. Identify the action-reaction pairs in the problem.

Answer

(a) The block is at rest on the floor. Its free-body diagram shows two forces on the block, the force of gravitational attraction by the earth equal to 2 × 10 = 20 N; and the normal force R of the floor on the block. By the First Law, the net force on the block must be zero i.e., R = 20 N. Using third law the action of the block (i.e. the force exerted on the floor by the block) is equal to 20 N and directed vertically downwards.

(b) The system (block + cylinder) accelerates downwards with 0.1 m s^{-2}. The free-body diagram of the system shows two forces on the system : the force of gravity due to the earth (270 N); and the normal force R' by the floor. Note, the free-body diagram of the system does not show the internal forces between the block and the cylinder. Applying the second law to the system,

$$270 - R' = 27 \quad 0.1\text{N}$$
$$\text{ie. } R' = 267.3 \text{ N}$$

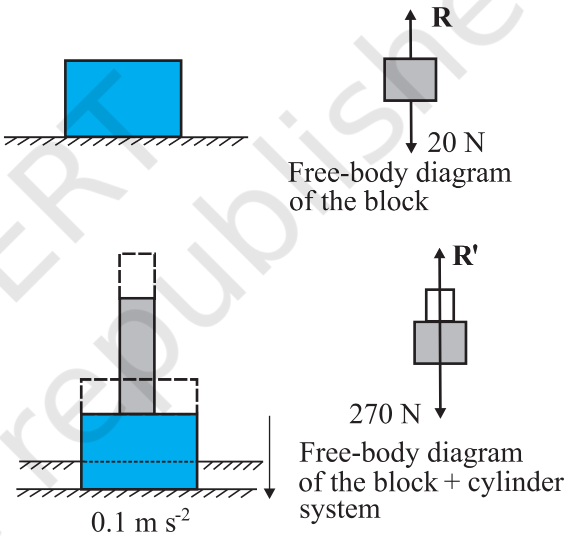


Fig. 4.15

By the third law, the action of the system on the floor is equal to 267.3 N vertically downward.

Action-reaction pairs

For (a): (i) the force of gravity (20 N) on the block by the earth (say, action); the force of gravity on the earth by the block (reaction) equal to 20 N directed upwards (not shown in the figure).
(ii) the force on the floor by the block (action); the force on the block by the floor (reaction).

For (b): (i) the force of gravity (270 N) on the system by the earth (say, action); the force of gravity on the earth by the system (reaction), equal to 270 N,

directed upwards (not shown in the figure).
(ii) the force on the floor by the system (action); the force on the system by the floor (reaction). In addition, for (b), the force on the block by the cylinder and the force on the cylinder by the block also constitute an action-reaction pair.

The important thing to remember is that an action-reaction pair consists of mutual forces which are always equal and opposite between two bodies. Two forces on the same body which happen to be equal and opposite can never constitute an action-reaction pair. The force of gravity on the mass in (a) or (b) and the normal force on the mass by the floor are not action-reaction pairs. These forces happen to be equal and opposite for (a) since the mass is at rest. They are not so for case (b), as seen already. The weight of the system is 270 N, while the normal force R' is 267.3 N. ◀

The practice of drawing free-body diagrams is of great help in solving problems in mechanics. It allows you to clearly define your system and consider all forces on the system due to objects that are not part of the system itself. A number of exercises in this and subsequent chapters will help you cultivate this practice.

SUMMARY

1. Aristotle's view that a force is necessary to keep a body in uniform motion is wrong. A force is necessary in practice to counter the opposing force of friction.
2. Galileo extrapolated simple observations on motion of bodies on inclined planes, and arrived at the law of inertia. Newton's first law of motion is the same law rephrased thus: *"Everybody continues to be in its state of rest or of uniform motion in a straight line, unless compelled by some external force to act otherwise"*. In simple terms, the First Law is **"If external force on a body is zero, its acceleration is zero"**.
3. Momentum ($\mathbf{p}$) of a body is the product of its mass *(m)* and velocity ($\mathbf{v}$) :
$$\mathbf{p} = m\mathbf{v}$$
4. Newton's second law of motion :
The rate of change of momentum of a body is proportional to the applied force and takes place in the direction in which the force acts. Thus
$$\mathbf{F} = k\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = k\,m\,\mathbf{a}$$
where $\mathbf{F}$ is the net external force on the body and $\mathbf{a}$ its acceleration. We set the constant of proportionality $k = 1$ in SI units. Then
$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\mathbf{a}$$
The SI unit of force is newton : 1 N = 1 kg m s^{-2}.
(a) The second law is consistent with the First Law ($\mathbf{F} = 0$ implies $\mathbf{a} = 0$)
(b) It is a vector equation
(c) It is applicable to a particle, and also to a body or a system of particles, provided $\mathbf{F}$ is the total external force on the system and $\mathbf{a}$ is the acceleration of the system as a whole.
(d) $\mathbf{F}$ at a point at a certain instant determines $\mathbf{a}$ at the same point at that instant. That is the Second Law is a local law; $\mathbf{a}$ at an instant does not depend on the history of motion.
4. Impulse is the product of force and time which equals change in momentum.
The notion of impulse is useful when a large force acts for a short time to produce a measurable change in momentum. Since the time of action of the force is very short, one can assume that there is no appreciable change in the position of the body during the action of the impulsive force.
6. Newton's third law of motion:
To every action, there is always an equal and opposite reaction

In simple terms, the law can be stated thus :

Forces in nature always occur between pairs of bodies. Force on a body A by body B is equal and opposite to the force on the body B by A.

Action and reaction forces are simultaneous forces. There is no cause-effect relation between action and reaction. Any of the two mutual forces can be called action and the other reaction. Action and reaction act on different bodies and so they cannot be cancelled out. The internal action and reaction forces between different parts of a body do, however, sum to zero.

7. *Law of Conservation of Momentum*
   The total momentum of an isolated system of particles is conserved. The law follows from the second and third law of motion.
8. *Friction*
   Frictional force opposes (impending or actual) relative motion between two surfaces in contact. It is the component of the contact force along the common tangent to the surface in contact. Static friction f_s opposes impending relative motion; kinetic friction f_k opposes actual relative motion. They are independent of the area of contact and satisfy the following approximate laws :

$$f_s \le \left(f_s\right)_{\max} = \mu_s R$$

$$f_k = \mu_k R$$

μ_s (co-efficient of static friction) and μ_k (co-efficient of kinetic friction) are constants characteristic of the pair of surfaces in contact. It is found experimentally that μ_k is less than μ_s .

Quantity	Symbol	Units	Dimensions	Remarks
Momentum	**p**	kg m s^{-1} or N s	[MLT^{-1}]	Vector
Force	**F**	N	[MLT^{-2}]	**F** = m **a** Second Law
Impulse		kg m s^{-1} or N s	[M LT^{-1}]	Impulse = force × time = change in momentum
Static friction	$\mathbf{f}_s$	N	[MLT^{-2}]	$\mathbf{f}_s \le \mu_s \mathbf{N}$
Kinetic friction	$\mathbf{f}_k$	N	[MLT^{-2}]	$\mathbf{f}_k = \mu_k \mathbf{N}$

POINTS TO PONDER

1. Force is not always in the direction of motion. Depending on the situation, **F** may be along **v**, opposite to **v**, normal to **v** or may make some other angle with **v**. In every case, it is parallel to acceleration.
2. If **v** = 0 at an instant, i.e. if a body is momentarily at rest, it does not mean that force or acceleration are necessarily zero at that instant. For example, when a ball thrown upward reaches its maximum height, **v** = 0 but the force continues to be its weight *mg* and the acceleration is not zero but *g*.
3. Force on a body at a given time is determined by the situation at the location of the body at that time. Force is not 'carried' by the body from its earlier history of motion. The moment after a stone is released out of an accelerated train, there is no horizontal force (or acceleration) on the stone, if the effects of the surrounding air are neglected. The stone then has only the vertical force of gravity.
4. In the second law of motion **F** = m **a**, **F** stands for the net force due to all material agencies external to the body. **a** is the effect of the force. m**a** should not be regarded as yet another force, besides **F**.

5. The centripetal force should not be regarded as yet another kind of force. It is simply a name given to the force that provides inward radial acceleration to a body in circular motion. We should always look for some material force like tension, gravitational force, electrical force, friction, etc as the centripetal force in any circular motion.
6. Static friction is a self-adjusting force up to its limit $\mu_s N$ ($f_s \le \mu_s N$). Do not put $f_s = \mu_s N$ without being sure that the maximum value of static friction is coming into play.
7. The familiar equation $mg = R$ for a body on a table is true only if the body is in equilibrium. The two forces mg and R can be different (e.g. a body in an accelerated lift). The equality of mg and R has no connection with the third law.
8. The terms 'action' and 'reaction' in the third Law of Motion simply stand for simultaneous mutual forces between a pair of bodies. Unlike their meaning in ordinary language, action does not precede or cause reaction. Action and reaction act on different bodies.
9. The different terms like 'friction', 'normal reaction' 'tension', 'air resistance', 'viscous drag', 'thrust', 'buoyancy', 'weight', 'centripetal force' all stand for 'force' in different contexts. For clarity, every force and its equivalent terms encountered in mechanics should be reduced to the phrase 'force on *A* by *B*'.
10. For applying the second law of motion, there is no conceptual distinction between inanimate and animate objects. An animate object such as a human also requires an external force to accelerate. For example, without the external force of friction, we cannot walk on the ground.
11. The objective concept of force in physics should not be confused with the subjective concept of the 'feeling of force'. On a merry-go-around, all parts of our body are subject to an inward force, but we have a feeling of being pushed outward – the direction of impending motion.

EXERCISES

(For simplicity in numerical calculations, take $g = 10 \text{ m s}^{-2}$)

4.1 Give the magnitude and direction of the net force acting on
(a) a drop of rain falling down with a constant speed,
(b) a cork of mass 10 g floating on water,
(c) a kite skillfully held stationary in the sky,
(d) a car moving with a constant velocity of 30 km/h on a rough road,
(e) a high-speed electron in space far from all material objects, and free of electric and magnetic fields.

4.2 A pebble of mass 0.05 kg is thrown vertically upwards. Give the direction and magnitude of the net force on the pebble,
(a) during its upward motion,
(b) during its downward motion,
(c) at the highest point where it is momentarily at rest. Do your answers change if the pebble was thrown at an angle of 45° with the horizontal direction?

Ignore air resistance.

4.3 Give the magnitude and direction of the net force acting on a stone of mass 0.1 kg,
(a) just after it is dropped from the window of a stationary train,
(b) just after it is dropped from the window of a train running at a constant velocity of 36 km/h,
(c) just after it is dropped from the window of a train accelerating with 1 m s^{-2},
(d) lying on the floor of a train which is accelerating with 1 m s^{-2}, the stone being at rest relative to the train.

Neglect air resistance throughout.

4.4 One end of a string of length l is connected to a particle of mass m and the other to a small peg on a smooth horizontal table. If the particle moves in a circle with speed v the net force on the particle (directed towards the centre) is :

(i) T, (ii) $T - \frac{mv^2}{l}$, (iii) $T + \frac{mv^2}{l}$, (iv) 0

T is the tension in the string. [Choose the correct alternative].

4.5 A constant retarding force of 50 N is applied to a body of mass 20 kg moving initially with a speed of 15 m s^{-1}. How long does the body take to stop ?

4.6 A constant force acting on a body of mass 3.0 kg changes its speed from 2.0 m s^{-1} to 3.5 m s^{-1} in 25 s. The direction of the motion of the body remains unchanged. What is the magnitude and direction of the force ?

4.7 A body of mass 5 kg is acted upon by two perpendicular forces 8 N and 6 N. Give the magnitude and direction of the acceleration of the body.

4.8 The driver of a three-wheeler moving with a speed of 36 km/h sees a child standing in the middle of the road and brings his vehicle to rest in 4.0 s just in time to save the child. What is the average retarding force on the vehicle ? The mass of the three-wheeler is 400 kg and the mass of the driver is 65 kg.

4.9 A rocket with a lift-off mass 20,000 kg is blasted upwards with an initial acceleration of 5.0 m s^{-2}. Calculate the initial thrust (force) of the blast.

4.10 A body of mass 0.40 kg moving initially with a constant speed of 10 m s^{-1} to the north is subject to a constant force of 8.0 N directed towards the south for 30 s. Take the instant the force is applied to be $t = 0$, the position of the body at that time to be $x = 0$, and predict its position at t = –5 s, 25 s, 100 s.

4.11 A truck starts from rest and accelerates uniformly at 2.0 m s^{-2}. At $t = 10$ s, a stone is dropped by a person standing on the top of the truck (6 m high from the ground). What are the (a) velocity, and (b) acceleration of the stone at $t = 11$s ? (Neglect air resistance.)

4.12 A bob of mass 0.1 kg hung from the ceiling of a room by a string 2 m long is set into oscillation. The speed of the bob at its mean position is 1 m s^{-1}. What is the trajectory of the bob if the string is cut when the bob is (a) at one of its extreme positions, (b) at its mean position.

4.13 A man of mass 70 kg stands on a weighing scale in a lift which is moving
(a) upwards with a uniform speed of 10 m s^{-1},
(b) downwards with a uniform acceleration of 5 m s^{-2},
(c) upwards with a uniform acceleration of 5 m s^{-2}.
What would be the readings on the scale in each case?
(d) What would be the reading if the lift mechanism failed and it hurtled down freely under gravity ?

4.14 Figure 4.16 shows the position-time graph of a particle of mass 4 kg. What is the (a) force on the particle for $t < 0$, $t > 4$ s, $0 < t < 4$ s? (b) impulse at $t = 0$ and $t = 4$ s ? (Consider one-dimensional motion only).

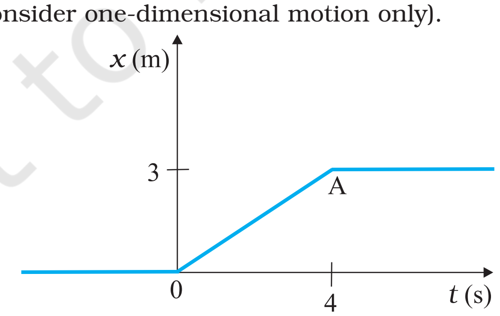


Fig. 4.16

4.15 Two bodies of masses 10 kg and 20 kg respectively kept on a smooth, horizontal surface are tied to the ends of a light string. A horizontal force F = 600 N is applied to (i) A, (ii) B along the direction of string. What is the tension in the string in each case?

4.16 Two masses 8 kg and 12 kg are connected at the two ends of a light inextensible string that goes over a frictionless pulley. Find the acceleration of the masses, and the tension in the string when the masses are released.

4.17 A nucleus is at rest in the laboratory frame of reference. Show that if it disintegrates into two smaller nuclei the products must move in opposite directions.

4.18 Two billiard balls each of mass 0.05 kg moving in opposite directions with speed 6 m s^{-1} collide and rebound with the same speed. What is the impulse imparted to each ball due to the other ?

4.19 A shell of mass 0.020 kg is fired by a gun of mass 100 kg. If the muzzle speed of the shell is 80 m s^{-1}, what is the recoil speed of the gun ?

4.20 A batsman deflects a ball by an angle of 45° without changing its initial speed which is equal to 54 km/h. What is the impulse imparted to the ball ? (Mass of the ball is 0.15 kg.)

4.21 A stone of mass 0.25 kg tied to the end of a string is whirled round in a circle of radius 1.5 m with a speed of 40 rev./min in a horizontal plane. What is the tension in the string ? What is the maximum speed with which the stone can be whirled around if the string can withstand a maximum tension of 200 N ?

4.22 If, in Exercise 4.21, the speed of the stone is increased beyond the maximum permissible value, and the string breaks suddenly, which of the following correctly describes the trajectory of the stone after the string breaks :
(a) the stone moves radially outwards,
(b) the stone flies off tangentially from the instant the string breaks,
(c) the stone flies off at an angle with the tangent whose magnitude depends on the speed of the particle ?

4.23 Explain why
(a) a horse cannot pull a cart and run in empty space,
(b) passengers are thrown forward from their seats when a speeding bus stops suddenly,
(c) it is easier to pull a lawn mower than to push it,
(d) a cricketer moves his hands backwards while holding a catch.

CHAPTER FIVE

WORK, ENERGY AND POWER

5.1 INTRODUCTION

The terms 'work', 'energy' and 'power' are frequently used in everyday language. A farmer ploughing the field, a construction worker carrying bricks, a student studying for a competitive examination, an artist painting a beautiful landscape, all are said to be working. In physics, however, the word 'Work' covers a definite and precise meaning. Somebody who has the capacity to work for 14-16 hours a day is said to have a large stamina or energy. We admire a long distance runner for her stamina or energy. Energy is thus our capacity to do work. In Physics too, the term 'energy' is related to work in this sense, but as said above the term 'work' itself is defined much more precisely. The word 'power' is used in everyday life with different shades of meaning. In karate or boxing we talk of 'powerful' punches. These are delivered at a great speed. This shade of meaning is close to the meaning of the word 'power' used in physics. We shall find that there is at best a loose correlation between the physical definitions and the physiological pictures these terms generate in our minds. The aim of this chapter is to develop an understanding of these three physical quantities. Before we proceed to this task, we need to develop a mathematical prerequisite, namely the scalar product of two vectors.

5.1.1 The Scalar Product

We have learnt about vectors and their use in Chapter 3. Physical quantities like displacement, velocity, acceleration, force etc. are vectors. We have also learnt how vectors are added or subtracted. We now need to know how vectors are multiplied. There are two ways of multiplying vectors which we shall come across : one way known as the scalar product gives a scalar from two vectors and the other known as the vector product produces a new vector from two vectors. We shall look at the vector product in Chapter 6. Here we take up the scalar product of two vectors. The scalar product or dot product of any two vectors **A** and **B**, denoted as **A**·**B** (read

A dot **B**) is defined as

$$\mathbf{A \cdot B} = A\,B \cos\theta \qquad (5.1a)$$

where θ is the angle between the two vectors as shown in Fig. 5.1(a). Since A, B and $\cos\theta$ are scalars, the dot product of **A** and **B** is a scalar quantity. Each vector, **A** and **B**, has a direction but their scalar product does not have a direction.

From Eq. (5.1a), we have

$$\begin{aligned}\mathbf{A \cdot B} &= A\,(B\cos\theta)\\ &= B\,(A\cos\theta)\end{aligned}$$

Geometrically, $B\cos\theta$ is the projection of **B** onto **A** in Fig.5.1 (b) and $A\cos\theta$ is the projection of **A** onto **B** in Fig. 5.1 (c). So, **A·B** is the product of the magnitude of **A** and the component of **B** along **A**. Alternatively, it is the product of the magnitude of **B** and the component of **A** along **B**.

Equation (5.1a) shows that the scalar product follows the commutative law :

$$\mathbf{A \cdot B} = \mathbf{B \cdot A}$$

Scalar product obeys the **distributive law**:

$$\mathbf{A \cdot (B + C)} = \mathbf{A \cdot B} + \mathbf{A \cdot C}$$

Further, $\mathbf{A} \cdot (\lambda \mathbf{B}) = \lambda\,(\mathbf{A \cdot B})$

where λ is a real number.

The proofs of the above equations are left to you as an exercise.

For unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ we have

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

Given two vectors

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

their scalar product is

$$\mathbf{A \cdot B} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\cdot\left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)$$

$$= A_xB_x + A_yB_y + A_zB_z \qquad (5.1b)$$

From the definition of scalar product and (Eq. 5.1b) we have :

(i) $\mathbf{A \cdot A} = A_xA_x + A_yA_y + A_zA_z$

Or, $A^2 = A_x^2 + A_y^2 + A_z^2 \qquad (5.1c)$

since $\mathbf{A \cdot A} = |\mathbf{A}|\,|\mathbf{A}| \cos 0 = A^2$.

(ii) $\mathbf{A \cdot B} = 0$, if **A** and **B** are perpendicular.

▶ ***Example 5.1*** Find the angle between force $\mathbf{F} = (3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}} - 5\,\hat{\mathbf{k}})$ unit and displacement $\mathbf{d} = (5\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}})$ unit. Also find the projection of **F** on **d**.

Answer $\mathbf{F \cdot d} = F_xd_x + F_yd_y + F_zd_z$

$= 3\,(5) + 4\,(4) + (-5)\,(3)$

$= 16$ unit

Hence $\mathbf{F \cdot d} = F\,d\,\cos\theta = 16$ unit

Now $\mathbf{F \cdot F} = F^2 = F_x^2 + F_y^2 + F_z^2$

$= 9 + 16 + 25$

$= 50$ unit

and $\mathbf{d \cdot d} = d^2 = d_x^2 + d_y^2 + d_z^2$

$= 25 + 16 + 9$

$= 50$ unit

$$\therefore \cos\theta = \frac{16}{\sqrt{50}\sqrt{50}} = \frac{16}{50} = 0.32,$$

$$\theta = \cos^{-1} 0.32$$

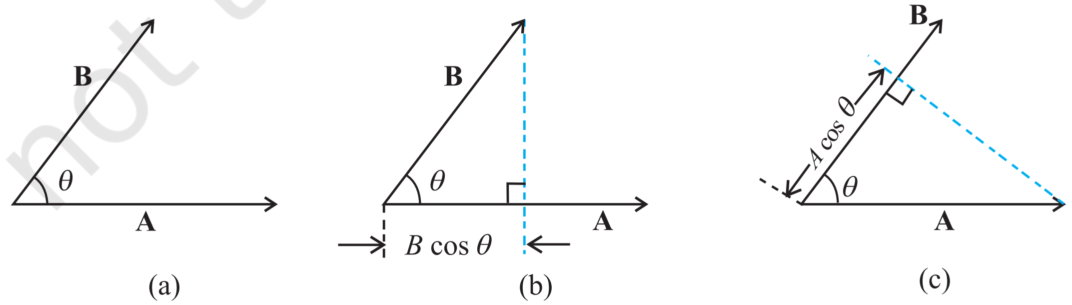


Fig. 5.1 (a) *The scalar product of two vectors* **A** *and* **B** *is a scalar :* $\mathbf{A \cdot B} = A\,B\cos\theta$. (b) $B\cos\theta$ *is the projection of* **B** *onto* **A**. (c) $A\cos\theta$ *is the projection of* **A** *onto* **B**.

5.2 NOTIONS OF WORK AND KINETIC ENERGY: THE WORK-ENERGY THEOREM

The following relation for rectilinear motion under constant acceleration a has been encountered in Chapter 3,

$$v^2 - u^2 = 2\,as \tag{5.2}$$

where u and v are the initial and final speeds and s the distance traversed. Multiplying both sides by $m/2$, we have

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mas = Fs \tag{5.2a}$$

where the last step follows from Newton's Second Law. We can generalise Eq. (5.2) to three dimensions by employing vectors

$$v^2 - u^2 = 2\,\mathbf{a.d}$$

Here **a** and **d** are acceleration and displacement vectors of the object respectively.
Once again multiplying both sides by $m/2$, we obtain

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = m\,\mathbf{a.d} = \mathbf{F.d} \tag{5.2b}$$

The above equation provides a motivation for the definitions of work and kinetic energy. The left side of the equation is the difference in the quantity 'half the mass times the square of the speed' from its initial value to its final value. We call each of these quantities the 'kinetic energy', denoted by K. The right side is a product of the displacement and the component of the force along the displacement. This quantity is called 'work' and is denoted by W. Eq. (5.2b) is then

$$K_f - K_i = W \tag{5.3}$$

where K_i and K_f are respectively the initial and final kinetic energies of the object. Work refers to the force and the displacement over which it acts. **Work is done by a force on the body over a certain displacement**.

Equation (5.2) is also a special case of the work-energy (WE) theorem : **The change in kinetic energy of a particle is equal to the work done on it by the net force**. We shall generalise the above derivation to a varying force in a later section.

▶ ***Example 5.2*** It is well known that a raindrop falls under the influence of the downward gravitational force and the opposing resistive force. The latter is known to be proportional to the speed of the drop but is otherwise undetermined. Consider a drop of mass 1.00 g falling from a height 1.00 km. It hits the ground with a speed of 50.0 m s^{-1}. (a) What is the work done by the gravitational force ? What is the work done by the unknown resistive force?

Answer (a) The change in kinetic energy of the drop is

$$\Delta K = \frac{1}{2}m\,v^2 - 0$$

$$= \frac{1}{2} \times 10^{-3} \times 50 \times 50$$

$$= 1.25\text{ J}$$

where we have assumed that the drop is initially at rest.

Assuming that g is a constant with a value 10 m/s^2, the work done by the gravitational force is,

$$W_g = mgh$$
$$= 10^{-3} \times 10 \times 10^3$$
$$= 10.0\text{ J}$$

(b) From the work-energy theorem

$$\Delta K = W_g + W_r$$

where W_r is the work done by the resistive force on the raindrop. Thus

$$W_r = \Delta K - W_g$$
$$= 1.25 - 10$$
$$= -\,8.75\text{ J}$$

is negative. ◀

5.3 WORK

As seen earlier, work is related to force and the displacement over which it acts. Consider a constant force **F** acting on an object of mass m. The object undergoes a displacement **d** in the positive x-direction as shown in Fig. 5.2.

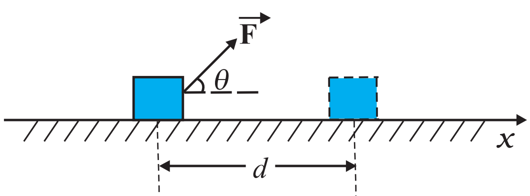


Fig. 5.2 *An object undergoes a displacement* **d** *under the influence of the force* **F**.

The work done by the force is defined to be the product of component of the force in the direction of the displacement and the magnitude of this displacement. Thus

$$W = (F\cos\theta)d = \mathbf{F.d} \qquad (5.4)$$

We see that if there is no displacement, there is no work done even if the force is large. Thus, when you push hard against a rigid brick wall, the force you exert on the wall does no work. Yet your muscles are alternatively contracting and relaxing and internal energy is being used up and you do get tired. Thus, the meaning of work in physics is different from its usage in everyday language.

No work is done if :

(i) the displacement is zero as seen in the example above. A weightlifter holding a 150 kg mass steadily on his shoulder for 30 s does no work on the load during this time.

(ii) the force is zero. A block moving on a smooth horizontal table is not acted upon by a horizontal force (since there is no friction), but may undergo a large displacement.

(iii) the force and displacement are mutually perpendicular. This is so since, for $\theta = \pi/2$ rad (= 90°), cos (π/2) = 0. For the block moving on a smooth horizontal table, the gravitational force *mg* does no work since it acts at right angles to the displacement. If we assume that the moon's orbits around the earth is perfectly circular then the earth's gravitational force does no work. The moon's instantaneous displacement is tangential while the earth's force is radially inwards and $\theta = \pi/2$.

Work can be both positive and negative. If θ is between 0° and 90°, cos θ in Eq. (5.4) is positive. If θ is between 90° and 180°, cos θ is negative. In many examples the frictional force opposes displacement and θ = 180°. Then the work done by friction is negative (cos 180° = –1).

From Eq. (5.4) it is clear that work and energy have the same dimensions, [ML^2T^{-2}]. The SI unit of these is joule (J), named after the famous British physicist James Prescott Joule (1811-1869). Since work and energy are so widely used as physical concepts, alternative units abound and some of these are listed in Table 5.1.

Table 5.1 Alternative Units of Work/Energy in J

erg	10^{-7} J
electron volt (eV)	1.6×10^{-19} J
calorie (cal)	4.186 J
kilowatt hour (kWh)	3.6×10^{6} J

▶ ***Example 5.3*** A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposed to the motion. (a) How much work does the road do on the cycle ? (b) How much work does the cycle do on the road ?

Answer Work done on the cycle by the road is the work done by the stopping (frictional) force on the cycle due to the road.

(a) The stopping force and the displacement make an angle of 180° (π rad) with each other. Thus, work done by the road,

$$W_r = Fd\cos\theta$$
$$= 200 \times 10 \times \cos\pi$$
$$= -2000 \text{ J}$$

It is this negative work that brings the cycle to a halt in accordance with WE theorem.

(b) From Newton's Third Law an equal and opposite force acts on the road due to the cycle. Its magnitude is 200 N. However, the road undergoes no displacement. Thus, work done by cycle on the road is zero. ◀

The lesson of Example 5.3 is that though the force on a body A exerted by the body B is always equal and opposite to that on B by A (Newton's Third Law); the work done on A by B is not necessarily equal and opposite to the work done on B by A.

5.4 KINETIC ENERGY

As noted earlier, if an object of mass *m* has velocity **v**, its kinetic energy *K* is

$$K = \frac{1}{2}m\,\mathbf{v.v} = \frac{1}{2}mv^2 \qquad (5.5)$$

Kinetic energy is a scalar quantity. The kinetic energy of an object is a measure of the work an

Table 5.2 **Typical kinetic energies (*K*)**

Object	Mass (kg)	Speed (m s^{-1})	*K* (J)
Car	2000	25	6.3×10^{5}
Running athlete	70	10	3.5×10^{3}
Bullet	5×10^{-2}	200	10^{3}
Stone dropped from 10 m	1	14	10^{2}
Rain drop at terminal speed	3.5×10^{-5}	9	1.4×10^{-3}
Air molecule	$\simeq 10^{-26}$	500	$\simeq 10^{-21}$

object can do by the virtue of its motion. This notion has been intuitively known for a long time. The kinetic energy of a fast flowing stream has been used to grind corn. Sailing ships employ the kinetic energy of the wind. Table 5.2 lists the kinetic energies for various objects.

Example 5.4 In a ballistics demonstration a police officer fires a bullet of mass 50.0 g with speed 200 m s^{-1} (see Table 5.2) on soft plywood of thickness 2.00 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet ?

Answer The initial kinetic energy of the bullet is $mv^2/2$ = 1000 J. It has a final kinetic energy of 0.1×1000 = 100 J. If v_f is the emergent speed of the bullet,

$$\frac{1}{2}mv_f^2 = 100\text{ J}$$

$$v_f = \sqrt{\frac{2\times100\text{ J}}{0.05\text{ kg}}}$$

$$= 63.2\text{ m s}^{-1}$$

The speed is reduced by approximately 68% (not 90%).

5.5 WORK DONE BY A VARIABLE FORCE

A constant force is rare. It is the variable force, which is more commonly encountered. Fig. 5.3 is a plot of a varying force in one dimension.

If the displacement Δx is small, we can take the force $F(x)$ as approximately constant and the work done is then

$$\Delta W = F(x)\,\Delta x$$

This is illustrated in Fig. 5.3(a). Adding successive rectangular areas in Fig. 5.3(a) we get the total work done as

$$W \cong \sum_{x_i}^{x_f} F(x)\Delta x \tag{5.6}$$

where the summation is from the initial position x_i to the final position x_f.

If the displacements are allowed to approach zero, then the number of terms in the sum increases without limit, but the sum approaches a definite value equal to the area under the curve in Fig. 5.3(b). Then the work done is

$$W = \lim_{\Delta x\to 0}\sum_{x_i}^{x_f} F(x)\Delta x$$

$$= \int_{x_i}^{x_f} F(x)\,\mathrm{d}x \tag{5.7}$$

where '*lim*' stands for the limit of the sum when Δx tends to zero. Thus, for a varying force the work done can be expressed as a definite integral of force over displacement (see also Appendix 3.1).

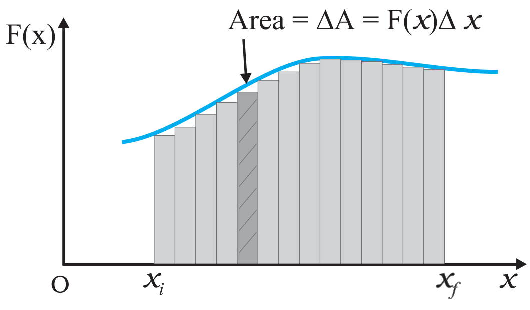


Fig. 5.3(a)

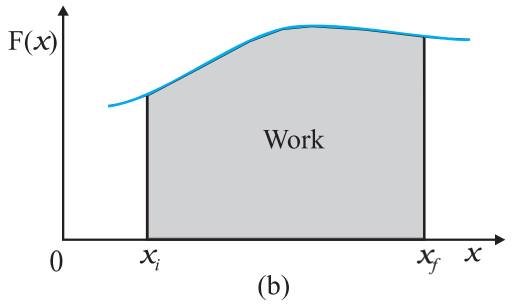


Fig. 5.3 *(a) The shaded rectangle represents the work done by the varying force F(x), over the small displacement Δx, ΔW = F(x) Δx. (b) adding the areas of all the rectangles we find that for $\Delta x \to 0$, the area under the curve is exactly equal to the work done by F(x).*

▶ ***Example 5.5*** A woman pushes a trunk on a railway platform which has a rough surface. She applies a force of 100 N over a distance of 10 m. Thereafter, she gets progressively tired and her applied force reduces linearly with distance to 50 N. The total distance through which the trunk has been moved is 20 m. Plot the force applied by the woman and the frictional force, which is 50 N versus displacement. Calculate the work done by the two forces over 20 m.

Answer

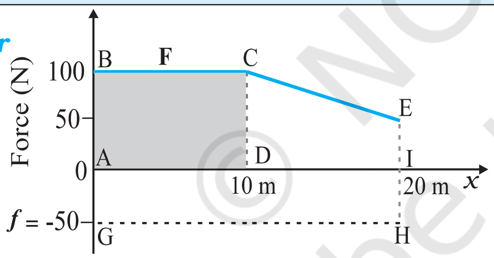


Fig. 5.4 *Plot of the force F applied by the woman and the opposing frictional force f versus* displacement.

The plot of the applied force is shown in Fig. 5.4. At x = 20 m, F = 50 N ($\neq$ 0). We are given that the frictional force f is $|\mathbf{f}|$ = 50 N. It opposes motion and acts in a direction opposite to **F**. It is therefore, shown on the negative side of the force axis.

The work done by the woman is

$W_F \to$ area of the rectangle ABCD + area of the trapezium CEID

$$W_F = 100 \times 10 + \frac{1}{2}(100+50) \times 10$$
$$= 1000 + 750$$
$$= 1750 \text{ J}$$

The work done by the frictional force is

$W_f \to$ area of the rectangle AGHI

$$W_f = (-50) \times 20$$
$$= -1000 \text{ J}$$

The area on the negative side of the force axis has a negative sign. ◀

5.6 THE WORK-ENERGY THEOREM FOR A VARIABLE FORCE

We are now familiar with the concepts of work and kinetic energy to prove the work-energy theorem for a variable force. We confine ourselves to one dimension. The time rate of change of kinetic energy is

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\,v^2\right)$$

$$= m\frac{\mathrm{d}v}{\mathrm{d}t}v$$

$$= F\,v \text{ (from Newton's Second Law)}$$

$$= F\frac{\mathrm{d}x}{\mathrm{d}t}$$

Thus

$$\mathrm{d}K = F\mathrm{d}x$$

Integrating from the initial position (x_i) to final position (x_f), we have

$$\int_{K_i}^{K_f} \mathrm{d}K = \int_{x_i}^{x_f} F\mathrm{d}x$$

where, K_i and K_f are the initial and final kinetic energies corresponding to x_i and x_f.

$$\text{or} \qquad K_f - K_i = \int_{x_i}^{x_f} F\mathrm{d}x \qquad (5.8a)$$

From Eq. (5.7), it follows that

$$K_f - K_i = W \qquad (5.8b)$$

Thus, the WE theorem is proved for a variable force.

While the WE theorem is useful in a variety of problems, it does not, in general, incorporate the complete dynamical information of Newton's second law. It is an integral form of Newton's second law. Newton's second law is a relation between acceleration and force at any instant of time. Work-energy theorem involves an integral over an interval of time. In this sense, the temporal (time) information contained in the statement of Newton's second law is 'integrated over' and is not available explicitly. Another observation is that Newton's second law for two or three dimensions is in vector form whereas the work-energy theorem is in scalar form. In the scalar form, information with respect to directions contained in Newton's second law is not present.

Example 5.6 A block of mass $m = 1$ kg, moving on a horizontal surface with speed $v_i = 2$ m s^{-1} enters a rough patch ranging from $x = 0.10$ m to $x = 2.01$ m. The retarding force F_r on the block in this range is inversely proportional to x over this range,

$$F_r = \frac{-k}{x} \text{ for } 0.1 < x < 2.01 \text{ m}$$

$= 0$ for $x < 0.1$m and $x > 2.01$ m

where $k = 0.5$ J. What is the final kinetic energy and speed v_f of the block as it crosses this patch ?

Answer From Eq. (5.8a)

$$K_f = K_i + \int_{0.1}^{2.01} \frac{(-k)}{x} \mathrm{d}x$$

$$= \frac{1}{2} m v_i^2 - k \ln(x)\Big|_{0.1}^{2.01}$$

$$= \frac{1}{2} m v_i^2 - k \ln(2.01/0.1)$$

$$= 2 - 0.5 \ln(20.1)$$

$$= 2 - 1.5 = 0.5 \text{ J}$$

$$v_f = \sqrt{2K_f/m} = 1 \text{ m s}^{-1}$$

Here, note that ln is a symbol for the natural logarithm to the base e and not the logarithm to the base 10 [ln X = $\log_e$ X = 2.303 $\log_{10}$ X]. ◀

5.7 THE CONCEPT OF POTENTIAL ENERGY

The word potential suggests possibility or capacity for action. The term potential energy brings to one's mind 'stored' energy. A stretched bow-string possesses potential energy. When it is released, the arrow flies off at a great speed. The earth's crust is not uniform, but has discontinuities and dislocations that are called fault lines. These fault lines in the earth's crust are like 'compressed springs'. They possess a large amount of potential energy. An earthquake results when these fault lines readjust. Thus, potential energy is the 'stored energy' by virtue of the position or configuration of a body. The body left to itself releases this stored energy in the form of kinetic energy. Let us make our notion of potential energy more concrete.

The gravitational force on a ball of mass m is mg . g may be treated as a constant near the earth surface. By 'near' we imply that the height h of the ball above the earth's surface is very small compared to the earth's radius R_E ($h << R_E$) so that we can ignore the variation of g near the earth's surface*. In what follows we have taken the upward direction to be positive. Let us raise the ball up to a height h. The work done by the external agency against the gravitational force is mgh. This work gets stored as potential energy. Gravitational potential energy of an object, as a function of the height h, is denoted by $V(h)$ and it is the negative of work done by the gravitational force in raising the object to that height.

$$V(h) = mgh$$

If h is taken as a variable, it is easily seen that the gravitational force F equals the negative of the derivative of $V(h)$ with respect to h. Thus,

$$F = -\frac{\mathrm{d}}{\mathrm{d}h} V(h) = -m\,g$$

The negative sign indicates that the gravitational force is downward. When released, the ball comes down with an increasing speed. Just before it hits the ground, its speed is given by the kinematic relation,

$$v^2 = 2gh$$

This equation can be written as

$$\frac{1}{2} m\, v^2 = m\,g\,h$$

which shows that the gravitational potential energy of the object at height h, when the object is released, manifests itself as kinetic energy of the object on reaching the ground.

Physically, the notion of potential energy is applicable only to the class of forces where work done against the force gets 'stored up' as energy. When external constraints are removed, it manifests itself as kinetic energy. Mathematically, (for simplicity, in one dimension) the potential

* *The variation of g with height is discussed in Chapter 7 on Gravitation.*

energy $V(x)$ is defined if the force $F(x)$ can be written as

$$F(x) = -\frac{dV}{dx}$$

This implies that

$$\int_{x_i}^{x_f} F(x)dx = -\int_{V_i}^{V_f} dV = V_i - V_f$$

The work done by a conservative force such as gravity depends on the initial and final positions only. In the previous chapter we have worked on examples dealing with inclined planes. If an object of mass m is released from rest, from the top of a smooth (frictionless) inclined plane of height h, its speed at the bottom is $\sqrt{2gh}$ irrespective of the angle of inclination. Thus, at the bottom of the inclined plane it acquires a kinetic energy, mgh. If the work done or the kinetic energy did depend on other factors such as the velocity or the particular path taken by the object, the force would be called non-conservative.

The dimensions of potential energy are $[ML^2T^{-2}]$ and the unit is joule (J), the same as kinetic energy or work. To reiterate, the change in potential energy, for a conservative force, ΔV is equal to the negative of the work done by the force

$$\Delta V = -F(x)\,\Delta x \tag{5.9}$$

In the example of the falling ball considered in this section we saw how potential energy was converted to kinetic energy. This hints at an important principle of conservation in mechanics, which we now proceed to examine.

5.8 THE CONSERVATION OF MECHANICAL ENERGY

For simplicity we demonstrate this important principle for one-dimensional motion. Suppose that a body undergoes displacement Δx under the action of a conservative force F. Then from the WE theorem we have,

$$\Delta K = F(x)\,\Delta x$$

If the force is conservative, the potential energy function $V(x)$ can be defined such that

$$-\Delta V = F(x)\,\Delta x$$

The above equations imply that

$$\Delta K + \Delta V = 0$$
$$\Delta(K + V) = 0 \tag{5.10}$$

which means that $K + V$, the sum of the kinetic and potential energies of the body is a constant. Over the whole path, x_i to x_f, this means that

$$K_i + V(x_i) = K_f + V(x_f) \tag{5.11}$$

The quantity $K + V(x)$, is called the total mechanical energy of the system. Individually the kinetic energy K and the potential energy $V(x)$ may vary from point to point, but the sum is a constant. The aptness of the term 'conservative force' is now clear.

Let us consider some of the definitions of a conservative force.

- A force $F(x)$ is conservative if it can be derived from a scalar quantity $V(x)$ by the relation given by Eq. (5.9). The three-dimensional generalisation requires the use of a vector derivative, which is outside the scope of this book.
- The work done by the conservative force depends only on the end points. This can be seen from the relation,
  $$W = K_f - K_i = V(x_i) - V(x_f)$$
  which depends on the end points.
- A third definition states that the work done by this force in a closed path is zero. This is once again apparent from Eq. (5.11) since $x_i = x_f$.

Thus, the principle of conservation of total mechanical energy can be stated as

The total mechanical energy of a system is conserved if the forces, doing work on it, are conservative.

The above discussion can be made more concrete by considering the example of the gravitational force once again and that of the spring force in the next section. Fig. 5.5 depicts a ball of mass m being dropped from a cliff of height H.

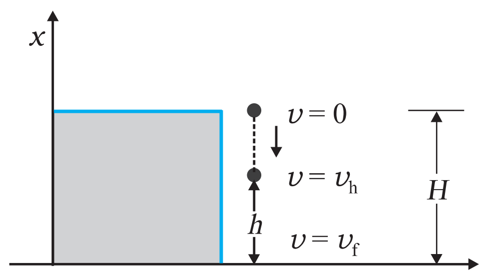


Fig. 5.5 *The conversion of potential energy to kinetic energy for a ball of mass m dropped from a height H.*

The total mechanical energies E_0, E_h, and E_H of the ball at the indicated heights zero (ground level), h and H, are

$$E_H = mgH \tag{5.11 a}$$

$$E_h = mgh + \frac{1}{2}mv_h^2 \tag{5.11 b}$$

$$E_0 = (1/2)\, mv_f^2 \tag{5.11 c}$$

The constant force is a special case of a spatially dependent force $F(x)$. Hence, the mechanical energy is conserved. Thus

$$E_H = E_0$$

or, $$mgH = \frac{1}{2}mv_f^2$$

$$v_f = \sqrt{2gH}$$

a result that was obtained in section 5.7 for a freely falling body.

Further,

$$E_H = E_h$$

which implies,

$$v_h^2 = 2g(H - h) \tag{5.11 d}$$

and is a familiar result from kinematics.

At the height H, the energy is purely potential. It is partially converted to kinetic at height h and is fully kinetic at ground level. This illustrates the conservation of mechanical energy.

▶ ***Example 5.7*** A bob of mass m is suspended by a light string of length L. It is imparted a horizontal velocity v_o at the lowest point A such that it completes a semi-circular trajectory in the vertical plane with the string becoming slack only on reaching the topmost point, C. This is shown in Fig. 5.6. Obtain an expression for (i) v_o; (ii) the speeds at points B and C; (iii) the ratio of the kinetic energies (K_B/K_C) at B and C. Comment on the nature of the trajectory of the bob after it reaches the point C.

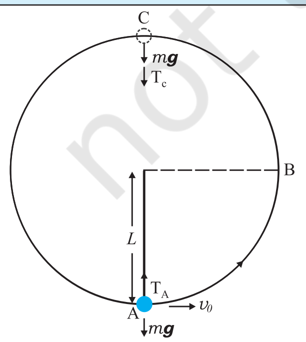


Fig. 5.6

Answer (i) There are two external forces on the bob : gravity and the tension (T) in the string. The latter does no work since the displacement of the bob is always normal to the string. The potential energy of the bob is thus associated with the gravitational force only. The total mechanical energy E of the system is conserved. We take the potential energy of the system to be zero at the lowest point A. Thus, at A :

$$E = \frac{1}{2}mv_0^2 \tag{5.12}$$

$$T_A - mg = \frac{mv_0^2}{L} \quad \text{[Newton's Second Law]}$$

where T_A is the tension in the string at A. At the highest point C, the string slackens, as the tension in the string (T_C) becomes zero.

Thus, at C

$$E = \frac{1}{2}mv_c^2 + 2mgL \tag{5.13}$$

$$mg = \frac{mv_c^2}{L} \quad \text{[Newton's Second Law]} \tag{5.14}$$

where v_C is the speed at C. From Eqs. (5.13) and (5.14)

$$E = \frac{5}{2}mgL$$

Equating this to the energy at A

$$\frac{5}{2}mgL = \frac{m}{2}v_0^2$$

or, $$v_0 = \sqrt{5gL}$$

(ii) It is clear from Eq. (5.14)

$$v_C = \sqrt{gL}$$

At B, the energy is

$$E = \frac{1}{2}mv_B^2 + mgL$$

Equating this to the energy at A and employing the result from (i), namely $v_0^2 = 5gL$,

$$\frac{1}{2}mv_B^2 + mgL = \frac{1}{2}mv_0^2$$

$$= \frac{5}{2}m\,g\,L$$

$$\therefore v_B = \sqrt{3gL}$$

(iii) The ratio of the kinetic energies at B and C is :

$$\frac{K_B}{K_C} = \frac{\frac{1}{2}mv_B^2}{\frac{1}{2}mv_C^2} = \frac{3}{1}$$

At point C, the string becomes slack and the velocity of the bob is horizontal and to the left. If the connecting string is cut at this instant, the bob will execute a projectile motion with horizontal projection akin to a rock kicked horizontally from the edge of a cliff. Otherwise the bob will continue on its circular path and complete the revolution. ◀

5.9 THE POTENTIAL ENERGY OF A SPRING

The spring force is an example of a variable force which is conservative. Fig. 5.7 shows a block attached to a spring and resting on a smooth horizontal surface. The other end of the spring is attached to a rigid wall. The spring is light and may be treated as massless. In an ideal spring, the spring force F_s is proportional to x where x is the displacement of the block from the equilibrium position. The displacement could be either positive [Fig. 5.7(b)] or negative [Fig. 5.7(c)]. This force law for the spring is called Hooke's law and is mathematically stated as

$$F_s = -kx$$

The constant k is called the spring constant. Its unit is N m^{-1}. The spring is said to be stiff if k is large and soft if k is small.

Suppose that we pull the block outwards as in Fig. 5.7(b). If the extension is x_m, the work done by the spring force is

$$W_s = \int_0^{x_m} F_s \, dx = -\int_0^{x_m} kx \, dx$$

$$= -\frac{k\,x_m^2}{2} \qquad (5.15)$$

This expression may also be obtained by considering the area of the triangle as in Fig. 5.7(d). Note that the work done by the external pulling force F is positive since it overcomes the spring force.

$$W = +\frac{k\,x_m^2}{2} \qquad (5.16)$$

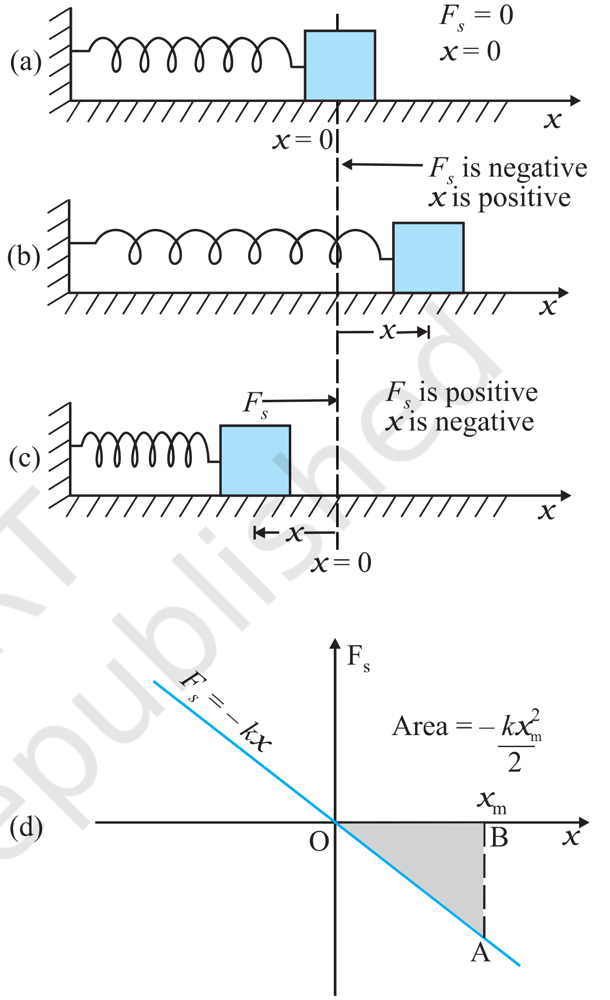


Fig. 5.7 *Illustration of the spring force with a block attached to the free end of the spring. (a) The spring force F_s is zero when the displacement x from the equilibrium position is zero. (b) For the stretched spring $x > 0$ and $F_s < 0$ (c) For the compressed spring $x < 0$ and $F_s > 0$.(d) The plot of F_s versus x. The area of the shaded triangle represents the work done by the spring force. Due to the opposing signs of F_s and x, this work done is negative, $W_s = -kx_m^2 / 2$.*

The same is true when the spring is compressed with a displacement x_c (< 0). The spring force does work $W_s = -kx_c^2/2$ while the external force F does work $+\, kx_c^2/2$. If the block is moved from an initial displacement x_i to a final displacement x_f, the work done by the spring force W_s is

$$W_s = -\int_{x_i}^{x_f} k\,x\,\mathrm{d}x = \frac{k\,x_i^2}{2} - \frac{k\,x_f^2}{2} \qquad (5.17)$$

Thus the work done by the spring force depends only on the end points. Specifically, if the block is pulled from x_i and allowed to return to x_i;

$$W_s = -\int_{x_i}^{x_i} k\,x\,\mathrm{d}x = \frac{k\,x_i^2}{2} - \frac{k\,x_i^2}{2}$$

$$= 0 \qquad (5.18)$$

The work done by the spring force in a cyclic process is zero. We have explicitly demonstrated that the spring force (i) is position dependent only as first stated by Hooke, ($F_s = -\,kx$); (ii) does work which only depends on the initial and final positions, e.g. Eq. (5.17). Thus, the spring force is a **conservative force**.

We define the potential energy $V(x)$ of the spring to be zero when block and spring system is in the equilibrium position. For an extension (or compression) x the above analysis suggests that

$$V(x) = \frac{kx^2}{2} \qquad (5.19)$$

You may easily verify that $-\,\mathrm{d}V/\mathrm{d}x = -\,k\,x$, the spring force. If the block of mass m in Fig. 5.7 is extended to x_m and released from rest, then its total mechanical energy at any arbitrary point x, where x lies between $-x_m$ and $+x_m$ will be given by

$$\frac{1}{2}k\,x_m^2 = \frac{1}{2}k\,x^2 + \frac{1}{2}m\,v^2$$

where we have invoked the conservation of mechanical energy. This suggests that the speed and the kinetic energy will be maximum at the equilibrium position, $x = 0$, i.e.,

$$\frac{1}{2}m\,v_m^2 = \frac{1}{2}k\,x_m^2$$

where v_m is the maximum speed.

or $$v_m = \sqrt{\frac{k}{m}}\,x_m$$

Note that k/m has the dimensions of [T^{-2}] and our equation is dimensionally correct. The kinetic energy gets converted to potential energy and vice versa, however, the total mechanical energy remains constant. This is graphically depicted in Fig. 5.8.

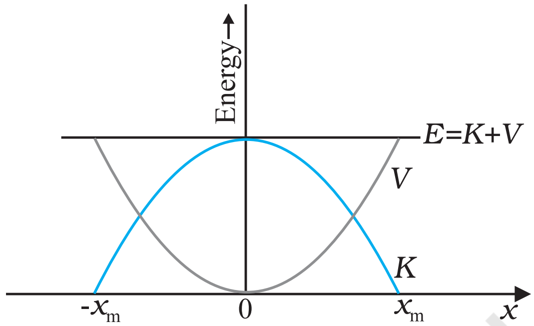


Fig. 5.8 *Parabolic plots of the potential energy V and kinetic energy K of a block attached to a spring obeying Hooke's law. The two plots are complementary, one decreasing as the other increases. The total mechanical energy E = K + V remains constant.*

▶ ***Example 5.8*** To simulate car accidents, auto manufacturers study the collisions of moving cars with mounted springs of different spring constants. Consider a typical simulation with a car of mass 1000 kg moving with a speed 18.0 km/h on a smooth road and colliding with a horizontally mounted spring of spring constant 5.25×10^3 N m^{-1}. What is the maximum compression of the spring ?

Answer At maximum compression the kinetic energy of the car is converted entirely into the potential energy of the spring.

The kinetic energy of the moving car is

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2} \times 10^3 \times 5 \times 5$$

$$K = 1.25 \times 10^4 \text{ J}$$

where we have converted 18 km h^{-1} to 5 m s^{-1} **[It is useful to remember that 36 km h^{-1} = 10 m s^{-1}]**. At maximum compression x_m, the potential energy V of the spring is equal to the kinetic energy K of the moving car from the principle of conservation of mechanical energy.

$$V = \frac{1}{2}k\,x_m^2$$

$$= 1.25 \times 10^4 \text{ J}$$

We obtain

$$x_m = 2.00 \text{ m}$$

We note that we have idealised the situation. The spring is considered to be massless. The surface has been considered to possess negligible friction. ◀

We conclude this section by making a few remarks on conservative forces.

(i) Information on time is absent from the above discussions. In the example considered above, we can calculate the compression, but not the time over which the compression occurs. A solution of Newton's Second Law for this system is required for temporal information.

(ii) Not all forces are conservative. Friction, for example, is a non-conservative force. The principle of conservation of energy will have to be modified in this case. This is illustrated in Example 5.9.

(iii) The zero of the potential energy is arbitrary. It is set according to convenience. For the spring force we took $V(x) = 0$, at $x = 0$, i.e. the unstretched spring had zero potential energy. For the constant gravitational force mg, we took $V = 0$ on the earth's surface. In a later chapter we shall see that for the force due to the universal law of gravitation, the zero is best defined at an infinite distance from the gravitational source. However, once the zero of the potential energy is fixed in a given discussion, it must be consistently adhered to throughout the discussion. You cannot change horses in midstream !

▶ ***Example 5.9*** Consider Example 5.8 taking the coefficient of friction, μ, to be 0.5 and calculate the maximum compression of the spring.

Answer In presence of friction, both the spring force and the frictional force act so as to oppose the compression of the spring as shown in Fig. 5.9.

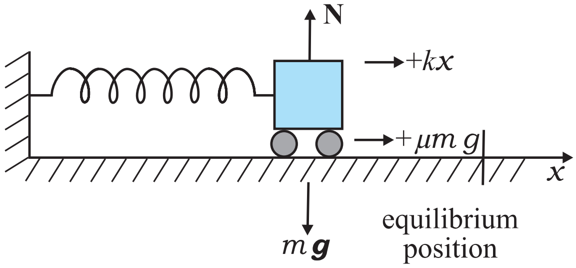


Fig. 5.9 *The forces acting on the car.*

We invoke the work-energy theorem, rather than the conservation of mechanical energy.

The change in kinetic energy is

$$\Delta K = K_f - K_i = 0 - \frac{1}{2} m v^2$$

The work done by the net force is

$$W = -\frac{1}{2} k x_m^2 - \mu m g x_m$$

Equating we have

$$\frac{1}{2} m v^2 = \frac{1}{2} k x_m^2 + \mu m g x_m$$

Now $\mu mg = 0.5 \times 10^3 \times 10 = 5 \times 10^3$ N (taking $g = 10.0 \text{ m s}^{-2}$). After rearranging the above equation we obtain the following quadratic equation in the unknown x_m.

$$k x_m^2 + 2\mu m g x_m - m v^2 = 0$$

$$x_m = \frac{-\mu m g + \left[\mu^2 m^2 g^2 + m k v^2\right]^{1/2}}{k}$$

where we take the positive square root since x_m is positive. Putting in numerical values we obtain

$$x_m = 1.35 \text{ m}$$

which, as expected, is less than the result in Example 5.8.

If the two forces on the body consist of a conservative force F_c and a non-conservative force F_{nc}, the conservation of mechanical energy formula will have to be modified. By the WE theorem

$$(F_c + F_{nc}) \Delta x = \Delta K$$

But $\quad F_c \Delta x = -\Delta V$

Hence, $\quad \Delta(K + V) = F_{nc} \Delta x$

$$\Delta E = F_{nc} \Delta x$$

where E is the total mechanical energy. Over the path this assumes the form

$$E_f - E_i = W_{nc}$$

where W_{nc} is the total work done by the non-conservative forces over the path. Note that unlike the conservative force, W_{nc} depends on the particular path i to f. ◀

5.10 POWER

Often it is interesting to know not only the work done on an object, but also the rate at which this work is done. We say a person is physically fit if he not only climbs four floors of a building but climbs them fast. **Power** is defined as the time rate at which work is done or energy is transferred.

The average power of a force is defined as the ratio of the work, W, to the total time t taken

$$P_{av} = \frac{W}{t}$$

The instantaneous power is defined as the limiting value of the average power as time interval approaches zero,

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{5.20}$$

The work dW done by a force F for a displacement $d\mathbf{r}$ is $\mathrm{d}W = \mathbf{F}.d\mathbf{r}$. The instantaneous power can also be expressed as

$$P = \mathbf{F}.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$$

$$= \mathbf{F.v} \tag{5.21}$$

where $\mathbf{v}$ is the instantaneous velocity when the force is $\mathbf{F}$.

Power, like work and energy, is a scalar quantity. Its dimensions are $[ML^2T^{-3}]$. In the SI, its unit is called a watt (W). The watt is 1 J s^{-1}. The unit of power is named after James Watt, one of the innovators of the steam engine in the eighteenth century.

There is another unit of power, namely the horse-power (hp)

1 hp = 746 W

This unit is still used to describe the output of automobiles, motorbikes, etc.

We encounter the unit watt when we buy electrical goods such as bulbs, heaters and refrigerators. A 100 watt bulb which is on for 10 hours uses 1 kilowatt hour (kWh) of energy.

100 (watt) × 10 (hour)
= 1000 watt hour
=1 kilowatt hour (kWh)
= 10^3 (W) × 3600 (s)
= 3.6×10^6 J

Our electricity bills carry the energy consumption in units of kWh. Note that kWh is a unit of energy and not of power.

▶ ***Example 5.10*** An elevator can carry a maximum load of 1800 kg (elevator + passengers) is moving up with a constant speed of 2 m s^{-1}. The frictional force opposing the motion is 4000 N. Determine the minimum power delivered by the motor to the elevator in watts as well as in horse power.

Answer The downward force on the elevator is

$$F = m\,g + F_f = (1800 \times 10) + 4000 = 22000 \text{ N}$$

The motor must supply enough power to balance this force. Hence,

$P = \mathbf{F}.\ \mathbf{v} = 22000 \times 2 = 44000$ W = 59 hp ◀

5.11 COLLISIONS

In physics we study motion (change in position). At the same time, we try to discover physical quantities, which do not change in a physical process. The laws of momentum and energy conservation are typical examples. In this section we shall apply these laws to a commonly encountered phenomena, namely collisions. Several games such as billiards, marbles or carrom involve collisions.We shall study the collision of two masses in an idealised form.

Consider two masses m_1 and m_2. The particle m_1 is moving with speed v_{1i}, the subscript 'i' implying initial. We can cosider m_2 to be at rest. No loss of generality is involved in making such a selection. In this situation the mass m_1 collides with the stationary mass m_2 and this is depicted in Fig. 5.10.

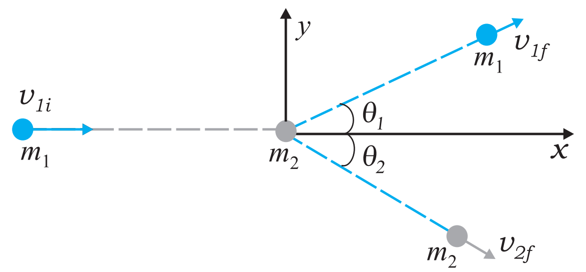


Fig. 5.10 *Collision of mass m_1, with a stationary mass m_2.*

The masses m_1 and m_2 fly-off in different directions. We shall see that there are relationships, which connect the masses, the velocities and the angles.

5.11.1 Elastic and Inelastic Collisions

In all collisions the total linear momentum is conserved; the initial momentum of the system is equal to the final momentum of the system. One can argue this as follows. When two objects collide, the mutual impulsive forces acting over the collision time Δt cause a change in their respective momenta :

$$\Delta \mathbf{p}_1 = \mathbf{F}_{12}\ \Delta t$$
$$\Delta \mathbf{p}_2 = \mathbf{F}_{21}\ \Delta t$$

where $\mathbf{F}_{12}$ is the force exerted on the first particle by the second particle. $\mathbf{F}_{21}$ is likewise the force exerted on the second particle by the first particle. Now from Newton's third law, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This implies

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = \mathbf{0}$$

The above conclusion is true even though the forces vary in a complex fashion during the collision time Δt. Since the third law is true at every instant, the total impulse on the first object is equal and opposite to that on the second.

On the other hand, the total kinetic energy of the system is not necessarily conserved. The impact and deformation during collision may generate heat and sound. Part of the initial kinetic energy is transformed into other forms of energy. A useful way to visualise the deformation during collision is in terms of a 'compressed spring'. If the 'spring' connecting the two masses regains its original shape without loss in energy, then the initial kinetic energy is equal to the final kinetic energy but the kinetic energy during the collision time Δt is not constant. Such a collision is called an **elastic collision**. On the other hand the deformation may not be relieved and the two bodies could move together after the collision. A collision in which the two particles move together after the collision is called a **completely inelastic collision**. The intermediate case where the deformation is partly relieved and some of the initial kinetic energy is lost is more common and is appropriately called an **inelastic collision**.

5.11.2 Collisions in One Dimension

Consider first a **completely inelastic collision** in one dimension. Then, in Fig. 5.10,

$$\theta_1 = \theta_2 = 0$$

$$m_1 v_{1i} = (m_1 + m_2) v_f \quad \text{(momentum conservation)}$$

$$v_f = \frac{m_1}{m_1 + m_2} v_{1i} \qquad (5.22)$$

The loss in kinetic energy on collision is

$$\Delta K = \frac{1}{2} m_1 v_{1i}^2 - \frac{1}{2}(m_1 + m_2) v_f^2$$

$$= \frac{1}{2} m_1 v_{1i}^2 - \frac{1}{2}\frac{m_1^2}{m_1 + m_2} v_{1i}^2 \quad \text{[using Eq. (5.22)]}$$

$$= \frac{1}{2} m_1 v_{1i}^2 \left[1 - \frac{m_1}{m_1 + m_2}\right]$$

$$= \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_{1i}^2$$

which is a positive quantity as expected.

Consider next an elastic collision. Using the above nomenclature with $\theta_1 = \theta_2 = 0$, the momentum and kinetic energy conservation equations are

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \qquad (5.23)$$

$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2 \qquad (5.24)$$

From Eqs. (5.23) and (5.24) it follows that,

$$m_1 v_{1i}(v_{2f} - v_{1i}) = m_1 v_{1f}(v_{2f} - v_{1f})$$

or, $$v_{2f}(v_{1i} - v_{1f}) = v_{1i}^2 - v_{1f}^2$$

$$= (v_{1i} - v_{1f})(v_{1i} + v_{1f})$$

Hence, $$\therefore v_{2f} = v_{1i} + v_{1f} \qquad (5.25)$$

Substituting this in Eq. (5.23), we obtain

$$v_{1f} = \frac{(m_1 - m_2)}{m_1 + m_2} v_{1i} \qquad (5.26)$$

and $$v_{2f} = \frac{2 m_1 v_{1i}}{m_1 + m_2} \qquad (5.27)$$

Thus, the 'unknowns' $\{v_{1f}, v_{2f}\}$ are obtained in terms of the 'knowns' $\{m_1, m_2, v_{1i}\}$. Special cases of our analysis are interesting.

Case I : If the two masses are equal

$$v_{1f} = 0$$
$$v_{2f} = v_{1i}$$

The first mass comes to rest and pushes off the second mass with its initial speed on collision.

Case II : If one mass dominates, e.g. $m_2 > > m_1$

$$v_{1f} \simeq -v_{1i} \qquad v_{2f} \simeq 0$$

The heavier mass is undisturbed while the lighter mass reverses its velocity.

Example 5.11 **Slowing down of neutrons**: In a nuclear reactor a neutron of high speed (typically 10^7 m s^{-1}) must be slowed to 10^3 m s^{-1} so that it can have a high probability of interacting with isotope $^{235}_{92}U$ and causing it to fission. Show that a neutron can lose most of its kinetic energy in an elastic collision with a light nuclei like deuterium or carbon which has a mass of only a few times the neutron mass. The material making up the light nuclei, usually heavy water (D_2O) or graphite, is called a moderator.

Answer The initial kinetic energy of the neutron is

$$K_{1i} = \frac{1}{2} m_1 v_{1i}^2$$

while its final kinetic energy from Eq. (5.26)

$$K_{1f} = \frac{1}{2} m_1 v_{1f}^2 \quad = \frac{1}{2} m_1 \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 v_{1i}^2$$

The fractional kinetic energy lost is

$$f_1 = \frac{K_{1f}}{K_{1i}} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2$$

while the fractional kinetic energy gained by the moderating nuclei K_{2f}/K_{1i} is

$$f_2 = 1 - f_1 \text{ (elastic collision)}$$

$$= \frac{4 m_1 m_2}{(m_1 + m_2)^2}$$

One can also verify this result by substituting from Eq. (5.27).

For deuterium $m_2 = 2m_1$ and we obtain $f_1 = 1/9$ while $f_2 = 8/9$. Almost 90% of the neutron's energy is transferred to deuterium. For carbon $f_1 = 71.6\%$ and $f_2 = 28.4\%$. In practice, however, this number is smaller since head-on collisions are rare.

If the initial velocities and final velocities of both the bodies are along the same straight line, then it is called a one-dimensional collision, or **head-on collision.** In the case of small spherical bodies, this is possible if the direction of travel of body 1 passes through the centre of body 2 which is at rest. In general, the collision is two-dimensional, where the initial velocities and the final velocities lie in a plane.

5.11.3 Collisions in Two Dimensions

Fig. 5.10 also depicts the collision of a moving mass m_1 with the stationary mass m_2. Linear momentum is conserved in such a collision. Since momentum is a vector this implies three equations for the three directions {x, y, z}. Consider the plane determined by the final velocity directions of m_1 and m_2 and choose it to be the x-y plane. The conservation of the z-component of the linear momentum implies that the entire collision is in the x-y plane. The x- and y-component equations are

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2 \quad (5.28)$$

$$0 = m_1 v_{1f} \sin\theta_1 - m_2 v_{2f} \sin\theta_2 \quad (5.29)$$

One knows {m_1, m_2, v_{1i}} in most situations. There are thus four unknowns {v_{1f}, v_{2f}, θ_1 *and* θ_2}, and only two equations. If $\theta_1 = \theta_2 = 0$, we regain Eq. (5.23) for one dimensional collision.

If, further the collision is elastic,

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (5.30)$$

We obtain an additional equation. That still leaves us one equation short. At least one of the four unknowns, say θ_1, must be made known for the problem to be solvable. For example, θ_1 can be determined by moving a detector in an angular fashion from the x to the y axis. Given {m_1, m_2, v_{1i}, θ_1} we can determine {v_{1f}, v_{2f}, θ_2} from Eqs. (5.28)-(5.30).

Example 5.12 Consider the collision depicted in Fig. 5.10 to be between two billiard balls with equal masses $m_1 = m_2$. The first ball is called the cue while the second ball is called the target. The billiard player wants to 'sink' the target ball in a corner pocket, which is at an angle $\theta_2 = 37$. Assume that the collision is elastic and that friction and rotational motion are not important. Obtain θ_1.

Answer From momentum conservation, since the masses are equal

$$\mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f}$$

or

$$v_{1i}^2 = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f})$$

$$= v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

$$= \left\{ v_{1f}^{\;2} + v_{2f}^{\;2} + 2v_{1f}v_{2f}\cos\left(\theta_1 + 37°\right) \right\} \qquad (5.31)$$

Since the collision is elastic and $m_1 = m_2$ it follows from conservation of kinetic energy that

$$v_{1i}^{\;2} = v_{1f}^{\;2} + v_{2f}^{\;2} \qquad (5.32)$$

Comparing Eqs. (5.31) and (5.32), we get

$$\cos(\theta_1 + 37°) = 0$$

or $\quad \theta_1 + 37° = 90°$

Thus, $\quad \theta_1 = 53°$

This proves the following result : when two equal masses undergo a glancing elastic collision with one of them at rest, after the collision, they will move at right angles to each other. ◀

The matter simplifies greatly if we consider spherical masses with smooth surfaces, and assume that collision takes place only when the bodies touch each other. This is what happens in the games of marbles, carrom and billiards.

In our everyday world, collisions take place only when two bodies touch each other. But consider a comet coming from far distances to the sun, or alpha particle coming towards a nucleus and going away in some direction. Here we have to deal with forces involving action at a distance. Such an event is called scattering. The velocities and directions in which the two particles go away depend on their initial velocities as well as the type of interaction between them, their masses, shapes and sizes.

SUMMARY

1. The *work-energy theorem* states that the change in kinetic energy of a body is the work done by the net force on the body.

$$K_f - K_i = W_{net}$$

2. A force is *conservative* if (i) work done by it on an object is path independent and depends only on the end points $\{x_i, x_j\}$, or (ii) the work done by the force is zero for an arbitrary closed path taken by the object such that it returns to its initial position.
3. For a conservative force in one dimension, we may define a *potential energy* function $V(x)$ such that

$$F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$$

or

$$V_i - V_f = \int_{x_i}^{x_f} F(x)\,\mathrm{d}x$$

4. The principle of conservation of mechanical energy states that the total mechanical energy of a body remains constant if the only forces that act on the body are conservative.
5. The *gravitational potential energy* of a particle of mass m at a height x about the earth's surface is

$$V(x) = m\,g\,x$$

where the variation of g with height is ignored.

5. The elastic potential energy of a spring of force constant k and extension x is

$$V(x) = \frac{1}{2}k\,x^2$$

7. The scalar or dot product of two vectors **A** and **B** is written as **A.B** and is a scalar quantity given by :**A·B** = $AB\cos\theta$, where θ is the angle between **A** and **B**. It can be positive, negative or zero depending upon the value of θ. The scalar product of two vectors can be interpreted as the product of magnitude of one vector and component of the other vector along the first vector. For unit vectors :

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1 \text{ and } \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0$$

Scalar products obey the commutative and the distributive laws.

Physical Quantity	Symbol	Dimensions	Units	Remarks
Work	W	$[ML^2T^{-2}]$	J	W = **F.d**
Kinetic energy	K	$[ML^2T^{-2}]$	J	$K = \frac{1}{2}mv^2$
Potential energy	$V(x)$	$[ML^2T^{-2}]$	J	$F(x) = -\frac{dV(x)}{dx}$
Mechanical energy	E	$[ML^2T^{-2}]$	J	$E = K + V$
Spring constant	k	$[MT^{-2}]$	$N\ m^{-1}$	$F = -kx$ $V(x) = \frac{1}{2}kx^2$
Power	P	$[ML^2T^{-3}]$	W	P = **F.v** $P = \frac{dW}{dt}$

POINTS TO PONDER

1. The phrase 'calculate the work done' is incomplete. We should refer (or imply clearly by context) to the work done by a specific force or a group of forces on a given body over a certain displacement.
2. Work done is a scalar quantity. It can be positive or negative unlike mass and kinetic energy which are positive scalar quantities. The work done by the friction or viscous force on a moving body is negative.
3. For two bodies, the sum of the mutual forces exerted between them is zero from Newton's Third Law,
$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$$
But the sum of the work done by the two forces need not *always cancel, i.e.*
$$W_{12} + W_{21} \neq 0$$
However, it *may sometimes be true.*
4. The work done by a force can be calculated sometimes even if the exact nature of the force is not known. This is clear from Example 5.2 where the WE theorem is used in such a situation.
5. The WE theorem is not independent of Newton's Second Law. The WE theorem may be viewed as a scalar form of the Second Law. The principle of conservation of mechanical energy may be viewed as a consequence of the WE theorem for conservative forces.
5. The WE theorem holds in all inertial frames. It can also be extended to non-inertial frames provided we include the pseudoforces in the calculation of the net force acting on the body under consideration.
7. The potential energy of a body subjected to a conservative force is always undetermined upto a constant. For example, the point where the potential energy is zero is a matter of choice. For the gravitational potential energy mgh, the zero of the potential energy is chosen to be the ground. For the spring potential energy $kx^2/2$, the zero of the potential energy is the equilibrium position of the oscillating mass.
8. Every force encountered in mechanics does not have an associated potential energy. For example, work done by friction over a closed path is not zero and no potential energy can be associated with friction.
9. During a collision : (a) the total linear momentum is conserved at each instant of the collision ; (b) the kinetic energy conservation (even if the collision is elastic) applies after the collision is over and does not hold at every instant of the collision. In fact the two colliding objects are deformed and may be momentarily at rest with respect to each other.

EXERCISES

5.1 The sign of work done by a force on a body is important to understand. State carefully if the following quantities are positive or negative:

(a) work done by a man in lifting a bucket out of a well by means of a rope tied to the bucket.

(b) work done by gravitational force in the above case,

(c) work done by friction on a body sliding down an inclined plane,

(d) work done by an applied force on a body moving on a rough horizontal plane with uniform velocity,

(e) work done by the resistive force of air on a vibrating pendulum in bringing it to rest.

5.2 A body of mass 2 kg initially at rest moves under the action of an applied horizontal force of 7 N on a table with coefficient of kinetic friction = 0.1. Compute the

(a) work done by the applied force in 10 s,

(b) work done by friction in 10 s,

(c) work done by the net force on the body in 10 s,

(d) change in kinetic energy of the body in 10 s,

and interpret your results.

5.3 Given in Fig. 5.11 are examples of some potential energy functions in one dimension. The total energy of the particle is indicated by a cross on the ordinate axis. In each case, specify the regions, if any, in which the particle cannot be found for the given energy. Also, indicate the minimum total energy the particle must have in each case. Think of simple physical contexts for which these potential energy shapes are relevant.

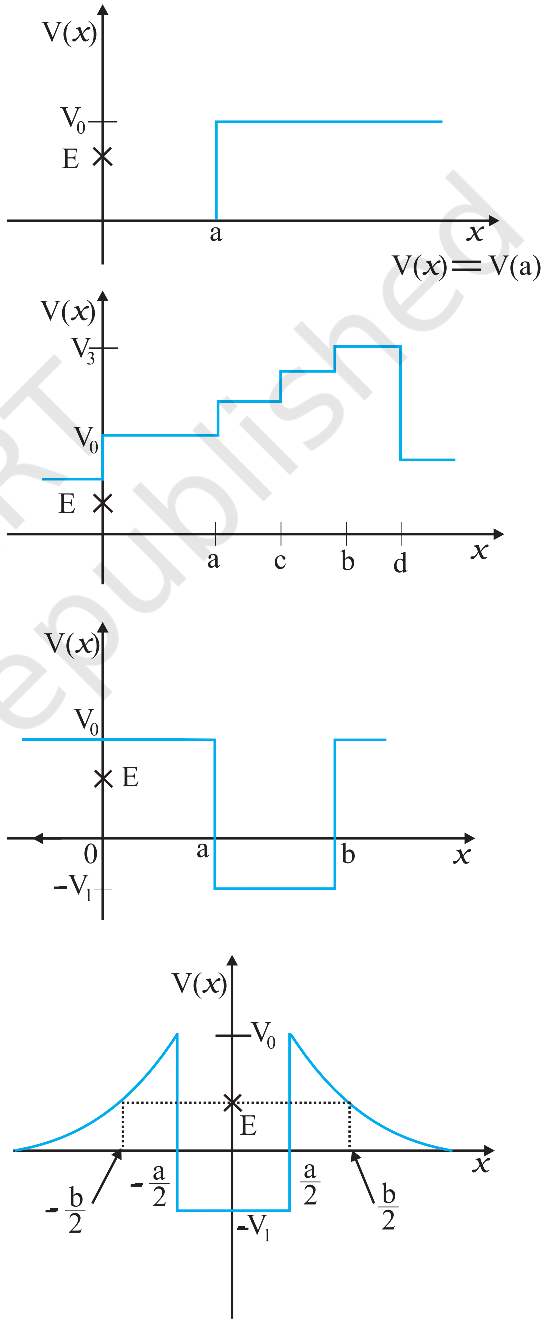


Fig. 5.11

5.4 The potential energy function for a particle executing linear simple harmonic motion is given by $V(x) = kx^2/2$, where k is the force constant of the oscillator. For $k = 0.5$ N m^{-1}, the graph of $V(x)$ versus x is shown in Fig. 5.12. Show that a particle of total energy 1 J moving under this potential must 'turn back' when it reaches $x = \pm 2$ m.

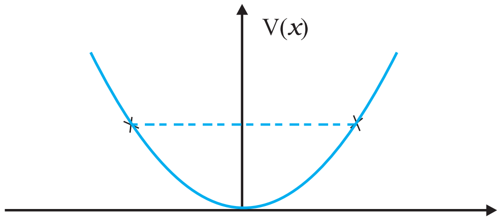
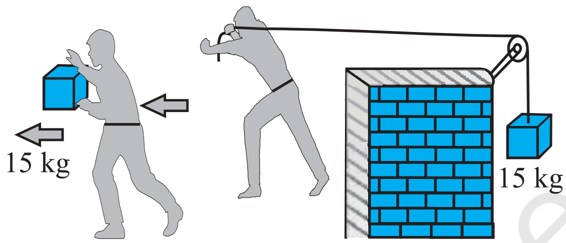


Fig. 5.12

5.5 Answer the following :

(a) The casing of a rocket in flight burns up due to friction. At whose expense is the heat energy required for burning obtained? The rocket or the atmosphere?

(b) Comets move around the sun in highly elliptical orbits. The gravitational force on the comet due to the sun is not normal to the comet's velocity in general. Yet the work done by the gravitational force over every complete orbit of the comet is zero. Why ?



(i) **Fig. 5.13** (ii)

(c) An artificial satellite orbiting the earth in very thin atmosphere loses its energy gradually due to dissipation against atmospheric resistance, however small. Why then does its speed increase progressively as it comes closer and closer to the earth ?

(d) In Fig. 5.13(i) the man walks 2 m carrying a mass of 15 kg on his hands. In Fig. 5.13(ii), he walks the same distance pulling the rope behind him. The rope goes over a pulley, and a mass of 15 kg hangs at its other end. In which case is the work done greater ?

5.6 Underline the correct alternative :

(a) When a conservative force does positive work on a body, the potential energy of the body increases/decreases/remains unaltered.

(b) Work done by a body against friction always results in a loss of its kinetic/potential energy.

(c) The rate of change of total momentum of a many-particle system is proportional to the external force/sum of the internal forces on the system.

(d) In an inelastic collision of two bodies, the quantities which do not change after the collision are the total kinetic energy/total linear momentum/total energy of the system of two bodies.

5.7 State if each of the following statements is true or false. Give reasons for your answer.

(a) In an elastic collision of two bodies, the momentum and energy of each body is conserved.

(b) Total energy of a system is always conserved, no matter what internal and external forces on the body are present.

(c) Work done in the motion of a body over a closed loop is zero for every force in nature.

(d) In an inelastic collision, the final kinetic energy is always less than the initial kinetic energy of the system.

5.8 Answer carefully, with reasons :

(a) In an elastic collision of two billiard balls, is the total kinetic energy conserved during the short time of collision of the balls (i.e. when they are in contact) ?

(b) Is the total linear momentum conserved during the short time of an elastic collision of two balls ?

(c) What are the answers to (a) and (b) for an inelastic collision ?

(d) If the potential energy of two billiard balls depends only on the separation distance between their centres, is the collision elastic or inelastic ? (Note, we are talking here of potential energy corresponding to the force during collision, not gravitational potential energy).

5.9 A body is initially at rest. It undergoes one-dimensional motion with constant acceleration. The power delivered to it at time t is proportional to

(i) $t^{1/2}$ (ii) t (iii) $t^{3/2}$ (iv) t^2

5.10 A body is moving unidirectionally under the influence of a source of constant power. Its displacement in time t is proportional to

(i) $t^{1/2}$ (ii) t (iii) $t^{3/2}$ (iv) t^2

5.11 A body constrained to move along the z-axis of a coordinate system is subject to a constant force **F** given by

$$\mathbf{F} = -\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}\ \text{N}$$

where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors along the x-, y- and z-axis of the system respectively. What is the work done by this force in moving the body a distance of 4 m along the z-axis ?

5.12 An electron and a proton are detected in a cosmic ray experiment, the first with kinetic energy 10 keV, and the second with 100 keV. Which is faster, the electron or the proton ? Obtain the ratio of their speeds. (electron mass = 9.11×10^{-31} kg, proton mass = 1.67×10^{-27} kg, 1 eV = 1.60×10^{-19} J).

5.13 A rain drop of radius 2 mm falls from a height of 500 m above the ground. It falls with decreasing acceleration (due to viscous resistance of the air) until at half its original height, it attains its maximum (terminal) speed, and moves with uniform speed thereafter. What is the work done by the gravitational force on the drop in the first and second half of its journey ? What is the work done by the resistive force in the entire journey if its speed on reaching the ground is 10 m s^{-1} ?

5.14 A molecule in a gas container hits a horizontal wall with speed 200 m s^{-1} and angle 30° with the normal, and rebounds with the same speed. Is momentum conserved in the collision ? Is the collision elastic or inelastic ?

5.15 A pump on the ground floor of a building can pump up water to fill a tank of volume 30 m^3 in 15 min. If the tank is 40 m above the ground, and the efficiency of the pump is 30%, how much electric power is consumed by the pump ?

5.16 Two identical ball bearings in contact with each other and resting on a frictionless table are hit head-on by another ball bearing of the same mass moving initially with a speed V. If the collision is elastic, which of the following (Fig. 5.14) is a possible result after collision ?

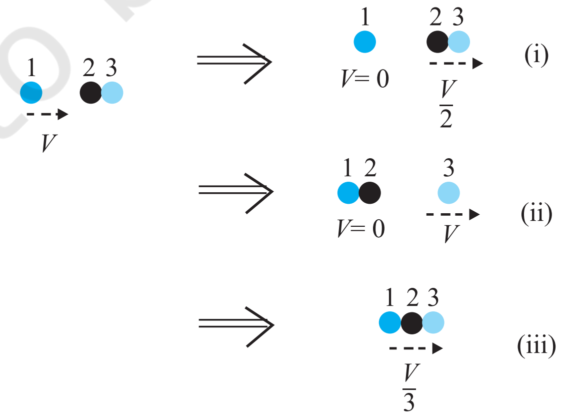


Fig. 5.14

5.17 The bob A of a pendulum released from 30° to the vertical hits another bob B of the same mass at rest on a table as shown in Fig. 5.15. How high does the bob A rise after the collision ? Neglect the size of the bobs and assume the collision to be elastic.

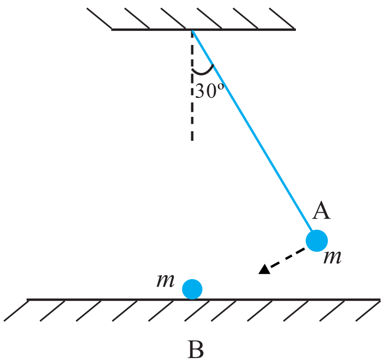


Fig. 5.15

5.18 The bob of a pendulum is released from a horizontal position. If the length of the pendulum is 1.5 m, what is the speed with which the bob arrives at the lowermost point, given that it dissipated 5% of its initial energy against air resistance ?

5.19 A trolley of mass 300 kg carrying a sandbag of 25 kg is moving uniformly with a speed of 27 km/h on a frictionless track. After a while, sand starts leaking out of a hole on the floor of the trolley at the rate of 0.05 kg s^{-1}. What is the speed of the trolley after the entire sand bag is empty ?

5.20 A body of mass 0.5 kg travels in a straight line with velocity $v = a\,x^{3/2}$ where $a = 5\ m^{-1/2}\ s^{-1}$. What is the work done by the net force during its displacement from $x = 0$ to $x = 2$ m ?

5.21 The blades of a windmill sweep out a circle of area A. (a) If the wind flows at a velocity v perpendicular to the circle, what is the mass of the air passing through it in time t ? (b) What is the kinetic energy of the air ? (c) Assume that the windmill converts 25% of the wind's energy into electrical energy, and that $A = 30\ m^2$, $v = 36$ km/h and the density of air is 1.2 kg m^{-3}. What is the electrical power produced ?

5.22 A person trying to lose weight (dieter) lifts a 10 kg mass, one thousand times, to a height of 0.5 m each time. Assume that the potential energy lost each time she lowers the mass is dissipated. (a) How much work does she do against the gravitational force ? (b) Fat supplies 3.8×10^7J of energy per kilogram which is converted to mechanical energy with a 20% efficiency rate. How much fat will the dieter use up?

5.23 A family uses 8 kW of power. (a) Direct solar energy is incident on the horizontal surface at an average rate of 200 W per square meter. If 20% of this energy can be converted to useful electrical energy, how large an area is needed to supply 8 kW? (b) Compare this area to that of the roof of a typical house.

Chapter Six

Systems of Particles and Rotational Motion

6.1 INTRODUCTION

In the earlier chapters we primarily considered the motion of a single particle. (A particle is ideally represented as a point mass having no size.) We applied the results of our study even to the motion of bodies of finite size, assuming that motion of such bodies can be described in terms of the motion of a particle.

Any real body which we encounter in daily life has a finite size. In dealing with the motion of extended bodies (bodies of finite size) often the idealised model of a particle is inadequate. In this chapter we shall try to go beyond this inadequacy. We shall attempt to build an understanding of the motion of extended bodies. An extended body, in the first place, is a system of particles. We shall begin with the consideration of motion of the system as a whole. The centre of mass of a system of particles will be a key concept here. We shall discuss the motion of the centre of mass of a system of particles and usefulness of this concept in understanding the motion of extended bodies.

A large class of problems with extended bodies can be solved by considering them to be rigid bodies. **Ideally a rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.** It is evident from this definition of a rigid body that no real body is truly rigid, since real bodies deform under the influence of forces. But in many situations the deformations are negligible. In a number of situations involving bodies such as wheels, tops, steel beams, molecules and planets on the other hand, we can ignore that they warp (twist out of shape), bend or vibrate and treat them as rigid.

6.1.1 What kind of motion can a rigid body have?

Let us try to explore this question by taking some examples of the motion of rigid bodies. Let us begin with a rectangular

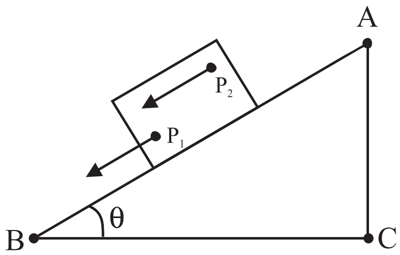


Fig 6.1 *Translational (sliding) motion of a block down an inclined plane.*
(Any point like P_1 or P_2 of the block moves with the same velocity at any instant of time.)

block sliding down an inclined plane without any sidewise movement. The block is taken as a rigid body. Its motion down the plane is such that all the particles of the body are moving together, i.e. they have the same velocity at any instant of time. The rigid body here is in pure translational motion (Fig. 6.1).

In pure translational motion at any instant of time, all particles of the body have the same velocity.

Consider now the rolling motion of a solid metallic or wooden cylinder down the same inclined plane (Fig. 6.2). The rigid body in this problem, namely the cylinder, shifts from the top to the bottom of the inclined plane, and thus, seems to have translational motion. But as Fig. 6.2 shows, all its particles are not moving with the same velocity at any instant. The body, therefore, is not in pure translational motion. Its motion is translational plus 'something else.'

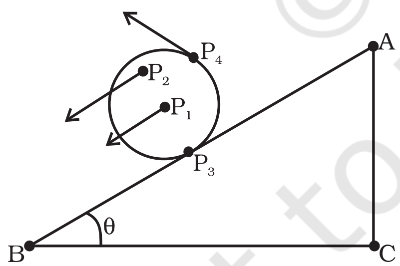


Fig. 6.2 *Rolling motion of a cylinder. It is not pure translational motion. Points P_1, P_2, P_3 and P_4 have different velocities (shown by arrows) at any instant of time. In fact, the velocity of the point of contact P_3 is zero at any instant, if the cylinder rolls without slipping.*

In order to understand what this 'something else' is, let us take a rigid body so constrained that it cannot have translational motion. The most common way to constrain a rigid body so that it does not have translational motion is to fix it along a straight line. The only possible motion of such a rigid body is **rotation**. The line or fixed axis about which the body is rotating is its **axis of rotation**. If you look around, you will come across many examples of rotation about an axis, a ceiling fan, a potter's wheel, a giant wheel in a fair, a merry-go-round and so on (Fig 6.3(a) and (b)).

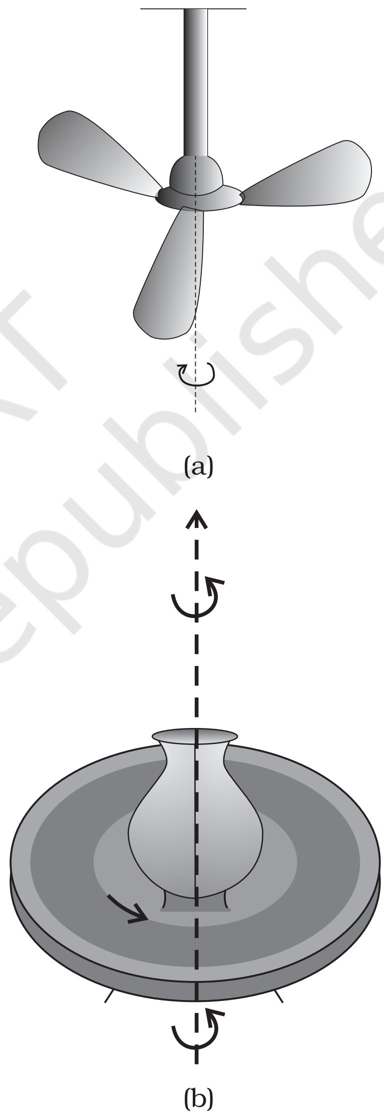


Fig. 6.3 *Rotation about a fixed axis*
(a) A ceiling fan
(b) A potter's wheel.

Let us try to understand what rotation is, what characterises rotation. You may notice that **in rotation of a rigid body about a fixed axis,**

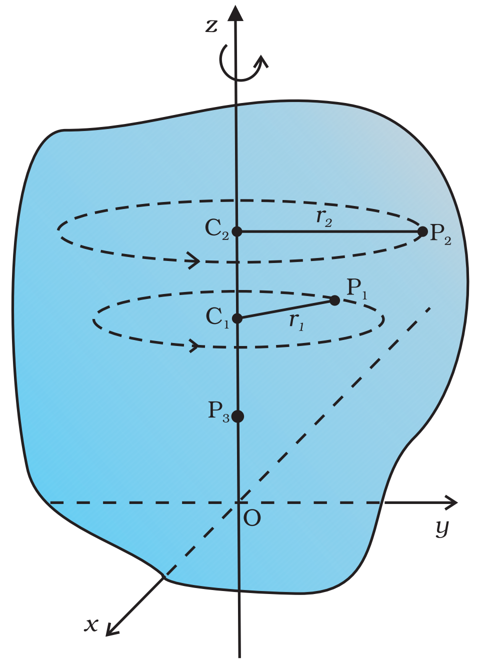


Fig. 6.4 *A rigid body rotation about the z-axis (Each point of the body such as P_1 or P_2 describes a circle with its centre (C_1 or C_2) on the axis of rotation. The radius of the circle (r_1 or r_2) is the perpendicular distance of the point (P_1 or P_2) from the axis. A point on the axis like P_3 remains stationary).*

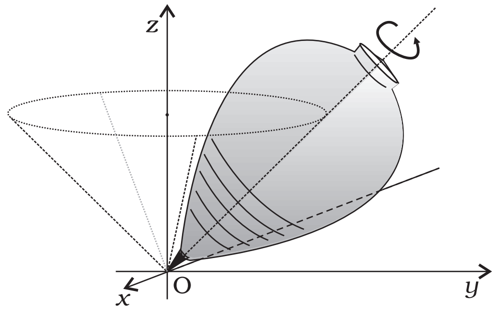


Fig. 6.5 (a) *A spinning top (The point of contact of the top with the ground, its tip O, is fixed.)*

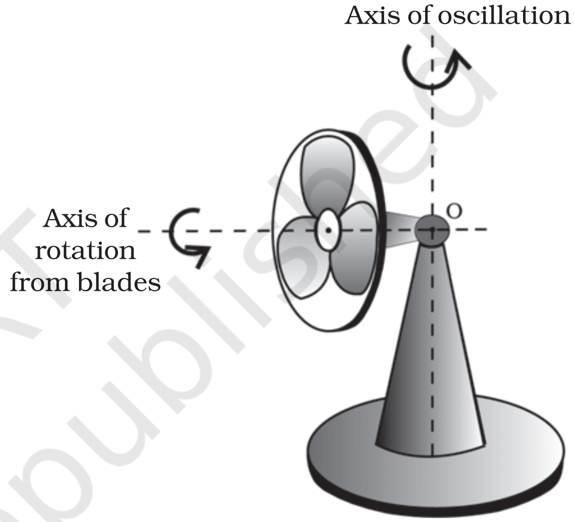


Fig. 6.5 (b) *An oscillating table fan with rotating blades. The pivot of the fan, point O, is fixed. The blades of the fan are under rotational motion, whereas, the axis of rotation of the fan blades is oscillating.*

every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis. Fig. 6.4 shows the rotational motion of a rigid body about a fixed axis (the *z*-axis of the frame of reference). Let P_1 be a particle of the rigid body, arbitrarily chosen and at a distance r_1 from fixed axis. The particle P_1 describes a circle of radius r_1 with its centre C_1 on the fixed axis. The circle lies in a plane perpendicular to the axis. The figure also shows another particle P_2 of the rigid body, P_2 is at a distance r_2 from the fixed axis. The particle P_2 moves in a circle of radius r_2 and with centre C_2 on the axis. This circle, too, lies in a plane perpendicular to the axis. Note that the circles described by P_1 and P_2 may lie in different planes; both these planes, however, are perpendicular to the fixed axis. For any particle on the axis like P_3, $r = 0$. Any such particle remains stationary while the body rotates. This is expected since the axis of rotation is fixed.

In some examples of rotation, however, the axis may not be fixed. A prominent example of this kind of rotation is a top spinning in place [Fig. 6.5(a)]. (We assume that the top does not slip from place to place and so does not have translational motion.) We know from experience that the axis of such a spinning top moves around the vertical through its point of contact with the ground, sweeping out a cone as shown in Fig. 6.5(a). (This movement of the axis of the top around the vertical is termed **precession**.) Note, the **point of contact of the top with ground is fixed**. The axis of rotation of the top at any instant passes through the point of contact. Another simple example of this kind of rotation is the oscillating table fan or a pedestal fan [Fig.6.5(b)]. You may have observed that the axis of rotation of such a fan has an oscillating (sidewise) movement in a horizontal plane about the vertical through the point at which the axis is pivoted (point O in Fig. 6.5(b)).

While the fan rotates and its axis moves sidewise, this point is fixed. Thus, in more general cases of rotation, such as the rotation of a top or a pedestal fan, **one point and not one line,** of the rigid body is fixed. In this case the axis is not fixed, though it always passes through the fixed point. In our study, however, we mostly deal with the simpler and special case of rotation in which one line (i.e. the axis) is fixed.

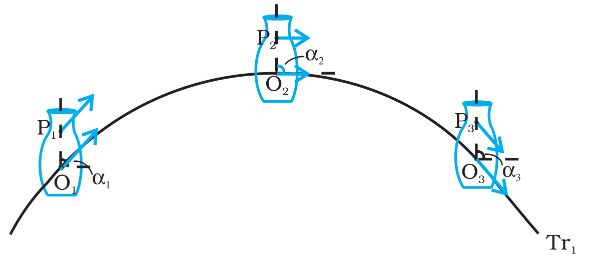


Fig. 6.6(a) *Motion of a rigid body which is pure translation.*

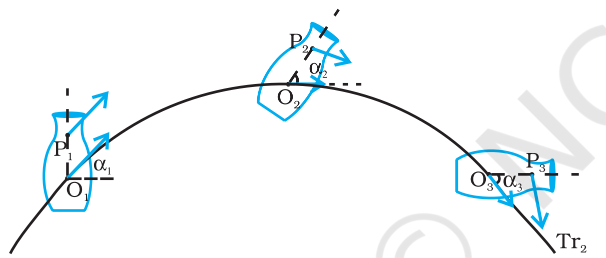


Fig. 6.6(b) *Motion of a rigid body which is a combination of translation and rotation.*

Fig 6.6 (a) and 6.6 (b) illustrate different motions of the same body. Note P is an arbitrary point of the body; O is the centre of mass of the body, which is defined in the next section. Suffice to say here that the trajectories of O are the translational trajectories Tr_1 *and* Tr_2 *of the body. The positions O and P at three different instants of time are shown by* O_1, O_2, *and* O_3, *and* P_1, P_2 *and* P_3, *respectively, in both Figs. 6.6 (a) and (b) . As seen from Fig. 6.6(a), at any instant the velocities of any particles like O and P of the body are the same in pure translation. Notice, in this case the orientation of OP, i.e. the angle OP makes with a fixed direction, say the horizontal, remains the same, i.e.* $\alpha_1 = \alpha_2 = \alpha_3$. *Fig. 6.6 (b) illustrates a case of combination of translation and rotation. In this case, at any instants the velocities of O and P differ. Also,* α_1, α_2 *and* α_3 *may all be different.*

Thus, for us rotation will be about a fixed axis only unless stated otherwise.

The rolling motion of a cylinder down an inclined plane is a combination of rotation about a fixed axis and translation. Thus, the 'something else' in the case of rolling motion which we referred to earlier is rotational motion. You will find Fig. 6.6(a) and (b) instructive from this point of view. Both these figures show motion of the same body along identical translational trajectory. In one case, Fig. 6.6(a), the motion is a pure translation; in the other case [Fig. 6.6(b)] it is a combination of translation and rotation. (You may try to reproduce the two types of motion shown, using a rigid object like a heavy book.)

We now recapitulate the most important observations of the present section: **The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation. The motion of a rigid body which is pivoted or fixed in some way is rotation.** The rotation may be about an axis that is fixed (e.g. a ceiling fan) or moving (e.g. an oscillating table fan [Fig.6.5(b)]). We shall, in the present chapter, consider rotational motion about a fixed axis only.

6.2 CENTRE OF MASS

We shall first see what the centre of mass of a system of particles is and then discuss its significance. For simplicity we shall start with a two particle system. We shall take the line joining the two particles to be the *x*- axis.

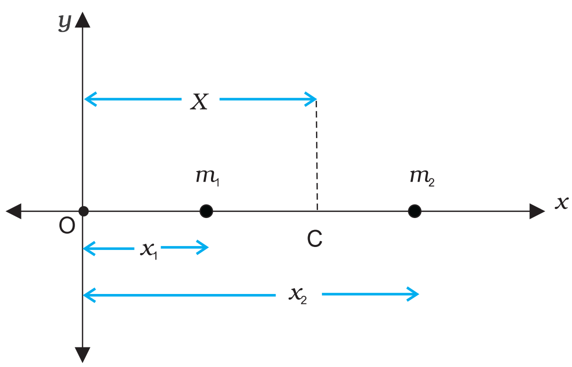


Fig. 6.7

Let the distances of the two particles be x_1 and x_2 respectively from some origin O. Let m_1 and m_2 be respectively the masses of the two particles. The centre of mass of the system is that point C which is at a distance X from O, where X is given by

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{6.1}$$

In Eq. (6.1), X can be regarded as the mass-weighted mean of x_1 and x_2. If the two particles have the same mass $m_1 = m_2 = m$ then

$$X = \frac{m x_1 + m x_2}{2m} = \frac{x_1 + x_2}{2}$$

Thus, for two particles of equal mass the centre of mass lies exactly midway between them.

If we have n particles of masses m_1, m_2, ...m_n respectively, along a straight line taken as the x- axis, then by definition the position of the centre of the mass of the system of particles is given by.

$$X = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum m_i x_i}{\sum m_i} \tag{6.2}$$

where x_1, x_2,...x_n are the distances of the particles from the origin; X is also measured from the same origin. The symbol $\sum$ (the Greek letter sigma) denotes summation, in this case over n particles. The sum

$$\sum m_i = M$$

is the total mass of the system.

Suppose that we have three particles, not lying in a straight line. We may define x– and y–axes in the plane in which the particles lie and represent the positions of the three particles by coordinates (x_1,y_1), (x_2,y_2) and (x_3,y_3) respectively. Let the masses of the three particles be m_1, m_2 and m_3 respectively. The centre of mass C of the system of the three particles is defined and located by the coordinates (X, Y) given by

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \tag{6.3a}$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \tag{6.3b}$$

For the particles of equal mass $m = m_1 = m_2 = m_3$,

$$X = \frac{m(x_1 + x_2 + x_3)}{3m} = \frac{x_1 + x_2 + x_3}{3}$$

$$Y = \frac{m(y_1 + y_2 + y_3)}{3m} = \frac{y_1 + y_2 + y_3}{3}$$

Thus, for three particles of equal mass, the centre of mass coincides with the centroid of the triangle formed by the particles.

Results of Eqs. (6.3a) and (6.3b) are generalised easily to a system of n particles, not necessarily lying in a plane, but distributed in space. The centre of mass of such a system is at (X, Y, Z), where

$$X = \frac{\sum m_i x_i}{M} \tag{6.4a}$$

$$Y = \frac{\sum m_i y_i}{M} \tag{6.4b}$$

and $$Z = \frac{\sum m_i z_i}{M} \tag{6.4c}$$

Here $M = \sum m_i$ is the total mass of the system. The index i runs from 1 to n; m_i is the mass of the i^{th} particle and the position of the i^{th} particle is given by (x_i, y_i, z_i).

Eqs. (6.4a), (6.4b) and (6.4c) can be combined into one equation using the notation of position vectors. Let $\mathbf{r}_i$ be the position vector of the i^{th} particle and $\mathbf{R}$ be the position vector of the centre of mass:

$$\mathbf{r}_i = x_i\,\hat{\mathbf{i}} + y_i\,\hat{\mathbf{j}} + z_i\,\hat{\mathbf{k}}$$

and $$\mathbf{R} = X\,\hat{\mathbf{i}} + Y\,\hat{\mathbf{j}} + Z\hat{\mathbf{k}}$$

Then $$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M} \tag{6.4d}$$

The sum on the right hand side is a vector sum.

Note the economy of expressions we achieve by use of vectors. If the origin of the frame of reference (the coordinate system) is chosen to be the centre of mass then $\sum m_i \mathbf{r}_i = 0$ for the given system of particles.

A rigid body, such as a metre stick or a flywheel, is a system of closely packed particles; Eqs. (6.4a), (6.4b), (6.4c) and (6.4d) are therefore, applicable to a rigid body. The number of particles (atoms or molecules) in such a body is so large that it is impossible to carry out the summations over individual particles in these equations. Since the spacing of the particles is small, we can treat the body as a continuous distribution of mass. We subdivide the body into n small elements of mass; $\Delta m_1, \Delta m_2 \ldots \Delta m_n$; the i^{th} element Δm_i is taken to be located about the point (x_i, y_i, z_i). The coordinates of the centre of mass are then approximately given by

$$X = \frac{\sum(\Delta m_i)x_i}{\sum \Delta m_i}, Y = \frac{\sum(\Delta m_i)y_i}{\sum \Delta m_i}, Z = \frac{\sum(\Delta m_i)z_i}{\sum \Delta m_i}$$

As we make n bigger and bigger and each Δm_i smaller and smaller, these expressions become exact. In that case, we denote the sums over i by integrals. Thus,

$$\sum \Delta m_i \rightarrow \int \mathrm{d}m = M,$$

$$\sum (\Delta m_i)x_i \rightarrow \int x\,\mathrm{d}m,$$

$$\sum (\Delta m_i)y_i \rightarrow \int y\,\mathrm{d}m,$$

and $$\sum (\Delta m_i)z_i \rightarrow \int z\,\mathrm{d}m$$

Here M is the total mass of the body. The coordinates of the centre of mass now are

$$X = \frac{1}{M}\int x\,\mathrm{d}m, Y = \frac{1}{M}\int y\,\mathrm{d}m \text{ and } Z = \frac{1}{M}\int z\,\mathrm{d}m \qquad (6.5a)$$

The vector expression equivalent to these three scalar expressions is

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,\mathrm{d}m \qquad (6.5b)$$

If we choose, the centre of mass as the origin of our coordinate system,

$$\mathbf{R} = \mathbf{0}$$

i.e., $\int \mathbf{r}\,\mathrm{d}m = \mathbf{0}$

or $$\int x\,\mathrm{d}m = \int y\,\mathrm{d}m = \int z\,\mathrm{d}m = 0 \qquad (6.6)$$

Often we have to calculate the centre of mass of homogeneous bodies of regular shapes like rings, discs, spheres, rods etc. (By a homogeneous body we mean a body with uniformly distributed mass.) By using symmetry consideration, we can easily show that the centres of mass of these bodies lie at their geometric centres.

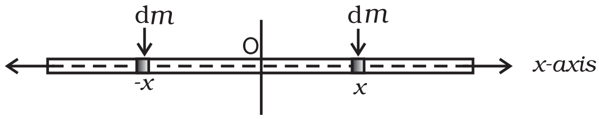


Fig. 6.8 *Determining the CM of a thin rod.*

Let us consider a thin rod, whose width and breath (in case the cross section of the rod is rectangular) or radius (in case the cross section of the rod is cylindrical) is much smaller than its length. Taking the origin to be at the geometric centre of the rod and x-axis to be along the length of the rod, we can say that on account of reflection symmetry, for every element dm of the rod at x, there is an element of the same mass dm located at $-x$ (Fig. 6.8).

The net contribution of every such pair to the integral and hence the integral $\int x\,\mathrm{d}m$ itself is zero. From Eq. (6.6), the point for which the integral itself is zero, is the centre of mass. Thus, the centre of mass of a homogenous thin rod coincides with its geometric centre. This can be understood on the basis of reflection symmetry.

The same symmetry argument will apply to homogeneous rings, discs, spheres, or even thick rods of circular or rectangular cross section. For all such bodies you will realise that for every element dm at a point (x, y, z) one can always take an element of the same mass at the point $(-x, -y, -z)$. (In other words, the origin is a point of reflection symmetry for these bodies.) As a result, the integrals in Eq. (6.5 a) all are zero. This means that for all the above bodies, their centre of mass coincides with their geometric centre.

Example 6.1 Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

Answer

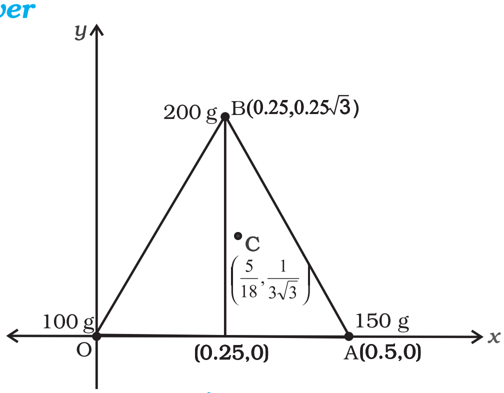


Fig. 6.9

With the x–and y–axes chosen as shown in Fig. 6.9, the coordinates of points O, A and B forming the equilateral triangle are respectively (0,0), (0.5,0), $(0.25, 0.25\sqrt{3})$. Let the masses 100 g, 150g and 200g be located at O, A and B be respectively. Then,

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{100(0) + 150(0.5) + 200(0.25) \quad \text{g m}}{(100 + 150 + 200) \quad \text{g}}$$

$$= \frac{75 + 50}{450}\text{ m} = \frac{125}{450}\text{ m} = \frac{5}{18}\text{ m}$$

$$Y = \frac{100(0) + 150(0) + 200(0.25\sqrt{3}) \quad \text{g m}}{450 \text{ g}}$$

$$= \frac{50\sqrt{3}}{450}\text{ m} = \frac{\sqrt{3}}{9}\text{ m} = \frac{1}{3\sqrt{3}}\text{ m}$$

The centre of mass C is shown in the figure. Note that it is not the geometric centre of the triangle OAB. Why? ◀

▶ ***Example 6.2*** Find the centre of mass of a triangular lamina.

Answer The lamina (ΔLMN) may be subdivided into narrow strips each parallel to the base (MN) as shown in Fig. 6.10

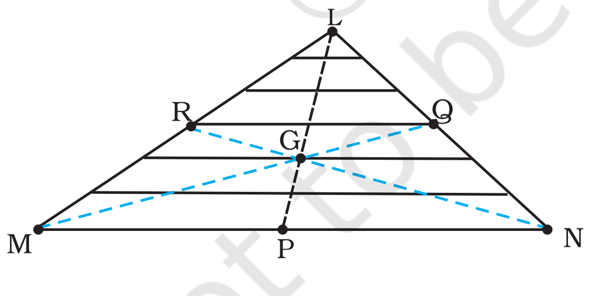


Fig. 6.10

By symmetry each strip has its centre of mass at its midpoint. If we join the midpoint of all the strips we get the median LP. The centre of mass of the triangle as a whole therefore, has to lie on the median LP. Similarly, we can argue that it lies on the median MQ and NR. This means the centre of mass lies on the point of concurrence of the medians, i.e. on the centroid G of the triangle. ◀

▶ ***Example 6.3*** Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The mass of the lamina is 3 kg.

Answer Choosing the X and Y axes as shown in Fig. 6.11 we have the coordinates of the vertices of the L-shaped lamina as given in the figure. We can think of the L-shape to consist of 3 squares each of length 1m. The mass of each square is 1kg, since the lamina is uniform. The centres of mass C_1, C_2 and C_3 of the squares are, by symmetry, their geometric centres and have coordinates (1/2,1/2), (3/2,1/2), (1/2,3/2) respectively. We take the masses of the squares to be concentrated at these points. The centre of mass of the whole L shape (X, Y) is the centre of mass of these mass points.

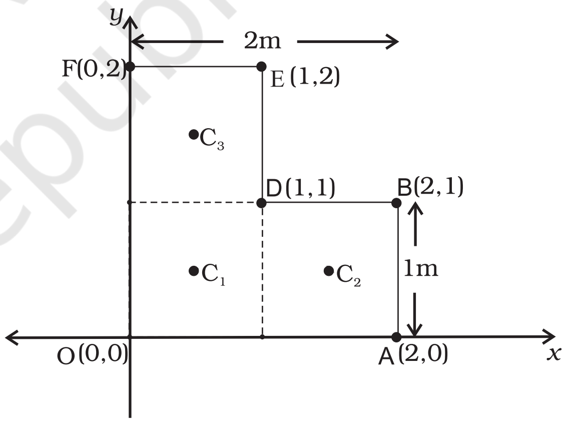


Fig. 6.11

Hence

$$X = \frac{[1(1/2) + 1(3/2) + 1(1/2)]\text{kg m}}{(1+1+1)\text{kg}} = \frac{5}{6}\text{m}$$

$$Y = \frac{[1(1/2) + 1(1/2) + 1(3/2)] \quad \text{kg m}}{(1+1+1)\text{kg}} = \frac{5}{6}\text{m}$$

The centre of mass of the L-shape lies on the line OD. We could have guessed this without calculations. Can you tell why? Suppose, the three squares that make up the L shaped lamina of Fig. 6.11 had different masses. How will you then determine the centre of mass of the lamina? ◀

6.3 MOTION OF CENTRE OF MASS

Equipped with the definition of the centre of mass, we are now in a position to discuss its physical importance for a system of n particles. We may rewrite Eq.(6.4d) as

$$M\mathbf{R} = \sum m_i \mathbf{r}_i = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n \qquad (6.7)$$

Differentiating the two sides of the equation with respect to time we get

$$M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = m_1\frac{\mathrm{d}\mathbf{r}_1}{\mathrm{d}t} + m_2\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}t} + ... + m_n\frac{\mathrm{d}\mathbf{r}_n}{dt}$$

or

$$M\mathbf{V} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + ... + m_n\mathbf{v}_n \qquad (6.8)$$

where $\mathbf{v}_1 \left(= \mathrm{d}\mathbf{r}_1/\mathrm{d}t\right)$ is the velocity of the first particle $\mathbf{v}_2\left(= d\mathbf{r}_2/dt\right)$ is the velocity of the second particle etc. and $\mathbf{V} = \mathrm{d}\mathbf{R}/\mathrm{d}t$ is the velocity of the centre of mass. Note that we assumed the masses m_1, m_2, ... etc. do not change in time. We have therefore, treated them as constants in differentiating the equations with respect to time.

Differentiating Eq.(6.8) with respect to time, we obtain

$$M\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = m_1\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} + m_2\frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}t} + ... + m_n\frac{\mathrm{d}\mathbf{v}_n}{\mathrm{d}t}$$

or

$$M\mathbf{A} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + ... + m_n\mathbf{a}_n \qquad (6.9)$$

where $\mathbf{a}_1 \left(= \mathrm{d}\mathbf{v}_1/\mathrm{d}t\right)$ is the acceleration of the first particle, $\mathbf{a}_2\left(= \mathrm{d}\mathbf{v}_2/\mathrm{d}t\right)$ is the acceleration of the second particle etc. and $\mathbf{A}\left(= \mathrm{d}\mathbf{V}/\mathrm{d}t\right)$ is the acceleration of the centre of mass of the system of particles.

Now, from Newton's second law, the force acting on the first particle is given by $\mathbf{F}_1 = m_1\mathbf{a}_1$. The force acting on the second particle is given by $\mathbf{F}_2 = m_2\mathbf{a}_2$ and so on. Eq. (6.9) may be written as

$$M\mathbf{A} = \mathbf{F}_1 + \mathbf{F}_2 + ... + \mathbf{F}_n \qquad (6.10)$$

Thus, the total mass of a system of particles times the acceleration of its centre of mass is the vector sum of all the forces acting on the system of particles.

Note when we talk of the force $\mathbf{F}_1$ on the first particle, it is not a single force, but the vector sum of all the forces on the first particle; likewise for the second particle etc. Among these forces on each particle there will be **external** forces exerted by bodies outside the system and also **internal** forces exerted by the particles on one another. We know from Newton's third law that these internal forces occur in equal and opposite pairs and in the sum of forces of Eq. (6.10), their contribution is zero. Only the external forces contribute to the equation. We can then rewrite Eq. (6.10) as

$$M\mathbf{A} = \mathbf{F}_{ext} \qquad (6.11)$$

where $\mathbf{F}_{ext}$ represents the sum of all external forces acting on the particles of the system.

Eq. (6.11) states that **the centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.**

Notice, to determine the motion of the centre of mass no knowledge of internal forces of the system of particles is required; for this purpose we need to know only the external forces.

To obtain Eq. (6.11) we did not need to specify the nature of the system of particles. The system may be a collection of particles in which there may be all kinds of internal motions, or it may be a rigid body which has either pure translational motion or a combination of translational and rotational motion. Whatever is the system and the motion of its individual particles, the centre of mass moves according to Eq. (6.11).

Instead of treating extended bodies as single particles as we have done in earlier chapters, we can now treat them as systems of particles. We can obtain the translational component of their motion, i.e. the motion of the centre of mass of the system, by taking the mass of the whole system to be concentrated at the centre of mass and all the external forces on the system to be acting at the centre of mass.

This is the procedure that we followed earlier in analysing forces on bodies and solving problems without explicitly outlining and justifying the procedure. We now realise that in earlier studies we assumed, without saying so, that rotational motion and/or internal motion of the particles were either absent or negligible. We no longer need to do this. We have not only found the justification of the procedure we followed earlier; but we also have found how to describe and separate the translational motion of (1) a rigid body which may be rotating as well, or (2) a system of particles with all kinds of internal motion.

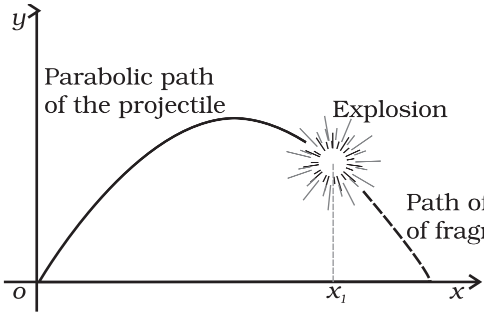


Fig. 6.12 *The centre of mass of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion.*

Figure 6.12 is a good illustration of Eq. (6.11). A projectile, following the usual parabolic trajectory, explodes into fragments midway in air. The forces leading to the explosion are internal forces. They contribute nothing to the motion of the centre of mass. The total external force, namely, the force of gravity acting on the body, is the same before and after the explosion. The centre of mass under the influence of the external force continues, therefore, along the same parabolic trajectory as it would have followed if there were no explosion.

6.4 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

Let us recall that the linear momentum of a particle is defined as

$$\mathbf{p} = m\,\mathbf{v} \tag{6.12}$$

Let us also recall that Newton's second law written in symbolic form for a single particle is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{6.13}$$

where **F** is the force on the particle. Let us consider a system of n particles with masses m_1, m_2,...m_n respectively and velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots \mathbf{v}_n$ respectively. The particles may be interacting and have external forces acting on them. The linear momentum of the first particle is $m_1\mathbf{v}_1$, of the second particle is $m_2\mathbf{v}_2$ and so on.

For the system of n particles, the linear momentum of the system is defined to be the vector sum of all individual particles of the system,

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n$$

$$= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \tag{6.14}$$

Comparing this with Eq. (6.8)

$$\mathbf{P} = M\,\mathbf{V} \tag{6.15}$$

Thus, **the total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass**. Differentiating Eq. (6.15) with respect to time,

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = M\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = M\mathbf{A} \tag{6.16}$$

Comparing Eq.(6.16) and Eq. (6.11),

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F}_{ext} \tag{6.17}$$

This is the statement of **Newton's second law of motion extended to a system of particles.**

Suppose now, that the sum of external forces acting on a system of particles is zero. Then from Eq.(6.17)

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0 \quad \text{or} \quad \mathbf{P} = \text{Constant} \tag{6.18a}$$

Thus, when the total external force acting on a system of particles is zero, the total linear momentum of the system is constant. This is the law of conservation of the total linear momentum of a system of particles. Because of Eq. (6.15), this also means that when the total external force on the system is zero the velocity of the centre of mass remains constant. (We assume throughout the discussion on systems of particles in this chapter that the total mass of the system remains constant.)

Note that on account of the internal forces, i.e. the forces exerted by the particles on one another, the individual particles may have complicated trajectories. Yet, if the total external force acting on the system is zero, the centre of mass moves with *a* constant velocity, i.e., moves uniformly in a straight line like a free particle.

The vector Eq. (6.18a) is equivalent to three scalar equations,

$P_x = c_1$, $P_y = c_2$ and $P_z = c_3$ (6.18 b)

Here P_x, P_y and P_z are the components of the total linear momentum vector **P** along the *x*–, *y*– and *z*–axes respectively; c_1, c_2 and c_3 are constants.

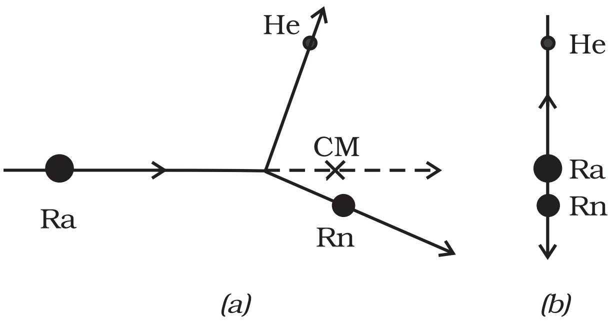


Fig. 6.13 (a) *A heavy nucleus radium (Ra) splits into a lighter nucleus radon (Rn) and an alpha particle (nucleus of helium atom). The CM of the system is in uniform motion.*
(b) *The same spliting of the heavy nucleus radium (Ra) with the centre of mass at rest. The two product particles fly back to back.*

As an example, let us consider the radioactive decay of a moving unstable particle, like the nucleus of radium. A radium nucleus disintegrates into a nucleus of radon and an alpha particle. The forces leading to the decay are internal to the system and the external forces on the system are negligible. So the total linear momentum of the system is the same before and after decay. The two particles produced in the decay, the radon nucleus and the alpha particle, move in different directions in such a way that their centre of mass moves along the same path along which the original decaying radium nucleus was moving [Fig. 6.13(a)].

If we observe the decay from the frame of reference in which the centre of mass is at rest, the motion of the particles involved in the decay looks particularly simple; the product particles

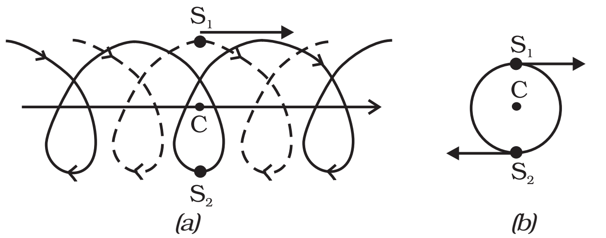


Fig. 6.14 (a) *Trajectories of two stars, S_1 (dotted line) and S_2 (solid line) forming a binary system with their centre of mass C in uniform motion.*
(b) *The same binary system, with the centre of mass C at rest.*

move back to back with their centre of mass remaining at rest as shown in Fig.6.13 (b).

In many problems on the system of particles, as in the above radioactive decay problem, it is convenient to work in the centre of mass frame rather than in the laboratory frame of reference.

In astronomy, binary (double) stars is a common occurrence. If there are no external forces, the centre of mass of a double star moves like a free particle, as shown in Fig.6.14 (a). The trajectories of the two stars of equal mass are also shown in the figure; they look complicated. If we go to the centre of mass frame, then we find that there the two stars are moving in a circle, about the centre of mass, which is at rest. Note that the position of the stars have to be diametrically opposite to each other [Fig. 6.14(b)]. Thus in our frame of reference, the trajectories of the stars are a combination of (i) uniform motion in a straight line of the centre of mass and (ii) circular orbits of the stars about the centre of mass.

As can be seen from the two examples, **separating** the motion of different parts of a system into motion **of the centre of mass and motion about the centre of mass** is a very useful technique that helps in understanding the motion of the system.

6.5 VECTOR PRODUCT OF TWO VECTORS

We are already familiar with vectors and their use in physics. In chapter 5 (Work, Energy, Power) we defined the scalar product of two vectors. An important physical quantity, work, is defined as a scalar product of two vector quantities, force and displacement.

We shall now define another product of two vectors. This product is a vector. Two important quantities in the study of rotational motion, namely, moment of a force and angular momentum, are defined as vector products.

Definition of Vector Product

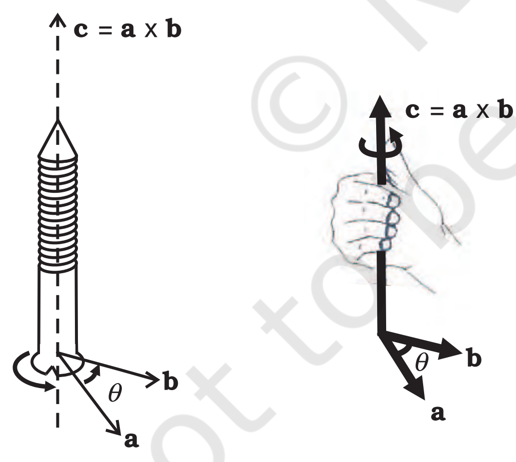
A vector product of two vectors **a** and **b** is a vector **c** such that

(i) magnitude of **c** = $c = ab\sin\theta$ where a and b are magnitudes of **a** and **b** and θ is the angle between the two vectors.

(ii) **c** is perpendicular to the plane containing **a** and **b**.

(iii) if we take a right handed screw with its head lying in the plane of **a** and **b** and the screw perpendicular to this plane, and if we turn the head in the direction from **a** to **b**, then the tip of the screw advances in the direction of **c**. This right handed screw rule is illustrated in Fig. 6.15a.

Alternately, if one curls up the fingers of right hand around a line perpendicular to the plane of the vectors **a** and **b** and if the fingers are curled up in the direction from **a** to **b**, then the stretched thumb points in the direction of **c**, as shown in Fig. 6.15b.



(a) (b)

Fig. 6.15 (a) *Rule of the right handed screw for defining the direction of the vector product of two vectors.*
(b) *Rule of the right hand for defining the direction of the vector product.*

A simpler version of the right hand rule is the following : Open up your right hand palm and curl the fingers pointing from **a** to **b**. Your stretched thumb points in the direction of **c**.

It should be remembered that there are two angles between any two vectors **a** and **b** . In Fig. 6.15 (a) or (b) they correspond to θ (as shown) and ($360^0 - \theta$). While applying either of the above rules, the rotation should be taken through the smaller angle (<180^0) between **a** and **b**. It is θ here.

Because of the cross (×) used to denote the vector product, it is also referred to as cross product.

- Note that scalar product of two vectors is commutative as said earlier, **a**.**b** = **b**.**a**

The vector product, however, is not commutative, i.e. **a** × **b** ≠ **b** × **a**

The magnitude of both **a** × **b** and **b** × **a** is the same ($ab\sin\theta$); also, both of them are perpendicular to the plane of **a** and **b**. But the rotation of the right-handed screw in case of **a** × **b** is from **a** to **b**, whereas in case of **b** × **a** it is from **b** to **a**. This means the two vectors are in opposite directions. We have

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$$

- Another interesting property of a vector product is its behaviour under reflection. Under reflection (i.e. on taking the plane mirror image) we have $x \to -x, y \to -y$ and $z \to -z$. As a result all the components of a vector change sign and thus $a \to -a,\ b \to -b$. What happens to **a** × **b** under reflection?

$$\mathbf{a}\times\mathbf{b} \to (-\mathbf{a})\times(-\mathbf{b}) = \mathbf{a}\times\mathbf{b}$$

Thus, **a** × **b** does not change sign under reflection.

- Both scalar and vector products are distributive with respect to vector addition. Thus,

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$$

- We may write **c** = **a** × **b** in the component form. For this we first need to obtain some elementary cross products:

(i) **a** × **a** = **0** (**0** is a null vector, i.e. a vector with zero magnitude)

This follows since magnitude of **a** × **a** is $a^2\sin 0° = 0$.

From this follow the results

(i) $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}, \; \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0}, \; \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$

(ii) $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$

Note that the magnitude of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is $\sin 90^0$ or 1, since $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ both have unit magnitude and the angle between them is 90^0. Thus, $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is a unit vector. A unit vector perpendicular to the plane of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and related to them by the right hand screw rule is $\hat{\mathbf{k}}$. Hence, the above result. You may verify similarly,

$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$

From the rule for commutation of the cross product, it follows:

$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \; \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \; \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$

Note if $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ occur cyclically in the above vector product relation, the vector product is positive. If $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ do not occur in cyclic order, the vector product is negative.

Now,

$\mathbf{a} \times \mathbf{b} = (a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} + a_z \hat{\mathbf{k}}) \times (b_x \hat{\mathbf{i}} + b_y \hat{\mathbf{j}} + b_z \hat{\mathbf{k}})$

$= a_x b_y \hat{\mathbf{k}} - a_x b_z \hat{\mathbf{j}} - a_y b_x \hat{\mathbf{k}} + a_y b_z \hat{\mathbf{i}} + a_z b_x \hat{\mathbf{j}} - a_z b_y \hat{\mathbf{i}}$

$= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$

We have used the elementary cross products in obtaining the above relation. The expression for $\mathbf{a} \times \mathbf{b}$ can be put in a determinant form which is easy to remember.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Example 6.4 *Find the scalar and vector products of two vectors.* $\mathbf{a} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ *and* $\mathbf{b} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$

Answer

$$\mathbf{a} \cdot \mathbf{b} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \cdot (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$
$$= -6 - 4 - 15$$
$$= -25$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix} = 7\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

Note $\mathbf{b} \times \mathbf{a} = -7\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ ◀

6.6 ANGULAR VELOCITY AND ITS RELATION WITH LINEAR VELOCITY

In this section we shall study what is angular velocity and its role in rotational motion. We have seen that every particle of a rotating body moves in a circle. The linear velocity of the particle is related to the angular velocity. The relation between these two quantities involves a vector product which we learnt about in the last section.

Let us go back to Fig. 6.4. As said above, in rotational motion of a rigid body about a fixed axis, every particle of the body moves in a circle,

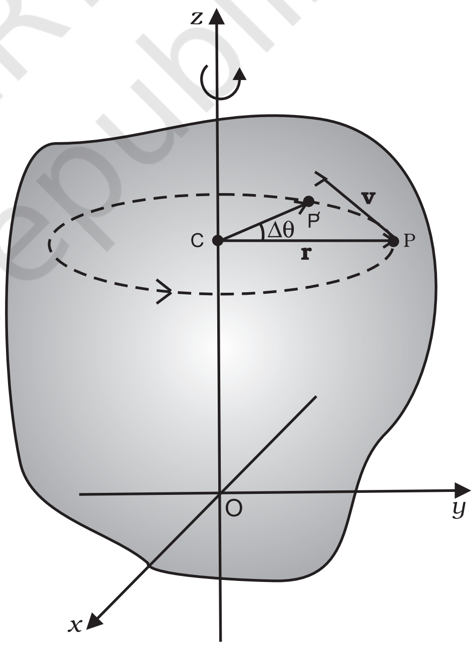


Fig. 6.16 *Rotation about a fixed axis. (A particle (P) of the rigid body rotating about the fixed (z-) axis moves in a circle with centre (C) on the axis.)*

which lies in a plane perpendicular to the axis and has its centre on the axis. In Fig. 6.16 we redraw Fig. 6.4, showing a typical particle (at a point P) of the rigid body rotating about a fixed axis (taken as the z-axis). The particle describes a circle with a centre C on the axis. The radius of the circle is r, the perpendicular distance of the point P from the axis. We also show the linear velocity vector **v** of the particle at P. It is along the tangent at P to the circle.

Let P′ be the position of the particle after an interval of time Δt (Fig. 6.16). The angle PCP′ describes the angular displacement $\Delta\theta$ of the particle in time Δt. The average angular velocity of the particle over the interval Δt is $\Delta\theta/\Delta t$. As Δt tends to zero (i.e. takes smaller and smaller values), the ratio $\Delta\theta/\Delta t$ approaches a limit which is the instantaneous angular velocity $d\theta/dt$ of the particle at the position P. We denote the **instantaneous angular velocity** by ω (the Greek letter omega). We know from our study of circular motion that the magnitude of linear velocity v of a particle moving in a circle is related to the angular velocity of the particle ω by the simple relation $\upsilon = \omega r$, where r is the radius of the circle.

We observe that at any given instant the relation $\upsilon = \omega r$ applies to all particles of the rigid body. Thus for a particle at a perpendicular distance r_i from the fixed axis, the linear velocity at a given instant v_i is given by

$$\upsilon_i = \omega r_i \qquad (6.19)$$

The index i runs from 1 to n, where n is the total number of particles of the body.

For particles on the axis, $r = 0$, and hence $v = \omega r = 0$. Thus, particles on the axis are stationary. This verifies that the axis is *fixed*.

Note that we use the same angular velocity ω for all the particles. **We therefore, refer to $\boldsymbol{\omega}$ as the angular velocity of the whole body**.

We have characterised pure translation of a body by all parts of the body having the same velocity at any instant of time. Similarly, we may characterise pure rotation by all parts of the body having the same angular velocity at any instant of time. Note that this characterisation of the rotation of a rigid body about a fixed axis is **just another way** of saying as in Sec. 6.1 that each particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has the centre on the axis.

In our discussion so far the angular velocity appears to be a scalar. In fact, it is a vector. We shall not justify this fact, but we shall accept it. For rotation about a fixed axis, the angular velocity vector lies along the axis of rotation, and points out in the direction in which a right handed screw would advance, if the head of the screw is rotated with the body. (See Fig. 6.17a).

The magnitude of this vector is $\omega = d\theta/dt$ referred as above.

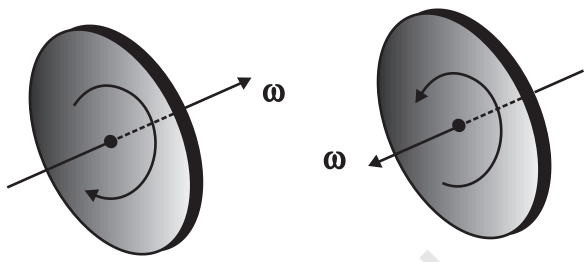


Fig. 6.17 (a) If the head of a right handed screw rotates with the body, the screw advances in the direction of the angular velocity $\boldsymbol{\omega}$. If the sense (clockwise or anticlockwise) of rotation of the body changes, so does the direction of $\boldsymbol{\omega}$.

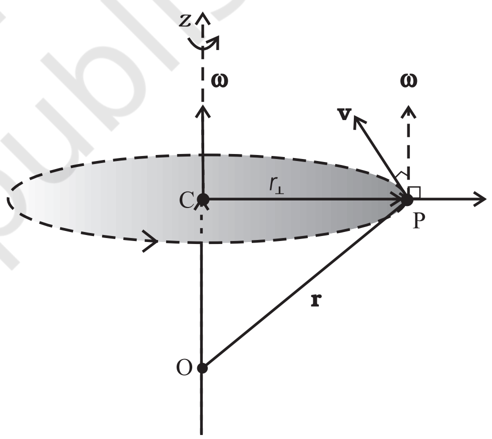


Fig. 6.17 (b) The angular velocity vector $\boldsymbol{\omega}$ is directed along the fixed axis as shown. The linear velocity of the particle at P is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. It is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$ and is directed along the tangent to the circle described by the particle.

We shall now look at what the vector product $\boldsymbol{\omega} \times \mathbf{r}$ corresponds to. Refer to Fig. 6.17(b) which is a part of Fig. 6.16 reproduced to show the path of the particle P. The figure shows the vector $\boldsymbol{\omega}$ directed along the fixed (z–) axis and also the position vector $\mathbf{r} = \mathbf{OP}$ of the particle at P of the rigid body with respect to the origin O. Note that the origin is chosen to be on the axis of rotation.

Now $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{OP} = \boldsymbol{\omega} \times (\mathbf{OC} + \mathbf{CP})$

But $\boldsymbol{\omega} \times \mathbf{OC} = 0$ as $\boldsymbol{\omega}$ is along $\mathbf{OC}$

Hence $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{CP}$

The vector $\boldsymbol{\omega} \times \mathbf{CP}$ is perpendicular to $\boldsymbol{\omega}$, i.e. to the z-axis and also to **CP**, the radius of the circle described by the particle at P. It is therefore, along the tangent to the circle at P. Also, the magnitude of $\boldsymbol{\omega} \times \mathbf{CP}$ is ω (CP) since $\boldsymbol{\omega}$ and **CP** are perpendicular to each other. We shall denote **CP** by $\mathbf{r}_\perp$ and not by **r**, as we did earlier.

Thus, $\boldsymbol{\omega} \times \mathbf{r}$ is a vector of magnitude $\omega r_\perp$ and is along the tangent to the circle described by the particle at P. The linear velocity vector **v** at P has the same magnitude and direction. Thus,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{6.20}$$

In fact, the relation, Eq. (6.20), holds good even for rotation of a rigid body with one point fixed, such as the rotation of the top [Fig. 6.6(a)]. In this case **r** represents the position vector of the particle with respect to the fixed point taken as the origin.

We note that **for rotation about a fixed axis, the direction of the vector $\boldsymbol{\omega}$ does not change with time. Its magnitude may, however, change from instant to instant. For the more general rotation, both the magnitude and the direction of $\boldsymbol{\omega}$ may change from instant to instant.**

6.6.1 Angular acceleration

You may have noticed that we are developing the study of rotational motion along the lines of the study of translational motion with which we are already familiar. Analogous to the kinetic variables of linear displacement (**s**) and velocity (**v**) in translational motion, we have angular displacement ($\boldsymbol{\theta}$) and angular velocity ($\boldsymbol{\omega}$) in rotational motion. It is then natural to define in rotational motion the concept of angular acceleration in analogy with linear acceleration defined as the time rate of change of velocity in translational motion. We define angular acceleration $\boldsymbol{\alpha}$ as the time rate of change of angular velocity. Thus,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} \tag{6.21}$$

If the axis of rotation is fixed, the direction of $\boldsymbol{\omega}$ and hence, that of $\boldsymbol{\alpha}$ is fixed. In this case the vector equation reduces to a scalar equation

$$\alpha = \frac{d\omega}{dt} \tag{6.22}$$

6.7 TORQUE AND ANGULAR MOMENTUM

In this section, we shall acquaint ourselves with two physical quantities (torque and angular momentum) which are defined as vector products of two vectors. These as we shall see, are especially important in the discussion of motion of systems of particles, particularly rigid bodies.

6.7.1 Moment of force (Torque)

We have learnt that the motion of a rigid body, in general, is a combination of rotation and translation. If the body is fixed at a point or along a line, it has only rotational motion. We know that force is needed to change the translational state of a body, i.e. to produce linear acceleration. We may then ask, what is the analogue of force in the case of rotational motion? To look into the question in a concrete situation let us take the example of opening or closing of a door. A door is a rigid body which can rotate about a fixed vertical axis passing through the hinges. What makes the door rotate? It is clear that unless a force is applied the door does not rotate. But any force does not do the job. A force applied to the hinge line cannot produce any rotation at all, whereas a force of given magnitude applied at right angles to the door at its outer edge is most effective in producing rotation. It is not the force alone, but how and where the force is applied is important in rotational motion.

The rotational analogue of force in linear motion is **moment of force**. It is also referred to as **torque** or **couple**. (We shall use the words moment of force and torque interchangeably.) We shall first define the moment of force for the special case of a single particle. Later on we shall extend the concept to systems of particles including rigid bodies. We shall also relate it to a change in the state of rotational motion, i.e. is angular acceleration of a rigid body.

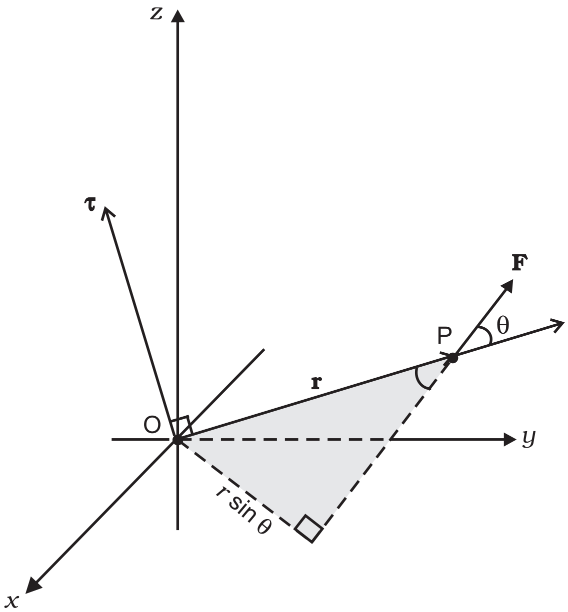


Fig. 6.18 *$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, τ is perpendicular to the plane containing* **r** *and* **F**, *and its direction is given by the right handed screw rule.*

If a force acts on a single particle at a point P whose position with respect to the origin O is given by the position vector **r** (Fig. 6.18), the moment of the force acting on the particle with respect to the origin O is defined as the vector product

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{6.23}$$

The moment of force (or torque) is a vector quantity. The symbol $\boldsymbol{\tau}$ stands for the Greek letter *tau*. The magnitude of $\boldsymbol{\tau}$ is

$$\tau = r F \sin\theta \tag{6.24a}$$

where r is the magnitude of the position vector **r**, i.e. the length OP, F is the magnitude of force **F** and θ is the angle between **r** and **F** as shown.

Moment of force has dimensions M L^2 T^{-2}. Its dimensions are the same as those of work or energy. It is, however, a very different physical quantity than work. Moment of a force is a vector, while work is a scalar. The SI unit of moment of force is newton metre (N m). The magnitude of the moment of force may be written

$$\tau = (r \sin\theta)F = r_\perp F \tag{6.24b}$$

$$\text{or } \tau = r F \sin\theta = rF_\perp \tag{6.24c}$$

where $r_\perp = r\sin\theta$ is the perpendicular distance of the line of action of **F** from the origin and $F_\perp (= F\sin\theta)$ is the component of **F** in the direction perpendicular to **r**. Note that $\tau = 0$ if $r = 0$, $F = 0$ or $\theta = 0^0$ or 180^0. Thus, the moment of a force vanishes if either the magnitude of the force is zero, or if the line of action of the force passes through the origin.

One may note that since $\mathbf{r} \times \mathbf{F}$ is a vector product, properties of a vector product of two vectors apply to it. If the direction of **F** is reversed, the direction of the moment of force is reversed. If directions of both **r** and **F** are reversed, the direction of the moment of force remains the same.

6.7.2 Angular momentum of a particle

Just as the moment of a force is the rotational analogue of force in linear motion, the quantity angular momentum is the rotational analogue of linear momentum. We shall first define angular momentum for the special case of a single particle and look at its usefulness in the context of single particle motion. We shall then extend the definition of angular momentum to systems of particles including rigid bodies.

Like moment of a force, angular momentum is also a vector product. It could also be referred to as moment of (linear) momentum. From this term one could guess how angular momentum is defined.

Consider a particle of mass m and linear momentum **p** at a position **r** relative to the origin O. The angular momentum ***l*** of the particle with respect to the origin O is defined to be

$$\boldsymbol{l} = \mathbf{r} \times \mathbf{p} \tag{6.25a}$$

The magnitude of the angular momentum vector is

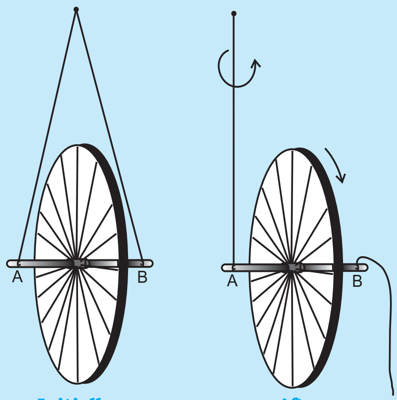
$$l = r\,p \sin\theta \tag{6.26a}$$

where p is the magnitude of **p** and θ is the angle between **r** and **p**. We may write

$$l = r\,p_\perp \text{ or } r_\perp p \tag{6.26b}$$

where $r_\perp$ $(= r\sin\theta)$ is the perpendicular distance of the directional line of **p** from the origin and $p_\perp (= p\sin\theta)$ is the component of **p** in a direction perpendicular to **r**. We expect the angular momentum to be zero ($l = 0$), if the linear momentum vanishes ($p = 0$), if the particle is at the origin ($r = 0$), or if the directional line of **p** passes through the origin $\theta = 0^0$ or 180^0.

An experiment with the bicycle rim



Initially After

Take a bicycle rim and extend its axle on both sides. Tie two strings at both ends A and B, as shown in the adjoining figure. Hold both the strings together in one hand such that the rim is vertical. If you leave one string, the rim will tilt. Now keeping the rim in vertical position with both the strings in one hand, put the wheel in fast rotation around the axle with the other hand. Then leave one string, say B, from your hand, and observe what happens.

The rim keeps rotating in a vertical plane and the plane of rotation turns around the string A which you are holding. We say that the axis of rotation of the rim or equivalently its angular momentum precesses about the string A.

The rotating rim gives rise to an angular momentum. Determine the direction of this angular momentum. When you are holding the rotating rim with string A, a torque is generated. (We leave it to you to find out how the torque is generated and what its direction is.) The effect of the torque on the angular momentum is to make it precess around an axis perpendicular to both the angular momentum and the torque. Verify all these statements.

The physical quantities, moment of a force and angular momentum, have an important relation between them. It is the rotational analogue of the relation between force and linear momentum. For deriving the relation in the context of a single particle, we differentiate $\boldsymbol{l} = \mathbf{r} \times \mathbf{p}$ with respect to time,

$$\frac{d\boldsymbol{l}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$

Applying the product rule for differentiation to the right hand side,

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

Now, the velocity of the particle is $\mathbf{v} = d\mathbf{r}/dt$ and $\mathbf{p} = m\,\mathbf{v}$

Because of this $\frac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{v} \times m\,\mathbf{v} = 0$,

as the vector product of two parallel vectors vanishes. Further, since $d\mathbf{p}/dt = \mathbf{F}$,

$$\mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{t}$$

Hence $\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \tau$

or $$\frac{d\boldsymbol{l}}{dt} = \tau \tag{6.27}$$

Thus, the time rate of change of the angular momentum of a particle is equal to the torque acting on it. This is the rotational analogue of the equation $\mathbf{F} = d\mathbf{p}/dt$, which expresses Newton's second law for the translational motion of a single particle.

Torque and angular momentum for a system of particles

To get the total angular momentum of a system of particles about a given point we need to add vectorially the angular momenta of individual particles. Thus, for a system of n particles,

$$\mathbf{L} = \boldsymbol{l}_1 + \boldsymbol{l}_2 + \ldots + \boldsymbol{l}_n = \sum_{i=1}^{n} \boldsymbol{l}_i$$

The angular momentum of the i^{th} particle is given by

$\boldsymbol{l}_i = \mathbf{r}_i \ \ \mathbf{p}_i$

where $\mathbf{r}_i$ is the position vector of the i^{th} particle with respect to a given origin and $\mathbf{p} = (m_i\mathbf{v}_i)$ is the linear momentum of the particle. (The particle has mass m_i and velocity $\mathbf{v}_i$) We may write the total angular momentum of a system of particles as

$$\mathbf{L} = \sum \boldsymbol{l}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i \tag{6.25b}$$

This is a generalisation of the definition of angular momentum (Eq. 6.25a) for a single particle to a system of particles.

Using Eqs. (6.23) and (6.25b), we get

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\sum \boldsymbol{l}_i\right) = \sum_i \frac{d\boldsymbol{l}}{dt} = \sum_i \tau_i \qquad (6.28a)$$

where $\boldsymbol{\tau}_i$ is the torque acting on the i^{th} particle;

$\tau_i = \mathbf{r}_i \times \mathbf{F}_i$

The force $\mathbf{F}_i$ on the i^{th} particle is the vector sum of external forces $\mathbf{F}_i^{ext}$ acting on the particle and the internal forces $\mathbf{F}_i^{\text{int}}$ exerted on it by the other particles of the system. We may therefore separate the contribution of the external and the internal forces to the total torque

$\tau = \sum_i \tau_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i$ as

$$\tau = \tau_{ext} + \tau_{\text{int}},$$

where $\tau_{ext} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext}$

and $\tau_{\text{int}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}}$

We shall assume not only Newton's third law of motion, i.e. the forces between any two particles of the system are equal and opposite, but also that these forces are directed along the line joining the two particles. In this case the contribution of the internal forces to the total torque on the system is zero, since the torque resulting from each action-reaction pair of forces is zero. We thus have, $\boldsymbol{\tau}_{\text{int}} = \mathbf{0}$ and therefore $\boldsymbol{\tau} = \boldsymbol{\tau}_{ext}$.

Since $\tau = \sum \tau_i$, it follows from Eq. (6.28a) that

$$\frac{d\mathbf{L}}{dt} = \tau_{ext} \qquad (6.28\text{ b})$$

Thus, **the time rate of the total angular momentum of a system of particles about a point** (taken as the origin of our frame of reference) **is equal to the sum of the external torques** (i.e. the torques due to external forces) **acting on the system taken about the same point**. Eq. (6.28 b) is the generalisation of the single particle case of Eq. (6.23) to a system of particles. Note that when we have only one particle, there are no internal forces or torques. Eq.(6.28 b) is the rotational analogue of

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{ext} \qquad (6.17)$$

Note that like Eq.(6.17), Eq.(6.28b) holds good for any system of particles, whether it is a rigid body or its individual particles have all kinds of internal motion.

Conservation of angular momentum

If $\boldsymbol{\tau}_{ext} = \mathbf{0}$, Eq. (6.28b) reduces to

$$\frac{d\mathbf{L}}{dt} = 0$$

or $\quad \mathbf{L}$ = constant. (6.29a)

Thus, if the total external torque on a system of particles is zero, then the total angular momentum of the system is conserved, i.e. remains constant. Eq. (6.29a) is equivalent to three scalar equations,

$L_x = K_1,\ L_y = K_2$ *and* $L_z = K_3$ (6.29 b)

Here K_1, K_2 and K_3 are constants; L_x, L_y and L_z are the components of the total angular momentum vector $\mathbf{L}$ along the *x,y* and *z* axes respectively. The statement that the total angular momentum is conserved means that each of these three components is conserved.

Eq. (6.29a) is the rotational analogue of Eq. (6.18a), i.e. the conservation law of the total linear momentum for a system of particles. Like Eq. (6.18a), it has applications in many practical situations. We shall look at a few of the interesting applications later on in this chapter.

Example 6.5 Find the torque of a force $7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ about the origin. The force acts on a particle whose position vector is $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Answer Here $\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$

and $\mathbf{F} = 7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$.

We shall use the determinant rule to find the torque $\tau = r \times F$

$$\tau = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{vmatrix} = (5-3)\hat{\mathbf{i}} - (-5-7)\hat{\mathbf{j}} + (3-(-7))\hat{\mathbf{k}}$$

or $\tau = 2\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$ ◀

Example 6.6 Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

Answer Let the particle with velocity **v** be at point P at some instant *t*. We want to calculate the angular momentum of the particle about an arbitrary point O.

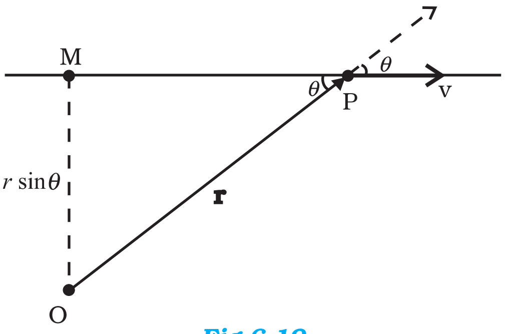


Fig 6.19

The angular momentum is **l** = **r** × *m***v**. Its magnitude is *mvr* sinθ, where θ is the angle between **r** and **v** as shown in Fig. 6.19. Although the particle changes position with time, the line of direction of **v** remains the same and hence OM = $r \sin \theta$. is a constant.

Further, the direction of **l** is perpendicular to the plane of **r** and **v**. It is into the page of the figure.This direction does not change with time.

Thus, **l** remains the same in magnitude and direction and is therefore conserved. Is there any external torque on the particle? ◀

6.8 EQUILIBRIUM OF A RIGID BODY

We are now going to concentrate on the motion of rigid bodies rather than on the motion of general systems of particles.

We shall recapitulate what effect the external forces have on a rigid body. (Henceforth we shall omit the adjective 'external' because unless stated otherwise, we shall deal with only external forces and torques.) The forces change the translational state of the motion of the rigid body, i.e. they change its total linear momentum in accordance with Eq. (6.17). But this is not the only effect the forces have. The total torque on the body may not vanish. Such a torque changes the rotational state of motion of the rigid body, i.e. it changes the total angular momentum of the body in accordance with Eq. (6.28 b).

A rigid body is said to be in mechanical equilibrium, if both its linear momentum and angular momentum are not changing with time, or equivalently, the body has neither linear acceleration nor angular acceleration. This means

(1) the total force, i.e. the vector sum of the forces, on the rigid body is zero;

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{0} \qquad (6.30a)$$

If the total force on the body is zero, then the total linear momentum of the body does not change with time. Eq. (6.30a) gives the condition for the translational equilibrium of the body.

(2) The total torque, i.e. the vector sum of the torques on the rigid body is zero,

$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \cdots + \boldsymbol{\tau}_n = \sum_{i=1}^{n} \boldsymbol{\tau}_i = \mathbf{0} \qquad (6.30b)$$

If the total torque on the rigid body is zero, the total angular momentum of the body does not change with time. Eq. (6.30 b) gives the condition for the rotational equilibrium of the body.

One may raise a question, whether the rotational equilibrium condition [Eq. 6.30(b)] remains valid, if the origin with respect to which the torques are taken is shifted. One can show that if the translational equilibrium condition [Eq. 6.30(a)] holds for a rigid body, then such a shift of origin does not matter, i.e. the rotational equilibrium condition is independent of the location of the origin about which the torques are taken. Example 6.7 gives a proof of this result in a special case of a couple, i.e. two forces acting on a rigid body in translational equilibrium. The generalisation of this result to *n* forces is left as an exercise.

Eq. (6.30a) and Eq. (6.30b), both, are vector equations. They are equivalent to three scalar equations each. Eq. (6.30a) corresponds to

$$\sum_{i=1}^{n} F_{ix} = 0\,,\ \sum_{i=1}^{n} F_{iy} = 0 \ \text{ and } \ \sum_{i=1}^{n} F_{iz} = 0 \qquad (6.31a)$$

where F_{ix}, F_{iy} and F_{iz} are respectively the *x, y* and *z* components of the forces $\mathbf{F}_i$. Similarly, Eq. (6.30b) is equivalent to three scalar equations

$$\sum_{i=1}^{n} \tau_{ix} = 0,\ \sum_{i=1}^{n} \tau_{iy} = 0 \ \text{ and } \sum_{i=1}^{n} \tau_{iz} = 0 \qquad (6.31b)$$

where τ_{ix}, τ_{iy} and τ_{iz} are respectively the *x, y* and *z* components of the torque $\boldsymbol{\tau}_i$.

Eq. (6.31a) and (6.31b) give six independent conditions to be satisfied for mechanical equilibrium of a rigid body. In a number of problems all the forces acting on the body are coplanar. Then we need only three conditions to be satisfied for mechanical equilibrium. Two of these conditions correspond to translational equilibrium; the sum of the components of the forces along any two perpendicular axes in the plane must be zero. The third condition corresponds to rotational equilibrium. The sum of the components of the torques along any axis perpendicular to the plane of the forces must be zero.

The conditions of equilibrium of a rigid body may be compared with those for a particle, which we considered in earlier chapters. Since consideration of rotational motion does not apply to a particle, only the conditions for translational equilibrium (Eq. 6.30 a) apply to a particle. Thus, for equilibrium of a particle the vector sum of all the forces on it must be zero. Since all these forces act on the single particle, they must be concurrent. Equilibrium under concurrent forces was discussed in the earlier chapters.

A body may be in partial equilibrium, i.e., it may be in translational equilibrium and not in rotational equilibrium, or it may be in rotational equilibrium and not in translational equilibrium.

Consider a light (i.e. of negligible mass) rod (AB) as shown in Fig. 6.20(a). At the two ends (A and B) of which two parallel forces, both equal in magnitude and acting along same direction are applied perpendicular to the rod.

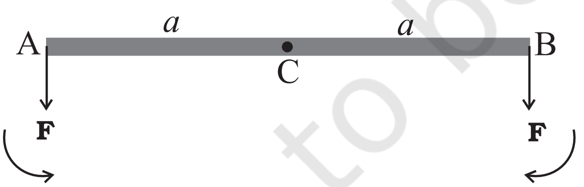


Fig. 6.20 (a)

Let C be the midpoint of AB, CA = CB = a. the moment of the forces at A and B will both be equal in magnitude (aF), but opposite in sense as shown. The net moment on the rod will be zero. The system will be in rotational equilibrium, but it will not be in translational equilibrium; $\sum \mathbf{F} \neq \mathbf{0}$

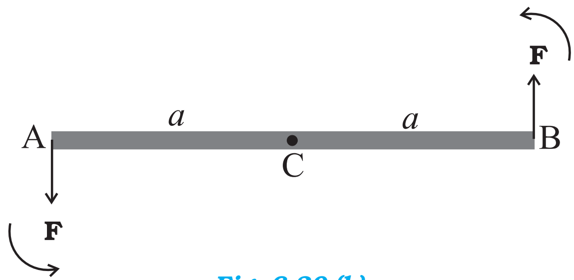


Fig. 6.20 (b)

The force at B in Fig. 6.20(a) is reversed in Fig. 6.20(b). Thus, we have the same rod with two forces of equal magnitude but acting in opposite diretions applied perpendicular to the rod, one at end A and the other at end B. Here the moments of both the forces are equal, but they are not opposite; they act in the same sense and cause anticlockwise rotation of the rod. The total force on the body is zero; so the body is in translational equilibrium; but it is not in rotational equilibrium. Although the rod is not fixed in any way, it undergoes pure rotation (i.e. rotation without translation).

A pair of forces of equal magnitude but acting in opposite directions with different lines of action is known as a **couple** or **torque**. A couple produces rotation without translation.

When we open the lid of a bottle by turning it, our fingers are applying a couple to the lid [Fig. 6.21(a)]. Another known example is a compass needle in the earth's magnetic field as shown in the Fig. 6.21(b). The earth's magnetic field exerts equal forces on the north and south poles. The force on the North Pole is towards the north, and the force on the South Pole is toward the south. Except when the needle points in the north-south direction; the two forces do not have the same line of action. Thus there is a **couple** acting on the needle due to the earth's magnetic field.

Fig. 6.21(a) *Our fingers apply a couple to turn the lid.*

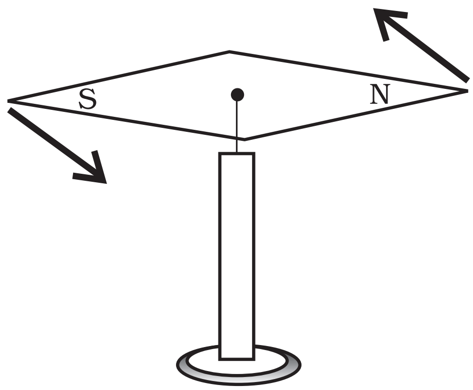


Fig. 6.21(b) *The Earth's magnetic field exerts equal and opposite forces on the poles of a compass needle. These two forces form a couple.*

Example 6.7 Show that moment of a couple does not depend on the point about which you take the moments.

Answer

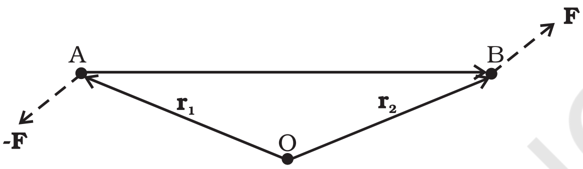


Fig. 6.22

Consider a couple as shown in Fig. 6.22 acting on a rigid body. The forces **F** and -**F** act respectively at points B and A. These points have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to origin O. Let us take the moments of the forces about the origin.

The moment of the couple = sum of the moments of the two forces making the couple

$= \mathbf{r}_1 \times (-\mathbf{F}) + \mathbf{r}_2 \times \mathbf{F}$

$= \mathbf{r}_2 \times \mathbf{F} - \mathbf{r}_1 \times \mathbf{F}$

$= (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}$

But $\mathbf{r}_1 + \mathbf{AB} = \mathbf{r}_2$, and hence $\mathbf{AB} = \mathbf{r}_2 - \mathbf{r}_1$.

The moment of the couple, therefore, is $\mathbf{AB} \times \mathbf{F}$.

Clearly this is independent of the origin, the point about which we took the moments of the forces. ◀

6.8.1 Principle of moments

An ideal lever is essentially a light (i.e. of negligible mass) rod pivoted at a point along its length. This point is called the fulcrum. A see-saw on the children's playground is a typical example of a lever. Two forces F_1 and F_2, parallel to each other and usually perpendicular to the lever, as shown here, act on the lever at distances d_1 and d_2 respectively from the fulcrum as shown in Fig. 6.23.

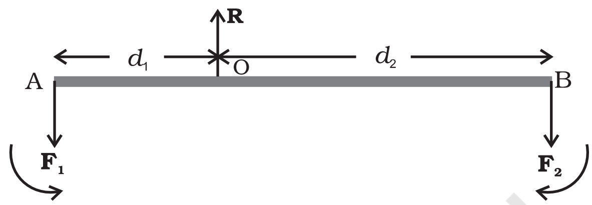


Fig. 6.23

The lever is a system in mechanical equilibrium. Let **R** be the reaction of the support at the fulcrum; **R** is directed opposite to the forces F_1 and F_2. For translational equilibrium,

$$R - F_1 - F_2 = 0 \qquad \text{(i)}$$

For considering rotational equilibrium we take the moments about the fulcrum; the sum of moments must be zero,

$$d_1 F_1 - d_2 F_2 = 0 \qquad \text{(ii)}$$

Normally the anticlockwise (clockwise) moments are taken to be positive (negative). Note R acts at the fulcrum itself and has zero moment about the fulcrum.

In the case of the lever force F_1 is usually some weight to be lifted. It is called the *load* and its distance from the fulcrum d_1 is called the *load arm*. Force F_2 is the *effort* applied to lift the load; distance d_2 of the effort from the fulcrum is the *effort arm*.

Eq. (ii) can be written as

$$d_1 F_1 = d_2 F_2 \qquad \text{(6.32a)}$$

or load arm × load = effort arm × effort

The above equation expresses the principle of moments for a lever. Incidentally the ratio F_1/F_2 is called the Mechanical Advantage (M.A.);

$$\text{M.A.} = \frac{F_1}{F_2} = \frac{d_2}{d_1} \qquad \text{(6.32b)}$$

If the effort arm d_2 is larger than the load arm, the mechanical advantage is greater than one. Mechanical advantage greater than one means that a small effort can be used to lift a large load. There are several examples of a lever around you besides the see-saw. The beam of a balance is a lever. Try to find more such examples and identify the fulcrum, the effort and effort arm, and the load and the load arm of the lever in each case.

You may easily show that the principle of moment holds even when the parallel forces F_1 and F_2 are not perpendicular, but act at some angle, to the lever.

6.8.2 Centre of gravity

Many of you may have the experience of balancing your notebook on the tip of a finger. Figure 6.24 illustrates a similar experiment that you can easily perform. Take an irregular-shaped cardboard having mass M and a narrow tipped object like a pencil. You can locate by trial and error a point G on the cardboard where it can be balanced on the tip of the pencil. (The cardboard remains horizontal in this position.) This point of balance is the centre of gravity (CG) of the cardboard. The tip of the pencil provides a vertically upward force due to which the cardboard is in mechanical equilibrium. As shown in the Fig. 6.24, the reaction of the tip is equal and opposite to $M\mathbf{g}$ and hence the cardboard is in translational equilibrium. It is also in rotational equilibrium; if it were not so, due to the unbalanced torque it would tilt and fall. There are torques on the card board due to the forces of gravity like $m_1\mathbf{g}$, $m_2\mathbf{g}$ …. etc, acting on the individual particles that make up the cardboard.

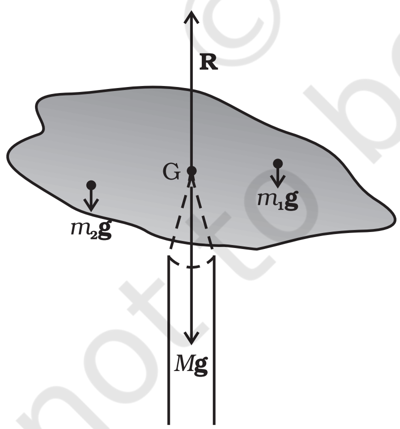


Fig. 6.24 *Balancing a cardboard on the tip of a pencil. The point of support, G, is the centre of gravity.*

The CG of the cardboard is so located that the total torque on it due to the forces $m_1\mathbf{g}$, $m_2\mathbf{g}$ …. etc. is zero.

If $\mathbf{r}_i$ is the position vector of the ith particle of an extended body with respect to its CG, then the torque about the CG, due to the force of gravity on the particle is $\boldsymbol{\tau}_i = \mathbf{r}_i \times m_i\mathbf{g}$. The total gravitational torque about the CG is zero, i.e.

$$\boldsymbol{\tau}_g = \sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times m_i\mathbf{g} = \mathbf{0} \qquad (6.33)$$

We may therefore, define the CG of a body as that point where the total gravitational torque on the body is zero.

We notice that in Eq. (6.33), $\mathbf{g}$ is the same for all particles, and hence it comes out of the summation. This gives, since $\mathbf{g}$ is non-zero, $\sum m_i\mathbf{r}_i = \mathbf{0}$. Remember that the position vectors ($\mathbf{r}_i$) are taken with respect to the CG. Now, in accordance with the reasoning given below Eq. (6.4a) in Sec. 6.2, if the sum is zero, the origin must be the centre of mass of the body. Thus, the centre of gravity of the body coincides with the centre of mass in uniform gravity or gravity-

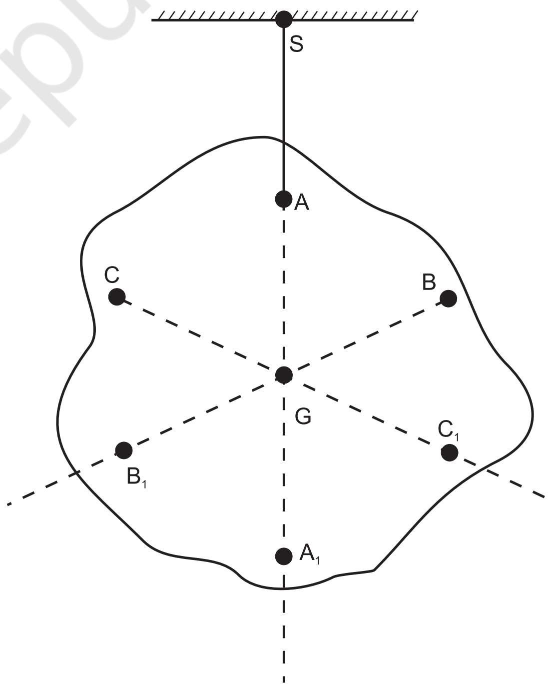


Fig. 6.25 *Determining the centre of gravity of a body of irregular shape. The centre of gravity G lies on the vertical AA_1 through the point of suspension of the body A.*

free space. We note that this is true because the body being small, **g** does not vary from one point of the body to the other. If the body is so extended that **g** varies from part to part of the body, then the centre of gravity and centre of mass will not coincide. Basically, the two are different concepts. The centre of mass has nothing to do with gravity. It depends only on the distribution of mass of the body.

In Sec. 6.2 we found out the position of the centre of mass of several regular, homogeneous objects. Obviously the method used there gives us also the centre of gravity of these bodies, if they are small enough.

Figure 6.25 illustrates another way of determining the CG of an irregular shaped body like a cardboard. If you suspend the body from some point like A, the vertical line through A passes through the CG. We mark the vertical AA_1. We then suspend the body through other points like B and C. The intersection of the verticals gives the CG. Explain why the method works. Since the body is small enough, the method allows us to determine also its centre of mass.

Example 6.8 A metal bar 70 cm long and 4.00 kg in mass supported on two knife-edges placed 10 cm from each end. A 6.00 kg load is suspended at 30 cm from one end. Find the reactions at the knife-edges. (Assume the bar to be of uniform cross section and homogeneous.)

Answer

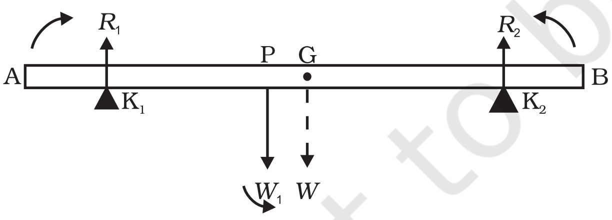


Fig. 6.26

Figure 6.26 shows the rod AB, the positions of the knife edges K_1 and K_2 , the centre of gravity of the rod at G and the suspended load at P.

Note the weight of the rod W acts at its centre of gravity G. The rod is uniform in cross section and homogeneous; hence G is at the centre of the rod; AB = 70 cm. AG = 35 cm, AP = 30 cm, PG = 5 cm, $AK_1 = BK_2$ = 10 cm and $K_1G = K_2G$ = 25 cm. Also, W= weight of the rod = 4.00 kg and W_1= suspended load = 6.00 kg; R_1 and R_2 are the normal reactions of the support at the knife edges.

For translational equilibrium of the rod,

$R_1 + R_2 - W_1 - W = 0$ (i)

Note W_1 and W act vertically down and R_1 and R_2 act vertically up.

For considering rotational equilibrium, we take moments of the forces. A convenient point to take moments about is G. The moments of R_2 and W_1 are anticlockwise (+ve), whereas the moment of R_1 is clockwise (-ve).

For rotational equilibrium,

$-R_1\ (K_1G) + W_1\ (PG) + R_2\ (K_2G) = 0$ (ii)

It is given that $W = 4.00g$ N and $W_1 = 6.00g$ N, where g = acceleration due to gravity. We take $g = 9.8$ m/s^2.

With numerical values inserted, from (i)

$R_1 + R_2 - 4.00g - 6.00g = 0$

or $R_1 + R_2 = 10.00g$ N (iii)

= 98.00 N

From (ii), $-\,0.25\ R_1 + 0.05\ W_1 + 0.25\ R_2 = 0$

or $R_1 - R_2 = 1.2g$ N = 11.76 N (iv)

From (iii) and (iv), R_1 = 54.88 N,

R_2 = 43.12 N

Thus the reactions of the support are about 55 N at K_1 and 43 N at K_2.

Example 6.9 A 3m long ladder weighing 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in Fig.6.27. Find the reaction forces of the wall and the floor.

Answer

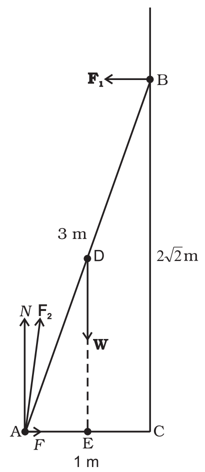


Fig. 6.27

The ladder AB is 3 m long, its foot A is at distance AC = 1 m from the wall. From Pythagoras theorem, BC = $2\sqrt{2}$ m. The forces on the ladder are its weight W acting at its centre of gravity D, reaction forces F_1 and F_2 of the wall and the floor respectively. Force F_1 is perpendicular to the wall, since the wall is frictionless. Force F_2 is resolved into two components, the normal reaction N and the force of friction F. Note that F prevents the ladder from sliding away from the wall and is therefore directed toward the wall.

For translational equilibrium, taking the forces in the vertical direction,

$$N - W = 0 \qquad \text{(i)}$$

Taking the forces in the horizontal direction,

$$F - F_1 = 0 \qquad \text{(ii)}$$

For rotational equilibrium, taking the moments of the forces about A,

$$2\sqrt{2}\,F_1 - (1/2)\,W = 0 \qquad \text{(iii)}$$

Now $\quad W = 20\text{ g} = 20 \quad 9.8\text{ N} = 196.0\text{ N}$

From (i) $\quad N = 196.0\text{ N}$

From (iii) $\quad F_1 = W/4\sqrt{2} = 196.0/4\sqrt{2} = 34.6\text{ N}$

From (ii) $\quad F = F_1 = 34.6\text{ N}$

$$F_2 = \sqrt{F^2 + N^2} = 199.0\text{ N}$$

The force F_2 makes an angle α with the horizontal,

$$\tan\alpha = N/F = 4\sqrt{2}\,, \quad \alpha = \tan^{-1}(4\sqrt{2}) \approx 80^\circ$$

6.9 MOMENT OF INERTIA

We have already mentioned that we are developing the study of rotational motion parallel to the study of translational motion with which we are familiar. We have yet to answer one major question in this connection. **What is the analogue of mass in rotational motion?** We shall attempt to answer this question in the present section. To keep the discussion simple, we shall consider rotation about a fixed axis only. Let us try to get an expression for *the* **kinetic energy of a rotating body**. We know that for a body rotating about a fixed axis, each particle of the body moves in a circle with linear velocity given by Eq. (6.19). (Refer to Fig. 6.16). For a particle at a distance from the axis, the linear velocity is $\upsilon_i = r_i\omega$. The kinetic energy of motion of this particle is

$$k_i = \frac{1}{2}m_i\upsilon_i^2 = \frac{1}{2}m_i r_i^2\omega^2$$

where m_i is the mass of the particle. The total kinetic energy K of the body is then given by the sum of the kinetic energies of individual particles,

$$K = \sum_{i=1}^{n} k_i = \frac{1}{2}\sum_{i=1}^{n}(m_i r_i^2\omega^2)$$

Here n is the number of particles in the body. Note ω is the same for all particles. Hence, taking ω out of the sum,

$$K = \frac{1}{2}\omega^2\left(\sum_{i=1}^{n} m_i r_i^2\right)$$

We define a new parameter characterising the rigid body, called the moment of inertia I, given by

$$I = \sum_{i=1}^{n} m_i r_i^2 \qquad (6.34)$$

With this definition,

$$K = \frac{1}{2}I\omega^2 \qquad (6.35)$$

Note that the parameter I is independent of the magnitude of the angular velocity. It is a characteristic of the rigid body and the axis about which it rotates.

Compare Eq. (6.35) for the kinetic energy of a rotating body with the expression for the kinetic energy of a body in linear (translational) motion,

$$K = \frac{1}{2}m\,\upsilon^2$$

Here, m is the mass of the body and v is its velocity. We have already noted the analogy between angular velocity ω (in respect of rotational motion about a fixed axis) and linear velocity v (in respect of linear motion). It is then evident that the parameter, moment of inertia I, is the desired rotational analogue of mass in linear motion. In rotation (about a fixed axis), the moment of inertia plays a similar role as mass does in linear motion.

We now apply the definition Eq. (6.34), to calculate the moment of inertia in two simple cases.

(a) Consider a thin ring of radius R and mass M, rotating in its own plane around its centre with angular velocity ω. Each mass element of the ring is at a distance R from the axis, and moves with a speed $R\omega$. The kinetic energy is therefore,

$$K = \frac{1}{2}M\upsilon^2 = \frac{1}{2}MR^2\omega^2$$

Comparing with Eq. (6.35) we get $I = MR^2$ for the ring.

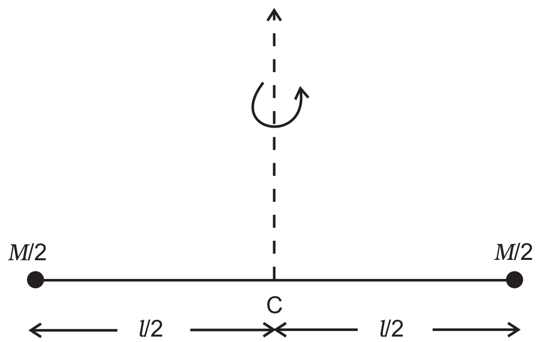


Fig. 6.28 *A light rod of length l with a pair of masses rotating about an axis through the centre of mass of the system and perpendicular to the rod. The total mass of the system is M.*

(b) Next, take a rigid rod of negligible mass of length of length l with a pair of small masses, rotating about an axis through the centre of mass perpendicular to the rod (Fig. 6.28). Each mass $M/2$ is at a distance $l/2$ from the axis. The moment of inertia of the masses is therefore given by

$$(M/2)\,(l/2)^2 + (M/2)(l/2)^2$$

Thus, for the pair of masses, rotating about the axis through the centre of mass perpendicular to the rod

$$I = Ml^2 / 4$$

Table 6.1 simply gives the moment of inertia of various familiar regular shaped bodies about specific axes. (The derivations of these expressions are beyond the scope of this textbook and you will study them in higher classes.)

As the mass of a body resists a change in its state of linear motion, it is a measure of its inertia in linear motion. Similarly, as the moment of inertia about a given axis of rotation resists a change in its rotational motion, it can be regarded as a measure of rotational inertia of the body; it is a measure of the way in which different parts of the body are distributed at different distances from the axis. Unlike the mass of a body, the moment of inertia is not a fixed quantity but depends on distribution of mass about the axis of rotation, and the orientation and position of the axis of rotation with respect to the body as a whole. As a measure of the way in which the mass of a rotating rigid body is distributed with respect to the axis of rotation, we can define a new parameter, the **radius of gyration**. It is related to the moment of inertia and the total mass of the body.

Notice from the Table 6.1 that in all cases, we can write I = Mk^2, where k has the dimension of length. For a rod, about the perpendicular axis at its midpoint, $k^2 = L^2/12$, i.e. $k = L/\sqrt{12}$. Similarly, $k = R/2$ for the circular disc about its diameter. The length k is a geometric property of the body and axis of rotation. It is called the **radius of gyration**. **The radius of gyration of a body about an axis** may be defined as the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

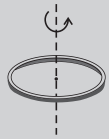
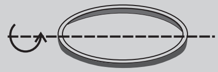
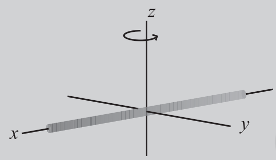
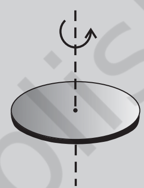
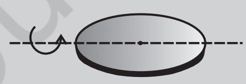
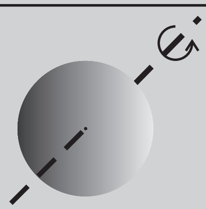
Thus, the moment of inertia of a rigid body depends on the mass of the body, its shape and size; distribution of mass about the axis of rotation, and the position and orientation of the axis of rotation.

From the definition, Eq. (6.34), we can infer that the dimensions of moments of inertia are ML^2 and its SI units are kg m^2.

The property of this extremely important quantity I, as a measure of rotational inertia of the body, has been put to a great practical use. The machines, such as steam engine and the automobile engine, etc., that produce rotational motion have a disc with a large moment of inertia, called a **flywheel**. Because of its large moment of inertia, the flywheel resists the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motions, thereby ensuring a smooth ride for the passengers on the vehicle.

Reprint 2025-26

Table 6.1 **Moments of inertia of some regular shaped bodies about specific axes**

Z	Body	Axis	Figure	I
(1)	Thin circular ring, radius R	Perpendicular to plane, at centre		MR^2
(2)	Thin circular ring, radius R	Diameter		$MR^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$ML^2/12$
(4)	Circular disc, radius R	Perpendicular to disc at centre		$MR^2/2$
(5)	Circular disc, radius R	Diameter		$MR^2/4$
(6)	Hollow cylinder, radius R	Axis of cylinder		MR^2
(7)	Solid cylinder, radius R	Axis of cylinder		$MR^2/2$
(8)	Solid sphere, radius R	Diameter		$2MR^2/5$

6.10 KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

We have already indicated the analogy between rotational motion and translational motion. For example, the angular velocity $\boldsymbol{\omega}$ plays the same role in rotation as the linear velocity $\mathbf{v}$ in translation. We wish to take this analogy further. In doing so we shall restrict the discussion only to rotation about fixed axis. This case of motion involves only one degree of freedom, i.e., needs only one independent variable to describe the motion. This in translation corresponds to linear motion. This section is limited only to kinematics. We shall turn to dynamics in later sections.

We recall that for specifying the angular displacement of the rotating body we take any particle like P (Fig.6.29) of the body. Its angular displacement θ in the plane it moves is the angular displacement of the whole body; θ is measured from a fixed direction in the plane of motion of P, which we take to be the x'-axis, chosen parallel to the x-axis. Note, as shown, the axis of rotation is the z – axis and the plane of the motion of the particle is the x - y plane. Fig. 6.29 also shows θ_0, the angular displacement at $t = 0$.

We also recall that the angular velocity is the time rate of change of angular displacement, $\omega = d\theta/dt$. Note since the axis of rotation is fixed, there is no need to treat angular velocity as a vector. Further, the angular acceleration, $\alpha = d\omega/dt$.

The kinematical quantities in rotational motion, angular displacement (θ), angular velocity (ω) and angular acceleration (α) respectively are analogous to kinematic quantities in linear motion, displacement (x), velocity (v) and acceleration (a). We know the kinematical equations of linear motion with uniform (i.e. constant) acceleration:

$$v = v_0 + at \qquad \text{(a)}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \text{(b)}$$

$$v^2 = v_0^2 + 2ax \qquad \text{(c)}$$

where x_0 = initial displacement and v_0= initial velocity. The word 'initial' refers to values of the quantities at $t = 0$

The corresponding kinematic equations for rotational motion with uniform angular acceleration are:

$$\omega = \omega_0 + \alpha t \qquad (6.36)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \qquad (6.37)$$

$$\text{and } \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad (6.38)$$

where θ_0= initial angular displacement of the rotating body, and ω_0 = initial angular velocity of the body.

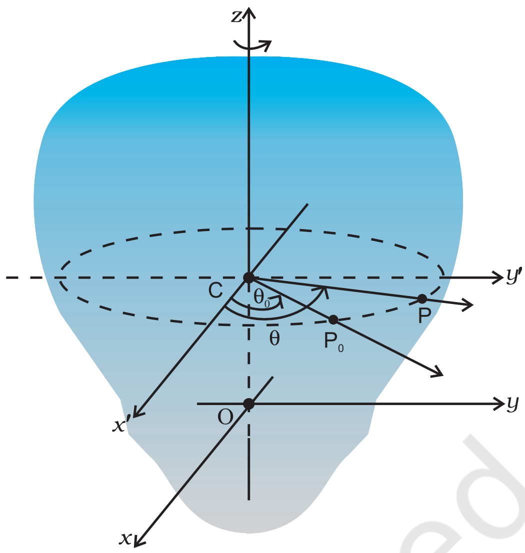


Fig.6.29 *Specifying the angular position of a rigid body.*

▶ ***Example 6.10*** Obtain Eq. (6.36) from first principles.

Answer The angular acceleration is uniform, hence

$$\frac{d\omega}{dt} = \alpha = constant \qquad \text{(i)}$$

Integrating this equation,

$$\omega = \int \alpha\, dt + c$$

$$= \alpha t + c \qquad (\text{as } \alpha \text{ is constant})$$

At $t = 0$, $\omega = \omega_0$ (given)

From (i) we get at $t = 0$, $\omega = c = \omega_0$

Thus, $\omega = \alpha t + \omega_0$ as required.

With the definition of $\omega = d\theta/dt$ we may integrate Eq. (6.36) to get Eq. (6.37). This derivation and the derivation of Eq. (6.38) is left as an exercise.

▶ ***Example 6.11*** The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform? (ii) How many revolutions does the engine make during this time?

Answer

(i) We shall use $\omega = \omega_0 + \alpha t$

ω_0 = initial angular speed in rad/s

$$= 2\pi \times \text{angular speed in rev/s}$$

$$= \frac{2\pi \times \text{angular speed in rev/min}}{60 \text{ s/min}}$$

$$= \frac{2\pi \times 1200}{60} \text{rad/s}$$

$$= 40\pi \text{ rad/s}$$

Similarly ω = final angular speed in rad/s

$$= \frac{2\pi \times 3120}{60} \text{rad/s}$$

$$= 2\pi \times 52 \text{ rad/s}$$

$$= 104\,\pi \text{ rad/s}$$

$\therefore$ Angular acceleration

$$\alpha = \frac{\omega - \omega_0}{t} \qquad = 4\,\pi \text{ rad/s}^2$$

The angular acceleration of the engine $= 4\pi$ rad/s^2

(ii) The angular displacement in time t is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= (40\pi \times 16 + \frac{1}{2} \times 4\pi \times 16^2) \text{ rad}$$

$$= (640\pi + 512\pi) \text{ rad}$$

$$= 1152\pi \text{ rad}$$

Number of revolutions $= \dfrac{1152\pi}{2\pi} = 576$ ◀

6.11 DYNAMICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

Table 6.2 lists quantities associated with linear motion and their analogues in rotational motion. We have already compared kinematics of the two motions. Also, we know that in rotational motion moment of inertia and torque play the same role as mass and force respectively in linear motion. Given this we should be able to guess what the other analogues indicated in the table are. For example, we know that in linear motion, work done is given by $F\,dx$, in rotational motion about a fixed axis it should be $\tau d\theta$, since we already know the correspondence $dx \rightarrow d\theta$ and $F \rightarrow \tau$. It is, however, necessary that these correspondences are established on sound dynamical considerations. This is what we now turn to.

Before we begin, we note *a* **simplification that arises in the case of rotational motion about a fixed axis**. Since the axis is fixed, only those components of torques, which are along the direction of the fixed axis need to be considered in our discussion. Only these components can cause the body to rotate about the axis. A component of the torque perpendicular to the axis of rotation will tend to turn the axis from its position. We specifically assume that there will arise necessary forces of constraint to cancel the effect of the perpendicular components of the (external) torques, so that the fixed position of the axis will be maintained. The perpendicular components of the torques, therefore need not be taken into account. This means that for our calculation of torques on a rigid body:

(1) We need to consider only those forces that lie in planes perpendicular to the axis. Forces which are parallel to the axis will give torques perpendicular to the axis and need not be taken into account.

(2) We need to consider only those components of the position vectors which are perpendicular to the axis. Components of position vectors along the axis will result in torques perpendicular to the axis and need not be taken into account.

Work done by a torque

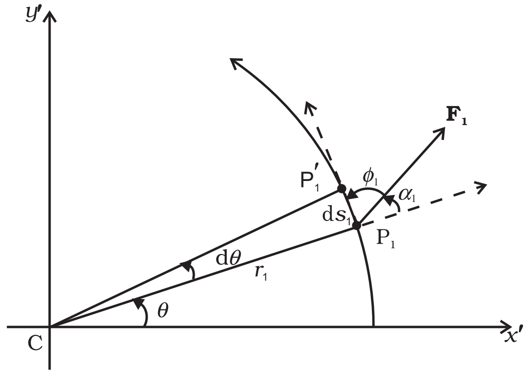


Fig. 6.30 *Work done by a force* $\mathbf{F}_1$ *acting on a particle of a body rotating about a fixed axis; the particle describes a circular path with centre C on the axis; arc* $P_1P'_1(ds_1)$ *gives the displacement of the particle.*

Table 6.2 Comparison of Translational and Rotational Motion

	Linear Motion	Rotational Motion about a Fixed Axis
1	Displacement x	Angular displacement θ
2	Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
3	Acceleration $a = dv/dt$	Angular acceleration $\alpha = d\omega/dt$
4	Mass M	Moment of inertia I
5	Force $F = Ma$	Torque $\tau = I\,\alpha$
6	Work $dW = F\,ds$	Work $W = \tau\, d\theta$
7	Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
8	Power $P = F\,v$	Power $P = \tau\omega$
9	Linear momentum $p = Mv$	Angular momentum $L = I\omega$

Figure 6.30 shows a cross-section of a rigid body rotating about a fixed axis, which is taken as the z-axis (perpendicular to the plane of the page; see Fig. 6.29). As said above we need to consider only those forces which lie in planes perpendicular to the axis. Let $\mathbf{F}_1$ be one such typical force acting as shown on a particle of the body at point P_1 with its line of action in a plane perpendicular to the axis. For convenience we call this to be the x'–y' plane (coincident with the plane of the page). The particle at P_1 describes a circular path of radius r_1 with centre C on the axis; $CP_1 = r_1$.

In time Δt, the point moves to the position P_1'. The displacement of the particle $d\mathbf{s}_1$, therefore, has magnitude $ds_1 = r_1 d\theta$ and direction tangential at P_1 to the circular path as shown. Here $d\theta$ is the angular displacement of the particle, $d\theta = \angle P_1CP_1'$ .The work done by the force on the particle is

$$dW_1 = \mathbf{F}_1 .\, d\mathbf{s}_1 = F_1 ds_1 \cos\phi_1 = F_1(r_1\, d\theta)\sin\alpha_1$$

where ϕ_1 is the angle between $\mathbf{F}_1$ and the tangent at P_1, and α_1 is the angle between $\mathbf{F}_1$ and the radius vector $\mathbf{OP}_1$; $\phi_1 + \alpha_1 = 90°$.

The torque due to $\mathbf{F}_1$ about the origin is $\mathbf{OP}_1 \times \mathbf{F}_1$. Now $\mathbf{OP}_1 = \mathbf{OC} + \mathbf{OP}_1$. [Refer to Fig. 6.17(b).] Since $\mathbf{OC}$ is along the axis, the torque resulting from it is excluded from our consideration. The effective torque due to $\mathbf{F}_1$ is $\boldsymbol{\tau}_1 = \mathbf{CP} \times \mathbf{F}_1$; it is directed along the axis of rotation and has a magnitude $\tau_1 = r_1 F_1 \sin\alpha$, Therefore,

$$dW_1 = \tau_1 d\theta$$

If there are more than one forces acting on the body, the work done by all of them can be added to give the total work done on the body. Denoting the magnitudes of the torques due to the different forces as $\tau_1, \tau_2, \ldots$ etc,

$$dW = (\tau_1 + \tau_2 + \ldots)d\theta$$

Remember, the forces giving rise to the torques act on different particles, but the angular displacement $d\theta$ is the same for all particles. Since all the torques considered are parallel to the fixed axis, the magnitude τ of the total torque is just the algebraic sum of the magnitudes of the torques, i.e., $\tau = \tau_1 + \tau_2 + \ldots..$ We, therefore, have

$$dW = \tau d\theta \qquad (6.39)$$

This expression gives the work done by the total (external) torque τ which acts on the body rotating about a fixed axis. Its similarity with the corresponding expression

$$dW = F ds$$

for linear (translational) motion is obvious.

Dividing both sides of Eq. (6.39) by dt gives

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

or $P = \tau\omega$ (6.40)

This is the instantaneous power. Compare this expression for power in the case of rotational motion about a fixed axis with that of power in the case of linear motion,

$$P = Fv$$

In a perfectly rigid body there is no internal motion. The work done by external torques is therefore, not dissipated and goes on to increase the kinetic energy of the body. The rate at which work is done on the body is given by Eq. (6.40). This is to be equated to the rate at which kinetic energy increases. The rate of increase of kinetic energy is

$$\frac{d}{dt}\left(\frac{I\omega^2}{2}\right) = I\frac{(2\omega)}{2}\frac{d\omega}{dt}$$

We assume that the moment of inertia does not change with time. This means that the mass of the body does not change, the body remains rigid and also the axis does not change its position with respect to the body.

Since $\alpha = d\omega/dt$, we get

$$\frac{d}{dt}\left(\frac{I\omega^2}{2}\right) = I\,\omega\,\alpha$$

Equating rates of work done and of increase in kinetic energy,

$$\tau\omega = I\,\omega\,\alpha$$

$$\tau = I\alpha \qquad (6.41)$$

Eq. (6.41) is similar to Newton's second law for linear motion expressed symbolically as

$F = ma$

Just as force produces acceleration, torque produces angular acceleration in a body. The angular acceleration is directly proportional to the applied torque and is inversely proportional to the moment of inertia of the body. In this respect, Eq.(6.41) can be called Newton's second law for rotational motion about a fixed axis.

Example 6.12 A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. 6.31. The flywheel is mounted on a horizontal axle with frictionless bearings.

(a) Compute the angular acceleration of the wheel.

(b) Find the work done by the pull, when 2m of the cord is unwound.

(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(d) Compare answers to parts (b) and (c).

Answer

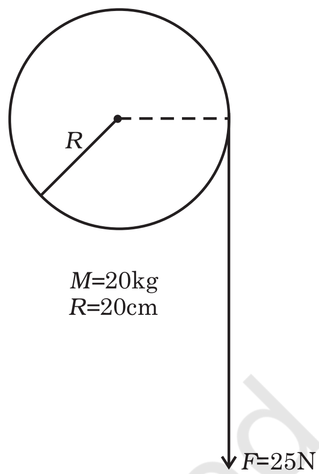


Fig. 6.31

(a) We use $I\alpha = \tau$

the torque $\tau = F R$

$= 25 \times 0.20$ Nm (as $R = 0.20$m)

$= 5.0$ Nm

I = Moment of inertia of flywheel about its axis $= \dfrac{MR^2}{2}$

$= \dfrac{20.0 \times (0.2)^2}{2} = 0.4$ kg m^2

α = angular acceleration

$= 5.0$ N m/0.4 kg m^2 = 12.5 s^{-2}

(b) Work done by the pull unwinding 2m of the cord

= 25 N × 2m = 50 J

(c) Let ω be the final angular velocity. The kinetic energy gained = $\dfrac{1}{2}I\omega^2$, since the wheel starts from rest. Now,

$\omega^2 = \omega_0^2 + 2\alpha\theta, \quad \omega_0 = 0$

The angular displacement θ = length of unwound string / radius of wheel

= 2m/0.2 m = 10 rad

$\omega^2 = 2 \times 12.5 \times 10.0 = 250\,(\text{rad/s})^2$

$\therefore$ K.E. gained $= \dfrac{1}{2} \times 0.4 \times 250 = 50$ J

(d) The answers are the same, i.e. the kinetic energy gained by the wheel = work done by the force. There is no loss of energy due to friction.

6.12 ANGULAR MOMENTUM IN CASE OF ROTATION ABOUT A FIXED AXIS

We have studied in section 6.7, the angular momentum of a system of particles. We already know from there that the time rate of total angular momentum of a system of particles about a point is equal to the total external torque on the system taken about the same point. When the total external torque is zero, the total angular momentum of the system is conserved.

We now wish to study the angular momentum in the special case of rotation about a fixed axis. The general expression for the total angular momentum of the system of n particles is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \qquad (6.25b)$$

We first consider the angular momentum of a typical particle of the rotating rigid body. We then sum up the contributions of individual particles to get **L** of the whole body.

For a typical particle $\boldsymbol{l} = \mathbf{r} \times \mathbf{p}$. As seen in the last section $\mathbf{r} = \mathbf{OP} = \mathbf{OC} + \mathbf{CP}$ [Fig. 6.17(b)]. With $\mathbf{p} = m\,\mathbf{v}$,

$$\boldsymbol{l} = (\mathbf{OC} \times m\,\mathbf{v}) + (\mathbf{CP} \times m\,\mathbf{v})$$

The magnitude of the linear velocity **v** of the particle at P is given by $v = \omega r_\perp$ where $r_\perp$ is the length of CP or the perpendicular distance of P from the axis of rotation. Further, **v** is tangential at P to the circle which the particle describes. Using the right-hand rule one can check that $\mathbf{CP} \times \mathbf{v}$ is parallel to the fixed axis. The unit vector along the fixed axis (chosen as the z-axis) is $\hat{\mathbf{k}}$. Hence

$$\mathbf{CP} \times m\,\mathbf{v} = r_\perp (mv)\,\hat{\mathbf{k}}$$

$$= mr_\perp^2 \omega\,\hat{\mathbf{k}} \quad (\text{since } \upsilon = \omega r_\perp)$$

Similarly, we can check that $\mathbf{OC} \times \mathbf{v}$ is perpendicular to the fixed axis. Let us denote the part of $\boldsymbol{l}$ along the fixed axis (i.e. the z-axis) by $\boldsymbol{l}_z$, then

$$\boldsymbol{l}_z = \mathbf{CP} \times m\,\mathbf{v} = mr_\perp^2 \omega\,\hat{\mathbf{k}}$$

and $\boldsymbol{l} = \boldsymbol{l}_z + \mathbf{OC} \times m\,\mathbf{v}$

We note that $\boldsymbol{l}_z$ is parallel to the fixed axis, but $\boldsymbol{l}$ is not. In general, for a particle, the angular momentum $\boldsymbol{l}$ is not along the axis of rotation, i.e. for a particle, $\boldsymbol{l}$ and $\boldsymbol{\omega}$ are not necessarily parallel. Compare this with the corresponding fact in translation. For a particle, **p** and **v** are always parallel to each other.

For computing the total angular momentum of the whole rigid body, we add up the contribution of each particle of the body.

Thus $\quad \mathbf{L} = \sum \boldsymbol{l}_i = \sum \boldsymbol{l}_{iz} + \sum \mathbf{OC}_i \times m_i \mathbf{v}_i$

We denote by $\mathbf{L}_\perp$ and $\mathbf{L}_z$ the components of $\mathbf{L}$ respectively perpendicular to the z-axis and along the z-axis;

$$\mathbf{L}_\perp = \sum \mathbf{OC}_i \times m_i \mathbf{v}_i \qquad (6.42a)$$

where m_i and $\mathbf{v}_i$ are respectively the mass and the velocity of the i^{th} particle and C_i is the centre of the circle described by the particle;

and $\quad \mathbf{L}_z = \sum \boldsymbol{l}_{iz} = \left(\sum_i m_i r_i^2 \right) w\,\mathbf{k}$

or $\quad \mathbf{L}_z = I\omega\,\hat{\mathbf{k}} \qquad (6.42b)$

The last step follows since the perpendicular distance of the i^{th} particle from the axis is r_i; and by definition the moment of inertia of the body about the axis of rotation is $I = \sum m_i r_i^2$.

Note $\quad \mathbf{L} = \mathbf{L}_z + \mathbf{L}_\perp \qquad (6.42c)$

The rigid bodies which we have mainly considered in this chapter are symmetric about the axis of rotation, i.e. the axis of rotation is one of their symmetry axes. For such bodies, for a given $\mathbf{OC}_i$, for every particle which has a velocity $\mathbf{v}_i$, there is another particle of velocity $-\mathbf{v}_i$ located diametrically opposite on the circle with centre C_i described by the particle. Together such pairs will contribute zero to $\mathbf{L}_\perp$ and as a result for symmetric bodies $\mathbf{L}_\perp$ is zero, and hence

$$\mathbf{L} = \mathbf{L}_z = I\omega\hat{\mathbf{k}} \qquad (6.42d)$$

For bodies, which are not symmetric about the axis of rotation, **L** is not equal to $\mathbf{L}_z$ and hence **L** does not lie along the axis of rotation.

Referring to Table 6.1, can you tell in which cases $\mathbf{L} = \mathbf{L}_z$ will not apply?

Let us differentiate Eq. (6.42b). Since $\hat{\mathbf{k}}$ is a fixed (constant) vector, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L}_z) = \left(\frac{\mathrm{d}}{\mathrm{d}t}(I\omega) \right) \hat{\mathbf{k}}$$

Now, Eq. (6.28b) states

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}$$

As we have seen in the last section, only those components of the external torques which are along the axis of rotation, need to be taken into account, when we discuss rotation about a fixed axis. This means we can take $\boldsymbol{\tau} = \tau\hat{\mathbf{k}}$. Since $\mathbf{L} = \mathbf{L}_z + \mathbf{L}_\perp$ and the direction of $\mathbf{L}_z$ (vector $\hat{\mathbf{k}}$) is fixed, it follows that for rotation about a fixed axis,

$$\frac{\mathrm{d}\mathbf{L}_z}{\mathrm{d}t} = \tau\hat{\mathbf{k}} \qquad (6.43a)$$

and $$\frac{\mathrm{d}\mathbf{L}_\perp}{\mathrm{d}t} = 0 \qquad (6.43b)$$

Thus, for rotation about a fixed axis, the component of angular momentum perpendicular to the fixed axis is constant. As $\mathbf{L}_z = I\omega\hat{\mathbf{k}}$, we get from Eq. (6.43a),

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\omega) = \tau \qquad (6.43c)$$

If the moment of inertia I does not change with time,

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\omega) = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\alpha$$

and we get from Eq. (6.43c),

$$\tau = I\alpha \qquad (6.41)$$

We have already derived this equation using the work - kinetic energy route.

6.12.1 Conservation of angular momentum

We are now in a position to revisit the principle of conservation of angular momentum in the context of rotation about a fixed axis. From Eq. (6.43c), if the external torque is zero,

$$L_z = I\omega = \text{constant} \qquad (6.44)$$

For symmetric bodies, from Eq. (6.42d), L_z may be replaced by L .(L and L_z are respectively the magnitudes of $\mathbf{L}$ and $\mathbf{L}_z$.)

This then is the required form, for fixed axis rotation, of Eq. (6.29a), which expresses the general law of conservation of angular momentum of a system of particles. Eq. (6.44) applies to many situations that we come across in daily life. You may do this experiment with your friend. Sit on a swivel chair (a chair with a seat, free to rotate about a pivot) with your arms folded and feet not resting on, i.e., away from, the ground. Ask your friend to rotate the chair rapidly. While the chair is rotating with considerable angular speed stretch your arms horizontally. What happens? Your angular speed is reduced. If you bring back your arms closer to your body, the angular speed increases again. This is a situation where the principle of conservation of angular momentum is applicable. If friction in the rotational

Fig 6.32 (a) *A demonstration of conservation of angular momentum. A girl sits on a swivel chair and stretches her arms/ brings her arms closer to the body.*

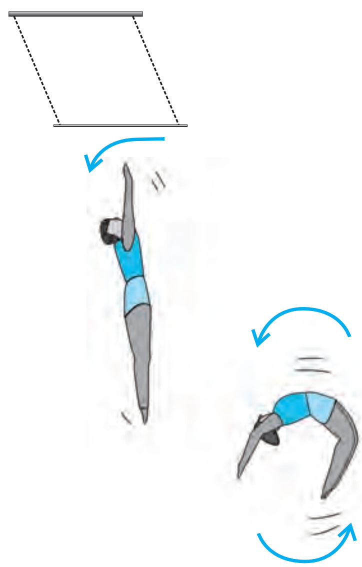


Fig 6.32 (b) *An acrobat employing the principle of conservation of angular momentum in her performance.*

mechanism is neglected, there is no external torque about the axis of rotation of the chair and hence $I\omega$ is constant. Stretching the arms increases I about the axis of rotation, resulting in decreasing the angular speed ω. Bringing the arms closer to the body has the opposite effect.

A circus acrobat and a diver take advantage of this principle. Also, skaters and classical, Indian or western, dancers performing a pirouette (a spinning about a tip–top) on the toes of one foot display 'mastery' over this principle. Can you explain?

SUMMARY

1. Ideally, a rigid body is one for which the distances between different particles of the body do not change, even though there are forces on them.
2. A rigid body fixed at one point or along a line can have only rotational motion. A rigid body not fixed in some way can have either pure translational motion or a combination of translational and rotational motions.
3. In rotation about a fixed axis, every particle of the rigid body moves in a circle which lies in a plane perpendicular to the axis and has its centre on the axis. Every Point in the rotating rigid body has the same angular velocity at any instant of time.
4. In pure translation, every particle of the body moves with the same velocity at any instant of time.
5. Angular velocity is a vector. Its magnitude is $\omega = d\theta/dt$ and it is directed along the axis of rotation. For rotation about a fixed axis, this vector $\boldsymbol{\omega}$ has a fixed direction.
6. The vector or cross product of two vector **a** and **b** is a vector written as $\mathbf{a}\times\mathbf{b}$. The magnitude of this vector is $ab\sin\theta$ and its direction is given by the right handed screw or the right hand rule.
7. The linear velocity of a particle of a rigid body rotating about a fixed axis is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where **r** is the position vector of the particle with respect to an origin along the fixed axis. The relation applies even to more general rotation of a rigid body with one point fixed. In that case **r** is the position vector of the particle with respect to the fixed point taken as the origin.
8. The centre of mass of a system of n particles is defined as the point whose position vector is

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{M}$$

9. Velocity of the centre of mass of a system of particles is given by $\mathbf{V} = \mathbf{P}/M$, where **P** is the linear momentum of the system. The centre of mass moves as if all the mass of the system is concentrated at this point and all the external forces act at it. If the total external force on the system is zero, then the total linear momentum of the system is constant.

10. The angular momentum of a system of n particles about the origin is

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i$$

The torque or moment of force on a system of n particles about the origin is

$$\boldsymbol{\tau} = \sum_{1} \mathbf{r}_i \times \mathbf{F}_i$$

The force $\mathbf{F}_i$ acting on the i^{th} particle includes the external as well as internal forces. Assuming Newton's third law of motion and that forces between any two particles act along the line joining the particles, we can show $\boldsymbol{\tau}_{int} = \mathbf{0}$ and

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_{ext}$$

11. A rigid body is in mechanical equilibrium if
   (1) it is in translational equilibrium, i.e., the total external force on it is zero : $\sum \mathbf{F}_i = \mathbf{0}$, and
   (2) it is in rotational equilibrium, i.e. the total external torque on it is zero : $\sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0}$.
12. The centre of gravity of an extended body is that point where the total gravitational torque on the body is zero.
13. The moment of intertia of a rigid body about an axis is defined by the formula $I = \sum m_i r_i^2$ where r_i is the perpendicular distance of the ith point of the body from the axis. The kinetic energy of rotation is $K = \frac{1}{2} I\omega^2$.

Quantity	Symbols	Dimensions	Units	Remarks
Angular velocity	ω	$[T^{-1}]$	rad s^{-}	$v = \omega \times r$
Angular momentum	L	$[ML^2 T^{-1}]$	J s	$L = r \times p$
Torque	τ	$[ML^2 T^{-2}]$	N m	$\tau = r \times F$
Moment of inertia	I	$[ML^2]$	kg m^2	$I = \sum m_i r_i^2$

POINTS TO PONDER

1. To determine the motion of the centre of mass of a system no knowledge of internal forces of the system is required. For this purpose we need to know only the external forces on the body.
2. Separating the motion of a system of particles as the motion of the centre of mass, (i.e., the translational motion of the system) and motion about (i.e. relative to) the centre of mass of the system is a useful technique in dynamics of a system of particles. One example of this technique is separating the kinetic energy of a system of particles K as the kinetic energy of the system about its centre of mass K' and the kinetic energy of the centre of mass $MV^2/2$,
$$K = K' + MV^2/2$$
3. Newton's Second Law for finite sized bodies (or systems of particles) is based in Newton's Second Law and also Newton's Third Law for particles.
4. To establish that the time rate of change of the total angular momentum of a system of particles is the total external torque in the system, we need not only Newton's second law for particles, but also Newton's third law with the provision that the forces between any two particles act along the line joining the particles.
5. The vanishing of the total external force and the vanishing of the total external torque are independent conditions. We can have one without the other. In a couple, total external force is zero, but total torque is non-zero.
6. The total torque on a system is independent of the origin if the total external force is zero.
7. The centre of gravity of a body coincides with its centre of mass only if the gravitational field does not vary from one part of the body to the other.

8. The angular momentum **L** and the angular velocity **ω** are not necessarily parallel vectors. However, for the simpler situations discussed in this chapter when rotation is about a fixed axis which is an axis of symmetry of the rigid body, the relation $\mathbf{L} = I\boldsymbol{\omega}$ holds good, where I is the moment of the inertia of the body about the rotation axis.

EXERCISES

6.1 Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body ?

6.2 In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

6.3 A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system ?

6.4 Show that the area of the triangle contained between the vectors **a** and **b** is one half of the magnitude of $\mathbf{a} \times \mathbf{b}$.

6.5 Show that $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})$ is equal in magnitude to the volume of the parallelepiped formed on the three vectors , **a**, **b** and **c**.

6.6 Find the components along the x, y, z axes of the angular momentum **l** of a particle, whose position vector is **r** with components x, y, z and momentum is **p** with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.

6.7 Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the angular momentum vector of the two particle system is the same whatever be the point about which the angular momentum is taken.

6.8 A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig.6.33. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.

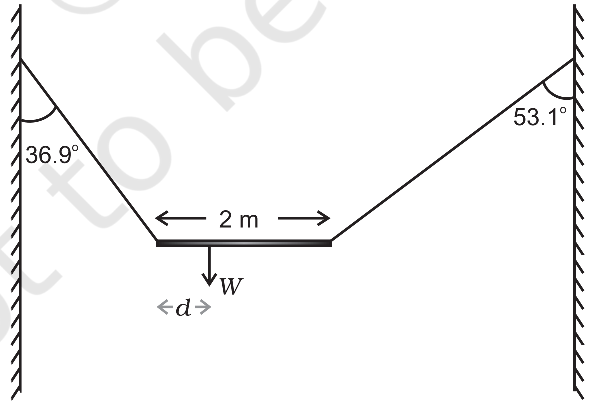


Fig. 6.33

6.9 A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

6.10 Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time.

6.11 A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

6.12 (a) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value ? Assume that the turntable rotates without friction.

(b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

6.13 A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N ? What is the linear acceleration of the rope ? Assume that there is no slipping.

6.14 To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 N m. What is the power required by the engine ? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

6.15 From a uniform disk of radius R, a circular hole of radius $R/2$ is cut out. The centre of the hole is at $R/2$ from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

6.16 A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

6.17 The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

CHAPTER SEVEN

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GRAVITATION

7.1 INTRODUCTION

Early in our lives, we become aware of the tendency of all material objects to be attracted towards the earth. Anything thrown up falls down towards the earth, going uphill is lot more tiring than going downhill, raindrops from the clouds above fall towards the earth and there are many other such phenomena. Historically it was the Italian Physicist Galileo (1564-1642) who recognised the fact that all bodies, irrespective of their masses, are accelerated towards the earth with a constant acceleration. It is said that he made a public demonstration of this fact. To find the truth, he certainly did experiments with bodies rolling down inclined planes and arrived at a value of the acceleration due to gravity which is close to the more accurate value obtained later.

A seemingly unrelated phenomenon, observation of stars, planets and their motion has been the subject of attention in many countries since the earliest of times. Observations since early times recognised stars which appeared in the sky with positions unchanged year after year. The more interesting objects are the planets which seem to have regular motions against the background of stars. The earliest recorded model for planetary motions proposed by Ptolemy about 2000 years ago was a 'geocentric' model in which all celestial objects, stars, the sun and the planets, all revolved around the earth. The only motion that was thought to be possible for celestial objects was motion in a circle. Complicated schemes of motion were put forward by Ptolemy in order to describe the observed motion of the planets. The planets were described as moving in circles with the centre of the circles themselves moving in larger circles. Similar theories were also advanced by Indian astronomers some 400 years later. However a more elegant model in which the Sun was the centre around which the planets revolved – the 'heliocentric' model – was already mentioned by Aryabhatta (5th century A.D.) in his treatise. A thousand years later, a Polish monk named Nicolas Copernicus (1473-1543) proposed a definitive model in which the planets moved in circles around a fixed central sun. His theory was discredited by the church, but notable amongst its supporters was Galileo who had to face prosecution from the state for his beliefs.

It was around the same time as Galileo, a nobleman called Tycho Brahe (1546-1601) hailing from Denmark, spent his entire lifetime recording observations of the planets with the naked eye. His compiled data were analysed later by his assistant Johannes Kepler (1571-1640). He could extract from the data three elegant laws that now go by the name of Kepler's laws. These laws were known to Newton and enabled him to make a great scientific leap in proposing his universal law of gravitation.

7.2 KEPLER'S LAWS

The three laws of Kepler can be stated as follows:

1. Law of orbits : All planets move in elliptical orbits with the Sun situated at one of the foci

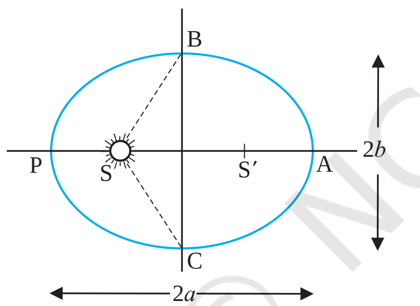


Fig. 7.1(a) *An ellipse traced out by a planet around the sun. The closest point is P and the farthest point is A, P is called the perihelion and A the aphelion. The semimajor axis is half the distance AP.*

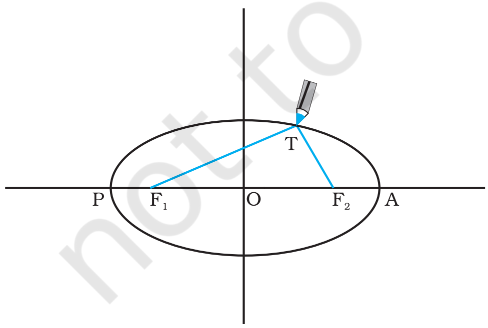


Fig. 7.1(b) *Drawing an ellipse. A string has its ends fixed at F_1 and F_2. The tip of a pencil holds the string taut and is moved around.*

of the ellipse (Fig. 7.1a). This law was a deviation from the Copernican model which allowed only circular orbits. The ellipse, of which the circle is a special case, is a closed curve which can be drawn very simply as follows.

Select two points F_1 and F_2. Take a length of a string and fix its ends at F_1 and F_2 by pins. With the tip of a pencil stretch the string taut and then draw a curve by moving the pencil keeping the string taut throughout.(Fig. 7.1(b)) The closed curve you get is called an ellipse. Clearly for any point T on the ellipse, the sum of the distances from F_1 and F_2 is a constant. F_1, F_2 are called the focii. Join the points F_1 and F_2 and extend the line to intersect the ellipse at points P and A as shown in Fig. 7.1(b). The midpoint of the line PA is the centre of the ellipse O and the length PO = AO is called the semi-major axis of the ellipse. For a circle, the two focii merge into one and the semi-major axis becomes the radius of the circle.

2. Law of areas : The line that joins any planet to the sun sweeps equal areas in equal intervals of time (Fig. 7.2). This law comes from the observations that planets appear to move slower when they are farther from the sun than when they are nearer.

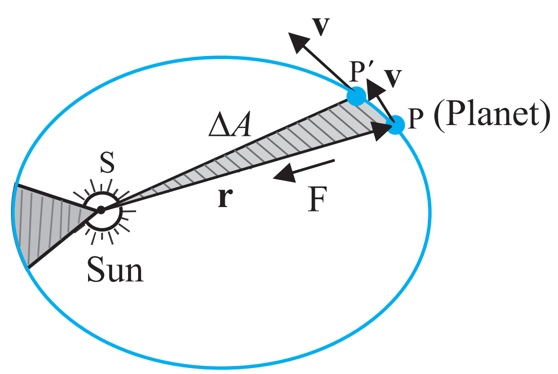


Fig. 7.2 *The planet P moves around the sun in an elliptical orbit. The shaded area is the area ΔA swept out in a small interval of time Δt.*

3. Law of periods : The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of the ellipse traced out by the planet.

Table 7.1 gives the approximate time periods of revolution of eight* planets around the sun along with values of their semi-major axes.

Table 7.1 **Data from measurement of planetary motions given below confirm Kepler's Law of Periods**

(a ≡ Semi-major axis in units of 10^{10} m.
T ≡ Time period of revolution of the planet in years(y).
Q ≡ The quotient (T^2/a^3) in units of 10^{-34} y^2 m^{-3}.)

Planet	a	T	Q
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84	2.98
Neptune	450	165	2.99

The law of areas can be understood as a consequence of conservation of angular momentum whch is valid for any central force . A central force is such that the force on the planet is along the vector joining the Sun and the planet. Let the Sun be at the origin and let the position and momentum of the planet be denoted by **r** and **p** respectively. Then the area swept out by the planet of mass m in time interval Δt is (Fig. 7.2) ΔA given by

$$\Delta \mathbf{A} = ½ \ (\mathbf{r} \times \mathbf{v}\Delta t) \qquad (7.1)$$

Hence

$$\Delta \mathbf{A} / \Delta t = ½ \ (\mathbf{r} \times \mathbf{p})/\text{m}, \ (\text{since } \mathbf{v} = \mathbf{p}/\text{m})$$
$$= \mathbf{L} / (2 \text{ m}) \qquad (7.2)$$

where **v** is the velocity, **L** is the angular momentum equal to (**r** × **p**). For a central force, which is directed along **r**, **L** is a constant as the planet goes around. Hence, $\Delta \mathbf{A} / \Delta t$ is a constant according to the last equation. This is the law of areas. Gravitation is a central force and hence the law of areas follows.

Example 7.1 Let the speed of the planet at the perihelion *P* in Fig. 7.1(a) be v_P and the Sun-planet distance SP be r_P. Relate {r_P, v_P} to the corresponding quantities at the aphelion {r_A, v_A}. Will the planet take equal times to traverse *BAC* and *CPB* ?

Answer The magnitude of the angular momentum at *P* is $L_p = m_p r_p v_p$, since inspection tells us that $\mathbf{r}_p$ and $\mathbf{v}_p$ are mutually perpendicular. Similarly, $L_A = m_p r_A v_A$. From angular momentum conservation

$$m_p r_p v_p = m_p r_A v_A$$

or
$$\frac{v_p}{v_A} = \frac{r_A}{r_p}$$

Since $r_A > r_p$, $v_p > v_A$.

The area *SBAC* bounded by the ellipse and the radius vectors *SB* and *SC* is larger than SBPC in Fig. 7.1. From Kepler's second law, equal areas are swept in equal times. Hence the planet will take a longer time to traverse *BAC* than *CPB*.

7.3 UNIVERSAL LAW OF GRAVITATION

Legend has it that observing an apple falling from a tree, Newton was inspired to arrive at an universal law of gravitation that led to an explanation of terrestrial gravitation as well as of Kepler's laws. Newton's reasoning was that the moon revolving in an orbit of radius R_m was subject to a centripetal acceleration due to earth's gravity of magnitude

$$a_m = \frac{V^2}{R_m} = \frac{4\pi^2 R_m}{T^2} \qquad (7.3)$$

where *V* is the speed of the moon related to the time period *T* by the relation $V = 2\pi R_m / T$. The time period *T* is about 27.3 days and R_m was already known then to be about 3.84 10^8m. If we substitute these numbers in Eq. (7.3), we get a value of a_m much smaller than the value of acceleration due to gravity g on the surface of the earth, arising also due to earth's gravitational attraction.

This clearly shows that the force due to earth's gravity decreases with distance. If one assumes that the gravitational force due to the earth decreases in proportion to the inverse square of the distance from the centre of the earth, we will have $a_m \alpha R_m^{-2}$; $g \alpha\ R_E^{-2}$ and we get

$$\frac{g}{a_m} = \frac{R_m^2}{R_E^2} \simeq 3600 \qquad (7.4)$$

in agreement with a value of $g \simeq 9.8$ m s^{-2} and the value of a_m from Eq. (7.3). These observations led Newton to propose the following Universal Law of Gravitation :

Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The quotation is essentially from Newton's famous treatise called 'Mathematical Principles of Natural Philosophy' (Principia for short).

Stated Mathematically, Newton's gravitation law reads : The force **F** on a point mass m_2 due to another point mass m_1 has the magnitude

$$|\mathbf{F}| = G\ \frac{m_1\ m_2}{r^2} \qquad (7.5)$$

Equation (7.5) can be expressed in vector form as

$$\mathbf{F} = G\ \frac{m_1\ m_2}{r^2}\left(-\hat{\mathbf{r}}\right) = -G\ \frac{m_1\ m_2}{r^2}\,\hat{\mathbf{r}}$$

$$= -G\ \frac{m_1\ m_2}{|\mathbf{r}|^3}\,\hat{\mathbf{r}}$$

where G is the universal gravitational constant, $\hat{\mathbf{r}}$ is the unit vector from m_1 to m_2 and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ as shown in Fig. 7.3.

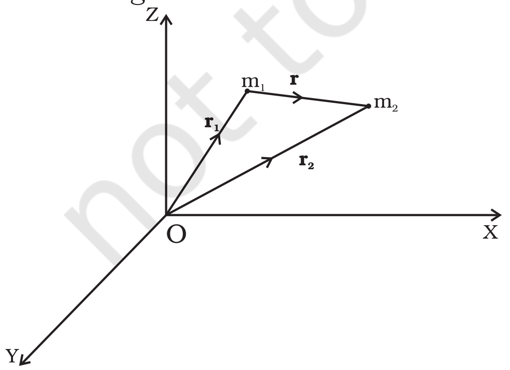


Fig. 7.3 *Gravitational force on m_1 due to m_2 is along* **r** *where the vector* **r** *is* ($\mathbf{r}_2 - \mathbf{r}_1$).

The gravitational force is attractive, i.e., the force **F** is along – **r**. The force on point mass m_1 due to m_2 is of course – **F** by Newton's third law. Thus, the gravitational force $\mathbf{F}_{12}$ on the body 1 due to 2 and $\mathbf{F}_{21}$ on the body 2 due to 1 are related as $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Before we can apply Eq. (7.5) to objects under consideration, we have to be careful since the law refers to **point** masses whereas we deal with extended objects which have finite size. If we have a collection of point masses, the force on any one of them is the vector sum of the gravitational forces exerted by the other point masses as shown in Fig 7.4.

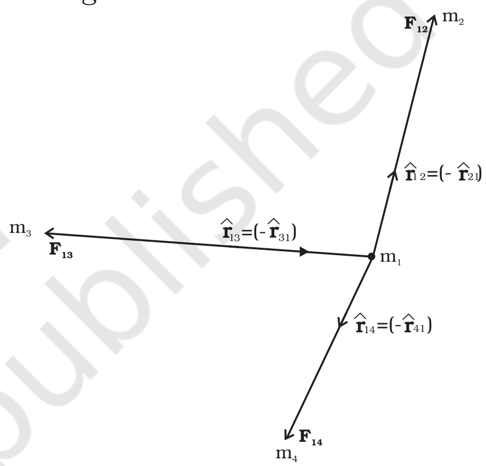


Fig. 7.4 *Gravitational force on point mass m_1 is the vector sum of the gravitational forces exerted by m_2, m_3 and m_4.*

The total force on m_1 is

$$\mathbf{F}_1 = \frac{Gm_2\,m_1}{r_{21}^2}\,\hat{\mathbf{r}}_{21} + \frac{Gm_3\,m_1}{r_{31}^2}\,\hat{\mathbf{r}}_{31} + \frac{Gm_4\,m_1}{r_{41}^2}\,\hat{\mathbf{r}}_{41}$$

Example 7.2 Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.
(a) What is the force acting on a mass $2m$ placed at the centroid G of the triangle?
(b) What is the force if the mass at the vertex A is doubled ?
Take AG = BG = CG = 1 m (see Fig. 7.5)

Answer (a) The angle between GC and the positive x-axis is 30° and so is the angle between GB and the negative x-axis. The individual forces in vector notation are

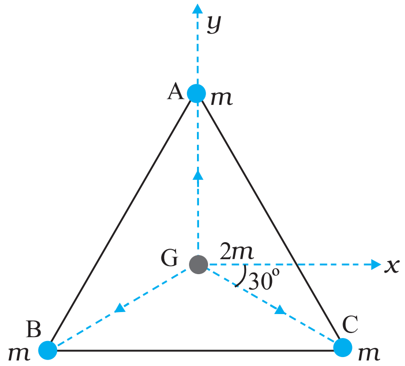


Fig. 7.5 *Three equal masses are placed at the three vertices of the Δ ABC. A mass 2m is placed at the centroid G.*

$$\mathbf{F}_{GA} = \frac{Gm(2m)}{1}\hat{\mathbf{j}}$$

$$\mathbf{F}_{GB} = \frac{Gm(2m)}{1}\left(-\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$

$$\mathbf{F}_{GC} = \frac{Gm(2m)}{1}\left(+\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$

From the principle of superposition and the law of vector addition, the resultant gravitational force $\mathbf{F}_R$ on $(2m)$ is

$$\mathbf{F}_R = \mathbf{F}_{GA} + \mathbf{F}_{GB} + \mathbf{F}_{GC}$$

$$\mathbf{F}_R = 2Gm^2\,\hat{\mathbf{j}} + 2Gm^2\left(-\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right) + 2Gm^2\left(\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right) = 0$$

Alternatively, one expects on the basis of symmetry that the resultant force ought to be zero.

(b) Now if the mass at vertex A is doubled then

$$F'_{GA} = \frac{G2m.2m}{1}\hat{j} = 4Gm^2\hat{j}$$

$$F'_{GB} = F_{GB} \text{ and } F'_{GC} = F_{GC}$$

$$F'_R = F'_{GA} + F'_{GB} + F'_{GC}$$

$$F'_R = 2\,Gm^2\,\hat{j}$$

For the gravitational force between an extended object (like the earth) and a point mass, Eq. (7.5) is not directly applicable. Each point mass in the extended object will exert a force on the given point mass and these force will not all be in the same direction. We have to add up these forces vectorially for all the point masses in the extended object to get the total force. This is easily done using calculus. For two special cases, a simple law results when you do that :

(1) **The force of attraction between a hollow spherical shell of uniform density and a point mass situated outside is just as if the entire mass of the shell is concentrated at the centre of the shell.**
Qualitatively this can be understood as follows: Gravitational forces caused by the various regions of the shell have components along the line joining the point mass to the centre as well as along a direction prependicular to this line. The components prependicular to this line cancel out when summing over all regions of the shell leaving only a resultant force along the line joining the point to the centre. The magnitude of this force works out to be as stated above.

(2) **The force of attraction due to a hollow spherical shell of uniform density, on a point mass situated inside it is zero.**
Qualitatively, we can again understand this result. Various regions of the spherical shell attract the point mass inside it in various directions. These forces cancel each other completely.

7.4 THE GRAVITATIONAL CONSTANT

The value of the gravitational constant G entering the Universal law of gravitation can be determined experimentally and this was first done by English scientist Henry Cavendish in 1798. The apparatus used by him is schematically shown in Fig.7.6

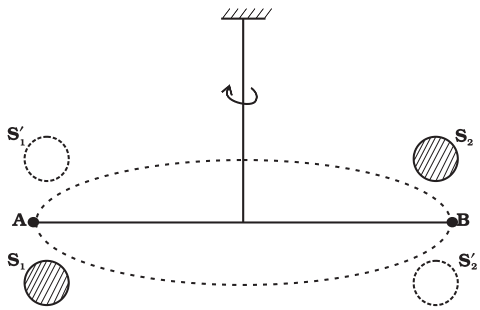


Fig. 7.6 *Schematic drawing of Cavendish's experiment. S_1 and S_2 are large spheres which are kept on either side (shown shades) of the masses at A and B. When the big spheres are taken to the other side of the masses (shown by dotted circles), the bar AB rotates a little since the torque reverses direction. The angle of rotation can be measured experimentally.*

The bar AB has two small lead spheres attached at its ends. The bar is suspended from a rigid support by a fine wire. Two large lead spheres are brought close to the small ones but on opposite sides as shown. The big spheres attract the nearby small ones by equal and opposite force as shown. There is no net force on the bar but only a torque which is clearly equal to F times the length of the bar,where F is the force of attraction between a big sphere and its neighbouring small sphere. Due to this torque, the suspended wire gets twisted till such time as the restoring torque of the wire equals the gravitational torque . If θ is the angle of twist of the suspended wire, the restoring torque is proportional to θ, equal to $\tau\theta$. Where τ is the restoring couple per unit angle of twist. τ can be measured independently e.g. by applying a known torque and measuring the angle of twist. The gravitational force between the spherical balls is the same as if their masses are concentrated at their centres. Thus if d is the separation between the centres of the big and its neighbouring small ball, M and m their masses, the gravitational force between the big sphere and its neighouring small ball is.

$$F = G\frac{Mm}{d^2} \tag{7.6}$$

If L is the length of the bar AB , then the torque arising out of F is F multiplied by L. At equilibrium, this is equal to the restoring torque and hence

$$G\frac{Mm}{d^2}L = \tau\,\theta \tag{7.7}$$

Observation of θ thus enables one to calculate G from this equation.

Since Cavendish's experiment, the measurement of G has been refined and the currently accepted value is

$$G = 6.67\times10^{-11}\ \mathrm{N\,m^2/kg^2} \tag{7.8}$$

7.5 ACCELERATION DUE TO GRAVITY OF THE EARTH

The earth can be imagined to be a sphere made of a large number of concentric spherical shells with the smallest one at the centre and the largest one at its surface. A point outside the earth is obviously outside all the shells. Thus, all the shells exert a gravitational force at the point outside just as if their masses are concentrated at their common centre according to the result stated in section 7.3. The total mass of all the shells combined is just the mass of the earth. Hence, at a point outside the earth, the gravitational force is just as if its entire mass of the earth is concentrated at its centre.

For a point inside the earth, the situation is different. This is illustrated in Fig. 7.7.

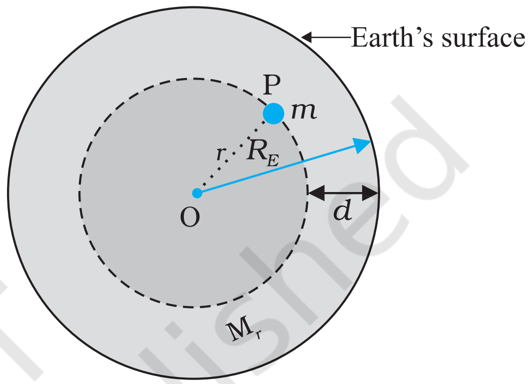


Fig. 7.7 *The mass m is in a mine located at a depth d below the surface of the Earth of mass M_E and radius R_E. We treat the Earth to be spherically symmetric.*

Again consider the earth to be made up of concentric shells as before and a point mass m situated at a distance r from the centre. The point P lies outside the sphere of radius r. For the shells of radius greater than r, the point P lies inside. Hence according to result stated in the last section, they exert no gravitational force on mass m kept at P. The shells with radius $\leq r$ make up a sphere of radius r for which the point P lies on the surface. This smaller sphere therefore exerts a force on a mass m at P as if its mass M_r is concentrated at the centre. Thus the force on the mass m at P has a magnitude

$$F = \frac{Gm\,(M_r)}{r^2} \tag{7.9}$$

We assume that the entire earth is of uniform density and hence its mass is $M_E = \frac{4\pi}{3}R_E^3\,\rho$ where M_E is the mass of the earth R_E is its radius and ρ is the density. On the other hand the mass of the sphere M_r of radius r is $\frac{4\pi}{3}\rho r^3$ and hence

$$F = Gm\left(\frac{4p}{3}r\right)\frac{r^3}{r^2} = Gm\left(\frac{M_E}{R_E^3}\right)\frac{r^3}{r^2}$$

$$= \frac{G\,m\,M_E}{{R_E}^3}\,r \qquad (7.10)$$

If the mass m is situated on the surface of earth, then $r = R_E$ and the gravitational force on it is, from Eq. (7.10)

$$F = G\,\frac{M_E m}{R_E^2} \qquad (7.11)$$

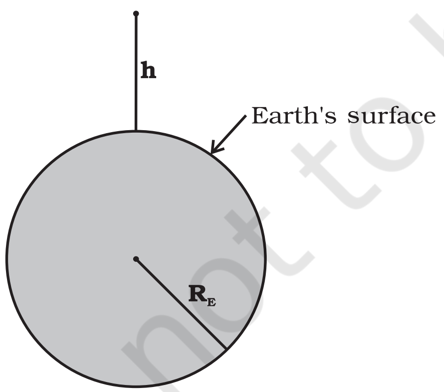
The acceleration experienced by the mass m, which is usually denoted by the symbol g is related to F by Newton's 2nd law by relation $F = mg$. Thus

$$g = \frac{F}{m} = \frac{GM_E}{R_E^2} \qquad (7.12)$$

Acceleration g is readily measurable. R_E is a known quantity. The measurement of G by Cavendish's experiment (or otherwise), combined with knowledge of g and R_E enables one to estimate M_E from Eq. (7.12). This is the reason why there is a popular statement regarding Cavendish : "Cavendish weighed the earth".

7.6 ACCELERATION DUE TO GRAVITY BELOW AND ABOVE THE SURFACE OF EARTH

Consider a point mass m at a height h above the surface of the earth as shown in Fig. 7.8(a). The radius of the earth is denoted by R_E. Since this point is outside the earth,



(a)

Fig. 7.8 (a) *g at a height h above the surface of the earth.*

its distance from the centre of the earth is $(R_E + h)$. If $F(h)$ denoted the magnitude of the force on the point mass m , we get from Eq. (7.5) :

$$F(h) = \frac{GM_E m}{(R_E + h)^2} \qquad (7.13)$$

The acceleration experienced by the point mass is $F(h)/m \equiv g(h)$ and we get

$$g(h) = \frac{F(h)}{m} = \frac{GM_E}{(R_E + h)^2}\,. \qquad (7.14)$$

This is clearly less than the value of g on the surface of earth : $g = \frac{GM_E}{R_E^2}$. For $h << R_E$, we can expand the RHS of Eq. (7.14) :

$$g(h) = \frac{GM_E}{R_E^2(1 + h/R_E)^2} = g\,(1 + h/R_E)^{-2}$$

For $\frac{h}{R_E} << 1$, using binomial expression,

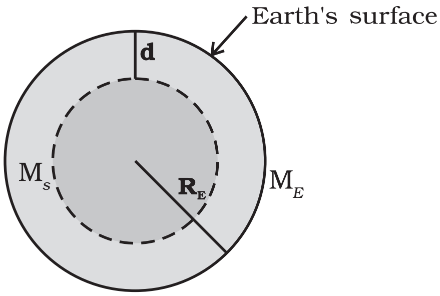
$$g(h) \cong g\left(1 - \frac{2h}{R_E}\right). \qquad (7.15)$$

Equation (7.15) thus tells us that for small heights h above the value of g decreases by a factor $(1 - 2h/R_E)$.

Now, consider a point mass m at a depth d below the surface of the earth (Fig. 7.8(b)), so that its distance from the centre of the earth is $(R_E - d)$ as shown in the figure. The earth can be thought of as being composed of a smaller sphere of radius $(R_E - d)$ and a spherical shell of thickness d. The force on m due to the outer shell of thickness d is zero because the result quoted in the previous section. As far as the smaller sphere of radius $(R_E - d)$ is concerned, the point mass is outside it and hence according to the result quoted earlier, the force due to this smaller sphere is just as if the entire mass of the smaller sphere is concentrated at the centre. If M_s is the mass of the smaller sphere, then,

$$M_s/M_E = (R_E - d)^3 \;/\; R_E^{\,3} \qquad (7.16)$$

Since mass of a sphere is proportional to be cube of its radius.



(b)

Fig. 7.8 (b) *g at a depth d. In this case only the smaller sphere of radius (R_E–d) contributes to g.*

Thus the force on the point mass is

$$F(d) = G\ M_s\ m\ /\ (R_E - d)^2 \qquad (7.17)$$

Substituting for M_s from above , we get

$$F(d) = G\ M_E\ m\ (R_E - d)\ /\ R_E^3 \qquad (7.18)$$

and hence the acceleration due to gravity at a depth d,

$$g(d) = \frac{F(d)}{m} \text{ is}$$

$$g(d) = \frac{F(d)}{m} = \frac{GM_E}{R_E^3}\ (R_E - d)$$

$$= g\frac{R_E - d}{R_E} = g(1 - d/R_E) \qquad (7.19)$$

Thus, as we go down below earth's surface, the acceleration due gravity decreases by a factor $(1 - d/R_E)$. The remarkable thing about acceleration due to earth's gravity is that it is maximum on its surface decreasing whether you go up or down.

7.7 GRAVITATIONAL POTENTIAL ENERGY

We had discussed earlier the notion of potential energy as being the energy stored in the body at its given position. If the position of the particle changes on account of forces acting on it, then the change in its potential energy is just the amount of work done on the body by the force. As we had discussed earlier, forces for which the work done is independent of the path are the conservative forces.

The force of gravity is a conservative force and we can calculate the potential energy of a body arising out of this force, called the gravitational potential energy. Consider points close to the surface of earth, at distances from the surface much smaller than the radius of the earth. In such cases, the force of gravity is practically a constant equal to mg, directed towards the centre of the earth. If we consider a point at a height h_1 from the surface of the earth and another point vertically above it at a height h_2 from the surface, the work done in lifting the particle of mass m from the first to the second position is denoted by W_{12}

$$W_{12} = \text{Force} \times \text{displacement}$$

$$= mg\ (h_2 - h_1) \qquad (7.20)$$

If we associate a potential energy $W(h)$ at a point at a height h above the surface such that

$$W(h) = mgh + W_o \qquad (7.21)$$

(where W_o = constant) ;

then it is clear that

$$W_{12} = W(h_2) - W(h_1) \qquad (7.22)$$

The work done in moving the particle is just the difference of potential energy between its final and initial positions.Observe that the constant W_o cancels out in Eq. (7.22). Setting $h = 0$ in the last equation, we get $W(h = 0) = W_o$. $h = 0$ means points on the surface of the earth. Thus, W_o is the potential energy on the surface of the earth.

If we consider points at arbitrary distance from the surface of the earth, the result just derived is not valid since the assumption that the gravitational force mg is a constant is no longer valid. However, from our discussion we know that a point outside the earth, the force of gravitation on a particle directed towards the centre of the earth is

$$F = \frac{G M_E m}{r^2} \qquad (7.23)$$

where M_E = mass of earth, m = mass of the particle and r its distance from the centre of the earth. If we now calculate the work done in lifting a particle from $r = r_1$ to $r = r_2$ ($r_2 > r_1$) along a vertical path, we get instead of Eq. (7.20)

$$W_{12} = \int_{r_1}^{r_2} \frac{G M m}{r^2} \mathrm{d}r$$

$$= -G M_E m \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad (7.24)$$

In place of Eq. (7.21), we can thus associate a potential energy $W(r)$ at a distance r, such that

$$W(r) = -\frac{G M_E m}{r} + W_1 , \qquad (7.25)$$

valid for $r > R$,

so that once again $W_{12} = W(r_2) - W(r_1)$. Setting r = infinity in the last equation, we get W (r = infinity) = W_1 . Thus, W_1 is the potential energy at infinity. One should note that only the difference of potential energy between two points has a definite meaning from Eqs. (7.22) and (7.24). One conventionally sets W_1 equal to zero, so that the potential energy at a point is just the amount of work done in displacing the particle from infinity to that point.

We have calculated the potential energy at a point of a particle due to gravitational forces on it due to the earth and it is proportional to the mass of the particle. The gravitational potential due to the gravitational force of the earth is defined as the potential energy of a particle of unit mass at that point. From the earlier discussion, we learn that the gravitational potential energy associated with two particles of masses m_1 and m_2 separated by distance by a distance r is given by

$$V = -\frac{G m_1 m_2}{r} \quad \text{(if we choose } V = 0 \text{ as } r \to \infty\text{)}$$

It should be noted that an isolated system of particles will have the total potential energy that equals the sum of energies (given by the above equation) for all possible pairs of its constituent particles. This is an example of the application of the superposition principle.

Example 7.3 Find the potential energy of a system of four particles placed at the vertices of a square of side l. Also obtain the potential at the centre of the square.

Answer Consider four masses each of mass m at the corners of a square of side l; See Fig. 7.9. We have four mass pairs at distance l and two diagonal pairs at distance $\sqrt{2}\, l$

Hence,

$$W(r) = -4\frac{G m^2}{l} - 2\frac{G m^2}{\sqrt{2}\, l}$$

$$= -\frac{2 G m^2}{l}\left(2 + \frac{1}{\sqrt{2}}\right) = -5.41\frac{G m^2}{l}$$

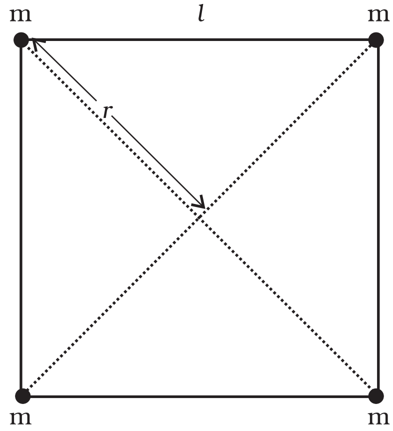


Fig. 7.9

The gravitational potential at the centre of the square $\left(r = \sqrt{2}\, l/2\right)$ is

$$U(r) = -4\sqrt{2}\,\frac{G m}{l}.$$

7.8 ESCAPE SPEED

If a stone is thrown by hand, we see it falls back to the earth. Of course using machines we can shoot an object with much greater speeds and with greater and greater initial speed, the object scales higher and higher heights. A natural query that arises in our mind is the following: 'can we throw an object with such high initial speeds that it does not fall back to the earth?'

The principle of conservation of energy helps us to answer this question. Suppose the object did reach infinity and that its speed there was V_f. The energy of an object is the sum of potential and kinetic energy. As before W_1 denotes that gravitational potential energy of the object at infinity. The total energy of the projectile at infinity then is

$$E(\infty) = W_1 + \frac{m V_f^2}{2} \qquad (7.26)$$

If the object was thrown initially with a speed V_i from a point at a distance $(h+R_E)$ from the centre of the earth (R_E = radius of the earth), its energy initially was

$$E(h + R_E) = \frac{1}{2} m V_i^2 - \frac{G m M_E}{(h + R_E)} + W_1 \qquad (7.27)$$

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By the principle of energy conservation Eqs. (7.26) and (7.27) must be equal. Hence

$$\frac{mV_i^2}{2} - \frac{GmM_E}{(h+R_E)} = \frac{mV_f^2}{2} \tag{7.28}$$

The R.H.S. is a positive quantity with a minimum value zero hence so must be the L.H.S. Thus, an object can reach infinity as long as V_i is such that

$$\frac{mV_i^2}{2} - \frac{GmM_E}{(h+R_E)} \geq 0 \tag{7.29}$$

The minimum value of V_i corresponds to the case when the L.H.S. of Eq. (7.29) equals zero. Thus, the minimum speed required for an object to reach infinity (i.e. escape from the earth) corresponds to

$$\frac{1}{2}m\left(V_i^2\right)_{\min} = \frac{GmM_E}{h+R_E} \tag{7.30}$$

If the object is thrown from the surface of the earth, $h = 0$, and we get

$$(V_i)_{\min} = \sqrt{\frac{2GM_E}{R_E}} \tag{7.31}$$

Using the relation $g = GM_E / R_E^2$, we get

$$(V_i)_{\min} = \sqrt{2gR_E} \tag{7.32}$$

Using the value of g and R_E, numerically $(V_i)_{\min} \approx 11.2$ km/s. This is called the escape speed, sometimes loosely called the escape velocity.

Equation (7.32) applies equally well to an object thrown from the surface of the moon with g replaced by the acceleration due to Moon's gravity on its surface and r_E replaced by the radius of the moon. Both are smaller than their values on earth and the escape speed for the moon turns out to be 2.3 km/s, about five times smaller. This is the reason that moon has no atmosphere. Gas molecules if formed on the surface of the moon having velocities larger than this will escape the gravitational pull of the moon.

Example 7.4 Two uniform solid spheres of equal radii R, but mass M and 4 M have a centre to centre separation 6 R, as shown in Fig. 7.10. The two spheres are held fixed. A projectile of mass m is projected from the surface of the sphere of mass M directly towards the centre of the second sphere. Obtain an expression for the minimum speed v of the projectile so that it reaches the surface of the second sphere.

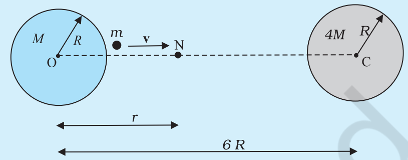

Fig. 7.10

Answer The projectile is acted upon by two mutually opposing gravitational forces of the two spheres. The neutral point N (see Fig. 7.10) is defined as the position where the two forces cancel each other exactly. If ON = r, we have

$$\frac{G\,M\,m}{r^2} = \frac{4\,G\,M\,m}{(6R-r)^2}$$

$(6R - r)^2 = 4r^2$

$6R - r = \pm 2r$

$r = 2R$ or $-6R$.

The neutral point $r = -6R$ does not concern us in this example. Thus ON = r = $2R$. It is sufficient to project the particle with a speed which would enable it to reach N. Thereafter, the greater gravitational pull of $4M$ would suffice. The mechanical energy at the surface of M is

$$E_i = \frac{1}{2}m\,v^2 - \frac{G\,M\,m}{R} - \frac{4\,G\,M\,m}{5\,R}.$$

At the neutral point N, the speed approaches zero. The mechanical energy at N is purely potential.

$$E_N = -\frac{G\,M\,m}{2\,R} - \frac{4\,G\,M\,m}{4\,R}.$$

From the principle of conservation of mechanical energy

$$\frac{1}{2}v^2 - \frac{GM}{R} - \frac{4GM}{5R} = -\frac{GM}{2R} - \frac{GM}{R}$$

or

$$v^2 = \frac{2GM}{R}\left(\frac{4}{5}-\frac{1}{2}\right)$$

$$v = \left(\frac{3GM}{5R}\right)^{1/2}$$

A point to note is that the speed of the projectile is zero at N, but is nonzero when it strikes the heavier sphere 4 *M*. The calculation of this speed is left as an exercise to the students.

7.9 EARTH SATELLITES

Earth satellites are objects which revolve around the earth. Their motion is very similar to the motion of planets around the Sun and hence Kepler's laws of planetary motion are equally applicable to them. In particular, their orbits around the earth are circular or elliptic. Moon is the only natural satellite of the earth with a near circular orbit with a time period of approximately 27.3 days which is also roughly equal to the rotational period of the moon about its own axis. Since, 1957, advances in technology have enabled many countries including India to launch artificial earth satellites for practical use in fields like telecommunication, geophysics and meteorology.

We will consider a satellite in a circular orbit of a distance $(R_E + h)$ from the centre of the earth, where R_E = radius of the earth. If *m* is the mass of the satellite and *V* its speed, the centripetal force required for this orbit is

$$\text{F(centripetal)} = \frac{mV^2}{(R_E + h)} \quad (7.33)$$

directed towards the centre. This centripetal force is provided by the gravitational force, which is

$$\text{F(gravitation)} = \frac{G\,m\,M_E}{(R_E + h)^2} \quad (7.34)$$

where M_E is the mass of the earth.

Equating R.H.S of Eqs. (7.33) and (7.34) and cancelling out *m*, we get

$$V^2 = \frac{G\,M_E}{(R_E + h)} \quad (7.35)$$

Thus *V* decreases as *h* increases. From equation (7.35),the speed *V* for $h = 0$ is

$$V^2 \ (h = 0) = GM/R_E = gR_E \quad (7.36)$$

where we have used the relation $g = GM/R_E^2$. In every orbit, the satellite traverses a distance $2\pi(R_E + h)$ with speed *V*. Its time period *T* therefore is

$$T = \frac{2\pi(R_E + h)}{V} = \frac{2\pi(R_E + h)^{3/2}}{\sqrt{G\,M_E}} \quad (7.37)$$

on substitution of value of *V* from Eq. (7.35). Squaring both sides of Eq. (7.37), we get

$$T^2 = k\ (R_E + h)^3 \quad (\text{where } k = 4\pi^2 / GM_E) \quad (7.38)$$

which is Kepler's law of periods, as applied to motion of satellites around the earth. For a satellite very close to the surface of earth *h* can be neglected in comparison to R_E in Eq. (7.38). Hence, for such satellites, *T* is T_o, where

$$T_0 = 2\pi\sqrt{R_E / g} \quad (7.39)$$

If we substitute the numerical values $g \simeq 9.8$ m s^{-2} and R_E = 6400 km., we get

$$T_0 = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}}\ \text{s}$$

Which is approximately 85 minutes.

Example 7.5 The planet Mars has two moons, phobos and delmos. (i) phobos has a period 7 hours, 39 minutes and an orbital radius of 9.4×10^3 km. Calculate the mass of mars. (ii) Assume that earth and mars move in circular orbits around the sun, with the martian orbit being 1.52 times the orbital radius of the earth. What is the length of the martian year in days ?

Answer (i) We employ Eq. (7.38) with the sun's mass replaced by the martian mass M_m

$$T^2 = \frac{4\pi^2}{GM_m} R^3$$

$$M_m = \frac{4\pi^2}{G}\frac{R^3}{T^2}$$

$$= \frac{4 \times (3.14)^2 \times (9.4)^3 \times 10^{18}}{6.67 \times 10^{-11} \times (459 \times 60)^2}$$

$$M_m = \frac{4 \times (3.14)^2 \times (9.4)^3 \times 10^{18}}{6.67 \times (4.59 \times 6)^2 \times 10^{-5}}$$

$= 6.48 \times 10^{23}$ kg.

(ii) Once again Kepler's third law comes to our aid,

$$\frac{T_M^2}{T_E^2} = \frac{R_{MS}^3}{R_{ES}^3}$$

where R_{MS} is the mars -sun distance and R_{ES} is the earth-sun distance.

$$\therefore\ T_M = (1.52)^{3/2} \times 365$$
$$= 684 \text{ days}$$

We note that the orbits of all planets except Mercury and Mars are very close to being circular. For example, the ratio of the semi-minor to semi-major axis for our Earth is, $b/a = 0.99986$. ◀

> ▶ **Example 7.6 Weighing the Earth** : You are given the following data: $g = 9.81$ ms^{-2}, $R_E = 6.37 \times 10^6$ m, the distance to the moon $R = 3.84 \times 10^8$ m and the time period of the moon's revolution is 27.3 days. Obtain the mass of the Earth M_E in two different ways.

Answer From Eq. (7.12) we have

$$M_E = \frac{g R_E^2}{G}$$

$$= \frac{9.81 \times \left(6.37 \times 10^6\right)^2}{6.67 \times 10^{-11}}$$
$$= 5.97 \times 10^{24} \text{ kg.}$$

The moon is a satellite of the Earth. From the derivation of Kepler's third law [see Eq. (7.38)]

$$T^2 = \frac{4\pi^2 R^3}{G M_E}$$

$$M_E = \frac{4\pi^2 R^3}{G T^2}$$

$$= \frac{4 \times 3.14 \times 3.14 \times (3.84)^3 \times 10^{24}}{6.67 \times 10^{-11} \times (27.3 \times 24 \times 60 \times 60)^2}$$

$$= 6.02 \times 10^{24} \text{ kg}$$

Both methods yield almost the same answer, the difference between them being less than 1%. ◀

> ▶ **Example 7.7** Express the constant k of Eq. (7.38) in days and kilometres. Given k = 10^{-13} s^2 m^{-3}. The moon is at a distance of 3.84×10^5 km from the earth. Obtain its time-period of revolution in days.

Answer Given

$$k = 10^{-13} \text{ s}^2 \text{ m}^{-3}$$

$$= 10^{-13}\left[\frac{1}{(24 \times 60 \times 60)^2}\text{d}^2\right]\left[\frac{1}{(1/1000)^3 \text{km}^3}\right]$$

$$= 1.33 \times 10^{-14} \text{ d}^2 \text{ km}^{-3}$$

Using Eq. (7.38) and the given value of k, the time period of the moon is

$$T^2 = (1.33 \times 10^{-14})(3.84 \times 10^5)^3$$
$$T = 27.3 \text{ d}$$ ◀

Note that Eq. (7.38) also holds for elliptical orbits if we replace (R_E+h) by the semi-major axis of the ellipse. The earth will then be at one of the foci of this ellipse.

7.10 ENERGY OF AN ORBITING SATELLITE

Using Eq. (7.35), the kinetic energy of the satellite in a circular orbit with speed v is

$$K{\bullet}E = \frac{1}{2} m v^2$$

$$= \frac{Gm M_E}{2(R_E + h)}, \tag{7.40}$$

Considering gravitational potential energy at infinity to be zero, the potential energy at distance (R_e+h) from the centre of the earth is

$$P.E = -\frac{G m M_E}{(R_E + h)} \tag{7.41}$$

The K.E is positive whereas the P.E is negative. However, in magnitude the K.E. is half the P.E, so that the total E is

$$E = K.E + P.E = -\frac{G m M_E}{2(R_E + h)} \tag{7.42}$$

The total energy of an circularly orbiting satellite is thus negative, with the potential energy being negative but twice is magnitude of the positive kinetic energy.

When the orbit of a satellite becomes elliptic, both the *K.E.* and *P.E.* vary from point to point. The total energy which remains constant is negative as in the circular orbit case. This is what we expect, since as we have discussed before if the total energy is positive or zero, the object escapes to infinity. Satellites are always at finite distance from the earth and hence their energies cannot be positive or zero.

Example 7.8 A 400 kg satellite is in a circular orbit of radius $2R_E$ about the Earth. How much energy is required to transfer it to a circular orbit of radius $4R_E$? What are the changes in the kinetic and potential energies ?

Answer Initially,

$$E_i = -\frac{G\,M_E\,m}{4\,R_E}$$

While finally

$$E_f = -\frac{G\,M_E\,m}{8\,R_E}$$

The change in the total energy is

$$\Delta E = E_f - E_i$$

$$= \frac{G\,M_E\,m}{8\,R_E} = \left(\frac{G\,M_E}{R_E^2}\right)\frac{m\,R_E}{8}$$

$$\Delta E = \frac{g\,m\,R_E}{8} = \frac{9.81\times 400\times 6.37\times 10^6}{8} = 3.13\times 10^9\,\text{J}$$

The kinetic energy is reduced and it mimics ΔE, namely, $\Delta K = K_f - K_i = -3.13\times 10^9$ J.

The change in potential energy is twice the change in the total energy, namely

$$\Delta V = V_f - V_i = -6.25\times 10^9\,\text{J}$$

SUMMARY

1. Newton's law of universal gravitation states that the gravitational force of attraction between any two particles of masses m_1 and m_2 separated by a distance r has the magnitude

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant, which has the value 6.672×10^{-11} N m^2 kg^{-2}.

2. If we have to find the resultant gravitational force acting on the particle m due to a number of masses $M_1, M_2, \ldots M_n$ etc. we use the principle of superposition. Let $F_1, F_2, \ldots F_n$ be the individual forces due to $M_1, M_2, \ldots M_n$, each given by the law of gravitation. From the principle of superposition each force acts independently and uninfluenced by the other bodies. The resultant force F_R is then found by vector addition

$$F_R = F_1 + F_2 + \ldots\ldots + F_n = \sum_{i=1}^{n} F_i$$

where the symbol 'Σ' stands for summation.

3. Kepler's laws of planetary motion state that
   (a) All planets move in elliptical orbits with the Sun at one of the focal points
   (b) The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals. This follows from the fact that the force of gravitation on the planet is central and hence angular momentum is conserved.
   (c) The square of the orbital period of a planet is proportional to the cube of the semi-major axis of the elliptical orbit of the planet

   The period T and radius R of the circular orbit of a planet about the Sun are related by

$$T^2 = \left(\frac{4\pi^2}{G\,M_s}\right)R^3$$

where M_s is the mass of the Sun. Most planets have nearly circular orbits about the Sun. For elliptical orbits, the above equation is valid if R is replaced by the semi-major axis, a.

4. The acceleration due to gravity.
   (a) at a height h above the earth's surface

$$g(h) = \frac{G\,M_E}{(R_E + h)^2}$$

$$\approx \frac{G\,M_E}{R_E^2}\left(1 - \frac{2h}{R_E}\right) \quad \text{for } h \ll R_E$$

$$g(h) = g(0)\left(1 - \frac{2h}{R_E}\right) \quad \text{where } g(0) = \frac{G\,M_E}{R_E^2}$$

(b) at depth d below the earth's surface is

$$g(d) = \frac{G\,M_E}{R_E^2}\left(1 - \frac{d}{R_E}\right) = g(0)\left(1 - \frac{d}{R_E}\right)$$

5. The gravitational force is a conservative force, and therefore a potential energy function can be defined. The *gravitational potential energy* associated with two particles separated by a distance r is given by

$$V = -\frac{G\,m_1\,m_2}{r}$$

where V is taken to be zero at $r \to \infty$. The total potential energy for a system of particles is the sum of energies for all pairs of particles, with each pair represented by a term of the form given by above equation. This prescription follows from the principle of superposition.

6. If an isolated system consists of a particle of mass m moving with a speed v in the vicinity of a massive body of mass M, the total mechanical energy of the particle is given by

$$E = \frac{1}{2}m\,v^2 - \frac{G\,M\,m}{r}$$

That is, the total mechanical energy is the sum of the kinetic and potential energies. The total energy is a constant of motion.

7. If m moves in a circular orbit of radius a about M, where $M >> m$, the total energy of the system is

$$E = -\frac{G\,M\,m}{2a}$$

with the choice of the arbitrary constant in the potential energy given in the point 5., above. The total energy is negative for any bound system, that is, one in which the orbit is closed, such as an elliptical orbit. The kinetic and potential energies are

$$K = \frac{G\,M\,m}{2a}$$

$$V = -\frac{G\,M\,m}{a}$$

8. The escape speed from the surface of the earth is

$$v_e = \sqrt{\frac{2\,G\,M_E}{R_E}} = \sqrt{2gR_E}$$

and has a value of 11.2 km s^{-1}.

9. If a particle is outside a uniform spherical shell or solid sphere with a spherically symmetric internal mass distribution, the sphere attracts the particle as though the mass of the sphere or shell were concentrated at the centre of the sphere.

10. If a particle is inside a uniform spherical shell, the gravitational force on the particle is zero. If a particle is inside a homogeneous solid sphere, the force on the particle acts toward the centre of the sphere. This force is exerted by the spherical mass interior to the particle.

Physical Quantity	**Symbol**	**Dimensions**	**Unit**	**Remarks**
Gravitational Constant	G	$[M^{-1}\,L^3\,T^{-2}]$	N m^2 kg^{-2}	6.67×10^{-11}
Gravitational Potential Energy	V(r)	$[M\,L^2\,T^{-2}]$	J	$-\frac{GMm}{r}$ (scalar)
Gravitational Potential	U(r)	$[L^2T^{-2}]$	J kg^{-1}	$-\frac{GM}{r}$ (scalar)
Gravitational Intensity	**E** or **g**	$[LT^{-2}]$	m s^{-2}	$\frac{GM}{r^2}\hat{r}$ (vector)

POINTS TO PONDER

1. In considering motion of an object under the gravitational influence of another object the following quantities are conserved:
   (a) Angular momentum
   (b) Total mechanical energy
   Linear momentum is **not** conserved
2. Angular momentum conservation leads to Kepler's second law. However, it is not special to the inverse square law of gravitation. It holds for any central force.
3. In Kepler's third law (see Eq. (7.1) and $T^2 = K_S R^3$. The constant K_S is the same for all planets in circular orbits. This applies to satellites orbiting the Earth [(Eq. (7.38)].
4. An astronaut experiences weightlessness in a space satellite. This is not because the gravitational force is small at that location in space. It is because both the astronaut and the satellite are in "free fall" towards the Earth.
5. The *gravitational potential energy* associated with two particles separated by a distance r is given by
$$V = -\frac{G m_1 m_2}{r} + \text{constant}$$
   The constant can be given any value. The simplest choice is to take it to be zero. With this choice
$$V = -\frac{G m_1 m_2}{r}$$
   This choice implies that $V \to 0$ as $r \to \infty$. Choosing location of zero of the gravitational energy is the same as choosing the arbitrary constant in the potential energy. Note that the gravitational force is not altered by the choice of this constant.
6. The total mechanical energy of an object is the sum of its kinetic energy (which is always positive) and the potential energy. Relative to infinity (i.e. if we presume that the potential energy of the object at infinity is zero), the gravitational potential energy of an object is negative. The total energy of a satellite is negative.
7. The commonly encountered expression $m g h$ for the potential energy is actually an approximation to the difference in the gravitational potential energy discussed in the point 6, above.
8. Although the gravitational force between two particles is central, the force between two finite rigid bodies is not necessarily along the line joining their centre of mass. For a spherically symmetric body however the force on a particle external to the body is as if the mass is concentrated at the centre and this force is therefore central.
9. The gravitational force on a particle inside a spherical shell is zero. However, (unlike a metallic shell which shields electrical forces) the shell does not shield other bodies outside it from exerting gravitational forces on a particle inside. *Gravitational shielding is not possible*.

EXERCISES

7.1 Answer the following :
(a) You can shield a charge from electrical forces by putting it inside a hollow conductor. Can you shield a body from the gravitational influence of nearby matter by putting it inside a hollow sphere or by some other means ?
(b) An astronaut inside a small space ship orbiting around the earth cannot detect gravity. If the space station orbiting around the earth has a large size, can he hope to detect gravity ?
(c) If you compare the gravitational force on the earth due to the sun to that due to the moon, you would find that the Sun's pull is greater than the moon's pull. (you can check this yourself using the data available in the succeeding exercises). However, the tidal effect of the moon's pull is greater than the tidal effect of sun. Why ?

7.2 Choose the correct alternative :

(a) Acceleration due to gravity increases/decreases with increasing altitude.

(b) Acceleration due to gravity increases/decreases with increasing depth (assume the earth to be a sphere of uniform density).

(c) Acceleration due to gravity is independent of mass of the earth/mass of the body.

(d) The formula $-G\,Mm(1/r_2 - 1/r_1)$ is more/less accurate than the formula $mg(r_2 - r_1)$ for the difference of potential energy between two points r_2 and r_1 distance away from the centre of the earth.

7.3 Suppose there existed a planet that went around the Sun twice as fast as the earth. What would be its orbital size as compared to that of the earth ?

7.4 Io, one of the satellites of Jupiter, has an orbital period of 1.769 days and the radius of the orbit is 4.22×10^8 m. Show that the mass of Jupiter is about one-thousandth that of the sun.

7.5 Let us assume that our galaxy consists of 2.5×10^{11} stars each of one solar mass. How long will a star at a distance of 50,000 ly from the galactic centre take to complete one revolution ? Take the diameter of the Milky Way to be 10^5 ly.

7.6 Choose the correct alternative:

(a) If the zero of potential energy is at infinity, the total energy of an orbiting satellite is negative of its kinetic/potential energy.

(b) The energy required to launch an orbiting satellite out of earth's gravitational influence is more/less than the energy required to project a stationary object at the same height (as the satellite) out of earth's influence.

7.7 Does the escape speed of a body from the earth depend on (a) the mass of the body, (b) the location from where it is projected, (c) the direction of projection, (d) the height of the location from where the body is launched?

7.8 A comet orbits the sun in a highly elliptical orbit. Does the comet have a constant (a) linear speed, (b) angular speed, (c) angular momentum, (d) kinetic energy, (e) potential energy, (f) total energy throughout its orbit? Neglect any mass loss of the comet when it comes very close to the Sun.

7.9 Which of the following symptoms is likely to afflict an astronaut in space (a) swollen feet, (b) swollen face, (c) headache, (d) orientational problem.

7.10 In the following two exercises, choose the correct answer from among the given ones: The gravitational intensity at the centre of a hemispherical shell of uniform mass density has the direction indicated by the arrow (see Fig 7.11) (i) a, (ii) b, (iii) c, (iv) 0.

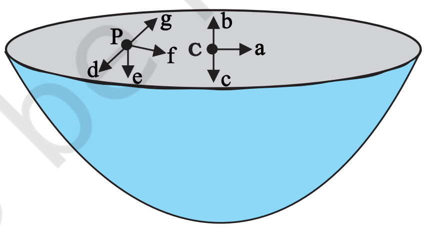

Fig. 7.11

7.11 For the above problem, the direction of the gravitational intensity at an arbitrary point P is indicated by the arrow (i) d, (ii) e, (iii) f, (iv) g.

7.12 A rocket is fired from the earth towards the sun. At what distance from the earth's centre is the gravitational force on the rocket zero ? Mass of the sun = 2×10^{30} kg, mass of the earth = 6×10^{24} kg. Neglect the effect of other planets etc. (orbital radius = 1.5×10^{11} m).

7.13 How will you 'weigh the sun', that is estimate its mass? The mean orbital radius of the earth around the sun is 1.5×10^8 km.

7.14 A saturn year is 29.5 times the earth year. How far is the saturn from the sun if the earth is 1.50×10^8 km away from the sun ?

7.15 A body weighs 63 N on the surface of the earth. What is the gravitational force on it due to the earth at a height equal to half the radius of the earth ?

7.16 Assuming the earth to be a sphere of uniform mass density, how much would a body weigh half way down to the centre of the earth if it weighed 250 N on the surface ?

7.17 A rocket is fired vertically with a speed of 5 km s^{-1} from the earth's surface. How far from the earth does the rocket go before returning to the earth ? Mass of the earth $= 6.0 \times 10^{24}$ kg; mean radius of the earth $= 6.4 \times 10^6$ m; $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

7.18 The escape speed of a projectile on the earth's surface is 11.2 km s^{-1}. A body is projected out with thrice this speed. What is the speed of the body far away from the earth? Ignore the presence of the sun and other planets.

7.19 A satellite orbits the earth at a height of 400 km above the surface. How much energy must be expended to rocket the satellite out of the earth's gravitational influence? Mass of the satellite = 200 kg; mass of the earth $= 6.0\times10^{24}$ kg; radius of the earth $= 6.4 \times 10^6$ m; $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

7.20 Two stars each of one solar mass ($= 2\times10^{30}$ kg) are approaching each other for a head on collision. When they are a distance 10^9 km, their speeds are negligible. What is the speed with which they collide ? The radius of each star is 10^4 km. Assume the stars to remain undistorted until they collide. (Use the known value of G).

7.21 Two heavy spheres each of mass 100 kg and radius 0.10 m are placed 1.0 m apart on a horizontal table. What is the gravitational force and potential at the mid point of the line joining the centres of the spheres ? Is an object placed at that point in equilibrium? If so, is the equilibrium stable or unstable ?

CHAPTER EIGHT

MECHANICAL PROPERTIES OF SOLIDS

8.1 INTRODUCTION

In Chapter 6, we studied the rotation of the bodies and then realised that the motion of a body depends on how mass is distributed within the body. We restricted ourselves to simpler situations of rigid bodies. A rigid body generally means a hard solid object having a definite shape and size. But in reality, bodies can be stretched, compressed and bent. Even the appreciably rigid steel bar can be deformed when a sufficiently large external force is applied on it. This means that solid bodies are not perfectly rigid.

A solid has definite shape and size. In order to change (or deform) the shape or size of a body, a force is required. If you stretch a helical spring by gently pulling its ends, the length of the spring increases slightly. When you leave the ends of the spring, it regains its original size and shape. The property of a body, by virtue of which it tends to regain its original size and shape when the applied force is removed, is known as **elasticity** and the deformation caused is known as **elastic** deformation. However, if you apply force to a lump of putty or mud, they have no gross tendency to regain their previous shape, and they get permanently deformed. Such substances are called **plastic** and this property is called **plasticity.** Putty and mud are close to ideal plastics.

The elastic behaviour of materials plays an important role in engineering design. For example, while designing a building, knowledge of elastic properties of materials like steel, concrete etc. is essential. The same is true in the design of bridges, automobiles, ropeways etc. One could also ask — Can we design an aeroplane which is very light but sufficiently strong? Can we design an artificial limb which is lighter but stronger? Why does a railway track have a particular shape like **I**? Why is glass brittle while brass is not? Answers to such questions begin with the study of how relatively simple kinds of loads or forces act to deform different solids bodies. In this chapter, we shall study the elastic behaviour and mechanical properties of solids which would answer many such questions.

8.2 STRESS AND STRAIN

When forces are applied on a body in such a manner that the body is still in static equilibrium, it is deformed to a small or large extent depending upon the nature of the material of the body and the magnitude of the deforming force. The deformation may not be noticeable visually in many materials but it is there. When a body is subjected to a deforming force, a restoring force is developed in the body. This restoring force is equal in magnitude but opposite in direction to the applied force. The restoring force per unit area is known as **stress.** If F is the force applied normal to the cross–section and A is the area of cross section of the body,

Magnitude of the stress = F/A (8.1)

The SI unit of stress is N m^{-2} or pascal (Pa) and its dimensional formula is [$ML^{-1}T^{-2}$].

There are three ways in which a solid may change its dimensions when an external force acts on it. These are shown in Fig. 8.1. In Fig.8.1(a), a cylinder is stretched by two equal forces applied normal to its cross-sectional area. The restoring force per unit area in this case is called **tensile stress**. If the cylinder is compressed under the action of applied forces, the restoring force per unit area is known as **compressive stress**. Tensile or compressive stress can also be termed as longitudinal stress.

In both the cases, there is a change in the length of the cylinder. The change in the length ΔL to the original length L of the body (cylinder in this case) is known as **longitudinal strain**.

$$\text{Longitudinal strain} = \frac{\Delta L}{L} \qquad (8.2)$$

However, if two equal and opposite deforming forces are applied parallel to the cross-sectional area of the cylinder, as shown in Fig. 8.1(b), there is relative displacement between the opposite faces of the cylinder. The restoring force per unit area developed due to the applied tangential force is known as **tangential** or **shearing stress**.

As a result of applied tangential force, there is a relative displacement Δx between opposite faces of the cylinder as shown in the Fig. 8.1(b). The strain so produced is known as **shearing strain** and it is defined as the ratio of relative displacement of the faces Δx to the length of the cylinder L.

$$\text{Shearing strain} = \frac{\Delta x}{L} = \tan\theta \qquad (8.3)$$

where θ is the angular displacement of the cylinder from the vertical (original position of the cylinder). Usually θ is very small, $\tan\theta$ is nearly equal to angle θ, (if $\theta = 10°$, for example, there is only 1% difference between θ and $\tan\theta$).

It can also be visualised, when a book is pressed with the hand and pushed horizontally, as shown in Fig. 8.2 (c).

Thus, shearing strain $= \tan\theta \approx \theta$ (8.4)

In Fig. 8.1 (d), a solid sphere placed in the fluid under high pressure is compressed uniformly on all sides. The force applied by the fluid acts in perpendicular direction at each point of the surface and the body is said to be under hydraulic compression. This leads to decrease

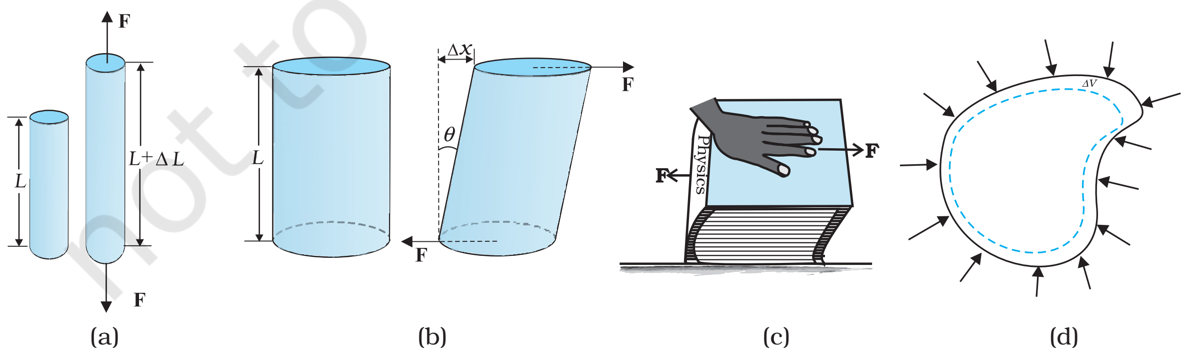

Fig. 8.1 *(a) A cylindrical body under tensile stress elongates by ΔL (b) Shearing stress on a cylinder deforming it by an angle θ (c) A body subjected to shearing stress (d) A solid body under a stress normal to the surface at every point (hydraulic stress). The volumetric strain is $\Delta V/V$, but there is no change in shape.*

in its volume without any change of its geometrical shape.

The body develops internal restoring forces that are equal and opposite to the forces applied by the fluid (the body restores its original shape and size when taken out from the fluid). The internal restoring force per unit area in this case is known as **hydraulic stress** and in magnitude is equal to the hydraulic pressure (applied force per unit area).

The strain produced by a hydraulic pressure is called **volume strain** and is defined as the ratio of change in volume (ΔV) to the original volume (V).

$$\text{Volume strain} = \frac{\Delta V}{V} \tag{8.5}$$

Since the strain is a ratio of change in dimension to the original dimension, it has no units or dimensional formula.

8.3 HOOKE'S LAW

Stress and strain take different forms in the situations depicted in the Fig. (8.1). For small deformations the stress and strain are proportional to each other. This is known as Hooke's law.

Thus,

stress $\propto$ strain

$$\text{stress} = k \times \text{strain} \tag{8.6}$$

where k is the proportionality constant and is known as modulus of elasticity.

Hooke's law is an empirical law and is found to be valid for most materials. However, there are some materials which do not exhibit this linear relationship.

8.4 STRESS-STRAIN CURVE

The relation between the stress and the strain for a given material under tensile stress can be found experimentally. In a standard test of tensile properties, a test cylinder or a wire is stretched by an applied force. The fractional change in length (the strain) and the applied force needed to cause the strain are recorded. The applied force is gradually increased in steps and the change in length is noted. A graph is plotted between the stress (which is equal in magnitude to the applied force per unit area) and the strain produced. A typical graph for a metal is shown in Fig. 8.2. Analogous graphs for compression and shear stress may also be obtained. The stress-strain curves vary from material to material. These curves help us to understand how a given material deforms with increasing loads. From the graph, we can see that in the region between O to A, the curve is linear. In this region, Hooke's law is obeyed. The body regains its original dimensions when the applied force is removed. In this region, the solid behaves as an elastic body.

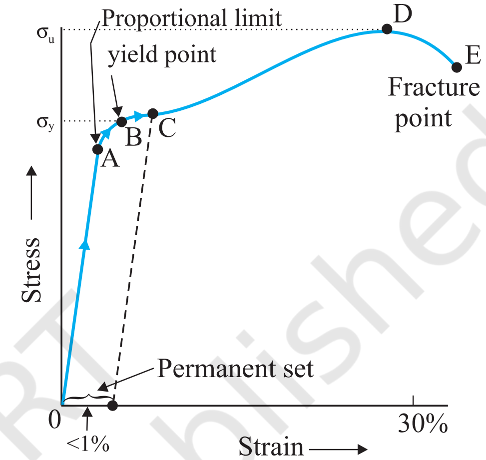

Fig. 8.2 *A typical stress-strain curve for a metal.*

In the region from A to B, stress and strain are not proportional. Nevertheless, the body still returns to its original dimension when the load is removed. The point B in the curve is known as **yield point** (also known as **elastic limit**) and the corresponding stress is known as **yield strength** (σ_y) of the material.

If the load is increased further, the stress developed exceeds the yield strength and strain increases rapidly even for a small change in the stress. The portion of the curve between B and D shows this. When the load is removed, say at some point C between B and D, the body does not regain its original dimension. In this case, even when the stress is zero, the strain is not zero. The material is said to have a **permanent set**. The deformation is said to be **plastic deformation**. The point D on the graph is the **ultimate tensile strength** (σ_u) of the material. Beyond this point, additional strain is produced even by a reduced applied force and fracture occurs at point E. If the ultimate strength and fracture points D and E are close, the material is said to be **brittle**. If they are far apart, the material is said to be **ductile**.

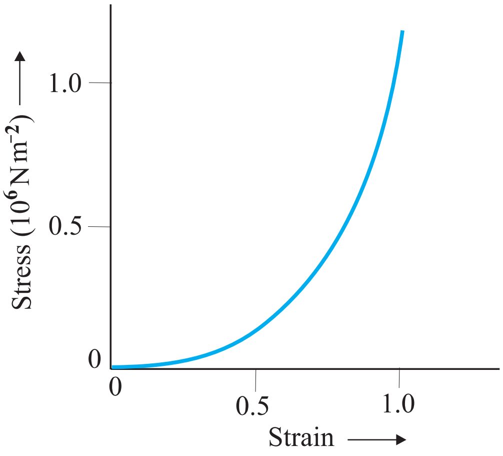

Fig. 8.3 *Stress-strain curve for the elastic tissue of Aorta, the large tube (vessel) carrying blood from the heart.*

As stated earlier, the stress-strain behaviour varies from material to material. For example, rubber can be pulled to several times its original length and still returns to its original shape. Fig. 8.3 shows stress-strain curve for the elastic tissue of aorta, present in the heart. Note that although elastic region is very large, the material does not obey Hooke's law over most of the region. Secondly, there is no well defined plastic region. Substances like tissue of aorta, rubber etc. which can be stretched to cause large strains are called **elastomers**.

8.5 ELASTIC MODULI

The proportional region within the elastic limit of the stress-strain curve (region OA in Fig. 8.2) is of great importance for structural and manufacturing engineering designs. The ratio of stress and strain, called **modulus of elasticity**, is found to be a characteristic of the material.

8.5.1 Young's Modulus

Experimental observation show that for a given material, the magnitude of the strain produced is same whether the stress is tensile or compressive. The ratio of tensile (or compressive) stress (σ) to the longitudinal strain (ε) is defined as **Young's modulus** and is denoted by the symbol Y.

$$Y = \frac{\sigma}{\varepsilon} \tag{8.7}$$

From Eqs. (8.1) and (8.2), we have

$$Y = (F/A)/(\Delta L/L)$$
$$= (F \times L)\ /(A \times \Delta L) \tag{8.8}$$

Since strain is a dimensionless quantity, the unit of Young's modulus is the same as that of stress *i.e.*, N m^{-2} or Pascal (Pa). Table 8.1 gives the values of Young's moduli and yield strengths of some material.

From the data given in Table 8.1, it is noticed that for metals Young's moduli are large.

Table 8.1 **Young's moduli and yield strenghs of some material**

Substance	Density ρ (kg m^{-3})	Young's modulus Y (10^9N m^{-2})	Ultimate strength, σ_u (10^6 N m^{-2})	Yield strength σ_y (10^6 N m^{-2})
Aluminium	2710	70	110	95
Copper	8890	110	400	200
Iron (wrought)	7800-7900	190	330	170
Steel	7860	200	400	250
Glass#	2190	65	50	—
Concrete	2320	30	40	—
Wood#	525	13	50	—
Bone#	1900	9.4	170	—
Polystyrene	1050	3	48	—

Substance tested under compression

Therefore, these materials require a large force to produce small change in length. To increase the length of a thin steel wire of 0.1 cm^2 cross-sectional area by 0.1%, a force of 2000 N is required. The force required to produce the same strain in aluminium, brass and copper wires having the same cross-sectional area are 690 N, 900 N and 1100 N respectively. It means that steel is more elastic than copper, brass and aluminium. It is for this reason that steel is preferred in heavy-duty machines and in structural designs. Wood, bone, concrete and glass have rather small Young's moduli.

Example 8.1 A structural steel rod has a radius of 10 mm and a length of 1.0 m. A 100 kN force stretches it along its length. Calculate (a) stress, (b) elongation, and (c) strain on the rod. Young's modulus, of structural steel is 2.0×10^{11} N m^{-2}.

Answer We assume that the rod is held by a clamp at one end, and the force F is applied at the other end, parallel to the length of the rod. Then the stress on the rod is given by

$$Stress = \frac{F}{A} = \frac{F}{\pi r^2}$$

$$= \frac{100 \times 10^3 \text{ N}}{3.14 \times \left(10^{-2} \text{ m}\right)^2}$$

$$= 3.18 \times 10^8 \text{ N m}^{-2}$$

The elongation,

$$\Delta L = \frac{(F/A)L}{Y}$$

$$= \frac{\left(3.18 \times 10^8 \text{ N m}^{-2}\right)(1 \text{m})}{2 \times 10^{11} \text{ N m}^{-2}}$$

$$= 1.59 \times 10^{-3} \text{ m}$$

$$= 1.59 \text{ mm}$$

The strain is given by

$$\text{Strain} = \Delta L/L$$

$$= (1.59 \times 10^{-3} \text{ m})/(1 \text{m})$$

$$= 1.59 \times 10^{-3}$$

$$= 0.16 \text{ \%}$$

Example 8.2 A copper wire of length 2.2 m and a steel wire of length 1.6 m, both of diameter 3.0 mm, are connected end to end. When stretched by a load, the net elongation is found to be 0.70 mm. Obtain the load applied.

Answer The copper and steel wires are under a tensile stress because they have the same tension (equal to the load W) and the same area of cross-section A. From Eq. (8.7) we have stress = strain × Young's modulus. Therefore

$$W/A = Y_c \times (\Delta L_c/L_c) = Y_s \times (\Delta L_s/L_s)$$

where the subscripts c and s refer to copper and stainless steel respectively. Or,

$$\Delta L_c/\Delta L_s = (Y_s/Y_c) \times (L_c/L_s)$$

Given $L_c = 2.2$ m, $L_s = 1.6$ m,

From Table 9.1 $Y_c = 1.1 \times 10^{11}$ N.m^{-2}, and

$Y_s = 2.0 \times 10^{11}$ N.m^{-2}.

$$\Delta L_c/\Delta L_s = (2.0 \times 10^{11}/1.1 \times 10^{11}) \times (2.2/1.6) = 2.5.$$

The total elongation is given to be

$$\Delta L_c + \Delta L_s = 7.0 \times 10^{-4} \text{ m}$$

Solving the above equations,

$\Delta L_c = 5.0 \times 10^{-4}$ m, and $\Delta L_s = 2.0 \times 10^{-4}$ m.

Therefore

$$W = (A \times Y_c \times \Delta L_c)/L_c$$

$$= \pi (1.5 \times 10^{-3})^2 \times [(5.0 \times 10^{-4} \times 1.1 \times 10^{11})/2.2]$$

$$= 1.8 \times 10^2 \text{ N}$$

Example 8.3 In a human pyramid in a circus, the entire weight of the balanced group is supported by the legs of a performer who is lying on his back (as shown in Fig. 8.4). The combined mass of all the persons performing the act, and the tables, plaques etc. involved is 280 kg. The mass of the performer lying on his back at the bottom of the pyramid is 60 kg. Each thighbone (femur) of this performer has a length of 50 cm and an effective radius of 2.0 cm. Determine the amount by which each thighbone gets compressed under the extra load.

Fig. 8.4 *Human pyramid in a circus.*

Answer Total mass of all the performers, tables, plaques etc. = 280 kg

Mass of the performer = 60 kg

Mass supported by the legs of the performer at the bottom of the pyramid

= 280 – 60 = 220 kg

Weight of this supported mass

= 220 kg wt. = 220 × 9.8 N = 2156 N.

Weight supported by each thighbone of the performer = ½ (2156) N = 1078 N.

From Table 9.1, the Young's modulus for bone is given by

$Y = 9.4 \times 10^9 \text{ N m}^{-2}$.

Length of each thighbone $L = 0.5$ m

the radius of thighbone = 2.0 cm

Thus the cross-sectional area of the thighbone

$A = \pi \times (2 \times 10^{-2})^2 \text{ m}^2 = 1.26 \times 10^{-3} \text{ m}^2$.

Using Eq. (9.8), the compression in each thighbone (ΔL) can be computed as

$$\Delta L = [(F \times L)/(Y \times A)]$$
$$= [(1078 \times 0.5)/(9.4 \times 10^9 \times 1.26 \times 10^{-3})]$$
$$= 4.55 \times 10^{-5} \text{ m or } 4.55 \times 10^{-3} \text{ cm.}$$

This is a very small change! The fractional decrease in the thighbone is $\Delta L/L = 0.000091$ or 0.0091%. ◀

8.5.2 Shear Modulus

The ratio of shearing stress to the corresponding shearing strain is called the *shear modulus* of the material and is represented by G. It is also called the *modulus of rigidity*.

$$G = \text{shearing stress } (\sigma_s)/\text{shearing strain}$$
$$G = (F/A)/(\Delta x/L)$$
$$= (F \times L)/(A \times \Delta x) \qquad (8.10)$$

Similarly, from Eq. (9.4)

$$G = (F/A)/\theta$$
$$= F/(A \times \theta) \qquad (8.11)$$

The shearing stress σ_s can also be expressed as

$$\sigma_s = G \times \theta \qquad (8.12)$$

SI unit of shear modulus is N m^{-2} or Pa. The shear moduli of a few common materials are given in Table 9.2. It can be seen that shear modulus (or modulus of rigidity) is generally less than Young's modulus (from Table 9.1). For most materials $G \approx Y/3$.

Table 8.2 **Shear moduli *(G)* of some common materials**

Material	G (10^9 Nm^{-2} or GPa)
Aluminium	25
Brass	36
Copper	42
Glass	23
Iron	70
Lead	5.6
Nickel	77
Steel	84
Tungsten	150
Wood	10

▶ ***Example 8.4*** A square lead slab of side 50 cm and thickness 10 cm is subject to a shearing force (on its narrow face) of 9.0×10^4 N. The lower edge is riveted to the floor. How much will the upper edge be displaced?

Answer The lead slab is fixed and the force is applied parallel to the narrow face as shown in Fig. 8.6. The area of the face parallel to which this force is applied is

$$A = 50 \text{ cm} \times 10 \text{ cm}$$
$$= 0.5 \text{ m} \times 0.1 \text{ m}$$
$$= 0.05 \text{ m}^2$$

Therefore, the stress applied is

$$= (9.4 \times 10^4 \text{ N}/0.05 \text{ m}^2)$$
$$= 1.80 \times 10^6 \text{ N.m}^{-2}$$

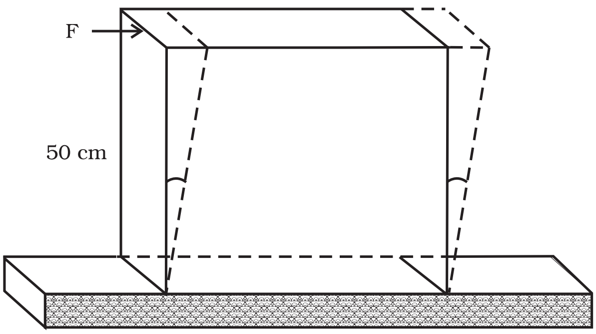

Fig. 8.5

We know that shearing strain = $(\Delta x/L)$= Stress / G.

Therefore the displacement Δx = (Stress × L)/G

$= (1.8 \times 10^6 \text{ N m}^{-2} \times 0.5\text{m})/(5.6 \times 10^9 \text{ N m}^{-2})$

$= 1.6 \times 10^{-4} \text{ m} = 0.16 \text{ mm}$ ◀

8.5.3 Bulk Modulus

In Section (8.3), we have seen that when a body is submerged in a fluid, it undergoes a hydraulic stress (equal in magnitude to the hydraulic pressure). This leads to the decrease in the volume of the body thus producing a strain called volume strain [Eq. (8.5)]. The ratio of hydraulic stress to the corresponding hydraulic strain is called *bulk modulus*. It is denoted by symbol B.

$$B = -p/(\Delta V/V) \qquad (8.12)$$

The negative sign indicates the fact that with an increase in pressure, a decrease in volume occurs. That is, if p is positive, ΔV is negative. Thus for a system in equilibrium, the value of bulk modulus B is always positive. SI unit of bulk modulus is the same as that of pressure *i.e.*, N m^{-2} or Pa. The bulk moduli of a few common materials are given in Table 8.3.

The reciprocal of the bulk modulus is called *compressibility* and is denoted by k. It is defined as the fractional change in volume per unit increase in pressure.

$$k = (1/B) = -(1/\Delta p) \times (\Delta V/V) \qquad (8.13)$$

It can be seen from the data given in Table 8.3 that the bulk moduli for solids are much larger than for liquids, which are again much larger than the bulk modulus for gases (air).

Table 8.3 Bulk moduli (*B*) of some common Materials

Material Solids	B (10^9 N m^{-2} or GPa)
Aluminium	72
Brass	61
Copper	140
Glass	37
Iron	100
Nickel	260
Steel	160
Liquids	
Water	2.2
Ethanol	0.9
Carbon disulphide	1.56
Glycerine	4.76
Mercury	25
Gases	
Air (at STP)	1.0×10^{-4}

Table 8.4 Stress, strain and various elastic moduli

Type of stress	Stress	Strain	Change in shape	Change in volume	Elastic Modulus	Name of Modulus	State of Matter
Tensile or compressive ($\sigma = F/A$)	Two equal and opposite forces perpendicular to opposite faces	Elongation or compression parallel to force direction ($\Delta L/L$) (longitudinal strain)	Yes	No	$Y = (F L)/(A \Delta L)$	Young's modulus	Solid
Shearing ($\sigma_s = F/A$)	Two equal and opposite forces parallel to oppoiste surfaces forces in each case such that total force and total torque on the body vanishes	Pure shear, θ	Yes	No	$G = F/(A\,\theta)$	Shear modulus or modulus of rigidity	Solid
Hydraulic	Forces perpendicular everywhere to the surface, force per unit area (pressure) same everywhere.	Volume change (compression or elongation) ($\Delta V/V$)	No	Yes	$B = -p/(\Delta V/V)$	Bulk modulus	Solid, liquid and gas

Thus, solids are the least compressible, whereas, gases are the most compressible. Gases are about a million times more compressible than solids! Gases have large compressibilities, which vary with pressure and temperature. The incompressibility of the solids is primarily due to the tight coupling between the neighbouring atoms. The molecules in liquids are also bound with their neighbours but not as strong as in solids. Molecules in gases are very poorly coupled to their neighbours.

Table 8.4 shows the various types of stress, strain, elastic moduli, and the applicable state of matter at a glance.

Example 8.5 The average depth of Indian Ocean is about 3000 m. Calculate the fractional compression, $\Delta V/V$, of water at the bottom of the ocean, given that the bulk modulus of water is 2.2×10^9 N m^{-2}. (Take g = 10 m s^{-2})

Answer The pressure exerted by a 3000 m column of water on the bottom layer

$$p = h\rho g = 3000 \text{ m} \times 1000 \text{ kg m}^{-3} \times 10 \text{ m s}^{-2}$$
$$= 3 \times 10^7 \text{ kg m}^{-1} \text{ s}^{-2}$$
$$= 3 \times 10^7 \text{ N m}^{-2}$$

Fractional compression $\Delta V/V$, is

$$\Delta V/V = \text{stress}/B = (3 \times 10^7 \text{ N m}^{-2})/(2.2 \times 10^9 \text{ N m}^{-2})$$
$$= 1.36 \times 10^{-2} \text{ or } 1.36 \ \%$$

8.5.4 Poisson's Ratio

The strain perpendicular to the applied force is called **lateral strain**. Simon Poisson pointed out that within the elastic limit, lateral strain is directly proportional to the longitudinal strain. The ratio of the lateral strain to the longitudinal strain in a stretched wire is called **Poisson's ratio.** If the original diameter of the wire is d and the contraction of the diameter under stress is Δd, the lateral strain is $\Delta d/d$. If the original length of the wire is L and the elongation under stress is ΔL, the longitudinal strain is $\Delta L/L$. Poisson's ratio is then $(\Delta d/d)/(\Delta L/L)$ or $(\Delta d/\Delta L)$ (L/d). Poisson's ratio is a ratio of two strains; it is a pure number and has no dimensions or units. Its value depends only on the nature of material. For steels the value is between 0.28 and 0.30, and for aluminium alloys it is about 0.33.

8.5.5 Elastic Potential Energy in a Stretched Wire

When a wire is put under a tensile stress, work is done against the inter-atomic forces. This work is stored in the wire in the form of elastic potential energy. When a wire of original length L and area of cross-section A is subjected to a deforming force F along the length of the wire, let the length of the wire be elongated by l. Then from Eq. (8.8), we have $F = YA \times (l/L)$. Here Y is the Young's modulus of the material of the wire. Now for a further elongation of infinitesimal small length dl, work done dW is $F\ dl$ or $YAl\,dl/L$. Therefore, the amount of work done (W) in increasing the length of the wire from L to $L + l$, that is from $l = 0$ to $l = l$ is

$$W = \int_0^l \frac{YAl}{L} dl = \frac{YA}{2} \times \frac{l^2}{L}$$

$$W = \frac{1}{2} \times Y \times \left(\frac{l}{L}\right)^2 \times AL$$

$$= \frac{1}{2} \times \text{Young's modulus} \times \text{strain}^2 \times \text{volume of the wire}$$

$$= \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume of the wire}$$

This work is stored in the wire in the form of elastic potential energy (U). Therefore the elastic potential energy per unit volume of the wire (u) is

$$u = \frac{1}{2} \sigma \varepsilon \qquad (8.14)$$

8.6 APPLICATIONS OF ELASTIC BEHAVIOUR OF MATERIALS

The elastic behaviour of materials plays an important role in everyday life. All engineering designs require precise knowledge of the elastic behaviour of materials. For example while designing a building, the structural design of the columns, beams and supports require knowledge of strength of materials used. Have you ever thought why the beams used in construction of bridges, as supports etc. have a cross-section of the type **I**? Why does a heap of sand or a hill have a pyramidal shape? Answers to these questions can be obtained from the study of structural engineering which is based on concepts developed here.

Cranes used for lifting and moving heavy loads from one place to another have a thick metal rope to which the load is attached. The rope is pulled up using pulleys and motors. Suppose we want to make a crane, which has a lifting capacity of 10 tonnes or metric tons (1 metric ton = 1000 kg). How thick should the steel rope be? We obviously want that the load does not deform the rope permanently. Therefore, the extension should not exceed the elastic limit. From Table 8.1, we find that mild steel has a yield strength (σ_y) of about 300×10^6 N m^{-2}. Thus, the area of cross-section (A) of the rope should at least be

$$A \geq W/\sigma_y = Mg/\sigma_y \qquad (8.15)$$
$$= (10^4 \text{ kg} \times 9.8 \text{ m s}^{-2})/(300 \times 10^6 \text{ N m}^{-2})$$
$$= 3.3 \times 10^{-4} \text{ m}^2$$

corresponding to a radius of about 1 cm for a rope of circular cross-section. Generally a large margin of safety (of about a factor of ten in the load) is provided. Thus a thicker rope of radius about 3 cm is recommended. A single wire of this radius would practically be a rigid rod. So the ropes are always made of a number of thin wires braided together, like in pigtails, for ease in manufacture, flexibility and strength.

A bridge has to be designed such that it can withstand the load of the flowing traffic, the force of winds and its own weight. Similarly, in the design of buildings the use of beams and columns is very common. In both the cases, the overcoming of the problem of bending of beam under a load is of prime importance. The beam should not bend too much or break. Let us consider the case of a beam loaded at the centre and supported near its ends as shown in Fig. 8.6. A bar of length l, breadth b, and depth d when loaded at the centre by a load W sags by an amount given by

$$\delta = W l^3/(4bd^3 Y) \qquad (8.16)$$

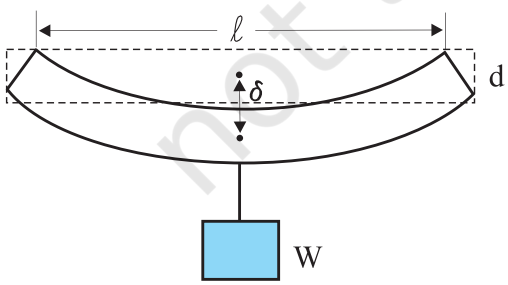

Fig. 8.6 *A beam supported at the ends and loaded at the centre.*

This relation can be derived using what you have already learnt and a little calculus. From Eq. (8.16), we see that to reduce the bending for a given load, one should use a material with a large Young's modulus Y. For a given material, increasing the depth d rather than the breadth b is more effective in reducing the bending, since δ is proportional to d^{-3} and only to b^{-1}(of course the length l of the span should be as small as possible). But on increasing the depth, unless the load is exactly at the right place (difficult to arrange in a bridge with moving traffic), the deep bar may bend as shown in Fig. 8.7(b). This is called buckling. To avoid this, a common compromise is the cross-sectional shape shown in Fig. 8.7(c). This section provides a large load-bearing surface and enough depth to prevent bending. This shape reduces the weight of the beam without sacrificing the strength and hence reduces the cost.

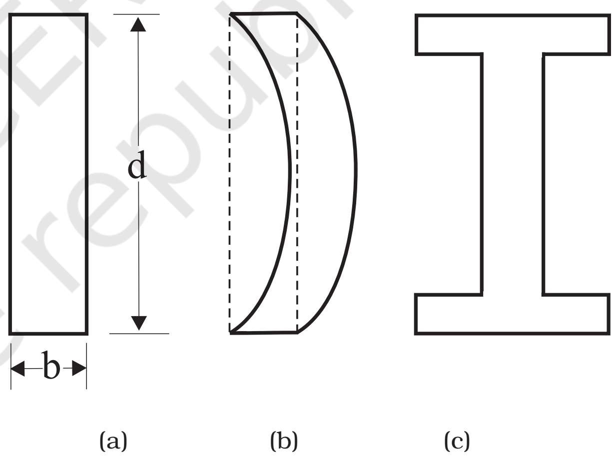

Fig. 8.7 *Different cross-sectional shapes of a beam. (a) Rectangular section of a bar; (b) A thin bar and how it can buckle; (c) Commonly used section for a load bearing bar.*

The use of pillars or columns is also very common in buildings and bridges. A pillar with rounded ends as shown in Fig. 8.9(a) supports less load than that with a distributed shape at the ends [Fig. 8.9(b)]. The precise design of a bridge or a building has to take into account the conditions under which it will function, the cost and long period, reliability of usable material, etc.

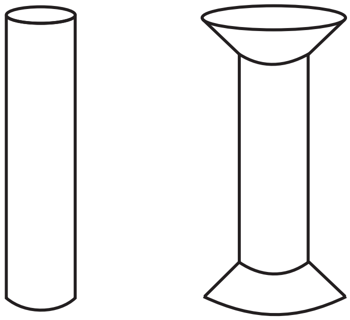

Fig. 8.8 *Pillars or columns: (a) a pillar with rounded ends, (b) Pillar with distributed ends.*

The answer to the question why the maximum height of a mountain on earth is ~10 km can also be provided by considering the elastic properties of rocks. A mountain base is not under uniform compression and this provides some shearing stress to the rocks under which they can flow. The stress due to all the material on the top should be less than the critical shearing stress at which the rocks flow.

At the bottom of a mountain of height h, the force per unit area due to the weight of the mountain is $h\rho g$ where ρ is the density of the material of the mountain and g is the acceleration due to gravity. The material at the bottom experiences this force in the vertical direction, and the sides of the mountain are free. Therefore, this is not a case of pressure or bulk compression. There is a shear component, approximately $h\rho g$ itself. Now the elastic limit for a typical rock is 30×10^7 N m^{-2}. Equating this to $h\rho g$, with $\rho = 3 \times 10^3$ kg m^{-3} gives

$$h\rho g = 30 \times 10^7 \text{ N m}^{-2} .$$

$$h = 30 \times 10^7 \text{ N m}^{-2}/(3 \times 10^3 \text{ kg m}^{-3} \times 10 \text{ m s}^{-2})$$

$$= 10 \text{ km}$$

which is more than the height of Mt. Everest!

SUMMARY

1. Stress is the restoring force per unit area and strain is the fractional change in dimension. In general there are three types of stresses (a) tensile stress — longitudinal stress (associated with stretching) or compressive stress (associated with compression), (b) shearing stress, and (c) hydraulic stress.
2. For small deformations, stress is directly proportional to the strain for many materials. This is known as Hooke's law. The constant of proportionality is called modulus of elasticity. Three elastic moduli *viz.*, Young's modulus, shear modulus and bulk modulus are used to describe the elastic behaviour of objects as they respond to deforming forces that act on them.
   A class of solids called elastomers does not obey Hooke's law.
3. When an object is under tension or compression, the Hooke's law takes the form
   $$F/A = Y\Delta L/L$$
   where $\Delta L/L$ is the tensile or compressive strain of the object, F is the magnitude of the applied force causing the strain, A is the cross-sectional area over which F is applied (perpendicular to A) and Y is the Young's modulus for the object. The stress is F/A.
4. A pair of forces when applied parallel to the upper and lower faces, the solid deforms so that the upper face moves sideways with respect to the lower. The horizontal displacement ΔL of the upper face is perpendicular to the vertical height L. This type of deformation is called shear and the corresponding stress is the shearing stress. This type of stress is possible only in solids.
   In this kind of deformation the Hooke's law takes the form
   $$F/A = G \times \Delta L/L$$
   where ΔL is the displacement of one end of object in the direction of the applied force F, and G is the shear modulus.
5. When an object undergoes hydraulic compression due to a stress exerted by a surrounding fluid, the Hooke's law takes the form
   $$p = B\,(\Delta V/V),$$
   where p is the pressure (hydraulic stress) on the object due to the fluid, $\Delta V/V$ (the volume strain) is the absolute fractional change in the object's volume due to that pressure and B is the bulk modulus of the object.

POINTS TO PONDER

1. In the case of a wire, suspended from celing and stretched under the action of a weight (F) suspended from its other end, the force exerted by the ceiling on it is equal and opposite to the weight. However, the tension at any cross-section A of the wire is just F and not $2F$. Hence, tensile stress which is equal to the tension per unit area is equal to F/A.
2. Hooke's law is valid only in the linear part of stress-strain curve.
3. The Young's modulus and shear modulus are relevant only for solids since only solids have lengths and shapes.
4. Bulk modulus is relevant for solids, liquid and gases. It refers to the change in volume when every part of the body is under the uniform stress so that the shape of the body remains unchanged.
5. Metals have larger values of Young's modulus than alloys and elastomers. A material with large value of Young's modulus requires a large force to produce small changes in its length.
6. In daily life, we feel that a material which stretches more is more elastic, but it a is misnomer. In fact material which stretches to a lesser extent for a given load is considered to be more elastic.
7. In general, a deforming force in one direction can produce strains in other directions also. The proportionality between stress and strain in such situations cannot be described by just one elastic constant. For example, for a wire under longitudinal strain, the lateral dimensions (radius of cross section) will undergo a small change, which is described by another elastic constant of the material (called *Poisson ratio*).
8. Stress is not a vector quantity since, unlike a force, the stress cannot be assigned a specific direction. Force acting on the portion of a body on a specified side of a section has a definite direction.

EXERCISES

8.1 A steel wire of length 4.7 m and cross-sectional area 3.0×10^{-5} m^2 stretches by the same amount as a copper wire of length 3.5 m and cross-sectional area of 4.0×10^{-5} m^2 under a given load. What is the ratio of the Young's modulus of steel to that of copper?

8.2 Figure 8.9 shows the strain-stress curve for a given material. What are (a) Young's modulus and (b) approximate yield strength for this material?

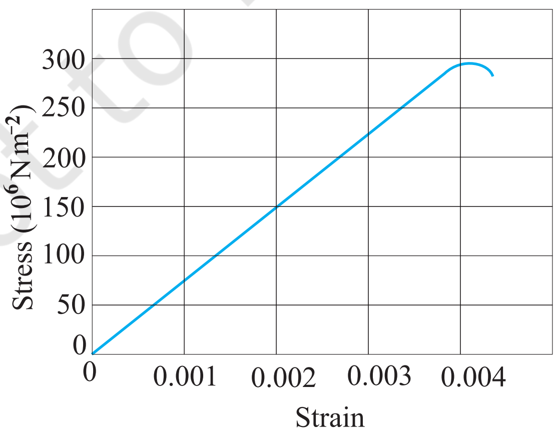

Fig. 8.9

8.3 The stress-strain graphs for materials *A* and *B* are shown in Fig. 8.10.

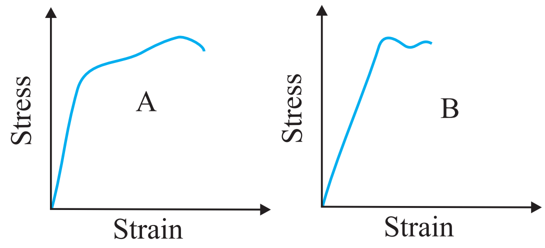

Fig. 8.10

The graphs are drawn to the same scale.
(a) Which of the materials has the greater Young's modulus?
(b) Which of the two is the stronger material?

8.4 Read the following two statements below carefully and state, with reasons, if it is true or false.
(a) The Young's modulus of rubber is greater than that of steel;
(b) The stretching of a coil is determined by its shear modulus.

8.5 Two wires of diameter 0.25 cm, one made of steel and the other made of brass are loaded as shown in Fig. 8.11. The unloaded length of steel wire is 1.5 m and that of brass wire is 1.0 m. Compute the elongations of the steel and the brass wires.

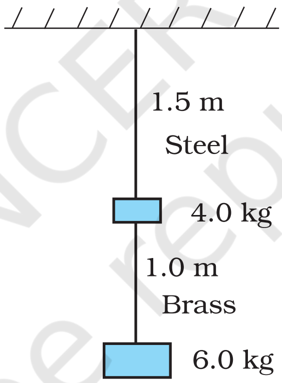

Fig. 8.11

8.6 The edge of an aluminium cube is 10 cm long. One face of the cube is firmly fixed to a vertical wall. A mass of 100 kg is then attached to the opposite face of the cube. The shear modulus of aluminium is 25 GPa. What is the vertical deflection of this face?

8.7 Four identical hollow cylindrical columns of mild steel support a big structure of mass 50,000 kg. The inner and outer radii of each column are 30 and 60 cm respectively. Assuming the load distribution to be uniform, calculate the compressional strain of each column.

8.8 A piece of copper having a rectangular cross-section of 15.2 mm × 19.1 mm is pulled in tension with 44,500 N force, producing only elastic deformation. Calculate the resulting strain?

8.9 A steel cable with a radius of 1.5 cm supports a chairlift at a ski area. If the maximum stress is not to exceed 10^8 N m^{-2}, what is the maximum load the cable can support ?

8.10 A rigid bar of mass 15 kg is supported symmetrically by three wires each 2.0 m long. Those at each end are of copper and the middle one is of iron. Determine the ratios of their diameters if each is to have the same tension.

8.11 A 14.5 kg mass, fastened to the end of a steel wire of unstretched length 1.0 m, is whirled in a vertical circle with an angular velocity of 2 rev/s at the bottom of the circle. The cross-sectional area of the wire is 0.065 cm^2. Calculate the elongation of the wire when the mass is at the lowest point of its path.

8.12 Compute the bulk modulus of water from the following data: Initial volume = 100.0 litre, Pressure increase = 100.0 atm (1 atm = 1.013×10^5 Pa), Final volume = 100.5 litre. Compare the bulk modulus of water with that of air (at constant temperature). Explain in simple terms why the ratio is so large.

8.13 What is the density of water at a depth where pressure is 80.0 atm, given that its density at the surface is 1.03×103 kg m^{-3}?

8.14 Compute the fractional change in volume of a glass slab, when subjected to a hydraulic pressure of 10 atm.

8.15 Determine the volume contraction of a solid copper cube, 10 cm on an edge, when subjected to a hydraulic pressure of 7.0×10^6 Pa.

8.16 How much should the pressure on a litre of water be changed to compress it by 0.10%? carry one quarter of the load.

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CHAPTER NINE

MECHANICAL PROPERTIES OF FLUIDS

9.1 INTRODUCTION

In this chapter, we shall study some common physical properties of liquids and gases. Liquids and gases can flow and are therefore, called fluids. It is this property that distinguishes liquids and gases from solids in a basic way.

Fluids are everywhere around us. Earth has an envelop of air and two-thirds of its surface is covered with water. Water is not only necessary for our existence; every mammalian body constitute mostly of water. All the processes occurring in living beings including plants are mediated by fluids. Thus understanding the behaviour and properties of fluids is important.

How are fluids different from solids? What is common in liquids and gases? Unlike a solid, a fluid has no definite shape of its own. Solids and liquids have a fixed volume, whereas a gas fills the entire volume of its container. We have learnt in the previous chapter that the volume of solids can be changed by stress. The volume of solid, liquid or gas depends on the stress or pressure acting on it. When we talk about fixed volume of solid or liquid, we mean its volume under atmospheric pressure. The difference between gases and solids or liquids is that for solids or liquids the change in volume due to change of external pressure is rather small. In other words solids and liquids have much lower compressibility as compared to gases.

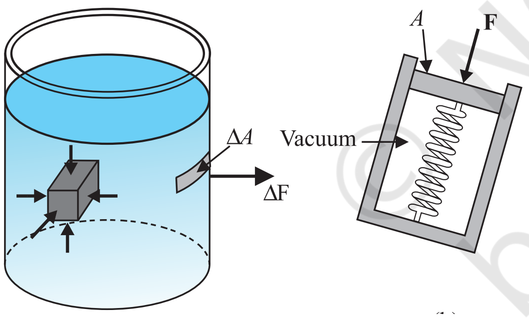
Shear stress can change the shape of a solid keeping its volume fixed. The key property of fluids is that they offer very little resistance to shear stress; their shape changes by application of very small shear stress. The shearing stress of fluids is about million times smaller than that of solids.

9.2 PRESSURE

A sharp needle when pressed against our skin pierces it. Our skin, however, remains intact when a blunt object with a wider contact area (say the back of a spoon) is pressed against it with the same force. If an elephant were to step on a man's chest, his ribs would crack. A circus performer across whose chest a large, light but strong wooden plank is placed first, is saved from this accident. Such everyday experiences convince us that both the force and its coverage area are important. Smaller the area on which the force acts, greater is the impact. This impact is known as pressure.

When an object is submerged in a fluid at rest, the fluid exerts a force on its surface. This force is always normal to the object's surface. This is so because if there were a component of force parallel to the surface, the object will also exert a force on the fluid parallel to it; as a consequence of Newton's third law. This force will cause the fluid to flow parallel to the surface. Since the fluid is at rest, this cannot happen. Hence, the force exerted by the fluid at rest has to be perpendicular to the surface in contact with it. This is shown in Fig.9.1(a).

The normal force exerted by the fluid at a point may be measured. An idealised form of one such pressure-measuring device is shown in Fig. 9.1(b). It consists of an evacuated chamber with a spring that is calibrated to measure the force acting on the piston. This device is placed at a point inside the fluid. The inward force exerted by the fluid on the piston is balanced by the outward spring force and is thereby measured.



(a) (b)

Fig. 9.1 *(a) The force exerted by the liquid in the beaker on the submerged object or on the walls is normal (perpendicular) to the surface at all points.*
(b) An idealised device for measuring pressure.

If F is the magnitude of this normal force on the piston of area A then the **average pressure** P_{av} is defined as the normal force acting per unit area.

$$P_{av} = \frac{F}{A} \tag{9.1}$$

In principle, the piston area can be made arbitrarily small. The pressure is then defined in a limiting sense as

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{9.2}$$

Pressure is a scalar quantity. We remind the reader that it is the component of the force normal to the area under consideration and not the (vector) force that appears in the numerator in Eqs. (9.1) and (9.2). Its dimensions are $[ML^{-1}T^{-2}]$. The SI unit of pressure is N m^{-2}. It has been named as pascal (Pa) in honour of the French scientist Blaise Pascal (1623-1662) who carried out pioneering studies on fluid pressure. A common unit of pressure is the atmosphere (atm), i.e. the pressure exerted by the atmosphere at sea level (1 atm = 1.013×10^5 Pa).

Another quantity, that is indispensable in describing fluids, is the density ρ. For a fluid of mass m occupying volume V,

$$\rho = \frac{m}{V} \tag{9.3}$$

The dimensions of density are $[ML^{-3}]$. Its SI unit is kg m^{-3}. It is a positive scalar quantity. A liquid is largely incompressible and its density is therefore, nearly constant at all pressures. Gases, on the other hand exhibit a large variation in densities with pressure.

The density of water at 4°C (277 K) is 1.0×10^3 kg m^{-3}. The relative density of a substance is the ratio of its density to the density of water at 4°C. It is a dimensionless positive scalar quantity. For example the relative density of aluminium is 2.7. Its density is 2.7×10^3 kg m^{-3}. The densities of some common fluids are displayed in Table 9.1.

Table 9.1 Densities of some common fluids at STP*

Fluid	**ρ (kg m^{-3})**
Water	1.00×10^3
Sea water	1.03×10^3
Mercury	13.6×10^3
Ethyl alcohol	0.806×10^3
Whole blood	1.06×10^3
Air	1.29
Oxygen	1.43
Hydrogen	9.0×10^{-2}
Interstellar space	$\approx 10^{-20}$

* *STP means standard temperature (0°C) and 1 atm pressure.*

Example 9.1 The two thigh bones (femurs), each of cross-sectional area 10 cm² support the upper part of a human body of mass 40 kg. Estimate the average pressure sustained by the femurs.

Answer Total cross-sectional area of the femurs is $A = 2 \times 10\ \text{cm}^2 = 20 \times 10^{-4}\ \text{m}^2$. The force acting on them is $F = 40$ kg wt = 400 N (taking $g = 10\ \text{m s}^{-2}$). This force is acting vertically down and hence, normally on the femurs. Thus, the average pressure is

$$P_{av} = \frac{F}{A} = 2 \times 10^5\ \text{N m}^{-2}$$

9.2.1 Pascal's Law

The French scientist Blaise Pascal observed that the pressure in a fluid at rest is the same at all points if they are at the same height. This fact may be demonstrated in a simple way.

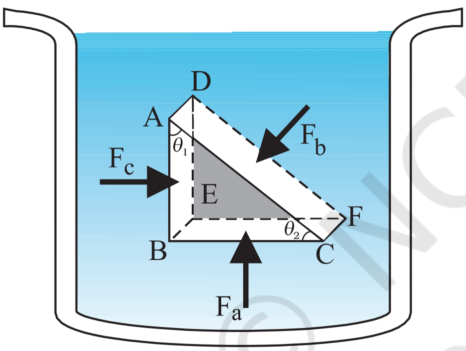


Fig. 9.2 *Proof of Pascal's law. ABC-DEF is an element of the interior of a fluid at rest. This element is in the form of a right-angled prism. The element is small so that the effect of gravity can be ignored, but it has been enlarged for the sake of clarity.*

Fig. 9.2 shows an element in the interior of a fluid at rest. This element ABC-DEF is in the form of a right-angled prism. In principle, this prismatic element is very small so that every part of it can be considered at the same depth from the liquid surface and therefore, the effect of the gravity is the same at all these points. But for clarity we have enlarged this element. The forces on this element are those exerted by the rest of the fluid and they must be normal to the surfaces of the element as discussed above. Thus, the fluid exerts pressures P_a, P_b and P_c on this element of area corresponding to the normal forces F_a, F_b and F_c as shown in Fig. 9.2 on the faces BEFC, ADFC and ADEB denoted by A_a, A_b and A_c respectively. Then

$F_b \sin\theta = F_c$, $\quad F_b \cos\theta = F_a$ (by equilibrium)

$A_b \sin\theta = A_c$, $\quad A_b \cos\theta = A_a$ (by geometry)

Thus,

$$\frac{F_b}{A_b} = \frac{F_c}{A_c} = \frac{F_a}{A_a}; \qquad P_b = P_c = P_a \tag{9.4}$$

Hence, pressure exerted is same in all directions in a fluid at rest. It again reminds us that like other types of stress, pressure is not a vector quantity. No direction can be assigned to it. The force against any area within (or bounding) a fluid at rest and under pressure is normal to the area, regardless of the orientation of the area.

Now consider a fluid element in the form of a horizontal bar of uniform cross-section. The bar is in equilibrium. The horizontal forces exerted at its two ends must be balanced or the pressure at the two ends should be equal. This proves that for a liquid in equilibrium the pressure is same at all points in a horizontal plane. Suppose the pressure were not equal in different parts of the fluid, then there would be a flow as the fluid will have some net force acting on it. Hence in the absence of flow the pressure in the fluid must be same everywhere in a horizontal plane.

9.2.2 Variation of Pressure with Depth

Consider a fluid at rest in a container. In Fig. 9.3 point 1 is at height h above a point 2. The pressures at points 1 and 2 are P_1 and P_2 respectively. Consider a cylindrical element of fluid having area of base A and height h. As the fluid is at rest the resultant horizontal forces should be zero and the resultant vertical forces should balance the weight of the element. The forces acting in the vertical direction are due to the fluid pressure at the top (P_1A) acting downward, at the bottom (P_2A) acting upward. If mg is weight of the fluid in the cylinder we have

$$(P_2 - P_1)\,A = mg \tag{9.5}$$

Now, if ρ is the mass density of the fluid, we have the mass of fluid to be $m = \rho V = \rho h A$ so that

$$P_2 - P_1 = \rho g h \tag{9.6}$$

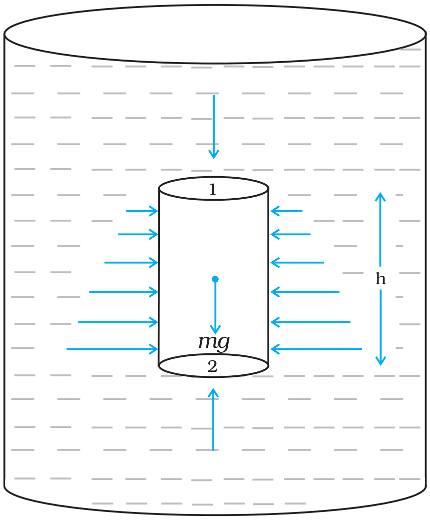


Fig.9.3 *Fluid under gravity. The effect of gravity is illustrated through pressure on a vertical cylindrical column.*

Pressure difference depends on the vertical distance h between the points (1 and 2), mass density of the fluid ρ and acceleration due to gravity g. If the point 1 under discussion is shifted to the top of the fluid (say, water), which is open to the atmosphere, P_1 may be replaced by atmospheric pressure (P_a) and we replace P_2 by P. Then Eq. (9.6) gives

$$P = P_a + \rho g h \tag{9.7}$$

Thus, the pressure P, at depth below the surface of a liquid open to the atmosphere is greater than atmospheric pressure by an amount ρgh. The excess of pressure, $P - P_a$, at depth h is called a **gauge pressure** at that point.

The area of the cylinder is not appearing in the expression of absolute pressure in Eq. (9.7). Thus, the height of the fluid column is important and not cross-sectional or base area or the shape of the container. The liquid pressure is the same at all points at the same horizontal level (same depth). The result is appreciated through the example of **hydrostatic paradox**. Consider three vessels A, B and C [Fig.9.4] of different shapes. They are connected at the bottom by a horizontal pipe. On filling with water, the level in the three vessels is the same, though they hold different amounts of water. This is so because water at the bottom has the same pressure below each section of the vessel.

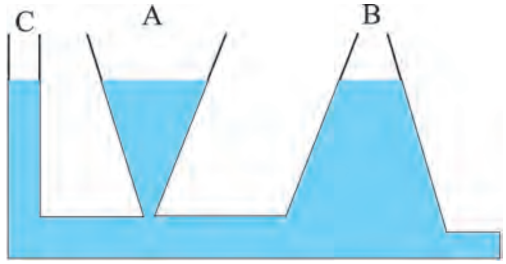


Fig 9.4 *Illustration of hydrostatic paradox. The three vessels A, B and C contain different amounts of liquids, all upto the same height.*

Example 9.2 What is the pressure on a swimmer 10 m below the surface of a lake?

Answer Here

h = 10 m *and* ρ = 1000 kg m^{-3}. Take g = 10 m s^{-2}

From Eq. (9.7)

$P = P_a + \rho gh$

$= 1.01 \times 10^5$ Pa + 1000 kg m^{-3} $\times$ 10 m s^{-2} $\times$ 10 m

$= 2.01 \times 10^5$ Pa

$\approx$ 2 atm

This is a 100% increase in pressure from surface level. At a depth of 1 km, the increase in pressure is 100 atm! Submarines are designed to withstand such enormous pressures.

9.2.3 Atmospheric Pressure and Gauge Pressure

The pressure of the atmosphere at any point is equal to the weight of a column of air of unit cross-sectional area extending from that point to the top of the atmosphere. At sea level, it is 1.013×10^5 Pa (1 atm). Italian scientist Evangelista Torricelli (1608–1647) devised for the first time a method for measuring atmospheric pressure. A long glass tube closed at one end and filled with mercury is inverted into a trough of mercury as shown in Fig.9.5 (a). This device is known as 'mercury barometer'. The space above the mercury column in the tube contains only mercury vapour whose pressure P is so small that it may be neglected. Thus, the pressure at Point A=0. The pressure inside the coloumn at Point B must be the same as the pressure at Point C, which is atmospheric pressure, P_a.

$$P_a = \rho g h \tag{9.8}$$

where ρ is the density of mercury and h is the height of the mercury column in the tube.

In the experiment it is found that the mercury column in the barometer has a height of about 76 cm at sea level equivalent to one atmosphere (1 atm). This can also be obtained using the value of ρ in Eq. (9.8). A common way of stating pressure is in terms of cm or mm of mercury (Hg). A pressure equivalent of 1 mm is called a torr (after Torricelli).

1 torr = 133 Pa.

The mm of Hg and torr are used in medicine and physiology. In meteorology, a common unit is the bar and millibar.

1 bar = 10^5 Pa

An open tube manometer is a useful instrument for measuring pressure differences. It consists of a U-tube containing a suitable liquid i.e., a low density liquid (such as oil) for measuring small pressure differences and a high density liquid (such as mercury) for large pressure differences. One end of the tube is open to the atmosphere and the other end is connected to the system whose pressure we want to measure [see Fig. 9.5 (b)]. The pressure P at A is equal to pressure at point B. What we normally measure is the gauge pressure, which is $P - P_a$, given by Eq. (9.8) and is proportional to manometer height h.

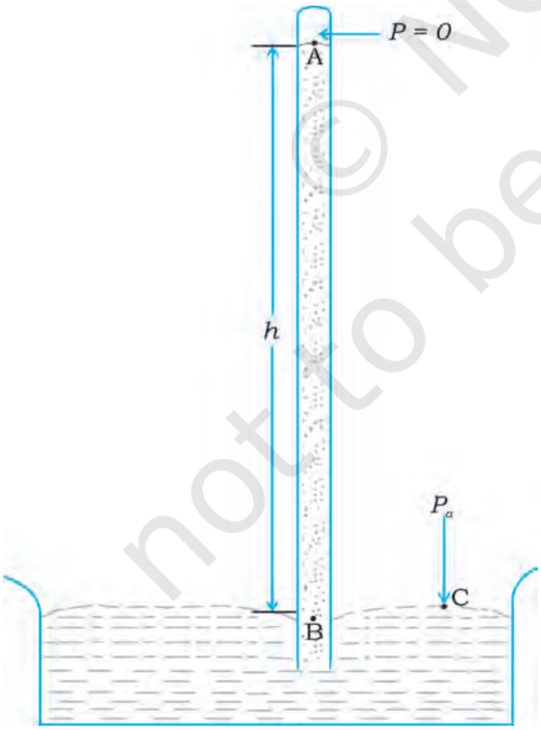
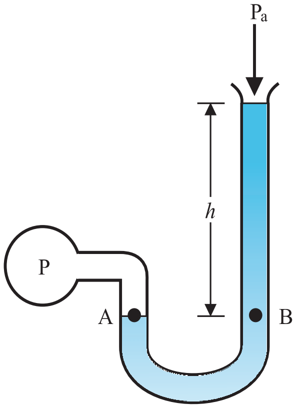


Fig 9.5 *(a) The mercury barometer.*



(b) The open tube manometer

Fig 9.5 *Two pressure measuring devices.*

Pressure is same at the same level on both sides of the U-tube containing a fluid. For liquids, the density varies very little over wide ranges in pressure and temperature and we can treat it safely as a constant for our present purposes. Gases on the other hand, exhibits large variations of densities with changes in pressure and temperature. Unlike gases, liquids are, therefore, largely treated as incompressible.

Example 9.3 The density of the atmosphere at sea level is 1.29 kg/m³. Assume that it does not change with altitude. Then how high would the atmosphere extend?

Answer We use Eq. (9.7)

$\rho g h = 1.29 \text{ kg m}^{-3} \times 9.8 \text{ m s}^2 \times h \text{ m} = 1.01 \times 10^5 \text{ Pa}$

$\therefore\ h = 7989 \text{ m} \approx 8 \text{ km}$

In reality the density of air decreases with height. So does the value of g. The atmospheric cover extends with decreasing pressure over 100 km. We should also note that the sea level atmospheric pressure is not always 760 mm of Hg. A drop in the Hg level by 10 mm or more is a sign of an approaching storm.

Example 9.4 At a depth of 1000 m in an ocean (a) what is the absolute pressure? (b) What is the gauge pressure? (c) Find the force acting on the window of area 20 cm × 20 cm of a submarine at this depth, the interior of which is maintained at sea-level atmospheric pressure. (The density of sea water is 1.03×10^3 kg m^{-3}, $g = 10$ m s^{-2}.)

Answer Here $h = 1000$ m and $\rho = 1.03 \times 10^3$ kg m^{-3}.

(a) From Eq. (9.6), absolute pressure

$P = P_a + \rho gh$

$= 1.01 \times 10^5$ Pa

$\quad + 1.03 \times 10^3$ kg m^{-3} $\times$ 10 m s^{-2} $\times$ 1000 m

$= 104.01 \times 10^5$ Pa

≈ 104 atm

(b) Gauge pressure is $P - P_a = \rho gh = P_g$

$P_g = 1.03 \times 10^3$ kg m^{-3} $\times$ 10 ms^2 $\times$ 1000 m

$= 103 \times 10^5$ Pa

≈ 103 atm

(c) The pressure outside the submarine is $P = P_a + \rho gh$ and the pressure inside it is P_a. Hence, the net pressure acting on the window is gauge pressure, $P_g = \rho gh$. Since the area of the window is $A = 0.04$ m^2, the force acting on it is

$F = P_g A = 103 \times 10^5 \text{ Pa} \times 0.04 \text{ m}^2 = 4.12 \times 10^5 \text{ N}$

9.2.4 Hydraulic Machines

Let us now consider what happens when we change the pressure on a fluid contained in a vessel. Consider a horizontal cylinder with a piston and three vertical tubes at different points [Fig. 9.6 (a)]. The pressure in the horizontal cylinder is indicated by the height of liquid column in the vertical tubes. It is necessarily the same in all. If we push the piston, the fluid level rises in all the tubes, again reaching the same level in each one of them.

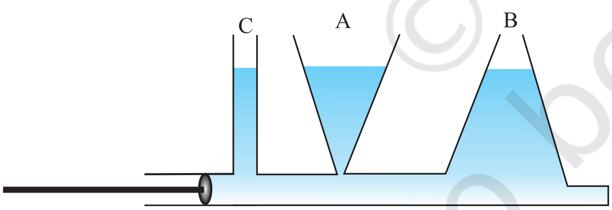


Fig 9.6 *(a) Whenever external pressure is applied on any part of a fluid in a vessel, it is equally transmitted in all directions.*

This indicates that when the pressure on the cylinder was increased, it was distributed uniformly throughout. We can say **whenever external pressure is applied on any part of a fluid contained in a vessel, it is transmitted undiminished and equally in all directions. This is another form of the Pascal's law and it has many applications in daily life.**

A number of devices, such as **hydraulic lift** and **hydraulic brakes,** are based on the Pascal's law. In these devices, fluids are used for transmitting pressure. In a hydraulic lift, as shown in Fig. 9.6 (b), two pistons are separated by the space filled with a liquid. A piston of small cross-section A_1 is used to exert a force F_1 directly on the liquid. The pressure $P = \dfrac{F_1}{A_1}$ is transmitted throughout the liquid to the larger cylinder attached with a larger piston of area A_2, which results in an upward force of $P \times A_2$. Therefore, the piston is capable of supporting a large force (large weight of, say a car, or a truck, placed on the platform) $F_2 = PA_2 = \dfrac{F_1 A_2}{A_1}$. By changing the force at A_1, the platform can be moved up or down. Thus, the applied force has been increased by a factor of $\dfrac{A_2}{A_1}$ and this factor is the mechanical advantage of the device. The example below clarifies it.

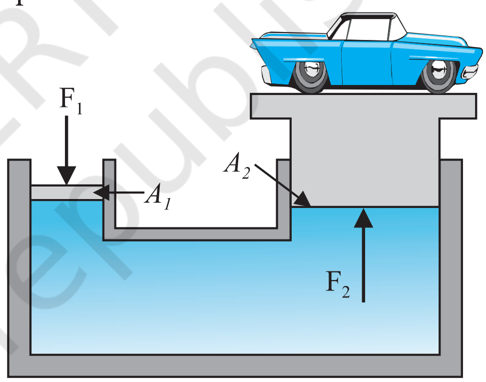


Fig 9.6 *(b) Schematic diagram illustrating the principle behind the hydraulic lift, a device used to lift heavy loads.*

Example 9.5 Two syringes of different cross-sections (without needles) filled with water are connected with a tightly fitted rubber tube filled with water. Diameters of the smaller piston and larger piston are 1.0 cm and 3.0 cm respectively. (a) Find the force exerted on the larger piston when a force of 10 N is applied to the smaller piston. (b) If the smaller piston is pushed in through 6.0 cm, how much does the larger piston move out?

Answer (a) Since pressure is transmitted undiminished throughout the fluid,

$$F_2 = \frac{A_2}{A_1} F_1 = \frac{\pi \left(3/2 \times 10^{-2}\,\text{m}\right)^2}{\pi \left(1/2 \times 10^{-2}\,\text{m}\right)^2} \times 10\,\text{N}$$

$$= 90\ \text{N}$$

(b) Water is considered to be perfectly incompressible. Volume covered by the movement of smaller piston inwards is equal to volume moved outwards due to the larger piston.

$$L_1 A_1 = L_2 A_2$$

$$L_2 = \frac{A_1}{A_2} L_1 = \frac{\pi \left(1/2 \times 10^{-2}\,\text{m}\right)^2}{\pi \left(3/2 \times 10^{-2}\,\text{m}\right)^2} \times 6 \times 10^{-2}\,\text{m}$$

$$\simeq 0.67 \times 10^{-2}\,\text{m} = 0.67\ \text{cm}$$

Note, atmospheric pressure is common to both pistons and has been ignored.

Example 9.6 In a car lift compressed air exerts a force F_1 on a small piston having a radius of 5.0 cm. This pressure is transmitted to a second piston of radius 15 cm (Fig 9.7). If the mass of the car to be lifted is 1350 kg, calculate F_1. What is the pressure necessary to accomplish this task? ($g = 9.8\ \text{ms}^{-2}$).

Answer Since pressure is transmitted undiminished throughout the fluid,

$$F_1 = \frac{A_1}{A_2} F_2 = \frac{\pi (5 \times 10^{-2}\,\text{m})^2}{\pi (15 \times 10^{-2}\,\text{m})^2} (1350\,\text{kg} \times 9.8\,m\,s^{-2})$$

$$= 1470\ \text{N}$$

$$\approx 1.5 \times 10^3\ \text{N}$$

The air pressure that will produce this force is

$$P = \frac{F_1}{A_1} = \frac{1.5 \times 10^3\,\text{N}}{\pi \left(5 \times 10^{-2}\right)^2\,\text{m}} = 1.9 \times 10^5\,\text{Pa}$$

This is almost double the atmospheric pressure.

Hydraulic brakes in automobiles also work on the same principle. When we apply a little force on the pedal with our foot the master piston moves inside the master cylinder, and the pressure caused is transmitted through the brake oil to act on a piston of larger area. A large force acts on the piston and is pushed down expanding the brake shoes against brake lining. In this way, a small force on the pedal produces a large retarding force on the wheel. An important advantage of the system is that the pressure set up by pressing pedal is transmitted equally to all cylinders attached to the four wheels so that the braking effort is equal on all wheels.

9.3 STREAMLINE FLOW

So far we have studied fluids at rest. The study of the fluids in motion is known as fluid dynamics. When a water tap is turned on slowly, the water flow is smooth initially, but loses its smoothness when the speed of the outflow is increased. In studying the motion of fluids, we focus our attention on what is happening to various fluid particles at a particular point in space at a particular time. The flow of the fluid is said to be **steady** if at any given point, the velocity of each passing fluid particle remains constant in time. This does not mean that the velocity at different points in space is same. The velocity of a particular particle may change as it moves from one point to another. That is, at some other point the particle may have a different velocity, but every other particle which passes the second point behaves exactly as the previous particle that has just passed that point. Each particle follows a smooth path, and the paths of the particles do not cross each other.

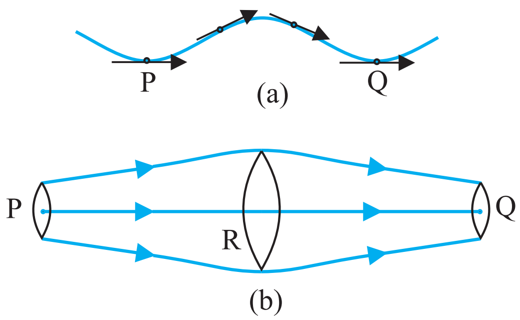


Fig. 9.7 *The meaning of streamlines. (a) A typical trajectory of a fluid particle. (b) A region of streamline flow.*

The path taken by a fluid particle under a steady flow is a **streamline**. It is defined as a curve whose tangent at any point is in the direction of the fluid velocity at that point. Consider the path of a particle as shown in Fig.9.7 (a), the curve describes how a fluid particle moves with time. The curve PQ is like a permanent map of fluid flow, indicating how the fluid streams. No two streamlines can cross, for if they do, an oncoming fluid particle can go either one way or the other and the flow would not be steady. Hence, in steady flow, the map of flow is stationary in time. How do we draw closely spaced streamlines ? If we intend to show streamline of every flowing particle, we would end up with a continuum of lines. Consider planes perpendicular to the direction of fluid flow e.g., at three points P, R and Q in Fig.9.7 (b). The plane pieces are so chosen that their boundaries be determined by the same set of streamlines. This means that number of fluid particles crossing the surfaces as indicated at P, R and Q is the same. If area of cross-sections at these points are A_P,A_R and A_Q and speeds of fluid particles are v_P, v_R and v_Q, then mass of fluid Δm_P crossing at A_P in a small interval of time Δt is $\rho_P A_P v_P \Delta t$. Similarly mass of fluid Δm_R flowing or crossing at A_R in a small interval of time Δt is $\rho_R A_R v_R \Delta t$ and mass of fluid Δm_Q is $\rho_Q A_Q v_Q \Delta t$ crossing at A_Q. The mass of liquid flowing out equals the mass flowing in, holds in all cases. Therefore,

$$\rho_P A_P v_P \Delta t = \rho_R A_R v_R \Delta t = \rho_Q A_Q v_Q \Delta t \qquad (9.9)$$

For flow of incompressible fluids

$$\rho_P = \rho_R = \rho_Q$$

Equation (9.9) reduces to

$$A_P v_P = A_R v_R = A_Q v_Q \qquad (9.10)$$

which is called the **equation of continuity** and it is a statement of conservation of mass in flow of incompressible fluids. In general

$$Av = \text{constant} \qquad (9.11)$$

Av gives the volume flux or flow rate and remains constant throughout the pipe of flow. Thus, at narrower portions where the streamlines are closely spaced, velocity increases and its vice versa. From (Fig 9.7b) it is clear that $A_R > A_Q$ or $v_R < v_Q$, the fluid is accelerated while passing from R to Q. This is associated with a change in pressure in fluid flow in horizontal pipes.

Steady flow is achieved at low flow speeds. Beyond a limiting value, called critical speed, this flow loses steadiness and becomes **turbulent**. One sees this when a fast flowing stream encounters rocks, small foamy whirlpool-like regions called 'white water rapids are formed.

Figure 9.8 displays streamlines for some typical flows. For example, Fig. 9.8(a) describes a laminar flow where the velocities at different points in the fluid may have different magnitudes but their directions are parallel. Figure 9.8 (b) gives a sketch of turbulent flow.

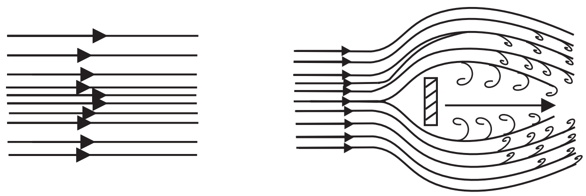


Fig. 9.8 *(a) Some streamlines for fluid flow. (b) A jet of air striking a flat plate placed perpendicular to it. This is an example of turbulent flow.*

9.4 BERNOULLI'S PRINCIPLE

Fluid flow is a complex phenomenon. But we can obtain some useful properties for steady or streamline flows using the conservation of energy.

Consider a fluid moving in a pipe of varying cross-sectional area. Let the pipe be at varying heights as shown in Fig. 9.9. We now suppose that an incompressible fluid is flowing through the pipe in a steady flow. Its velocity must change as a consequence of equation of continuity. A force is required to produce this acceleration, which is caused by the fluid surrounding it, the pressure must be different in different regions. Bernoulli's equation is a general expression that relates the pressure difference between two points in a pipe to both velocity changes (kinetic energy change) and elevation (height) changes (potential energy change). The Swiss Physicist Daniel Bernoulli developed this relationship in 1738.

Consider the flow at two regions 1 (i.e., BC) and 2 (i.e., DE). Consider the fluid initially lying between B and D. In an infinitesimal time interval Δt, this fluid would have moved. Suppose v_1 is the speed at B and v_2 at D, then fluid initially at B has moved a distance $v_1\Delta t$ to C ($v_1\Delta t$ is small enough to assume constant cross-section along BC). In the same interval Δt the fluid initially at D moves to E, a distance equal to $v_2\Delta t$. Pressures P_1 and P_2 act as shown on the plane faces of areas A_1 and A_2 binding the two regions. The work done on the fluid at left end (BC) is $W_1 = P_1A_1(v_1\Delta t) = P_1\Delta V$. Since the same volume ΔV passes through both the regions (from the equation of continuity) the work done by the fluid at the other end (DE) is $W_2 = P_2A_2(v_2\Delta t) = P_2\Delta V$ or, the work done on the fluid is $-P_2\Delta V$. So the total work done on the fluid is

$$W_1 - W_2 = (P_1 - P_2)\,\Delta V$$

Part of this work goes into changing the kinetic energy of the fluid, and part goes into changing the gravitational potential energy. If the density of the fluid is ρ and $\Delta m = \rho A_1 v_1 \Delta t = \rho\Delta V$ is the mass passing through the pipe in time Δt, then change in gravitational potential energy is

$$\Delta U = \rho g \Delta V\,(h_2 - h_1)$$

The change in its kinetic energy is

$$\Delta K = \frac{1}{2}\,\rho\,\Delta V\,(v_2^{\,2} - v_1^{\,2})$$

We can employ the work – energy theorem (Chapter 6) to this volume of the fluid and this yields

$$(P_1 - P_2)\,\Delta V = \frac{1}{2}\,\rho\,\Delta V (v_2^{\,2} - v_1^{\,2}) + \rho g \Delta V\,(h_2 - h_1)$$

We now divide each term by ΔV to obtain

$$(P_1 - P_2) = \frac{1}{2}\,\rho\,(v_2^{\,2} - v_1^{\,2}) + \rho g\,(h_2 - h_1)$$

We can rearrange the above terms to obtain

$$P_1 + \frac{1}{2}\,\rho v_1^{\,2} + \rho g h_1 = P_2 + \frac{1}{2}\,\rho v_2^{\,2} + \rho g h_2 \qquad (9.12)$$

This is **Bernoulli's equation**. Since 1 and 2 refer to any two locations along the pipeline, we may write the expression in general as

$$P + \frac{1}{2}\,\rho v^2 + \rho g h = \text{constant} \qquad (9.13)$$

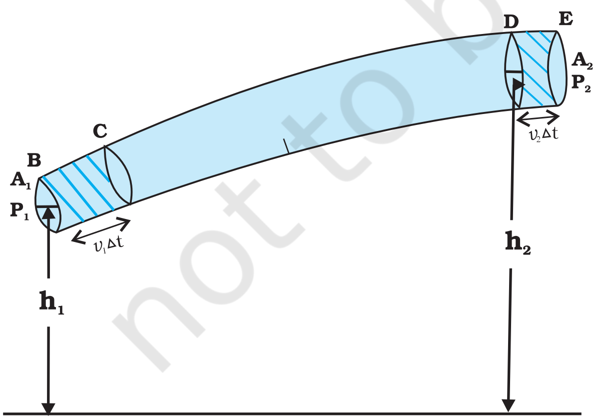


Fig. 9.9 *The flow of an ideal fluid in a pipe of varying cross section. The fluid in a section of length $v_1\Delta t$ moves to the section of length $v_2\Delta t$ in time Δt.*

In words, the Bernoulli's relation may be stated as follows: As we move along a streamline the sum of the pressure (P), the kinetic energy per unit volume $\left(\dfrac{\rho v^2}{2}\right)$ and the potential energy per unit volume ($\rho g h$) remains a constant.

Note that in applying the energy conservation principle, there is an assumption that no energy is lost due to friction. But in fact, when fluids flow, some energy does get lost due to internal friction. This arises due to the fact that in a fluid flow, the different layers of the fluid flow with different velocities. These layers exert frictional forces on each other resulting in a loss of energy. This property of the fluid is called viscosity and is discussed in more detail in a later section. The lost kinetic energy of the fluid gets converted into heat energy. Thus, Bernoulli's equation ideally applies to fluids with zero viscosity or non-viscous fluids. Another restriction on application of Bernoulli theorem is that the fluids must be incompressible, as the elastic energy of the fluid is also not taken into consideration. In practice, it has a large number of useful applications and can help explain a wide variety of phenomena for low viscosity incompressible fluids. Bernoulli's equation also does not hold for non-steady or turbulent flows, because in that situation velocity and pressure are constantly fluctuating in time.

When a fluid is at rest i.e., its velocity is zero everywhere, Bernoulli's equation becomes

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

$$(P_1 - P_2) = \rho g\,(h_2 - h_1)$$

which is same as Eq. (9.6).

9.4.1 Speed of Efflux: Torricelli's Law

The word efflux means fluid outflow. Torricelli discovered that the speed of efflux from an open tank is given by a formula identical to that of a freely falling body. Consider a tank containing a liquid of density ρ with a small hole in its side at a height y_1 from the bottom (see Fig. 9.10). The air above the liquid, whose surface is at height y_2, is at pressure P. From the equation of continuity [Eq. (9.10)] we have

$$v_1 A_1 = v_2 A_2$$

$$v_2 = \frac{A_1}{A_2} v_1$$

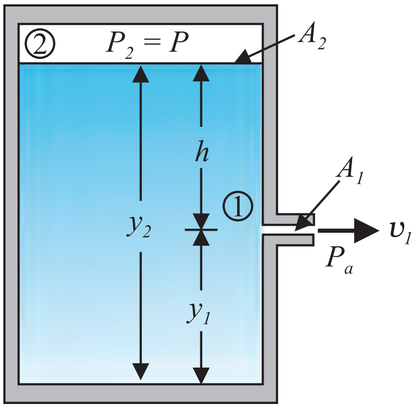


Fig. 9.10 *Torricelli's law. The speed of efflux, v_1, from the side of the container is given by the application of Bernoulli's equation. If the container is open at the top to the atmosphere then* $v_1 = \sqrt{2gh}$.

If the cross-sectional area of the tank A_2 is much larger than that of the hole ($A_2 >> A_1$), then we may take the fluid to be approximately at rest at the top, i.e., $v_2 = 0$. Now, applying the Bernoulli equation at points 1 and 2 and noting that at the hole $P_1 = P_a$, the atmospheric pressure, we have from Eq. (9.12)

$$P_a + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P + \rho g y_2$$

Taking $y_2 - y_1 = h$ we have

$$v_1 = \sqrt{2g\,h + \frac{2(P - P_a)}{\rho}} \quad (9.14)$$

When $P >> P_a$ and $2\,g\,h$ may be ignored, the speed of efflux is determined by the container pressure. Such a situation occurs in rocket propulsion. On the other hand, if the tank is open to the atmosphere, then $P = P_a$ and

$$v_1 = \sqrt{2gh} \quad (9.15)$$

This is also the speed of a freely falling body. Equation (9.15) represents **Torricelli's law**.

9.4.2 Dynamic Lift

Dynamic lift is the force that acts on a body, such as airplane wing, a hydrofoil or a spinning ball, by virtue of its motion through a fluid. In many games such as cricket, tennis, baseball, or golf, we notice that a spinning ball deviates from its parabolic trajectory as it moves through air. This deviation can be partly explained on the basis of Bernoulli's principle.

(i) **Ball moving without spin**: Fig. 9.11(a) shows the streamlines around a non-spinning ball moving relative to a fluid. From the symmetry of streamlines it is clear that the velocity of fluid (air) above and below the ball at corresponding points is the same resulting in zero pressure difference. The air therefore, exerts no upward or downward force on the ball.

(ii) **Ball moving with spin**: A ball which is spinning drags air along with it. If the surface is rough more air will be dragged. Fig 9.11(b) shows the streamlines of air for a ball which is moving and spinning at the same time. The ball is moving forward and relative to it the air is moving backwards. Therefore, the velocity of air above the ball relative to the ball is larger and below it is smaller (see Section 9.3). The stream lines, thus, get crowded above and rarified below.

This difference in the velocities of air results in the pressure difference between the lower and upper faces and there is a net upward force on the ball. This dynamic lift due to spining is called **Magnus effect**.

Aerofoil or lift on aircraft wing: Figure 9.11 (c) shows an aerofoil, which is a solid piece shaped to provide an upward dynamic lift when it moves horizontally through air. The cross-section of the wings of an aeroplane looks somewhat like the aerofoil shown in Fig. 9.11 (c) with streamlines around it. When the aerofoil moves against the wind, the orientation of the wing relative to flow direction causes the streamlines to crowd together above the wing more than those below it. The flow speed on top is higher than that below it. There is an upward force resulting in a dynamic lift of the wings and this balances the weight of the plane. The following example illustrates this.

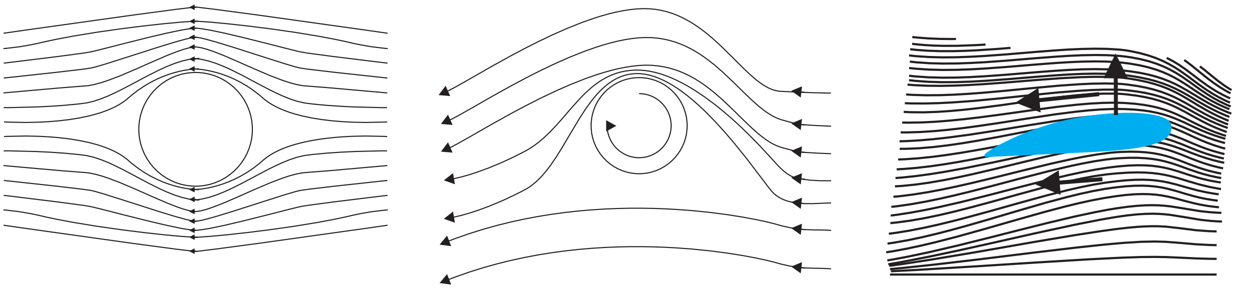


Fig 9.11 *(a) Fluid streaming past a static sphere. (b) Streamlines for a fluid around a sphere spinning clockwise. (c) Air flowing past an aerofoil.*

Example 9.7 A fully loaded Boeing aircraft has a mass of 3.3×10^5 kg. Its total wing area is 500 m^2. It is in level flight with a speed of 960 km/h. (a) Estimate the pressure difference between the lower and upper surfaces of the wings (b) Estimate the fractional increase in the speed of the air on the upper surface of the wing relative to the lower surface. [The density of air is ρ = 1.2 kg m^{-3}]

Answer (a) The weight of the Boeing aircraft is balanced by the upward force due to the pressure difference

$\Delta P \ \ A = 3.3 \times 10^5 \text{kg} \times 9.8$

$\Delta P = (3.3 \times 10^5 \text{ kg} \times 9.8 \text{ m s}^{-2}) \ / \ 500 \text{ m}^2$

$= 6.5 \times 10^3 \text{ Nm}^{-2}$

(b) We ignore the small height difference between the top and bottom sides in Eq. (9.12). The pressure difference between them is then

$$\Delta P = \frac{\rho}{2}\left(v_2^2 - v_1^2\right)$$

where v_2 is the speed of air over the upper surface and v_1 is the speed under the bottom surface.

$$\left(v_2 - v_1\right) = \frac{2\Delta P}{\rho\left(v_2 + v_1\right)}$$

Taking the average speed

$v_{av} = (v_2 + v_1)/2 = 960 \text{ km/h} = 267 \text{ m s}^{-1}$,

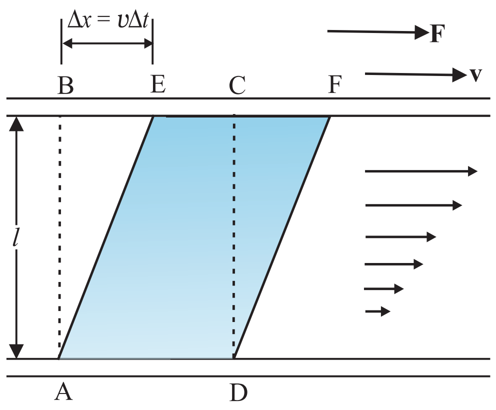
we have

$$\left(v_2 - v_1\right) / v_{av} = \frac{\Delta P}{\rho v_{av}^2} \approx 0.08$$

The speed above the wing needs to be only 8 % higher than that below.

9.5 VISCOSITY

Most of the fluids are not ideal ones and offer some resistance to motion. This resistance to fluid motion is like an internal friction analogous to friction when a solid moves on a surface. It is called viscosity. This force exists when there is relative motion between layers of the liquid. Suppose we consider a fluid like oil enclosed between two glass plates as shown in Fig. 9.12 (a). The bottom plate is fixed while the top plate is moved with a constant velocity **v** relative to the fixed plate. If oil is replaced by honey, a greater force is required to move the plate with the same velocity. Hence we say that honey is more viscous than oil. The fluid in contact with a surface has the same velocity as that of the surfaces. Hence, the layer of the liquid in contact with top surface moves with a velocity **v** and the layer of the liquid in contact with the fixed surface is stationary. The velocities of layers increase uniformly from bottom (zero velocity) to the top layer (velocity **v**). For any layer of liquid, its upper layer pulls it forward while lower layer pulls it backward. This results in force between the layers. This type of flow is known as laminar. The layers of liquid slide over one another as the pages of a book do when it is placed flat on a table and a horizontal force is applied to the top cover. When a fluid is flowing in a pipe or a tube, then velocity of the liquid layer along the axis of the tube is maximum and decreases gradually as we move towards the walls where it becomes zero, Fig. 9.12 (b). The velocity on a cylindrical surface in a tube is constant.



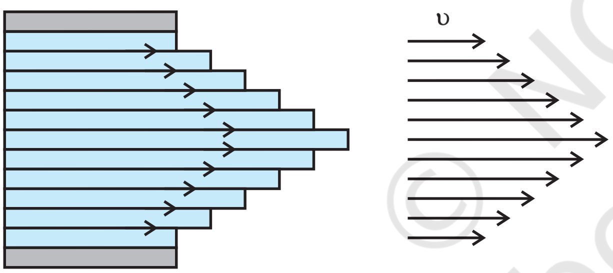


Fig 9.12 *(a) A layer of liquid sandwiched between two parallel glass plates, in which the lower plate is fixed and the upper one is moving to the right with velocity* **v** *(b) velocity distribution for viscous flow in a pipe.*

On account of this motion, a portion of liquid, which at some instant has the shape ABCD, take the shape of AEFD after short interval of time (Δt). During this time interval the liquid has undergone a shear strain of $\Delta x/l$. Since, the strain in a flowing fluid increases with time continuously. Unlike a solid, here the stress is found experimentally to depend on 'rate of change of strain' or 'strain rate' i.e. $\Delta x/(l\,\Delta t)$ or v/l instead of strain itself. The coefficient of viscosity (pronounced 'eta') for a fluid is defined as the ratio of shearing stress to the strain rate.

$$\eta = \frac{F/A}{v/l} = \frac{Fl}{vA} \tag{9.16}$$

The SI unit of viscosity is poiseiulle (Pl). Its other units are N s m^{-2} or Pa s. The dimensions of viscosity are [ML^{-1}T^{-1}]. Generally, thin liquids, like water, alcohol, etc., are less viscous than thick liquids, like coal tar, blood, glycerine, etc. The coefficients of viscosity for some common fluids are listed in Table 9.2. We point out two facts about blood and water that you may find interesting. As Table 9.2 indicates, blood is 'thicker' (more viscous) than water. Further, the relative viscosity (η/η_{water}) of blood remains constant between 0 °C and 37 °C.

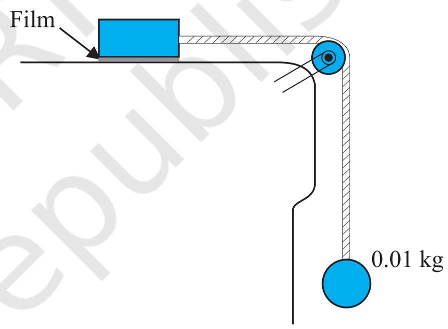


Fig. 9.13 *Measurement of the coefficient of viscosity of a liquid.*

The viscosity of liquids decreases with temperature, while it increases in the case of gases.

Example 9.8 A metal block of area 0.10 m^2 is connected to a 0.010 kg mass via a string that passes over an ideal pulley (considered massless and frictionless), as in Fig. 9.13. A liquid with a film thickness of 0.30 mm is placed between the block and the table. When released the block moves to the right with a constant speed of 0.085 m s^{-1}. Find the coefficient of viscosity of the liquid.

Answer The metal block moves to the right because of the tension in the string. The tension T is equal in magnitude to the weight of the suspended mass m. Thus, the shear force F is

$F = T = mg = 0.010$ kg × 9.8 m s^{-2} = 9.8 × 10^{-2} N

$$\text{Shear stress on the fluid} = F/A = \frac{9.8\times10^{-2}}{0.10}\text{N/m}^2$$

$$\text{Strain rate} = \frac{v}{l} = \frac{0.085}{0.30\times10^{-3}}$$

$$h = \frac{\text{stress}}{\text{strain rate}}\ \text{s}^{-1}$$

$$= \frac{(9.8\times10^{-2}\ \text{N})\ (0.30\times10^{-3}\ \text{m})}{(0.085\,\text{m s}^{-1})\ (0.10\,\text{m}^2)}$$

$= 3.46 \times 10^{-3}$ Pa s ◀

Table 9.2 The viscosities of some fluids

Fluid	T(°C)	Viscosity (mPl)
Water	20	1.0
	100	0.3
Blood	37	2.7
Machine Oil	16	113
	38	34
Glycerine	20	830
Honey	–	200
Air	0	0.017
	40	0.019

9.5.1 Stokes' Law

When a body falls through a fluid it drags the layer of the fluid in contact with it. A relative motion between the different layers of the fluid is set and, as a result, the body experiences a retarding force. Falling of a raindrop and swinging of a pendulum bob are some common examples of such motion. It is seen that the viscous force is proportional to the velocity of the object and is opposite to the direction of motion. The other quantities on which the force F depends are viscosity η of the fluid and radius a of the sphere. Sir George G. Stokes (1819–1903), an English scientist enunciated clearly the viscous drag force F as

$$F = 6\pi\eta a v \qquad (9.17)$$

This is known as Stokes' law. We shall not derive Stokes' law.

This law is an interesting example of retarding force, which is proportional to velocity. We can study its consequences on an object falling through a viscous medium. We consider a raindrop in air. It accelerates initially due to gravity. As the velocity increases, the retarding force also increases. Finally, when viscous force plus buoyant force becomes equal to the force due to gravity, the net force becomes zero and so does the acceleration. The sphere (raindrop) then descends with a constant velocity. Thus, in equilibrium, this terminal velocity v_t is given by

$$6\pi\eta a v_t = (4\pi/3)\ a^3\ (\rho\text{-}\sigma)g$$

where ρ and σ are mass densities of sphere and the fluid, respectively. We obtain

$$v_t = 2a^2\ (\rho\text{-}\sigma)g\ /\ (9\eta) \qquad (9.18)$$

So the terminal velocity v_t depends on the square of the radius of the sphere and inversely on the viscosity of the medium.

You may like to refer back to Example 6.2 in this context.

> ▶ ***Example 9.9*** The terminal velocity of a copper ball of radius 2.0 mm falling through a tank of oil at 20°C is 6.5 cm s^{-1}. Compute the viscosity of the oil at 20°C. Density of oil is 1.5 ×10^3 kg m^{-3}, density of copper is 8.9 × 10^3 kg m^{-3}.

Answer We have v_t = 6.5 × 10^{-2} ms^{-1}, a = 2 × 10^{-3} m, g = 9.8 ms^{-2}, ρ = 8.9 × 10^3 kg m^{-3},

σ =1.5 ×10^3 kg m^{-3}. From Eq. (9.18)

$$\eta = \frac{2}{9}\times\frac{(2\times10^{-3})^2\ \text{m}^2\times9.8\,\text{m s}^{-2}}{6.5\times10^{-2}\ \text{m s}^{-1}}\times7.4\times10^3\ \text{kg m}^{-3}$$

$= 9.9 \times 10^{-1}$ kg m^{-1} s^{-1} ◀

9.6 SURFACE TENSION

You must have noticed that, oil and water do not mix; water wets you and me but not ducks; mercury does not wet glass but water sticks to it, oil rises up a cotton wick, inspite of gravity, Sap and water rise up to the top of the leaves of the tree, hair of a paint brush do not cling together when dry and even when dipped in water but form a fine tip when taken out of it. All these and many more such experiences are related with the free surfaces of liquids. As liquids have no definite shape but have a definite volume, they acquire a free surface when poured in a container. These surfaces possess some additional energy. This phenomenon is known as surface tension and it is concerned with only liquid as gases do not have free surfaces. Let us now understand this phenomena.

9.6.1 Surface Energy

A liquid stays together because of attraction between molecules. Consider a molecule well inside a liquid. The intermolecular distances are such that it is attracted to all the surrounding molecules [Fig. 9.14(a)]. This attraction results in a negative potential energy for the molecule, which depends on the number and distribution of molecules around the chosen one. But the average potential energy of all the molecules is the same. This is supported by the fact that to take a collection of such molecules (the liquid) and to disperse them far away from each other in order to evaporate or vaporise, the heat of evaporation required is quite large. For water it is of the order of 40 kJ/mol.

Let us consider a molecule near the surface Fig. 9.14(b). Only lower half side of it is surrounded by liquid molecules. There is some negative potential energy due to these, but obviously it is less than that of a molecule in bulk, i.e., the one fully inside. Approximately it is half of the latter. Thus, molecules on a liquid surface have some extra energy in comparison to molecules in the interior. A liquid, thus, tends to have the least surface area which external conditions permit. Increasing surface area requires energy. Most surface phenomenon can be understood in terms of this fact. What is the energy required for having a molecule at the surface? As mentioned above, roughly it is half the energy required to remove it entirely from the liquid i.e., half the heat of evaporation.

Finally, what is a surface? Since a liquid consists of molecules moving about, there cannot be a perfectly sharp surface. The density of the liquid molecules drops rapidly to zero around $z = 0$ as we move along the direction indicated Fig 9.14 (c) in a distance of the order of a few molecular sizes.

9.6.2 Surface Energy and Surface Tension

As we have discussed that an extra energy is associated with surface of liquids, the creation of more surface (spreading of surface) keeping other things like volume fixed requires a

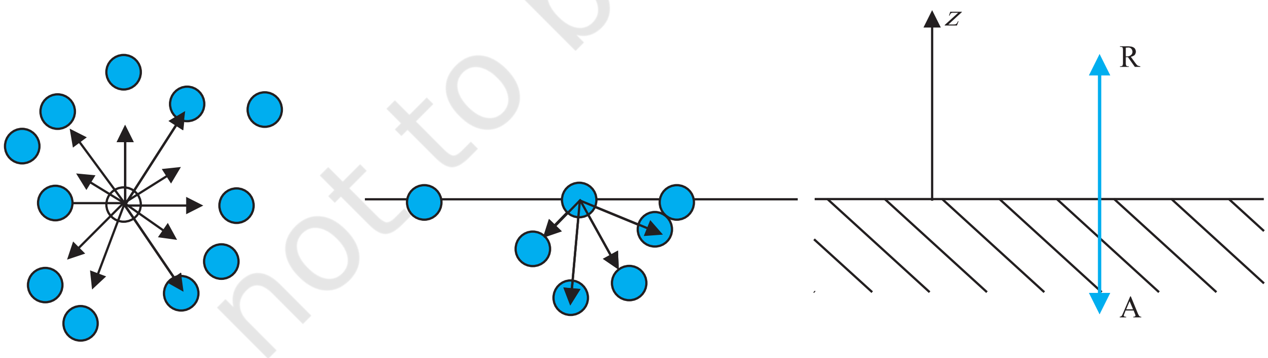


Fig. 9.14 Schematic picture of molecules in a liquid, at the surface and balance of forces. (a) Molecule inside a liquid. Forces on a molecule due to others are shown. Direction of arrows indicates attraction of repulsion. (b) Same, for a molecule at a surface. (c) Balance of attractive (AI and repulsive (R) forces.

horizontal liquid film ending in bar free to slide over parallel guides Fig (9.15).

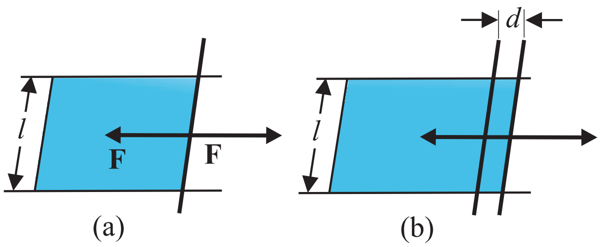


Fig. 9.15 *Stretching a film. (a) A film in equilibrium; (b) The film stretched an extra distance.*

Suppose that we move the bar by a small distance d as shown. Since the area of the surface increases, the system now has more energy, this means that some work has been done against an internal force. Let this internal force be **F**, the work done by the applied force is $\mathbf{F}\cdot\mathbf{d} = Fd$. From conservation of energy, this is stored as additional energy in the film. If the surface energy of the film is S per unit area, the extra area is $2dl$. A film has two sides and the liquid in between, so there are two surfaces and the extra energy is

$$S\,(2dl) = Fd \qquad (9.19)$$

$$\text{Or, } S = Fd/2dl = F/2l \qquad (9.20)$$

This quantity S is the magnitude of surface tension. It is equal to the surface energy per unit area of the liquid interface and is also equal to the force per unit length exerted by the fluid on the movable bar.

So far we have talked about the surface of one liquid. More generally, we need to consider fluid surface in contact with other fluids or solid surfaces. The surface energy in that case depends on the materials on both sides of the surface. For example, if the molecules of the materials attract each other, surface energy is reduced while if they repel each other the surface energy is increased. Thus, more appropriately, the surface energy is the energy of the interface between two materials and depends on both of them.

We make the following observations from above:

(i) Surface tension is a force per unit length (or surface energy per unit area) acting in the plane of the interface between the plane of the liquid and any other substance; it also is the extra energy that the molecules at the interface have as compared to molecules in the interior.

(ii) At any point on the interface besides the boundary, we can draw a line and imagine equal and opposite surface tension forces S per unit length of the line acting perpendicular to the line, in the plane of the interface. The line is in equilibrium. To be more specific, imagine a line of atoms or molecules at the surface. The atoms to the left pull the line towards them; those to the right pull it towards them! This line of atoms is in equilibrium under tension. If the line really marks the end of the interface, as in Figure 9.14 (a) and (b) there is only the force S per unit length acting inwards.

Table 9.3 gives the surface tension of various liquids. The value of surface tension depends on temperature. Like viscosity, the surface tension of a liquid usually falls with temperature.

Table 9.3 **Surface tension of some liquids at the temperatures indicated with the heats of the vaporisation**

Liquid	Temp (°C)	Surface Tension (N/m)	Heat of vaporisation (kJ/mol)
Helium	–270	0.000239	0.115
Oxygen	–183	0.0132	7.1
Ethanol	20	0.0227	40.6
Water	20	0.0727	44.16
Mercury	20	0.4355	63.2

A fluid will stick to a solid surface if the surface energy between fluid and the solid is smaller than the sum of surface energies between solid-air, and fluid-air. Now there is attraction between the solid surface and the liquid. It can be directly measured experimentaly as schematically shown in Fig. 9.16. A flat vertical glass plate, below which a vessel of some liquid is kept, forms one arm of the balance. The plate is balanced by weights on the other side, with its horizontal edge just over water. The vessel is raised slightly till the liquid just touches the glass plate and pulls it down a little because of surface tension. Weights are added till the plate just clears water.

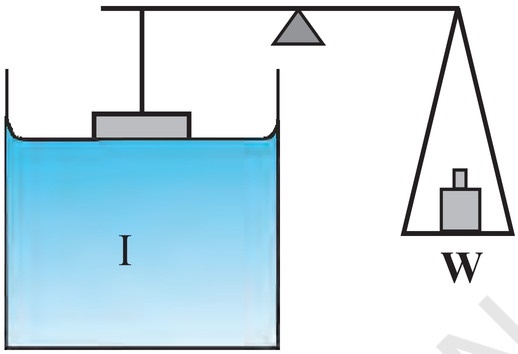


Fig. 9.16 *Measuring Surface Tension.*

Suppose the additional weight required is W. Then from Eq. 9.20 and the discussion given there, the surface tension of the liquid-air interface is

$$S_{la} = (W/2l) = (mg/2l) \qquad (9.21)$$

where m is the extra mass and l is the length of the plate edge. The subscript (la) emphasises the fact that the liquid-air interface tension is involved.

9.6.3 Angle of Contact

The surface of liquid near the plane of contact, with another medium is in general curved. The angle between tangent to the liquid surface at the point of contact and solid surface inside the liquid is termed as angle of contact. It is denoted by θ. It is different at interfaces of different pairs of liquids and solids. The value of θ determines whether a liquid will spread on the surface of a solid or it will form droplets on it. For example, water forms droplets on lotus leaf as shown in Fig. 9.17 (a) while spreads over a clean plastic plate as shown in Fig. 9.17(b).

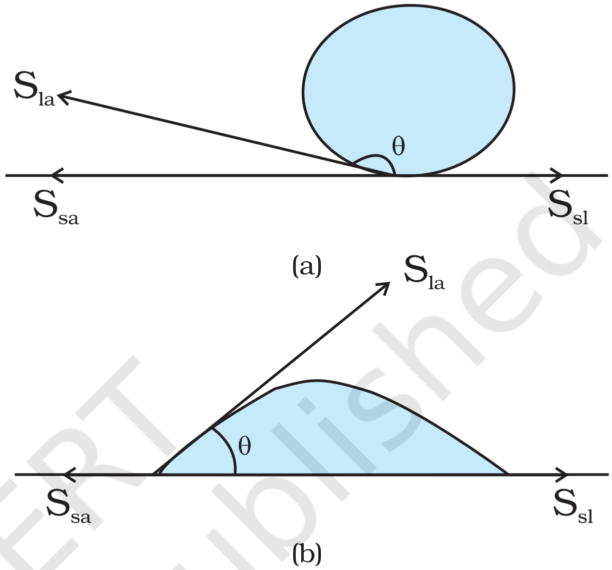


Fig. 9.17 *Different shapes of water drops with interfacial tensions (a) on a lotus leaf (b) on a clean plastic plate.*

We consider the three interfacial tensions at all the three interfaces, liquid-air, solid-air and solid-liquid denoted by S_{la}, S_{sa} and S_{sl}, respectively as given in Fig. 9.17 (a) and (b). At the line of contact, the surface forces between the three media must be in equilibrium. From the Fig. 9.17(b) the following relation is easily derived.

$$S_{la} \cos\theta + S_{sl} = S_{sa} \qquad (9.22)$$

The angle of contact is an obtuse angle if $S_{sl} > S_{la}$ as in the case of water-leaf interface while it is an acute angle if $S_{sl} < S_{la}$ as in the case of water-plastic interface. When θ is an obtuse angle then molecules of liquids are attracted strongly to themselves and weakly to those of solid, it costs a lot of energy to create a liquid-solid surface, and liquid then does not wet the solid. This is what happens with water on a waxy or oily surface, and with mercury on any surface. On the other hand, if the molecules of the liquid are strongly attracted to those of the solid, this will reduce S_{sl} and therefore, cos θ may increase or θ may decrease. In this case θ is an acute angle. This is what happens for water on glass or on plastic and for kerosene oil on virtually anything (it just spreads). Soaps, detergents and dying substances are wetting agents. When they are added the angle of contact becomes small so that these may penetrate well and become effective. Water proofing agents on the other hand are added to create a large angle of contact between the water and fibres.

9.6.4 Drops and Bubbles

One consequence of surface tension is that free liquid drops and bubbles are spherical if effects of gravity can be neglected. You must have seen this especially clearly in small drops just formed in a high-speed spray or jet, and in soap bubbles blown by most of us in childhood. Why are drops and bubbles spherical? What keeps soap bubbles stable?

As we have been saying repeatedly, a liquid-air interface has energy, so for a given volume the surface with minimum energy is the one with the least area. The sphere has this property. Though it is out of the scope of this book, but you can check that a sphere is better than at least a cube in this respect! So, if gravity and other forces (e.g. air resistance) were ineffective, liquid drops would be spherical.

Another interesting consequence of surface tension is that the pressure inside a spherical drop Fig. 9.18(a) is more than the pressure outside. Suppose a spherical drop of radius r is in equilibrium. If its radius increase by Δr. The extra surface energy is

$$[4\pi(r + \Delta r)^2 - 4\pi r^2]\, S_{la} = 8\pi r \,\Delta r\, S_{la} \qquad (9.23)$$

If the drop is in equilibrium this energy cost is balanced by the energy gain due to expansion under the pressure difference $(P_i - P_o)$ between the inside of the bubble and the outside. The work done is

$$W = (P_i - P_o)\, 4\pi r^2 \Delta r \qquad (9.24)$$

so that

$$(P_i - P_o) = (2\, S_{la} / r) \qquad (9.25)$$

In general, for a liquid-gas interface, the convex side has a higher pressure than the concave side. For example, an air bubble in a liquid, would have higher pressure inside it. See Fig 9.18 (b).

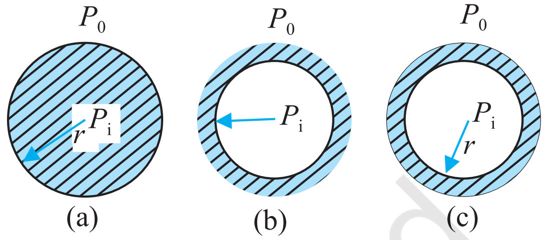


Fig. 9.18 Drop, cavity and bubble of radius r.

A bubble Fig 9.18 (c) differs from a drop and a cavity; in this it has two interfaces. Applying the above argument we have for a bubble

$$(P_i - P_o) = (4\, S_{la} / r) \qquad (9.26)$$

This is probably why you have to blow hard, but not too hard, to form a soap bubble. A little extra air pressure is needed inside!

9.6.5 Capillary Rise

One consequence of the pressure difference across a curved liquid-air interface is the well-known effect that water rises up in a narrow tube in spite of gravity. The word capilla means hair in Latin; if the tube were hair thin, the rise would be very large. To see this, consider a vertical capillary tube of circular cross section (radius a) inserted into an open vessel of water (Fig. 9.19). The contact angle between water and

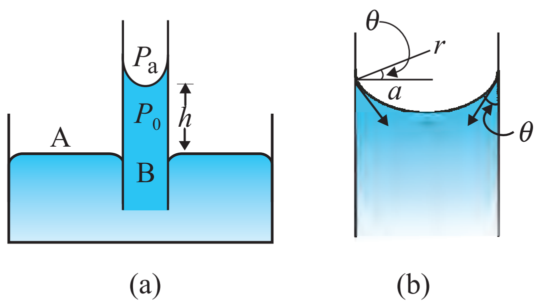


Fig. 9.19 Capillary rise, (a) Schematic picture of a narrow tube immersed water. (b) Enlarged picture near interface.

glass is acute. Thus the surface of water in **the capillary is concave. This means that there is a pressure difference between the two sides of the top surface. This is given by**

$$(P_i - P_o) = (2S/r) = 2S/(a \sec\theta)$$

$$= (2S/a)\cos\theta \qquad (9.27)$$

Thus the pressure of the water inside the tube, just at the meniscus (air-water interface) is less than the atmospheric pressure. Consider the two points A and B in Fig. 9.19(a). They must be at the same pressure, namely

$$P_0 + h\rho g = P_i = P_A \qquad (9.28)$$

where ρ is the density of water and h is called the capillary rise [Fig. 9.19(a)]. Using Eq. (9.27) and (9.28) we have

$$h\rho g = (P_i - P_0) = (2S\cos\theta)/a \qquad (9.29)$$

The discussion here, and the Eqs. (9.24) and (9.25) make it clear that the capillary rise is due to surface tension. It is larger, for a smaller a. Typically it is of the order of a few cm for fine capillaries. For example, if a = 0.05 cm, using the value of surface tension for water (Table 9.3), we find that

$$h = 2S/(\rho g a)$$

$$= \frac{2\times(0.073 \text{ N m}^{-1})}{(10^3 \text{ kg m}^{-3})(9.8 \text{ m s}^{-2})(5\times10^{-4}\text{ m})}$$

$$= 2.98\times10^{-2}\text{ m} = 2.98 \text{ cm}$$

Notice that if the liquid meniscus is convex, as for mercury, i.e., if $\cos\theta$ is negative then from Eq. (9.28) for example, it is clear that the liquid will be lower in the capillary !

Example 9.10 The lower end of a capillary tube of diameter 2.00 mm is dipped 8.00 cm below the surface of water in a beaker. What is the pressure required in the tube in order to blow a hemispherical bubble at its end in water? The surface tension of water at temperature of the experiments is 7.30×10^{-2} Nm^{-1}. 1 atmospheric pressure = 1.01×10^5 Pa, density of water = 1000 kg/m^3, g = 9.80 m s^{-2}. Also calculate the excess pressure.

Answer The excess pressure in a bubble of gas in a liquid is given by $2S/r$, where S is the surface tension of the liquid-gas interface. You should note there is only one liquid surface in this case. (For a bubble of liquid in a gas, there are two liquid surfaces, so the formula for excess pressure in that case is $4S/r$.) The radius of the bubble is r. Now the pressure outside the bubble P_o equals atmospheric pressure plus the pressure due to 8.00 cm of water column. That is

$$P_o = (1.01\times10^5 \text{ Pa} + 0.08 \text{ m}\times1000 \text{ kg m}^{-3} \times 9.80 \text{ m s}^{-2})$$

$$= 1.01784\times10^5 \text{ Pa}$$

Therefore, the pressure inside the bubble is

$$P_i = P_o + 2S/r$$
$$= 1.01784\times10^5 \text{ Pa} + (2\times7.3\times10^{-2} \text{ Pa m}/10^{-3}\text{ m})$$
$$= (1.01784 + 0.00146)\times10^5 \text{ Pa}$$
$$= 1.02\times10^5 \text{ Pa}$$

where the radius of the bubble is taken to be equal to the radius of the capillary tube, since the bubble is hemispherical ! (The answer has been rounded off to three significant figures.) The excess pressure in the bubble is 146 Pa.

SUMMARY

1. The basic property of a fluid is that it can flow. The fluid does not have any resistance to change of its shape. Thus, the shape of a fluid is governed by the shape of its container.
2. A liquid is incompressible and has a free surface of its own. A gas is compressible and it expands to occupy all the space available to it.
3. If F is the normal force exerted by a fluid on an area A then the average pressure P_{av} is defined as the ratio of the force to area

$$P_{av} = \frac{F}{A}$$

4. The unit of the pressure is the pascal (Pa). It is the same as N m^{-2}. Other common units of pressure are
   1 atm = 1.01×10^5 Pa
   1 bar = 10^5 Pa
   1 torr = 133 Pa = 0.133 kPa
   1 mm of Hg = 1 torr = 133 Pa
5. *Pascal's law* states that: Pressure in a fluid at rest is same at all points which are at the same height. A change in pressure applied to an enclosed fluid is transmitted undiminished to every point of the fluid and the walls of the containing vessel.
6. The pressure in a fluid varies with depth h according to the expression
   $P = P_a + \rho gh$
   where ρ is the density of the fluid, assumed uniform.
7. The volume of an incompressible fluid passing any point every second in a pipe of non uniform crossection is the same in the steady flow.
   $v\,A$ = constant (v is the velocity and A is the area of crossection)
   The equation is due to mass conservation in incompressible fluid flow.
8. *Bernoulli's principle* states that as we move along a streamline, the sum of the pressure (P), the kinetic energy per unit volume ($\rho v^2/2$) and the potential energy per unit volume (ρgy) remains a constant.
   $P + \rho v^2/2 + \rho gy$ = constant

   The equation is basically the conservation of energy applied to non viscuss fluid motion in steady state. There is no fluid which have zero viscosity, so the above statement is true only approximately. The viscosity is like friction and converts the kinetic energy to heat energy.
9. Though shear strain in a fluid does not require shear stress, when a shear stress is applied to a fluid, the motion is generated which causes a shear strain growing with time. The ratio of the shear stress to the time rate of shearing strain is known as coefficient of viscosity, η.

   where symbols have their usual meaning and are defined in the text.
10. *Stokes' law* states that the viscous drag force **F** on a sphere of radius a moving with velocity **v** through a fluid of viscosity is, $\mathbf{F} = 6\pi\eta a\mathbf{v}$.
11. Surface tension is a force per unit length (or surface energy per unit area) acting in the plane of interface between the liquid and the bounding surface. It is the extra energy that the molecules at the interface have as compared to the interior.

POINTS TO PONDER

1. Pressure is a *scalar quantity*. The definition of the pressure as "force per unit area" may give one false impression that pressure is a vector. The "force" in the numerator of the definition is the component of the force normal to the area upon which it is impressed. While describing fluids as a concept, shift from particle and rigid body mechanics is required. We are concerned with properties that vary from point to point in the fluid.
2. One should not think of pressure of a fluid as being exerted only on a solid like the walls of a container or a piece of solid matter immersed in the fluid. Pressure exists at all points in a fluid. An element of a fluid (such as the one shown in Fig. 9.4) is in equilibrium because the pressures exerted on the various faces are equal.

3. The expression for pressure
$P = P_a + \rho g h$
holds true if fluid is incompressible. Practically speaking it holds for liquids, which are largely incompressible and hence is a constant with height.
4. The gauge pressure is the difference of the actual pressure and the atmospheric pressure.
$P - P_a = P_g$
Many pressure-measuring devices measure the gauge pressure. These include the tyre pressure gauge and the blood pressure gauge (sphygmomanometer).
5. A streamline is a map of fluid flow. In a steady flow two streamlines do not intersect as it means that the fluid particle will have two possible velocities at the point.
6. Bernoulli's principle does not hold in presence of viscous drag on the fluid. The work done by this dissipative viscous force must be taken into account in this case, and P_2 [Fig. 9.9] will be lower than the value given by Eq. (9.12).
7. As the temperature rises the atoms of the liquid become more mobile and the coefficient of viscosity, η falls. In a gas the temperature rise increases the random motion of atoms and η increases.
8. Surface tension arises due to excess potential energy of the molecules on the surface in comparison to their potential energy in the interior. Such a surface energy is present at the interface separating two substances at least one of which is a fluid. It is not the property of a single fluid alone.

Physical Quantity	Symbol	Dimensions	Unit	Remarks
Pressure	P	$[M L^{-1} T^{-2}]$	pascal (Pa)	1 atm = 1.013×10^5 Pa, Scalar
Density	ρ	$[M L^{-3}]$	kg m^{-3}	Scalar
Specific Gravity		No	No	$\frac{\rho_{substance}}{\rho_{water}}$, Scalar
Co-efficient of viscosity	η	$[M L^{-1} T^{-1}]$	Pa s or poiseiulles (Pl)	Scalar
Surface Tension	S	$[M T^{-2}]$	N m^{-1}	Scalar

EXERCISES

9.1 Explain why
(a) The blood pressure in humans is greater at the feet than at the brain
(b) Atmospheric pressure at a height of about 6 km decreases to nearly half of its value at the sea level, though the height of the atmosphere is more than 100 km
(c) Hydrostatic pressure is a scalar quantity even though pressure is force divided by area.

9.2 Explain why
(a) The angle of contact of mercury with glass is obtuse, while that of water with glass is acute.
(b) Water on a clean glass surface tends to spread out while mercury on the same surface tends to form drops. (Put differently, water wets glass while mercury does not.)

(c) Surface tension of a liquid is independent of the area of the surface
(d) Water with detergent disolved in it should have small angles of contact.
(e) A drop of liquid under no external forces is always spherical in shape

9.3 Fill in the blanks using the word(s) from the list appended with each statement:
(a) Surface tension of liquids generally . . . with temperatures (increases / decreases)
(b) Viscosity of gases . . . with temperature, whereas viscosity of liquids . . . with temperature (increases / decreases)
(c) For solids with elastic modulus of rigidity, the shearing force is proportional to . . . , while for fluids it is proportional to . . . (shear strain / rate of shear strain)
(d) For a fluid in a steady flow, the increase in flow speed at a constriction follows (conservation of mass / Bernoulli's principle)
(e) For the model of a plane in a wind tunnel, turbulence occurs at a ... speed for turbulence for an actual plane (greater / smaller)

9.4 Explain why
(a) To keep a piece of paper horizontal, you should blow over, not under, it
(b) When we try to close a water tap with our fingers, fast jets of water gush through the openings between our fingers
(c) The size of the needle of a syringe controls flow rate better than the thumb pressure exerted by a doctor while administering an injection
(d) A fluid flowing out of a small hole in a vessel results in a backward thrust on the vessel
(e) A spinning cricket ball in air does not follow a parabolic trajectory

9.5 A 50 kg girl wearing high heel shoes balances on a single heel. The heel is circular with a diameter 1.0 cm. What is the pressure exerted by the heel on the horizontal floor ?

9.6 Toricelli's barometer used mercury. Pascal duplicated it using French wine of density 984 kg m^{-3}. Determine the height of the wine column for normal atmospheric pressure.

9.7 A vertical off-shore structure is built to withstand a maximum stress of 10^9 Pa. Is the structure suitable for putting up on top of an oil well in the ocean ? Take the depth of the ocean to be roughly 3 km, and ignore ocean currents.

9.8 A hydraulic automobile lift is designed to lift cars with a maximum mass of 3000 kg. The area of cross-section of the piston carrying the load is 425 cm^2. What maximum pressure would the smaller piston have to bear ?

9.9 A U-tube contains water and methylated spirit separated by mercury. The mercury columns in the two arms are in level with 10.0 cm of water in one arm and 12.5 cm of spirit in the other. What is the specific gravity of spirit ?

9.10 In the previous problem, if 15.0 cm of water and spirit each are further poured into the respective arms of the tube, what is the difference in the levels of mercury in the two arms ? (Specific gravity of mercury = 13.6)

9.11 Can Bernoulli's equation be used to describe the flow of water through a rapid in a river ? Explain.

9.12 Does it matter if one uses gauge instead of absolute pressures in applying Bernoulli's equation ? Explain.

9.13 Glycerine flows steadily through a horizontal tube of length 1.5 m and radius 1.0 cm. If the amount of glycerine collected per second at one end is 4.0×10^{-3} kg s^{-1}, what is the pressure difference between the two ends of the tube ? (Density of glycerine = 1.3×10^3 kg m^{-3} and viscosity of glycerine = 0.83 Pa s). [You may also like to check if the assumption of laminar flow in the tube is correct].

9.14 In a test experiment on a model aeroplane in a wind tunnel, the flow speeds on the upper and lower surfaces of the wing are 70 m $s^{-1}$and 63 m s^{-1} respectively. What is the lift on the wing if its area is 2.5 m^2 ? Take the density of air to be 1.3 kg m^{-3}.

9.15 Figures 9.20(a) and (b) refer to the steady flow of a (non-viscous) liquid. Which of the two figures is incorrect ? Why ?

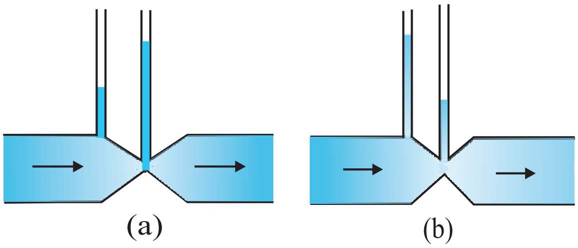


Fig. 9.20

9.16 The cylindrical tube of a spray pump has a cross-section of 8.0 cm^2 one end of which has 40 fine holes each of diameter 1.0 mm. If the liquid flow inside the tube is 1.5 m min^{-1}, what is the speed of ejection of the liquid through the holes ?

9.17 A U-shaped wire is dipped in a soap solution, and removed. The thin soap film formed between the wire and the light slider supports a weight of 1.5×10^{-2} N (which includes the small weight of the slider). The length of the slider is 30 cm. What is the surface tension of the film ?

9.18 Figure 9.21 (a) shows a thin liquid film supporting a small weight = 4.5×10^{-2} N. What is the weight supported by a film of the same liquid at the same temperature in Fig. (b) and (c) ? Explain your answer physically.

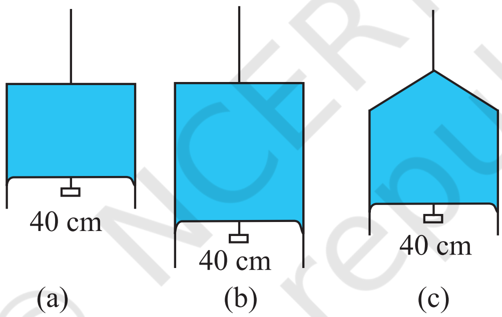


Fig. 9.21

9.19 What is the pressure inside the drop of mercury of radius 3.00 mm at room temperature ? Surface tension of mercury at that temperature (20 °C) is 4.65×10^{-1} N m^{-1}. The atmospheric pressure is 1.01×10^5 Pa. Also give the excess pressure inside the drop.

9.20 What is the excess pressure inside a bubble of soap solution of radius 5.00 mm, given that the surface tension of soap solution at the temperature (20 °C) is 2.50×10^{-2} N m^{-1} ? If an air bubble of the same dimension were formed at depth of 40.0 cm inside a container containing the soap solution (of relative density 1.20), what would be the pressure inside the bubble ? (1 atmospheric pressure is 1.01×10^5 Pa).

Chapter Ten

Thermal Properties of Matter

10.1 INTRODUCTION

We all have common sense notions of heat and temperature. Temperature is a measure of 'hotness' of a body. A kettle with boiling water is hotter than a box containing ice. In physics, we need to define the notion of heat, temperature, etc., more carefully. In this chapter, you will learn what heat is and how it is measured, and study the various proceses by which heat flows from one body to another. Along the way, you will find out why blacksmiths heat the iron ring before fitting on the rim of a wooden wheel of a horse cart and why the wind at the beach often reverses direction after the sun goes down. You will also learn what happens when water boils or freezes, and its temperature does not change during these processes even though a great deal of heat is flowing into or out of it.

10.2 TEMPERATURE AND HEAT

We can begin studying thermal properties of matter with definitions of temperature and heat. Temperature is a relative measure, or indication of hotness or coldness. A hot utensil is said to have a high temperature, and ice cube to have a low temperature. An object that has a higher temperature than another object is said to be hotter. Note that hot and cold are relative terms, like tall and short. We can perceive temperature by touch. However, this temperature sense is somewhat unreliable and its range is too limited to be useful for scientific purposes.

We know from experience that a glass of ice-cold water left on a table on a hot summer day eventually warms up whereas a cup of hot tea on the same table cools down. It means that when the temperature of body, ice-cold water or hot tea in this case, and its surrounding medium are different, heat transfer takes place between the system and the surrounding medium, until the body and the surrounding medium are at the same temperature. We also know that in the case of glass tumbler of ice-cold water, heat flows from the environment to the glass tumbler, whereas in the case of hot tea, it flows from the cup of hot tea to the environment. So, we can say that **heat is the form of energy transferred between two (or more) systems or a system and its surroundings by virtue of temperature difference.** The SI unit of heat energy transferred is expressed in joule (J) while SI unit of temperature is Kelvin (K), and degree Celsius (°C) is a commonly used unit of temperature. When an object is heated, many changes may take place. Its temperature may rise, it may expand or change state. We will study the effect of heat on different bodies in later sections.

10.3 MEASUREMENT OF TEMPERATURE

A measure of temperature is obtained using a thermometer. Many physical properties of materials change sufficiently with temperature. Some such properties are used as the basis for constructing thermometers. The commonly used property is variation of the volume of a liquid with temperature. For example, in common liquid–in–glass thermometers, mercury, alcohol etc., are used whose volume varies linearly with temperature over a wide range.

Thermometers are calibrated so that a numerical value may be assigned to a given temperature in an appropriate scale. For the definition of any standard scale, two fixed reference points are needed. Since all substances change dimensions with temperature, an absolute reference for expansion is not available. However, the necessary fixed points may be correlated to the physical phenomena that always occur at the same temperature. The ice point and the steam point of water are two convenient fixed points and are known as the freezing and boiling points, respectively. These two points are the temperatures at which pure water freezes and boils under standard pressure. The two familiar temperature scales are the Fahrenheit temperature scale and the Celsius temperature scale. The ice and steam point have values 32 °F and 212 °F, respectively, on the Fahrenheit scale and 0 C and 100 °C on the Celsius scale. On the Fahrenheit scale, there are 180 equal intervals between two reference points, and on the Celsius scale, there are 100.

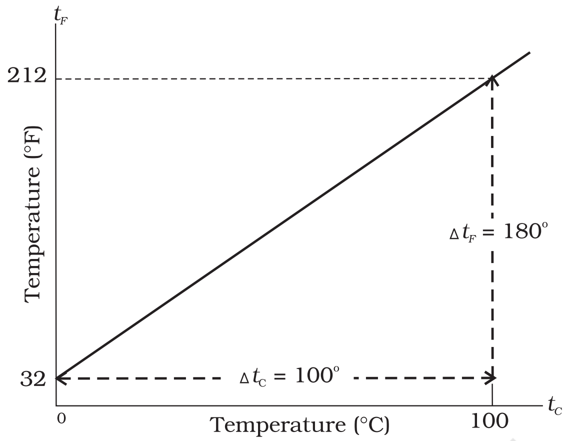


Fig. 10.1 A plot of Fahrenheit temperature (t_F) versus Celsius temperature (t_c).

A relationship for converting between the two scales may be obtained from a graph of Fahrenheit temperature (t_F) versus celsius temperature (t_C) in a straight line (Fig. 10.1), whose equation is

$$\frac{t_F - 32}{180} = \frac{t_C}{100} \tag{10.1}$$

10.4 IDEAL-GAS EQUATION AND ABSOLUTE TEMPERATURE

Liquid-in-glass thermometers show different readings for temperatures other than the fixed points because of differing expansion properties. A thermometer that uses a gas, however, gives the same readings regardless of which gas is used. Experiments show that all gases at low densities exhibit same expansion behaviour. The variables that describe the behaviour of a given quantity (mass) of gas are pressure, volume, and temperature (P, V, and T)(where $T = t + 273.15$; t is the temperature in °C). When temperature is held constant, the pressure and volume of a quantity of gas are related as PV = constant. This relationship is known as Boyle's law, after Robert Boyle (1627–1691), the English Chemist who discovered it. When the pressure is held constant, the volume of a quantity of the gas is related to the temperature as V/T = constant. This relationship is known as Charles' law, after French scientist Jacques Charles (1747–1823). Low-density gases obey these laws, which may be combined into a single

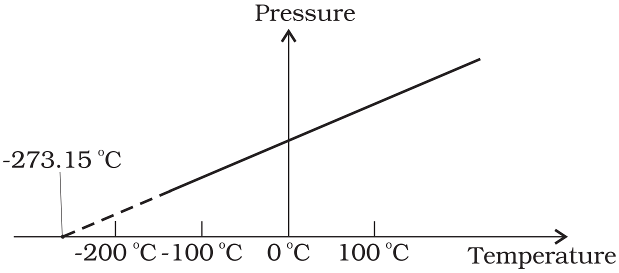


Fig. 10.2 *Pressure versus temperature of a low density gas kept at constant volume.*

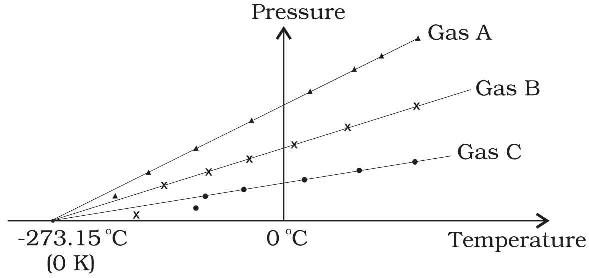


Fig. 10.3 *A plot of pressure versus temperature and extrapolation of lines for low density gases indicates the same absolute zero temperature.*

relationship. Notice that since PV = constant and V/T = constant for a given quantity of gas, then PV/T should also be a constant. This relationship is known as ideal gas law. It can be written in a more general form that applies not just to a given quantity of a single gas but to any quantity of any low-density gas and is known as **ideal-gas equation**:

$$\frac{PV}{T} = \mu R$$

$$\text{or } PV = \mu RT \qquad (10.2)$$

where, μ is the number of moles in the sample of gas and R is called universal gas constant:

R = 8.31 J mol^{-1} K^{-1}

In Eq. 10.2, we have learnt that the pressure and volume are directly proportional to temperature : $PV \propto T$. This relationship allows a gas to be used to measure temperature in a constant volume gas thermometer. Holding the volume of a gas constant, it gives $P \propto T$. Thus, with a constant-volume gas thermometer, temperature is read in terms of pressure. A plot of pressure versus temperature gives a straight line in this case, as shown in Fig. 10.2.

However, measurements on real gases deviate from the values predicted by the ideal gas law at low temperature. But the relationship is linear over a large temperature range, and it looks as though the pressure might reach zero with decreasing temperature if the gas continued to be a gas. The absolute minimum temperature for an ideal gas, therefore, inferred by extrapolating the straight line to the axis, as in Fig. 10.3. This temperature is found to be – 273.15 °C and is designated as **absolute zero**. Absolute zero is the foundation of the Kelvin temperature scale or absolute scale temperature named after the British scientist Lord Kelvin. On this scale, – 273.15 °C is taken as the zero point, that is 0 K (Fig. 10.4).

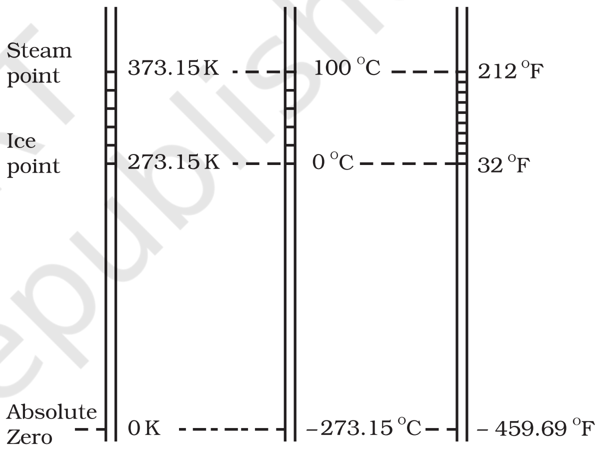


Fig. 10.4 *Comparision of the Kelvin, Celsius and Fahrenheit temperature scales.*

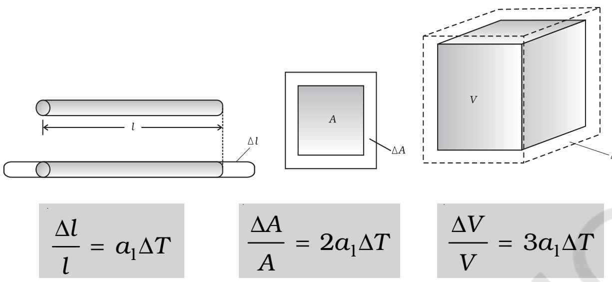
The size of unit in Kelvin and Celsius temperature scales is the same. So, temperature on these scales are related by

$$T = t_C + 273.15 \qquad (10.3)$$

10.5 THERMAL EXPANSION

You may have observed that sometimes sealed bottles with metallic lids are so tightly screwed that one has to put the lid in hot water for some time to open it. This would allow the metallic lid to expand, thereby loosening it to unscrew easily. In case of liquids, you may have observed that mercury in a thermometer rises, when the thermometer is put in slightly warm water. If we take out the thermometer from the warm water the level of mercury falls again. Similarly, in case of gases, a balloon partially inflated in a cool room may expand to full size when placed in warm water. On the other hand, a fully inflated balloon when immersed in cold water would start shrinking due to contraction of the air inside.

It is our common experience that most substances expand on heating and contract on cooling. A change in the temperature of a body causes change in its dimensions. The increase in the dimensions of a body due to the increase in its temperature is called thermal expansion. The expansion in length is called **linear expansion**. The expansion in area is called **area expansion**. The expansion in volume is called **volume expansion** (Fig. 10.5).



(a) Linear expansion (b) Area expansion (c) Volume expansion

Fig. 10.5 *Thermal Expansion.*

If the substance is in the form of a long rod, then for small change in temperature, ΔT, the fractional change in length, $\Delta l/l$, is directly proportional to ΔT.

$$\frac{\Delta l}{l} = \alpha_1 \, \Delta T \qquad (10.4)$$

where α_1 is known as the **coefficient of linear expansion** (or linear expansivity) and is characteristic of the material of the rod. In Table 10.1, typical average values of the coefficient of linear expansion for some material in the temperature range 0 °C to 100 C are given. From this Table, compare the value of α_1 for glass and copper. We find that copper expands about five times more than glass for the same rise in temperature. Normally, metals expand more and have relatively high values of α_1.

Table 10.1 **Values of coefficient of linear expansion for some material**

Material	α_1 (10^{-5} K^{-1})
Aluminium	2.5
Brass	1.8
Iron	1.2
Copper	1.7
Silver	1.9
Gold	1.4
Glass (pyrex)	0.32
Lead	0.29

Similarly, we consider the fractional change in volume, $\frac{\Delta V}{V}$, of a substance for temperature change ΔT and define the **coefficient of volume expansion (or volume expansivity)**, α_V as

$$\alpha_V = \left(\frac{\Delta V}{V}\right)\frac{1}{\Delta T} \qquad (10.5)$$

Here α_V is also a characteristic of the substance but is not strictly a constant. It depends in general on temperature (Fig 10.6). It is seen that α_V becomes constant only at a high temperature.

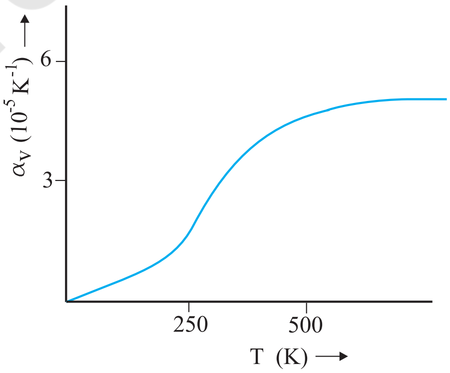


Fig. 10.6 *Coefficient of volume expansion of copper as a function of temperature.*

Table 10.2 gives the values of coefficient of volume expansion of some common substances in the temperature range 0–100 °C. You can see that thermal expansion of these substances (solids and liquids) is rather small, with material, like pyrex glass and invar (a special iron-nickel alloy) having particularly low values of α_V. From this Table we find that the value of α_v for alcohol (ethanol) is more than mercury and expands more than mercury for the same rise in temperature.

Table 10.2 **Values of coefficient of volume expansion for some substances**

Material	α_v (K^{-1})
Aluminium	7×10^{-5}
Brass	6×10^{-5}
Iron	3.55×10^{-5}
Paraffin	58.8×10^{-5}
Glass (ordinary)	2.5×10^{-5}
Glass (pyrex)	1×10^{-5}
Hard rubber	2.4×10^{-4}
Invar	2×10^{-6}
Mercury	18.2×10^{-5}
Water	20.7×10^{-5}
Alcohol (ethanol)	110×10^{-5}

Water exhibits an anomalous behaviour; it contracts on heating between 0 °C and 4 °C. The volume of a given amount of water decreases as it is cooled from room temperature, until its temperature reaches 4 °C, [Fig. 10.7(a)]. Below 4 °C, the volume increases, and therefore, the density decreases [Fig. 10.7(b)].

This means that water has the maximum density at 4 °C. This property has an important environmental effect: bodies of water, such as lakes and ponds, freeze at the top first. As a lake cools toward 4 °C, water near the surface loses energy to the atmosphere, becomes denser, and sinks; the warmer, less dense water near the bottom rises. However, once the colder water on top reaches temperature below 4 °C, it becomes less dense and remains at the surface, where it freezes. If water did not have this property, lakes and ponds would freeze from the bottom up, which would destroy much of their animal and plant life.

Gases, at ordinary temperature, expand more than solids and liquids. For liquids, the coefficient of volume expansion is relatively independent of the temperature. However, for gases it is dependent on temperature. For an ideal gas, the coefficient of volume expansion at constant pressure can be found from the ideal gas equation:

$PV = \mu RT$

At constant pressure

$P\Delta V = \mu R \Delta T$

$$\frac{\Delta V}{V} = \frac{\Delta T}{T}$$

$$\text{i.e., } \alpha_v = \frac{1}{T} \text{ for ideal gas} \qquad (10.6)$$

At 0 °C, α_v = 3.7×10^{-3} K^{-1}, which is much larger than that for solids and liquids. Equation (10.6) shows the temperature dependence of α_v; it decreases with increasing temperature. For a gas at room temperature and constant pressure, α_v is about 3300×10^{-6} K^{-1}, as

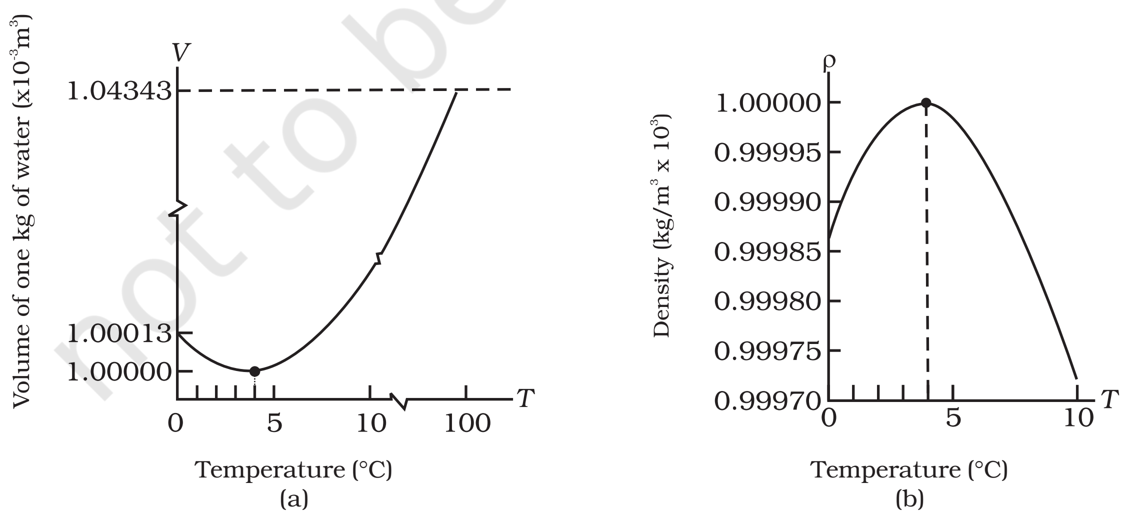


Fig. 10.7 *Thermal expansion of water.*

much as order(s) of magnitude larger than the coefficient of volume expansion of typical liquids.

There is a simple relation between the coefficient of volume expansion (α_v) and coefficient of linear expansion (α_l). Imagine a cube of length, l, that expands equally in all directions, when its temperature increases by ΔT. We have

$\Delta l = \alpha_l \, l \, \Delta T$

so, $\Delta V = (l+\Delta l)^3 - l^3 \simeq 3l^2 \, \Delta l$ (10.7)

In Equation (10.7), terms in $(\Delta l)^2$ and $(\Delta l)^3$ have been neglected since Δl is small compared to l. So

$$\Delta V = \frac{3V\,\Delta l}{l} = 3V\alpha_l \, \Delta T \tag{10.8}$$

which gives

$$\alpha_v = 3\alpha_l \tag{10.9}$$

What happens by preventing the thermal expansion of a rod by fixing its ends rigidly? Clearly, the rod acquires a compressive strain due to the external forces provided by the rigid support at the ends. The corresponding stress set up in the rod is called **thermal stress**. For example, consider a steel rail of length 5 m and area of cross-section 40 cm^2 that is prevented from expanding while the temperature rises by 10 °C. The coefficient of linear expansion of steel is $\alpha_{l(steel)} = 1.2 \times 10^{-5}$ K^{-1}. Thus, the compressive strain is $\frac{\Delta l}{l} = \alpha_{l(steel)} \Delta T = 1.2 \times 10^{-5} \times 10 = 1.2 \times 10^{-4}$. Youngs modulus of steel is $Y_{(steel)} = 2 \times 10^{11}$ N m^{-2}. Therefore, the thermal stress developed is $\frac{\Delta F}{A} = Y_{steel}\left(\frac{\Delta l}{l}\right) = 2.4 \times 10^{7}$ N m^{-2}, which corresponds to an external force of

$$\Delta F = AY_{steel}\left(\frac{\Delta l}{l}\right) = 2.4 \times 10^{7} \times 40 \times 10^{-4} \simeq 10^{5}\text{N}.$$

If two such steel rails, fixed at their outer ends, are in contact at their inner ends, a force of this magnitude can easily bend the rails.

> **Example 10.1** Show that the coefficient of area expansion, $(\Delta A/A)/\Delta T$, of a rectangular sheet of the solid is twice its linear expansivity, α_l.

Answer

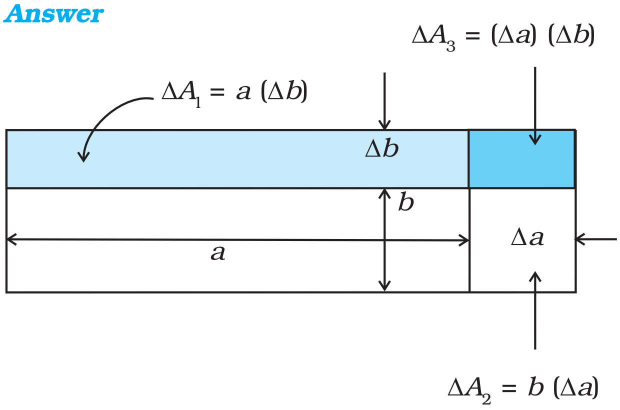


Fig. 10.8

Consider a rectangular sheet of the solid material of length a and breadth b (Fig. 10.8). When the temperature increases by ΔT, a increases by $\Delta a = \alpha_l a \Delta T$ and b increases by $\Delta b = \alpha_l b \Delta T$. From Fig. 10.8, the increase in area

$\Delta A = \Delta A_1 + \Delta A_2 + \Delta A_3$

$\Delta A = a\,\Delta b + b\,\Delta a + (\Delta a)\,(\Delta b)$

$= a\,\alpha_l b\,\Delta T + b\,\alpha_l\,a\,\Delta T + (\alpha_l)^2\,ab\,(\Delta T)^2$

$= \alpha_l\,ab\,\Delta T(2 + \alpha_l\,\Delta T) = \alpha_l\,A\,\Delta T(2 + \alpha_l\,\Delta T)$

Since $\alpha_l \simeq 10^{-5}$ K^{-1}, from Table 10.1, the product $\alpha_l \Delta T$ for fractional temperature is small in comparision with 2 and may be neglected. Hence,

$$\left(\frac{\Delta A}{A}\right)\frac{1}{\Delta T} \simeq 2\alpha_l$$

> **Example 10.2** A blacksmith fixes iron ring on the rim of the wooden wheel of a horse cart. The diameter of the rim and the iron ring are 5.243 m and 5.231 m, respectively at 27 °C. To what temperature should the ring be heated so as to fit the rim of the wheel?

Answer

Given, $T_1 = 27$ °C

$L_{T1} = 5.231$ m

$L_{T2} = 5.243$ m

So,

$L_{T2} = L_{T1}\,[1+\alpha_l\,(T_2 - T_1)]$

5.243 m = 5.231 m [1 + 1.20 10^{-5} K^{-1} (T_2–27 °C)]

or $T_2 = 218$ °C.

10.6 SPECIFIC HEAT CAPACITY

Take some water in a vessel and start heating it on a burner. Soon you will notice that bubbles begin to move upward. As the temperature is raised the motion of water particles increases till it becomes turbulent as water starts boiling. What are the factors on which the quantity of heat required to raise the temperature of a substance depend? In order to answer this question in the first step, heat a given quantity of water to raise its temperature by, say 20 °C and note the time taken. Again take the same amount of water and raise its temperature by 40 °C using the same source of heat. Note the time taken by using a stopwatch. You will find it takes about twice the time and therefore, double the quantity of heat required raising twice the temperature of same amount of water.

In the second step, now suppose you take double the amount of water and heat it, using the same heating arrangement, to raise the temperature by 20 °C, you will find the time taken is again twice that required in the first step.

In the third step, in place of water, now heat the same quantity of some oil, say mustard oil, and raise the temperature again by 20 °C. Now note the time by the same stopwatch. You will find the time taken will be shorter and therefore, the quantity of heat required would be less than that required by the same amount of water for the same rise in temperature.

The above observations show that the quantity of heat required to warm a given substance depends on its mass, m, the change in temperature, ΔT and the nature of substance. The change in temperature of a substance, when a given quantity of heat is absorbed or rejected by it, is characterised by a quantity called the **heat capacity** of that substance. We define heat capacity, S of a substance as

$$S = \frac{\Delta Q}{\Delta T} \tag{10.10}$$

where ΔQ is the amount of heat supplied to the substance to change its temperature from T to $T + \Delta T$.

You have observed that if equal amount of heat is added to equal masses of different substances, the resulting temperature changes will not be the same. It implies that every substance has a unique value for the amount of heat absorbed or given off to change the temperature of unit mass of it by one unit. This quantity is referred to as the **specific heat capacity** of the substance.

If ΔQ stands for the amount of heat absorbed or given off by a substance of mass m when it undergoes a temperature change ΔT, then the specific heat capacity, of that substance is given by

$$s = \frac{S}{m} = \frac{1}{m}\frac{\Delta Q}{\Delta T} \tag{10.11}$$

The **specific heat capacity** is the property of the substance which determines the change in the temperature of the substance (undergoing no phase change) when a given quantity of heat is absorbed (or given off) by it. It is defined as the amount of heat per unit mass absorbed or given off by the substance to change its temperature by one unit. It depends on the nature of the substance and its temperature. The SI unit of specific heat capacity is J kg^{-1} K^{-1}.

If the amount of substance is specified in terms of moles μ, instead of mass m in kg, we can define heat capacity per mole of the substance by

$$C = \frac{S}{\mu} = \frac{1}{\mu}\frac{\Delta Q}{\Delta T} \tag{10.12}$$

where C is known as **molar specific heat capacity** of the substance. Like S, C also depends on the nature of the substance and its temperature. The SI unit of molar specific heat capacity is J mol^{-1} K^{-1}.

However, in connection with specific heat capacity of gases, additional conditions may be needed to define C. In this case, heat transfer can be achieved by keeping either pressure or volume constant. If the gas is held under constant pressure during the heat transfer, then it is called the **molar specific heat capacity at constant pressure** and is denoted by C_p. On the other hand, if the volume of the gas is maintained during the heat transfer, then the corresponding molar specific heat capacity is called **molar specific heat capacity at constant volume** and is denoted by C_v. For details see Chapter 11. Table 10.3 lists measured specific heat capacity of some substances at atmospheric pressure and ordinary temperature while Table 10.4 lists molar specific heat capacities of some gases. From Table 10.3 you can note that water

Table 10.3 **Specific heat capacity of some substances at room temperature and atmospheric pressure**

Substance	Specific heat capacity ($J\ kg^{-1}\ K^{-1}$)	Substance	Specific heat capacity ($J\ kg^{-1}\ K^{-1}$)
Aluminium	900.0	Ice	2060
Carbon	506.5	Glass	840
Copper	386.4	Iron	450
Lead	127.7	Kerosene	2118
Silver	236.1	Edible oil	1965
Tungesten	134.4	Mercury	140
Water	4186.0		

has the highest specific heat capacity compared to other substances. For this reason water is also used as a coolant in automobile radiators, as well as, a heater in hot water bags. Owing to its high specific heat capacity, water warms up more slowly than land during summer, and consequently wind from the sea has a cooling effect. Now, you can tell why in desert areas, the earth surface warms up quickly during the day and cools quickly at night.

Table 10.4 **Molar specific heat capacities of some gases**

Gas	C_p ($J\ mol^{-1}K^{-1}$)	C_v($J\ mol^{-1}K^{-1}$)
He	20.8	12.5
H_2	28.8	20.4
N_2	29.1	20.8
O_2	29.4	21.1
CO_2	37.0	28.5

10.7 CALORIMETRY

A system is said to be isolated if no exchange or transfer of heat occurs between the system and its surroundings. When different parts of an isolated system are at different temperature, a quantity of heat transfers from the part at higher temperature to the part at lower temperature. The heat lost by the part at higher temperature is equal to the heat gained by the part at lower temperature.

Calorimetry means measurement of heat. When a body at higher temperature is brought in contact with another body at lower temperature, the heat lost by the hot body is equal to the heat gained by the colder body, provided no heat is allowed to escape to the surroundings. A device in which heat measurement can be done is called a **calorimeter**. It consists of a metallic vessel and stirrer of the same material, like copper or aluminium. The vessel is kept inside a wooden jacket, which contains heat insulating material, like glass wool etc. The outer jacket acts as a heat shield and reduces the heat loss from the inner vessel. There is an opening in the outer jacket through which a mercury thermometer can be inserted into the calorimeter (Fig. 10.20). The following example provides a method by which the specific heat capacity of a given solid can be determinated by using the principle, heat gained is equal to the heat lost.

Example 10.3 A sphere of 0.047 kg aluminium is placed for sufficient time in a vessel containing boiling water, so that the sphere is at 100 °C. It is then immediately transfered to 0.14 kg copper calorimeter containing 0.25 kg water at 20 °C. The temperature of water rises and attains a steady state at 23 °C. Calculate the specific heat capacity of aluminium.

Answer In solving this example, we shall use the fact that at a steady state, heat given by an aluminium sphere will be equal to the heat absorbed by the water and calorimeter.

Mass of aluminium sphere (m_1) = 0.047 kg
Initial temperature of aluminium sphere=100 °C
Final temperature = 23 °C
Change in temperature (ΔT)=(100 °C-23°C)= 77 °C
Let specific heat capacity of aluminium be s_{Al}.

The amount of heat lost by the aluminium sphere = $m_1 s_{Al} \Delta T = 0.047 \text{kg} \times s_{Al} \times 77\ °\text{C}$

Mass of water (m_2) = 0.25 kg

Mass of calorimeter (m_3) = 0.14 kg

Initial temperature of water and calorimeter=20 °C

Final temperature of the mixture = 23 °C

Change in temperature (ΔT_2) = 23 °C – 20 °C = 3 °C

Specific heat capacity of water (s_w)

$= 4.18 \times 10^3$ J kg^{-1} K^{-1}

Specific heat capacity of copper calorimeter

$= 0.386 \times 10^3$ J kg^{-1} K^{-1}

The amount of heat gained by water and calorimeter = $m_2\, s_w\, \Delta T_2 + m_3 s_{cu} \Delta T_2$

$= (m_2 s_w + m_3 s_{cu})\,(\Delta T_2)$

$= (0.25 \text{ kg} \times 4.18 \times 10^3 \text{ J kg}^{-1}\text{ K}^{-1} + 0.14 \text{ kg} \times 0.386 \times 10^3 \text{ J kg}^{-1}\text{ K}^{-1})\ (23\ °\text{C} - 20\ °\text{C})$

In the steady state heat lost by the aluminium sphere = heat gained by water + heat gained by calorimeter.

So, $0.047 \text{ kg} \times s_{Al} \times 77\ °\text{C}$

$= (0.25 \text{ kg} \times 4.18 \times 10^3 \text{ J kg}^{-1}\text{ K}^{-1} + 0.14 \text{ kg} \times 0.386 \times 10^3 \text{ J kg}^{-1}\text{ K}^{-1})(3\ °\text{C})$

$s_{Al} = 0.911$ kJ kg^{-1} K^{-1} ◀

10.8 CHANGE OF STATE

Matter normally exists in three states: solid, liquid and gas. A transition from one of these states to another is called a change of state. Two common changes of states are solid to liquid and liquid to gas (and, vice versa). These changes can occur when the exchange of heat takes place between the substance and its surroundings. To study the change of state on heating or cooling, let us perform the following activity.

Take some cubes of ice in a beaker. Note the temperature of ice. Start heating it slowly on a constant heat source. Note the temperature after every minute. Continuously stir the mixture of water and ice. Draw a graph between temperature and time (Fig. 10.9). You will observe no change in the temperature as long as there is ice in the beaker. In the above process, the temperature of the system does not change even though heat is being continuously supplied. The heat supplied is being utilised in changing the state from solid (ice) to liquid (water).

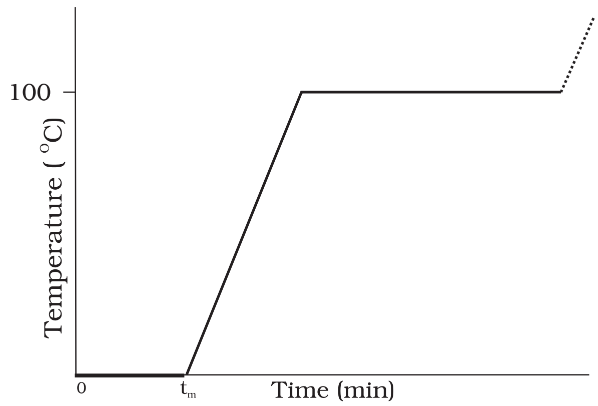


Fig. 10.9 *A plot of temperature versus time showing the changes in the state of ice on heating (not to scale).*

The change of state from solid to liquid is called **melting** or fusion and from liquid to solid is called **freezing**. It is observed that the temperature remains constant until the entire amount of the solid substance melts. That is, **both the solid and the liquid states of the substance coexist in thermal equilibrium during the change of states from solid to liquid.** The temperature at which the solid and the liquid states of the substance is in thermal equilibrium with each other is called its **melting point**. It is characteristic of the substance. It also depends on pressure. The melting point of a substance at standard atomspheric pressure is called its **normal melting point**. Let us do the following activity to understand the process of melting of ice.

Take a slab of ice. Take a metallic wire and fix two blocks, say 5 kg each, at its ends. Put the wire over the slab as shown in Fig. 10.10. You will observe that the wire passes through the ice slab. This happens due to the fact that just below the wire, ice melts at lower temperature due to increase in pressure. When the wire has passed, water above the wire freezes again. Thus, the wire passes through the slab and the slab does not split. This phenomenon of refreezing is called **regelation**. Skating is possible on snow due to the formation of water under the skates. Water is formed due to the increase of pressure and it acts as a lubricant.

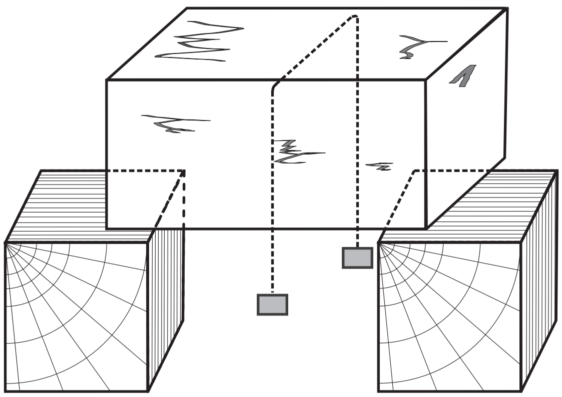


Fig. 10.10

After the whole of ice gets converted into water and as we continue further heating, we shall see that temperature begins to rise (Fig.10.9). The temperature keeps on rising till it reaches nearly 100 °C when it again becomes steady. The heat supplied is now being utilised to change water from liquid state to vapour or gaseous state.

The change of state from liquid to vapour (or gas) is called **vaporisation**. It is observed that the temperature remains constant until the entire amount of the liquid is converted into vapour. That is, both the liquid and vapour states of the substance coexist in thermal equilibrium, during the change of state from liquid to vapour. The temperature at which the liquid and the vapour states of the substance coexist is called its **boiling point**. Let us do the following activity to understand the process of boiling of water.

Take a round-bottom flask, more than half filled with water. Keep it over a burner and fix a

Triple Point

The temperature of a substance remains constant during its change of state (phase change). A graph between the temperature T and the Pressure P of the substance is called a phase diagram or $P - T$ diagram. The following figure shows the phase diagram of water and CO_2. Such a phase diagram divides the $P - T$ plane into a solid-region, the vapour-region and the liquid-region. The regions are separated by the curves such as sublimation curve (BO), **fusion curve** (AO) and **vaporisation curve** (CO). The points on **sublimation curve** represent states in which solid and vapour phases coexist. The point on the sublimation curve BO represent states in which the solid and vapour phases co-exist. Points on the fusion curve AO represent states in which solid and liquid phase coexist. Points on the vapourisation curve CO represent states in which the liquid and vapour phases coexist. The temperature and pressure at which the fusion curve, the vaporisation curve and the sublimation curve meet and all the three phases of a substance coexist is called the **triple point** of the substance. For example the triple point of water is represented by the temperature 273.16 K and pressure 6.11×10^{-3} Pa.

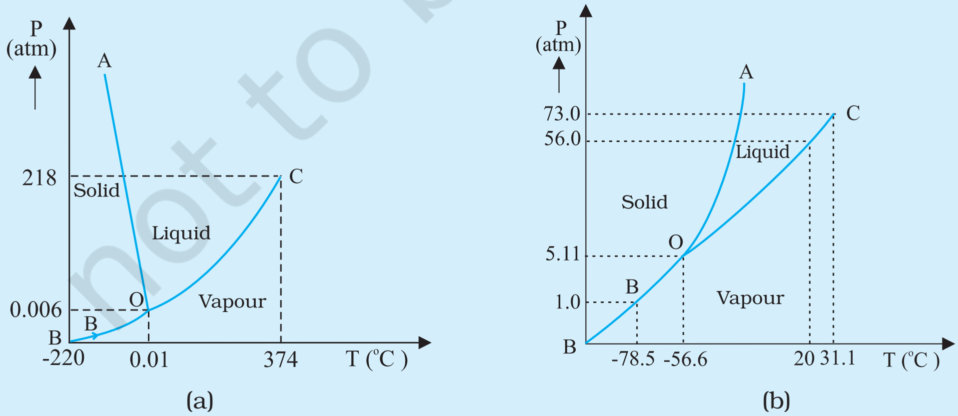


Figure : *Pressure-temperature phase diagrams for (a) water and (b) CO_2 (not to the scale).*

thermometer and steam outlet through the cork of the flask (Fig. 10.11). As water gets heated in the flask, note first that the air, which was dissolved in the water, will come out as small bubbles. Later, bubbles of steam will form at the bottom but as they rise to the cooler water near the top, they condense and disappear. Finally, as the temperature of the entire mass of the water reaches 100 °C, bubbles of steam reach the surface and boiling is said to occur. The steam in the flask may not be visible but as it comes out of the flask, it condenses as tiny droplets of water, giving a foggy appearance.

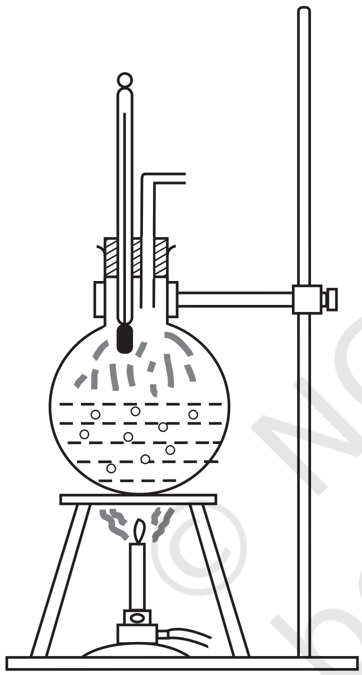


Fig. 10.11 *Boiling process.*

If now the steam outlet is closed for a few seconds to increase the pressure in the flask, you will notice that boiling stops. More heat would be required to raise the temperature (depending on the increase in pressure) before boiling begins again. Thus boiling point increases with increase in pressure.

Let us now remove the burner. Allow water to cool to about 80 °C. Remove the thermometer and steam outlet. Close the flask with the airtight cork. Keep the flask turned upside down on the stand. Pour ice-cold water on the flask. Water vapours in the flask condense reducing the pressure on the water surface inside the flask. Water begins to boil again, now at a lower temperature. Thus boiling point decreases with decrease in pressure.

This explains why cooking is difficult on hills. At high altitudes, atmospheric pressure is lower, reducing the boiling point of water as compared to that at sea level. On the other hand, boiling point is increased inside a pressure cooker by increasing the pressure. Hence cooking is faster. The boiling point of a substance at standard atmospheric pressure is called its **normal boiling point**.

However, all substances do not pass through the three states: solid-liquid-gas. There are certain substances which normally pass from the solid to the vapour state directly and vice versa. The change from solid state to vapour state without passing through the liquid state is called **sublimation**, and the substance is said to sublime. Dry ice (solid CO_2) sublimes, so also iodine. During the sublimation process both the solid and vapour states of a substance coexist in thermal equilibrium.

10.8.1 Latent Heat

In Section 10.8, we have learnt that certain amount of heat energy is transferred between a substance and its surroundings when it undergoes a change of state. The amount of heat per unit mass transferred during change of state of the substance is called latent heat of the substance for the process. For example, if heat is added to a given quantity of ice at –10 °C, the temperature of ice increases until it reaches its melting point (0 °C). At this temperature, the addition of more heat does not increase the temperature but causes the ice to melt, or changes its state. Once the entire ice melts, adding more heat will cause the temperature of the water to rise. A similar situation occurs during liquid gas change of state at the boiling point. Adding more heat to boiling water causes vaporisation, without increase in temperature.

Table 10.5 **Temperatures of the change of state and latent heats for various substances at 1 atm pressure**

Substance	Melting Point (C)	L_f (10^5J kg^{-1})	Boiling Point (C)	L_v (10^5J kg^{-1})
Ethanol	–114	1.0	78	8.5
Gold	1063	0.645	2660	15.8
Lead	328	0.25	1744	8.67
Mercury	–39	0.12	357	2.7
Nitrogen	–210	0.26	–196	2.0
Oxygen	–219	0.14	–183	2.1
Water	0	3.33	100	22.6

The heat required during a change of state depends upon the heat of transformation and the mass of the substance undergoing a change of state. Thus, if mass m of a substance undergoes a change from one state to the other, then the quantity of heat required is given by

$$Q = m\,L$$

or $$L = Q/m \qquad (10.13)$$

where L is known as latent heat and is a characteristic of the substance. Its SI unit is J kg^{-1}. The value of L also depends on the pressure. Its value is usually quoted at standard atmospheric pressure. The latent heat for a solid-liquid state change is called the **latent heat of fusion** (L_f), and that for a liquid-gas state change is called the **latent heat of vaporisation** (L_v). These are often referred to as the heat of fusion and the heat of vaporisation. A plot of temperature versus heat for a quantity of water is shown in Fig. 10.12. The latent heats of some substances, their freezing and boiling points, are given in Table 10.5.

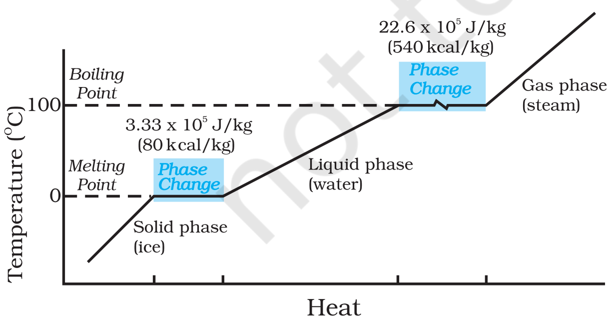


Fig. 10.12 Temperature versus heat for water at 1 atm pressure (not to scale).

Note that when heat is added (or removed) during a change of state, the temperature remains constant. Note in Fig. 10.12 that the slopes of the phase lines are not all the same, which indicate that specific heats of the various states are not equal. For water, the latent heat of fusion and vaporisation are $L_f = 3.33 \times 10^5$ J kg^{-1} and $L_v = 22.6 \times 10^5$ J kg^{-1}, respectively. That is, 3.33×10^5 J of heat is needed to melt 1 kg ice at 0 °C, and 22.6×10^5 J of heat is needed to convert 1 kg water into steam at 100 °C. So, steam at 100 °C carries 22.6×10^5 J kg^{-1} more heat than water at 100 °C. This is why burns from steam are usually more serious than those from boiling water.

Example 10.4 When 0.15 kg of ice at 0 °C is mixed with 0.30 kg of water at 50 °C in a container, the resulting temperature is 6.7 °C. Calculate the heat of fusion of ice. (s_{water} = 4186 J kg^{-1} K^{-1})

Answer

Heat lost by water = $ms_w\,(\theta_f – \theta_i)_w$

= (0.30 kg) (4186 J kg^{-1} K^{-1}) (50.0 °C – 6.7 °C)

= 54376.14 J

Heat required to melt ice = $m_2 L_f$ = (0.15 kg) L_f

Heat required to raise temperature of ice water to final temperature = $m_I s_w\,(\theta_f – \theta_i)_I$

= (0.15 kg) (4186 J kg^{-1} K^{-1}) (6.7 °C – 0 °C)

= 4206.93 J

Heat lost = heat gained

54376.14 J = (0.15 kg) L_f + 4206.93 J

$L_f = 3.34\ 10^5$ J kg^{-1}.

Example 10.5 Calculate the heat required to convert 3 kg of ice at –12 °C kept in a calorimeter to steam at 100 °C at atmospheric pressure. Given specific heat capacity of ice = 2100 J kg^{-1} K^{-1}, specific heat capacity of water = 4186 J kg^{-1} K^{-1}, latent heat of fusion of ice = 3.35×10^5 J kg^{-1} and latent heat of steam = 2.256×10^6 J kg^{-1}.

Answer We have

Mass of the ice, m = 3 kg

specific heat capacity of ice, s_{ice}
= 2100 J kg^{-1} K^{-1}

specific heat capacity of water, s_{water}
= 4186 J kg^{-1} K^{-1}

latent heat of fusion of ice, $L_{f\,ice}$
= 3.35×10^5 J kg^{-1}

latent heat of steam, L_{steam}
= 2.256×10^6 J kg^{-1}

Now, Q = heat required to convert 3 kg of ice at –12 °C to steam at 100 °C,

Q_1 = heat required to convert ice at –12 °C to ice at 0 °C.
= $m\, s_{ice}\, \Delta T_1$ = (3 kg) (2100 J kg^{-1} K^{-1}) [0–(–12)]°C = 75600 J

Q_2 = heat required to melt ice at 0 °C to water at 0 °C
= $m\, L_{f\,ice}$ = (3 kg) (3.35×10^5 J kg^{-1})
= 1005000 J

Q_3 = heat required to convert water at 0 °C to water at 100 °C.
= $ms_w\, \Delta T_2$ = (3kg) (4186J kg^{-1} K^{-1}) (100 °C)
= 1255800 J

Q_4 = heat required to convert water at 100 °C to steam at 100 °C.
= $m\, L_{steam}$ = (3 kg) (2.256×10^6 J kg^{-1})
= 6768000 J

So, Q = $Q_1 + Q_2 + Q_3 + Q_4$
= 75600J + 1005000 J + 1255800 J + 6768000 J
= 9.1×10^6 J

10.9 HEAT TRANSFER

We have seen that heat is energy transfer from one system to another or from one part of a system to another part, arising due to temperature difference. What are the different ways by which this energy transfer takes place? There are three distinct modes of heat transfer: conduction, convection and radiation (Fig. 10.13).

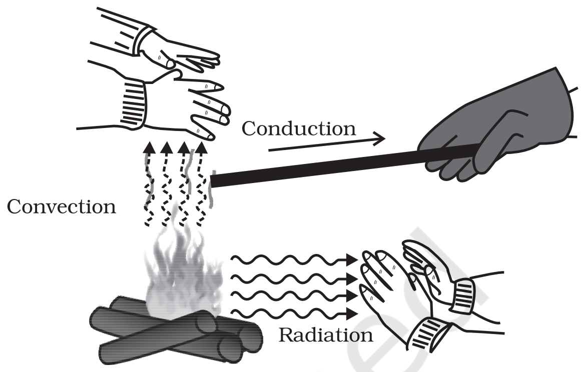


Fig. 10.13 *Heating by conduction, convection and radiation.*

10.9.1 Conduction

Conduction is the mechanism of transfer of heat between two adjacent parts of a body because of their temperature difference. Suppose, one end of a metallic rod is put in a flame, the other end of the rod will soon be so hot that you cannot hold it by your bare hands. Here, heat transfer takes place by conduction from the hot end of the rod through its different parts to the other end. Gases are poor thermal conductors, while liquids have conductivities intermediate between solids and gases.

Heat conduction may be described quantitatively as the time rate of heat flow in a material for a given temperature difference. Consider a metallic bar of length L and uniform cross-section A with its two ends maintained at different temperatures. This can be done, for example, by putting the ends in thermal contact with large reservoirs at temperatures, say, T_C and T_D, respectively (Fig. 10.14). Let us assume the ideal condition that the sides of the bar are fully insulated so that no heat is exchanged between the sides and the surroundings.

After sometime, a steady state is reached; the temperature of the bar decreases uniformly with distance from T_C to T_D; (T_C>T_D). The reservoir at C supplies heat at a constant rate, which transfers through the bar and is given out at the same rate to the reservoir at D. It is found

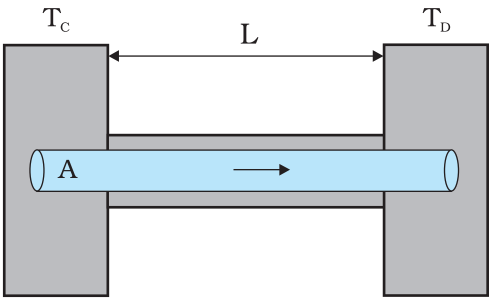


Fig. 10.14 *Steady state heat flow by conduction in a bar with its two ends maintained at temperatures T_C and T_D; ($T_C > T_D$).*

experimentally that in this steady state, the rate of flow of heat (or heat current) H is proportional to the temperature difference ($T_C - T_D$) and the area of cross-section A and is inversely proportional to the length L :

$$H = KA\frac{T_C - T_D}{L} \tag{10.14}$$

The constant of proportionality K is called the **thermal conductivity** of the material. The greater the value of K for a material, the more rapidly will it conduct heat. The SI unit of K is J s^{-1} m^{-1} K^{-1} or W m^{-1} K^{-1}. The thermal conductivities of various substances are listed in Table 10.6. These values vary slightly with temperature, but can be considered to be constant over a normal temperature range.

Compare the relatively large thermal conductivities of good thermal conductors and, metals, with the relatively small thermal conductivities of some good thermal insulators, such as wood and glass wool. You may have noticed that some cooking pots have copper coating on the bottom. Being a good conductor of heat, copper promotes the distribution of heat over the bottom of a pot for uniform cooking. Plastic foams, on the other hand, are good insulators, mainly because they contain pockets of air. Recall that gases are poor conductors, and note the low thermal conductivity of air in the Table 10.5. Heat retention and transfer are important in many other applications. Houses made of concrete roofs get very hot during summer days because thermal conductivity of concrete (though much smaller than that of a metal) is still not small enough. Therefore, people, usually, prefer to give a layer of earth or foam insulation on the ceiling so that heat transfer is prohibited and keeps the room cooler. In some situations, heat transfer is critical. In a nuclear reactor, for example, elaborate heat transfer systems need to be installed so that the enormous energy produced by nuclear fission in the core transits out sufficiently fast, thus preventing the core from overheating.

Table 10.6 Thermal conductivities of some material

Material	Thermal conductivity (J s^{-1} m^{-1} K^{-1})
Metals	
Silver	406
Copper	385
Aluminium	205
Brass	109
Steel	50.2
Lead	34.7
Mercury	8.3
Non-metals	
Insulating brick	0.15
Concrete	0.8
Body fat	0.20
Felt	0.04
Glass	0.8
Ice	1.6
Glass wool	0.04
Wood	0.12
Water	0.8
Gases	
Air	0.024
Argon	0.016
Hydrogen	0.14

Example 10.6 What is the temperature of the steel-copper junction in the steady state of the system shown in Fig. 10.15. Length of the steel rod = 15.0 cm, length of the copper rod = 10.0 cm, temperature of the furnace = 300 °C, temperature of the other end = 0 °C. The area of cross section of the steel rod is twice that of the copper rod. (Thermal conductivity of steel = 50.2 J s^{-1} m^{-1} K^{-1}; and of copper = 385 J s^{-1} m^{-1} K^{-1}).

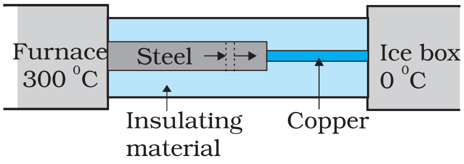


Fig. 10.15

Answer The insulating material around the rods reduces heat loss from the sides of the rods. Therefore, heat flows only along the length of the rods. Consider any cross section of the rod. In the steady state, heat flowing into the element must equal the heat flowing out of it; otherwise there would be a net gain or loss of heat by the element and its temperature would not be steady. Thus in the steady state, rate of heat flowing across a cross section of the rod is the same at every point along the length of the combined steel-copper rod. Let T be the temperature of the steel-copper junction in the steady state. Then,

$$\frac{K_1\, A_1\,(300-T)}{L_1}=\frac{K_2\, A_2\,(T-0)}{L_2}$$

where 1 and 2 refer to the steel and copper rod respectively. For $A_1 = 2\ A_2$, $L_1 = 15.0$ cm, $L_2 = 10.0$ cm, $K_1 = 50.2$ J s^{-1} m^{-1} K^{-1}, $K_2 = 385$ J s^{-1} m^{-1} K^{-1}, we have

$$\frac{50.2\times 2\,(300-T)}{15}=\frac{385T}{10}$$

which gives $T = 44.4$ °C

Example 10.7 An iron bar ($L_1 = 0.1$ m, $A_1 = 0.02$ m^2, $K_1 = 79$ W m^{-1} K^{-1}) and a brass bar ($L_2 = 0.1$ m, $A_2 = 0.02$ m^2, $K_2 = 109$ W m^{-1}K^{-1}) are soldered end to end as shown in Fig. 10.16. The free ends of the iron bar and brass bar are maintained at 373 K and 273 K respectively. Obtain expressions for and hence compute (i) the temperature of the junction of the two bars, (ii) the equivalent thermal conductivity of the compound bar, and (iii) the heat current through the compound bar.

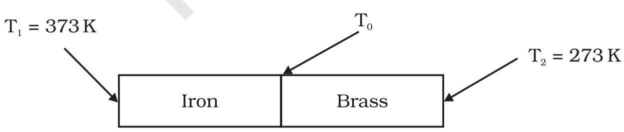


Fig 10.16

Answer

Given, $L_1 = L_2 = L = 0.1$ m, $A_1 = A_2 = A = 0.02$ m^2 $K_1 = 79$ W m^{-1} K^{-1}, $K_2 = 109$ W m^{-1} K^{-1}, $T_1 = 373$ K, and $T_2 = 273$ K.

Under steady state condition, the heat current (H_1) through iron bar is equal to the heat current (H_2) through brass bar.

So, $H = H_1 = H_2$

$$=\frac{K_1 A_1\,(T_1-T_0)}{L_1}=\frac{K_2 A_2\,(T_0-T_2)}{L_2}$$

For $A_1 = A_2 = A$ and $L_1 = L_2 = L$, this equation leads to

$K_1\,(T_1 - T_0) = K_2\,(T_0 - T_2)$

Thus, the junction temperature T_0 of the two bars is

$$T_0=\frac{(K_1T_1+K_2T_2)}{(K_1+K_2)}$$

Using this equation, the heat current H through either bar is

$$H=\frac{K_1 A(T_1-T_0)}{L}=\frac{K_2 A(T_0-T_2)}{L}$$

$$=\left(\frac{K_1K_2}{K_1+K_2}\right)\frac{A\,(T_1-T_0)}{L}=\frac{A\,(T_1-T_2)}{L\left(\frac{1}{K_1}+\frac{1}{K_2}\right)}$$

Using these equations, the heat current H' through the compound bar of length $L_1 + L_2 = 2L$ and the equivalent thermal conductivity K', of the compound bar are given by

$$H'=\frac{K'\ A\ (T_1-T_2)}{2\ L}=H$$

$$K'=\frac{2\ K_1\ K_2}{K_1+K_2}$$

(i) $$T_0=\frac{(K_1T_1+K_2T_2)}{(K_1+K_2)}$$

$$=\frac{(79\,\text{W m}^{-1}\text{K}^{-1})(373\,\text{K})+(109\,\text{W m}^{-1}\text{K}^{-1})(273\,\text{K})}{79\,\text{W m}^{-1}\text{K}^{-1}+109\ \text{W m}^{-1}\text{K}^{-1}}$$

$= 315$ K

(ii) $$K'=\frac{2K_1\ K_2}{K_1+K_2}$$

$$=\frac{2\times(79\,\text{W m}^{-1}\,\text{K}^{-1})\times(109\,\text{W m}^{-1}\,\text{K}^{-1})}{79\,\text{W m}^{-1}\,\text{K}^{-1}+109\,\text{W m}^{-1}\,\text{K}^{-1}}$$

$= 91.6$ W m^{-1} K^{-1}

(iii) $$H' = H = \frac{K'\ A\ (T_1 - T_2)}{2\ L}$$

$$= \frac{(91.6\ \mathrm{W\ m^{-1}\ K^{-1}}) \times (0.02\ \mathrm{m}^2) \times (373\,\mathrm{K} - 273\,\mathrm{K})}{2 \times (0.1\ \mathrm{m})}$$

$= 916.1$ W ◀

10.9.2 Convection

Convection is a mode of heat transfer by actual motion of matter. It is possible only in fluids. Convection can be natural or forced. In natural convection, gravity plays an important part. When a fluid is heated from below, the hot part expands and, therefore, becomes less dense. Because of buoyancy, it rises and the upper colder part replaces it. This again gets heated, rises up and is replaced by the relatively colder part of the fluid. The process goes on. This mode of heat transfer is evidently different from conduction. Convection involves bulk transport of different parts of the fluid.

In forced convection, material is forced to move by a pump or by some other physical means. The common examples of forced convection systems are forced-air heating systems in home, the human circulatory system, and the cooling system of an automobile engine. In the human body, the heart acts as the pump that circulates blood through different parts of the body, transferring heat by forced convection and maintaining it at a uniform temperature.

Natural convection is responsible for many familiar phenomena. During the day, the ground heats up more quickly than large bodies of water do. This occurs both because water has a greater specific heat capacity and because mixing currents disperse the absorbed heat throughout the great volume of water. The air in contact with the warm ground is heated by conduction. It expands, becoming less dense than the surrounding cooler air. As a result, the warm air rises (air currents) and the other air moves (winds) to fill the space-creating a sea breeze near a large body of water. Cooler air descends, and a thermal convection cycle is set up, which transfers heat away from the land. At night, the ground loses its heat more quickly, and the water surface is warmer than the land. As a result, the cycle is reveresed (Fig. 10.17).

The other example of natural convection is the steady surface wind on the earth blowing in from north-east towards the equator, the so-called trade wind. A resonable explanation is as follows: the equatorial and polar regions of the earth receive unequal solar heat. Air at the earth's surface near the equator is hot, while the air in the upper atmosphere of the poles is cool. In the absence of any other factor, a convection current would be set up, with the air at the equatorial surface rising and moving out towards the poles, descending and streaming in towards the equator. The rotation of the earth, however, modifies this convection current. Because of this, air close to the equator has an eastward speed of 1600 km/h, while it is zero close to the poles. As a result, the air descends not at the poles but at 30° N (North) latitude and returns to the equator. This is called **trade wind**.

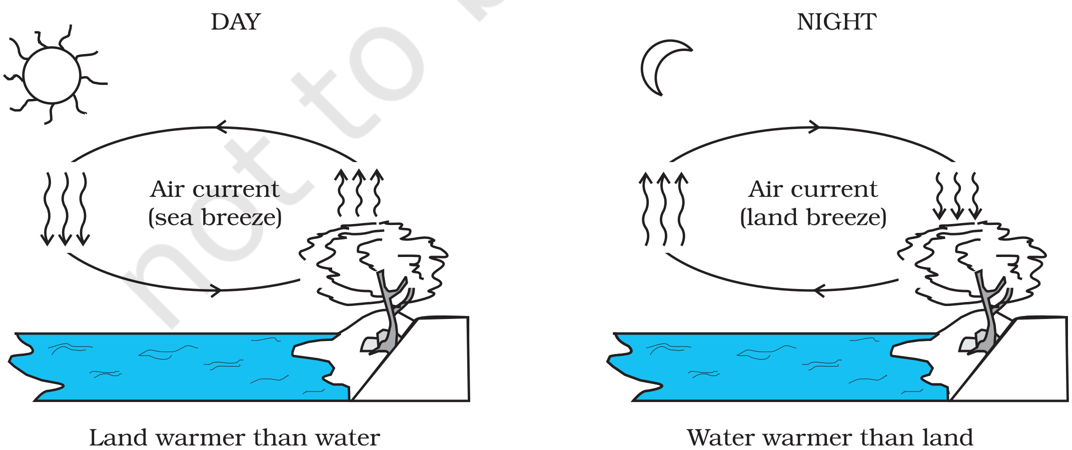


Fig. 10.17 *Convection cycles.*

10.9.3 Radiation

Conduction and convection require some material as a transport medium. These modes of heat transfer cannot operate between bodies separated by a distance in vacuum. But the earth does receive heat from the Sun across a huge distance. Similarly, we quickly feel the warmth of the fire nearby even though air conducts poorly and before convection takes some time to set in. The third mechanism for heat transfer needs no medium; it is called radiation and the energy so transferred by electromagnetic waves is called radiant energy. In an electromagnetic wave, electric and magnetic fields oscillate in space and time. Like any wave, electromagnetic waves can have different wavelengths and can travel in vacuum with the same speed, namely the speed of light i.e., 3×10^8 m s^{-1}. You will learn these matters in more detail later, but you now know why heat transfer by radiation does not need any medium and why it is so fast. This is how heat is transferred to the earth from the Sun through empty space. All bodies emit radiant energy, whether they are solid, liquid or gas. The electromagnetic radiation emitted by a body by virtue of its temperature, like radiation by a red hot iron or light from a filament lamp is called thermal radiation.

When this thermal radiation falls on other bodies, it is partly reflected and partly absorbed. The amount of heat that a body can absorb by radiation depends on the colour of the body.

We find that black bodies absorb and emit radiant energy better than bodies of lighter colours. This fact finds many applications in our daily life. We wear white or light coloured clothes in summer, so that they absorb the least heat from the Sun. However, during winter, we use dark coloured clothes, which absorb heat from the sun and keep our body warm. The bottoms of utensils for cooking food are blackened so that they absorb maximum heat from fire and transfer it to the vegetables to be cooked.

Similarly, a Dewar flask or thermos bottle is a device to minimise heat transfer between the contents of the bottle and outside. It consists of a double-walled glass vessel with the inner and outer walls coated with silver. Radiation from the inner wall is reflected back to the contents of the bottle. The outer wall similarly reflects back any incoming radiation. The space between the walls is evacuated to reduce conduction and convection losses and the flask is supported on an insulator, like cork. The device is, therefore, useful for preventing hot contents (like, milk) from getting cold, or alternatively, to store cold contents (like, ice).

10.9.4 Blackbody Radiation

We have so far not mentioned the wavelength content of thermal radiation. The important thing about thermal radiation at any temperature is that it is not of one (or a few) wavelength(s) but has a continuous spectrum from the small to the long wavelengths. The energy content of radiation, however, varies for different wavelengths. Figure 10.18 gives the experimental curves for radiation energy per unit area per unit wavelength emitted by a blackbody versus wavelength for different temperatures.

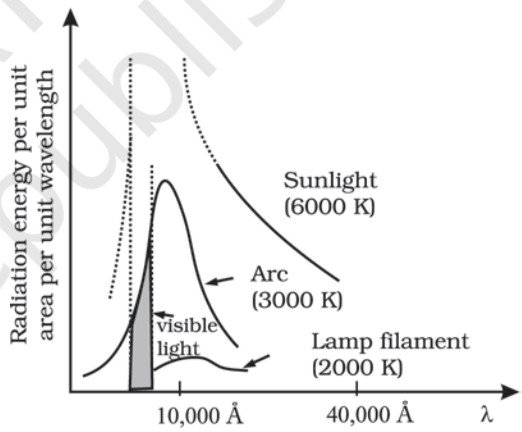


Fig. 10.18: *Energy emitted versus wavelength for a blackbody at different temperatures*

Notice that the wavelength λ_m for which energy is the maximum decreases with increasing temperature. The relation between λ_m and T is given by what is known as **Wien's Displacement Law**:

$$\lambda_m T = \text{constant} \qquad (10.15)$$

The value of the constant (Wien's constant) is 2.9×10^{-3} m K. This law explains why the colour of a piece of iron heated in a hot flame first becomes dull red, then reddish yellow, and finally white hot. Wien's law is useful for estimating the surface temperatures of celestial bodies like, the moon, Sun and other stars. Light from the moon is found to have a maximum intensity near the wavelength 14 μm. By Wien's law, the surface of the moon is estimated to have a temperature of 200 K. Solar radiation has a maximum at λ_m = 4753 Å. This corresponds to T = 6060 K. Remember, this is the temperature of the surface of the sun, not its interior.

The most significant feature of the blackbody radiation curves in Fig. 10.18 is that they are *universal*. They depend only on the temperature and not on the size, shape or material of the blackbody. Attempts to explain blackbody radiation theoretically, at the beginning of the twentieth century, spurred the quantum revolution in physics, as you will learn in later courses.

Energy can be transferred by radiation over large distances, without a medium (i.e., in vacuum). The total electromagnetic energy radiated by a body at absolute temperature T is proportional to its size, its ability to radiate (called emissivity) and most importantly to its temperature. For a body, which is a perfect radiator, the energy emitted per unit time (H) is given by

$$H = A\sigma T^4 \tag{10.16}$$

where A is the area and T is the absolute temperature of the body. This relation obtained experimentally by Stefan and later proved theoretically by Boltzmann is known as **Stefan-Boltzmann law** and the constant σ is called Stefan-Boltzmann constant. Its value in SI units is 5.67×10^{-8} W m^{-2} K^{-4}. Most bodies emit only a fraction of the rate given by Eq. 10.16. A substance like lamp black comes close to the limit. One, therefore, defines a dimensionless fraction e called *emissivity* and writes,

$$H = Ae\sigma T^4 \tag{10.17}$$

Here, $e = 1$ for a perfect radiator. For a tungsten lamp, for example, e is about 0.4. Thus, a tungsten lamp at a temperature of 3000 K and a surface area of 0.3 cm^2 radiates at the rate $H = 0.3 \times 10^{-4} \times 0.4 \times 5.67 \times 10^{-8} \times (3000)^4 = 60$ W.

A body at temperature T, with surroundings at temperatures T_s, emits, as well as, receives energy. For a perfect radiator, the net rate of loss of radiant energy is

$$H = \sigma A\,(T^4 - T_s^4)$$

For a body with emissivity e, the relation modifies to

$$H = e\sigma A\,(T^4 - T_s^4) \tag{10.18}$$

As an example, let us estimate the heat radiated by our bodies. Suppose the surface area of a person's body is about 1.9 m^2 and the room temperature is 22 °C. The internal body temperature, as we know, is about 37 °C. The skin temperature may be 28 °C (say). The emissivity of the skin is about 0.97 for the relevant region of electromagnetic radiation. The rate of heat loss is:

$$H = 5.67 \times 10^{-8} \times 1.9 \times 0.97 \times \{(301)^4 - (295)^4\}$$
$$= 66.4 \text{ W}$$

which is more than half the rate of energy production by the body at rest (120 W). To prevent this heat loss effectively (better than ordinary clothing), modern arctic clothing has an additional thin shiny metallic layer next to the skin, which reflects the body's radiation.

10.10 NEWTON'S LAW OF COOLING

We all know that hot water or milk when left on a table begins to cool, gradually. Ultimately it attains the temperature of the surroundings. To study how slow or fast a given body can cool on exchanging heat with its surroundings, let us perform the following activity.

Take some water, say 300 mL, in a calorimeter with a stirrer and cover it with a two-holed lid. Fix the stirrer through one hole and fix a thermometer through another hole in the lid and make sure that the bulb of thermometer is immersed in the water. Note the reading of the thermometer. This reading T_1 is the temperature of the surroundings. Heat the water kept in the calorimeter till it attains a temperature, say 40 °C above room temperature (i.e., temperature of the surroundings). Then, stop heating the water by removing the heat source. Start the stop-watch and note the reading of the thermometer after a fixed interval of time, say after every one minute of stirring gently with the stirrer. Continue to note the temperature (T_2) of water till it attains a temperature about 5 °C above that of the surroundings. Then, plot a graph by taking each value of temperature $\Delta T = T_2 - T_1$ along y-axis and the coresponding value of t along x-axis (Fig. 10.19).

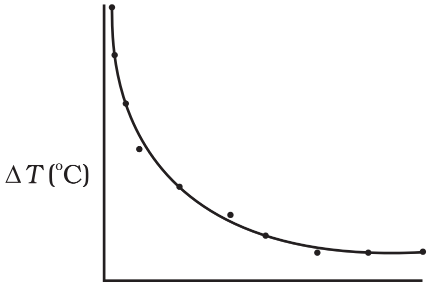


Fig. 10.19 *Curve showing cooling of hot water with time.*

From the graph you can infer how the cooling of hot water depends on the difference of its temperature from that of the surroundings. You will also notice that initially the rate of cooling is higher and decreases as the temperature of the body falls.

The above activity shows that a hot body loses heat to its surroundings in the form of heat radiation. The rate of loss of heat depends on the difference in temperature between the body and its surroundings. Newton was the first to study, in a systematic manner, the relation between the heat lost by a body in a given enclosure and its temperature.

According to Newton's law of cooling, the rate of loss of heat, $-\mathrm{d}Q/\mathrm{d}t$ of the body is directly proportional to the difference of temperature $\Delta T = (T_2 - T_1)$ of the body and the surroundings. The law holds good only for small difference of temperature. Also, the loss of heat by radiation depends upon the nature of the surface of the body and the area of the exposed surface. We can write

$$-\frac{dQ}{dt} = k(T_2 - T_1) \tag{10.19}$$

where k is a positive constant depending upon the area and nature of the surface of the body. Suppose a body of mass m and specific heat capacity s is at temperature T_2. Let T_1 be the temperature of the surroundings. If the temperature falls by a small amount $\mathrm{d}T_2$ in time $\mathrm{d}t$, then the amount of heat lost is

$\mathrm{d}Q = ms\,\mathrm{d}T_2$

$\therefore$ Rate of loss of heat is given by

$$\frac{dQ}{dt} = ms\frac{dT_2}{dt} \tag{10.20}$$

From Eqs. (10.15) and (10.16) we have

$$-ms\frac{dT_2}{dt} = k(T_2 - T_1)$$

$$\frac{dT_2}{T_2 - T_1} = -\frac{k}{ms}dt = -K\,dt \tag{10.21}$$

where $K = k/m\,s$

On integrating,

$$\log_e(T_2 - T_1) = -K\,t + c \tag{10.22}$$

$$\text{or } T_2 = T_1 + C'\,e^{-Kt};\ \text{where } C' = e^c \tag{10.23}$$

Equation (10.23) enables you to calculate the time of cooling of a body through a particular range of temperature.

For small temperature differences, the rate of cooling, due to conduction, convection, and radiation combined, is proportional to the difference in temperature. It is a valid approximation in the transfer of heat from a radiator to a room, the loss of heat through the wall of a room, or the cooling of a cup of tea on the table.

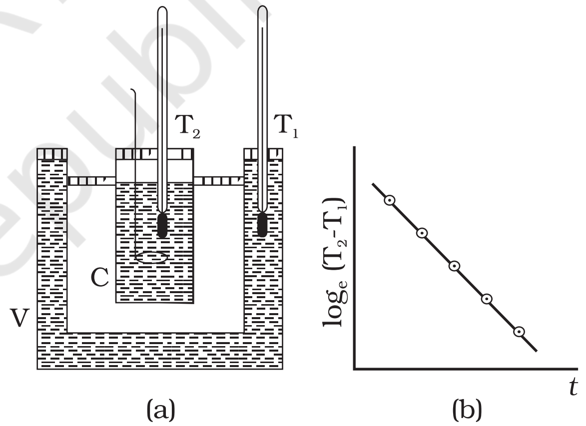


Fig. 10.20 *Verification of Newton's Law of cooling.*

Newton's law of cooling can be verified with the help of the experimental set-up shown in Fig. 10.20(a). The set-up consists of a double-walled vessel (V) containing water between the two walls. A copper calorimeter (C) containing hot water is placed inside the double-walled vessel. Two thermometers through the corks are used to note the temperatures T_2 of water in calorimeter and T_1 of hot water in between the double walls, respectively. Temperature of hot water in the calorimeter is noted after equal intervals of time. A graph is plotted between $\log_e(T_2 - T_1)$ [or $\ln(T_2 - T_1)$] and time (t). The nature of the graph is observed to be a straight line having a negative slope as shown in Fig. 10.20(b). This is in support of Eq. 10.22.

Example 10.8 A pan filled with hot food cools from 94 °C to 86 °C in 2 minutes when the room temperature is at 20 °C. How long will it take to cool from 71 °C to 69 °C?

Answer The average temperature of 94 °C and 86 °C is 90 °C, which is 70 °C above the room temperature. Under these conditions the pan cools 8 °C in 2 minutes.

Using Eq. (10.21), we have

$$\frac{\text{Change in temperature}}{\text{Time}} = K\Delta T$$

$$\frac{8\,°\text{C}}{2\text{ min}} = K(70\,°\text{C})$$

The average of 69 °C and 71 °C is 70 °C, which is 50 °C above room temperature. K is the same for this situation as for the original.

$$\frac{2\,°\text{C}}{\text{Time}} = K(50\,°\text{C})$$

When we divide above two equations, we have

$$\frac{8\,°\text{C}/2\text{ min}}{2\,°\text{C}/\text{time}} = \frac{K\,(70\,°\text{C})}{K\,(50\,°\text{C})}$$

Time = 0.7 min

= 42 s

SUMMARY

1. Heat is a form of energy that flows between a body and its surrounding medium by virtue of temperature difference between them. The degree of hotness of the body is quantitatively represented by temperature.
2. A temperature-measuring device (thermometer) makes use of some measurable property (called thermometric property) that changes with temperature. Different thermometers lead to different temperature scales. To construct a temperature scale, two fixed points are chosen and assigned some arbitrary values of temperature. The two numbers fix the origin of the scale and the size of its unit.
3. The Celsius temperature (t_C) and the Farenheit temperare (t_F)are related by

   $$t_F = (9/5)\, t_C + 32$$

4. The ideal gas equation connecting pressure (P), volume (V) and absolute temperature (T) is :

   $$PV = \mu RT$$

   where μ is the number of moles and R is the universal gas constant.
5. In the absolute temperature scale, the zero of the scale corresponds to the temperature where every substance in nature has the least possible molecular activity. The Kelvin absolute temperature scale (T) has the same unit size as the Celsius scale (T_c), but differs in the origin :

   $$T_C = T - 273.15$$

6. The coefficient of linear expansion (α_l) and volume expansion (α_V) are defined by the relations :

   $$\frac{\Delta l}{l} = \alpha_l \Delta T$$

   $$\frac{\Delta V}{V} = \alpha_V \Delta T$$

where Δl and ΔV denote the change in length l and volume V for a change of temperature ΔT. The relation between them is :

$$\alpha_v = 3\,\alpha_l$$

7. The specific heat capacity of a substance is defined by

$$s = \frac{1}{m}\frac{\Delta Q}{\Delta T}$$

where m is the mass of the substance and ΔQ is the heat required to change its temperature by ΔT. The molar specific heat capacity of a substance is defined by

$$C = \frac{1}{\mu}\frac{\Delta Q}{\Delta T}$$

where μ is the number of moles of the substance.

8. The latent heat of fusion (L_f) is the heat per unit mass required to change a substance from solid into liquid at the same temperature and pressure. The latent heat of vaporisation (L_v) is the heat per unit mass required to change a substance from liquid to the vapour state without change in the temperature and pressure.

9. The three modes of heat transfer are conduction, convection and radiation.

10. In conduction, heat is transferred between neighbouring parts of a body through molecular collisions, without any flow of matter. For a bar of length L and uniform cross section A with its ends maintained at temperatures T_C and T_D, the rate of flow of heat H is :

$$H = K\,A\,\frac{T_C - T_D}{L}$$

where K is the thermal conductivity of the material of the bar.

11. Newton's Law of Cooling says that the rate of cooling of a body is proportional to the excess temperature of the body over the surroundings :

$$\frac{dQ}{dt} = -k\,(T_2 - T_1)$$

Where T_1 is the temperature of the surrounding medium and T_2 is the temperature of the body.

Quantity	Symbol	Dimensions	Unit	Remark
Amount of substance	μ	[mol]	mol	
Celsius temperature	t_c	[K]	°C	
Kelvin absolute temperature	T	[K]	K	$t_c = T - 273.15$
Co-efficient of linear expansion	α_l	$[K^{-1}]$	K^{-1}	
Co-efficient of volume expansion	α_v	$[K^{-1}]$	K^{-1}	$\alpha_v = 3\,\alpha_l$
Heat supplied to a system	ΔQ	$[ML^2\,T^{-2}]$	J	Q is not a state variable
Specific heat capacity	s	$[L^2\,T^{-2}\,K^{-1}]$	$J\,kg^{-1}\,K^{-1}$	
Thermal Conductivity	K	$[M\,LT^{-3}K^{-1}]$	$J\,s^{-1}K^{-1}$	$H = -KA\,\frac{dT}{dx}$

POINTS TO PONDER

1. The relation connecting Kelvin temperature (T) and the Celsius temperature t_c

$$T = t_c + 273.15$$

and the assignment $T = 273.16$ K for the triple point of water are exact relations (by choice). With this choice, the Celsius temperature of the melting point of water and boiling point of water (both at 1 atm pressure) are very close to, but not exactly equal to 0 °C and 100 °C respectively. In the original Celsius scale, these latter fixed points were exactly at 0 °C and 100 °C (by choice), but now the triple point of water is the preferred choice for the fixed point, because it has a unique temperature.

2. A liquid in equilibrium with vapour has the same pressure and temperature throughout the system; the two phases in equilibrium differ in their molar volume (i.e. density). This is true for a system with any number of phases in equilibrium.
3. Heat transfer always involves temperature difference between two systems or two parts of the same system. Any energy transfer that does not involve temperature difference in some way is not heat.
4. Convection involves flow of matter *within a fluid* due to unequal temperatures of its parts. A hot bar placed under a running tap loses heat by conduction between the surface of the bar and water and not by convection within water.

EXERCISES

10.1 The triple points of neon and carbon dioxide are 24.57 K and 216.55 K respectively. Express these temperatures on the Celsius and Fahrenheit scales.

10.2 Two absolute scales A and B have triple points of water defined to be 200 A and 350 B. What is the relation between T_A and T_B ?

10.3 The electrical resistance in ohms of a certain thermometer varies with temperature according to the approximate law :

$$R = R_o [1 + \alpha (T - T_o)]$$

The resistance is 101.6 Ω at the triple-point of water 273.16 K, and 165.5 Ω at the normal melting point of lead (600.5 K). What is the temperature when the resistance is 123.4 Ω ?

10.4 Answer the following :

(a) The triple-point of water is a standard fixed point in modern thermometry. Why ? What is wrong in taking the melting point of ice and the boiling point of water as standard fixed points (as was originally done in the Celsius scale) ?

(b) There were two fixed points in the original Celsius scale as mentioned above which were assigned the number 0 °C and 100 °C respectively. On the absolute scale, one of the fixed points is the triple-point of water, which on the Kelvin absolute scale is assigned the number 273.16 K. What is the other fixed point on this (Kelvin) scale ?

(c) The absolute temperature (Kelvin scale) T is related to the temperature t_c on the Celsius scale by

$$t_c = T - 273.15$$

Why do we have 273.15 in this relation, and not 273.16 ?

(d) What is the temperature of the triple-point of water on an absolute scale whose unit interval size is equal to that of the Fahrenheit scale ?

10.5 Two ideal gas thermometers A and B use oxygen and hydrogen respectively. The following observations are made :

Temperature	Pressure thermometer A	Pressure thermometer B
Triple-point of water	1.250×10^5 Pa	0.200×10^5 Pa
Normal melting point of sulphur	1.797×10^5 Pa	0.287×10^5 Pa

(a) What is the absolute temperature of normal melting point of sulphur as read by thermometers A and B ?

(b) What do you think is the reason behind the slight difference in answers of thermometers A and B ? (The thermometers are not faulty). What further procedure is needed in the experiment to reduce the discrepancy between the two readings ?

10.6 A steel tape 1m long is correctly calibrated for a temperature of 27.0 °C. The length of a steel rod measured by this tape is found to be 63.0 cm on a hot day when the temperature is 45.0 °C. What is the actual length of the steel rod on that day ? What is the length of the same steel rod on a day when the temperature is 27.0 °C ? Coefficient of linear expansion of steel = 1.20×10^{-5} K^{-1} .

10.7 A large steel wheel is to be fitted on to a shaft of the same material. At 27 °C, the outer diameter of the shaft is 8.70 cm and the diameter of the central hole in the wheel is 8.69 cm. The shaft is cooled using 'dry ice'. At what temperature of the shaft does the wheel slip on the shaft? Assume coefficient of linear expansion of the steel to be constant over the required temperature range :
$\alpha_{steel} = 1.20 \times 10^{-5}$ K^{-1}.

10.8 A hole is drilled in a copper sheet. The diameter of the hole is 4.24 cm at 27.0 °C. What is the change in the diameter of the hole when the sheet is heated to 227 °C? Coefficient of linear expansion of copper = 1.70×10^{-5} K^{-1}.

10.9 A brass wire 1.8 m long at 27 °C is held taut with little tension between two rigid supports. If the wire is cooled to a temperature of –39 °C, what is the tension developed in the wire, if its diameter is 2.0 mm ? Co-efficient of linear expansion of brass = 2.0×10^{-5} K^{-1}; Young's modulus of brass = 0.91×10^{11} Pa.

10.10 A brass rod of length 50 cm and diameter 3.0 mm is joined to a steel rod of the same length and diameter. What is the change in length of the combined rod at 250 °C, if the original lengths are at 40.0 °C? Is there a 'thermal stress' developed at the junction ? The ends of the rod are free to expand (Co-efficient of linear expansion of brass = 2.0×10^{-5} K^{-1}, steel = 1.2×10^{-5} K^{-1}).

10.11 The coefficient of volume expansion of glycerine is 49×10^{-5} K^{-1}. What is the fractional change in its density for a 30 °C rise in temperature ?

10.12 A 10 kW drilling machine is used to drill a bore in a small aluminium block of mass 8.0 kg. How much is the rise in temperature of the block in 2.5 minutes, assuming 50% of power is used up in heating the machine itself or lost to the surroundings. Specific heat of aluminium = 0.91 J g^{-1} K^{-1}.

10.13 A copper block of mass 2.5 kg is heated in a furnace to a temperature of 500 °C and then placed on a large ice block. What is the maximum amount of ice that can melt? (Specific heat of copper = 0.39 J g^{-1} K^{-1}; heat of fusion of water = 335 J g^{-1}).

10.14 In an experiment on the specific heat of a metal, a 0.20 kg block of the metal at 150 °C is dropped in a copper calorimeter (of water equivalent 0.025 kg) containing 150 cm^3 of water at 27 °C. The final temperature is 40 °C. Compute the specific heat of the metal. If heat losses to the surroundings are not negligible, is your answer greater or smaller than the actual value for specific heat of the metal ?

10.15 Given below are observations on molar specific heats at room temperature of some common gases.

Gas	Molar specific heat (C_v) (cal mo1^{-1} K^{-1})
Hydrogen	4.87
Nitrogen	4.97
Oxygen	5.02
Nitric oxide	4.99
Carbon monoxide	5.01
Chlorine	6.17

The measured molar specific heats of these gases are markedly different from those for monatomic gases. Typically, molar specific heat of a monatomic gas is 2.92 cal/mol K. Explain this difference. What can you infer from the somewhat larger (than the rest) value for chlorine ?

10.16 A child running a temperature of 101°F is given an antipyrin (i.e. a medicine that lowers fever) which causes an increase in the rate of evaporation of sweat from his body. If the fever is brought down to 98 °F in 20 minutes, what is the average rate of extra evaporation caused, by the drug. Assume the evaporation mechanism to be the only way by which heat is lost. The mass of the child is 30 kg. The specific heat of human body is approximately the same as that of water, and latent heat of evaporation of water at that temperature is about 580 cal g^{-1}.

10.17 A 'thermacole' icebox is a cheap and an efficient method for storing small quantities of cooked food in summer in particular. A cubical icebox of side 30 cm has a thickness of 5.0 cm. If 4.0 kg of ice is put in the box, estimate the amount of ice remaining after 6 h. The outside temperature is 45 °C, and co-efficient of thermal conductivity of thermacole is 0.01 J s^{-1} m^{-1} K^{-1}. [Heat of fusion of water = 335 × 10^3 J kg^{-1}]

10.18 A brass boiler has a base area of 0.15 m^2 and thickness 1.0 cm. It boils water at the rate of 6.0 kg/min when placed on a gas stove. Estimate the temperature of the part of the flame in contact with the boiler. Thermal conductivity of brass = 109 J s^{-1} m^{-1} K^{-1} ; Heat of vaporisation of water = 2256 × 10^3 J kg^{-1}.

10.19 Explain why :
(a) a body with large reflectivity is a poor emitter
(b) a brass tumbler feels much colder than a wooden tray on a chilly day
(c) an optical pyrometer (for measuring high temperatures) calibrated for an ideal black body radiation gives too low a value for the temperature of a red hot iron piece in the open, but gives a correct value for the temperature when the same piece is in the furnace
(d) the earth without its atmosphere would be inhospitably cold
(e) heating systems based on circulation of steam are more efficient in warming a building than those based on circulation of hot water

10.20 A body cools from 80 °C to 50 °C in 5 minutes. Calculate the time it takes to cool from 60 °C to 30 °C. The temperature of the surroundings is 20 °C.

CHAPTER ELEVEN

THERMODYNAMICS

11.1 INTRODUCTION

In previous chapter we have studied thermal properties of matter. In this chapter we shall study laws that govern thermal energy. We shall study the processes where work is converted into heat and vice versa. In winter, when we rub our palms together, we feel warmer; here work done in rubbing produces the 'heat'. Conversely, in a steam engine, the 'heat' of the steam is used to do useful work in moving the pistons, which in turn rotate the wheels of the train.

In physics, we need to define the notions of heat, temperature, work, etc. more carefully. Historically, it took a long time to arrive at the proper concept of 'heat'. Before the modern picture, heat was regarded as a fine invisible fluid filling in the pores of a substance. On contact between a hot body and a cold body, the fluid (called caloric) flowed from the colder to the hotter body ! This is similar to what happens when a horizontal pipe connects two tanks containing water up to different heights. The flow continues until the levels of water in the two tanks are the same. Likewise, in the 'caloric' picture of heat, heat flows until the 'caloric levels' (i.e., the temperatures) equalise.

In time, the picture of heat as a fluid was discarded in favour of the modern concept of heat as a form of energy. An important experiment in this connection was due to Benjamin Thomson (also known as Count Rumford) in 1798. He observed that boring of a brass cannon generated a lot of heat, indeed enough to boil water. More significantly, the amount of heat produced depended on the work done (by the horses employed for turning the drill) but not on the sharpness of the drill. In the caloric picture, a sharper drill would scoop out more heat fluid from the pores; but this was not observed. A most natural explanation of the observations was that heat was a form of energy and the experiment demonstrated conversion of energy from one form to another–from work to heat.

Thermodynamics is the branch of physics that deals with the concepts of heat and temperature and the inter-conversion of heat and other forms of energy. Thermodynamics is a macroscopic science. It deals with bulk systems and does not go into the molecular constitution of matter. In fact, its concepts and laws were formulated in the nineteenth century before the molecular picture of matter was firmly established. Thermodynamic description involves relatively few macroscopic variables of the system, which are suggested by common sense and can be usually measured directly. A microscopic description of a gas, for example, would involve specifying the co-ordinates and velocities of the huge number of molecules constituting the gas. The description in kinetic theory of gases is not so detailed but it does involve molecular distribution of velocities. Thermodynamic description of a gas, on the other hand, avoids the molecular description altogether. Instead, the state of a gas in thermodynamics is specified by macroscopic variables such as pressure, volume, temperature, mass and composition that are felt by our sense perceptions and are measurable*.

The distinction between mechanics and thermodynamics is worth bearing in mind. In mechanics, our interest is in the motion of particles or bodies under the action of forces and torques. Thermodynamics is not concerned with the motion of the system as a whole. It is concerned with the internal macroscopic state of the body. When a bullet is fired from a gun, what changes is the mechanical state of the bullet (its kinetic energy, in particular), not its temperature. When the bullet pierces a wood and stops, the kinetic energy of the bullet gets converted into heat, changing the temperature of the bullet and the surrounding layers of wood. Temperature is related to the energy of the internal (disordered) motion of the bullet, not to the motion of the bullet as a whole.

11.2 THERMAL EQUILIBRIUM

Equilibrium in mechanics means that the net external force and torque on a system are zero. The term 'equilibrium' in thermodynamics appears in a different context : we say the state of a system is an equilibrium state if the macroscopic variables that characterise the system do not change in time. For example, a gas inside a closed rigid container, completely insulated from its surroundings, with fixed values of pressure, volume, temperature, mass and composition that do not change with time, is in a state of thermodynamic equilibrium.

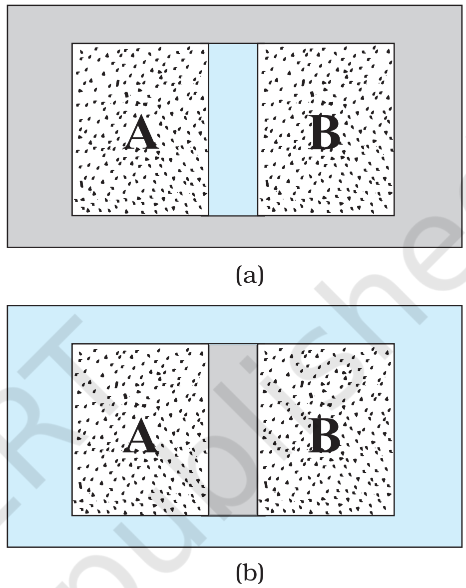


Fig. 11.1 *(a) Systems A and B (two gases) separated by an adiabatic wall – an insulating wall that does not allow flow of heat. (b) The same systems A and B separated by a diathermic wall – a conducting wall that allows heat to flow from one to another. In this case, thermal equilibrium is attained in due course.*

In general, whether or not a system is in a state of equilibrium depends on the surroundings and the nature of the wall that separates the system from the surroundings. Consider two gases *A* and *B* occupying two different containers. We know experimentally that pressure and volume of a given mass of gas can be chosen to be its two independent variables. Let the pressure and volume of the gases be (P_A, V_A) and (P_B, V_B) respectively. Suppose first that the two systems are put in proximity but are separated by an

* *Thermodynamics may also involve other variables that are not so obvious to our senses e.g. entropy, enthalpy, etc., and they are all macroscopic variables. However, a thermodynamic state is specified by five state variables viz., pressure, volume, temperature, internal energy and entropy. Entropy is a measure of disorderness in the system. Enthalpy is a measure of total heat content of the system.*

adiabatic wall – an insulating wall (can be movable) that does not allow flow of energy (heat) from one to another. The systems are insulated from the rest of the surroundings also by similar adiabatic walls. The situation is shown schematically in Fig. 11.1 (a). In this case, it is found that any possible pair of values (P_A, V_A) will be in equilibrium with any possible pair of values (P_B, V_B). Next, suppose that the adiabatic wall is replaced by a **diathermic wall** – a conducting wall that allows energy flow (heat) from one to another. It is then found that the macroscopic variables of the systems *A* and *B* change spontaneously until both the systems attain equilibrium states. After that there is no change in their states. The situation is shown in Fig. 11.1(b). The pressure and volume variables of the two gases change to (P_B', V_B') and (P_A', V_A') such that the new states of *A* and *B* are in equilibrium with each other*. There is no more energy flow from one to another. We then say that the system *A* is in thermal equilibrium with the system *B*.

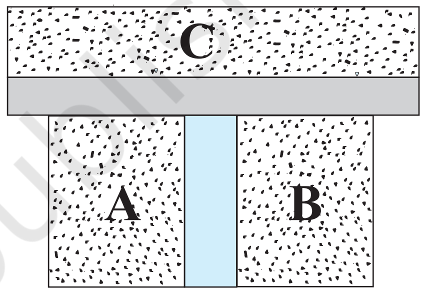
What characterises the situation of thermal equilibrium between two systems ? You can guess the answer from your experience. In thermal equilibrium, the temperatures of the two systems are equal. We shall see how does one arrive at the concept of temperature in thermodynamics? The Zeroth law of thermodynamics provides the clue.

11.3 ZEROTH LAW OF THERMODYNAMICS

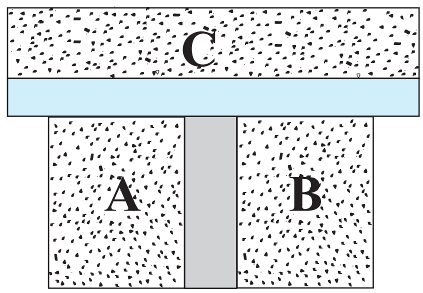
Imagine two systems *A* and *B*, separated by an adiabatic wall, while each is in contact with a third system *C*, via a conducting wall [Fig. 11.2(a)]. The states of the systems (i.e., their macroscopic variables) will change until both *A* and *B* come to thermal equilibrium with *C*. After this is achieved, suppose that the adiabatic wall between *A* and *B* is replaced by a conducting wall and *C* is insulated from *A* and *B* by an adiabatic wall [Fig.11.2(b)]. It is found that the states of *A* and *B* change no further i.e. they are found **to be in thermal equilibrium with each other**. This observation forms the basis of the **Zeroth Law of Thermodynamics**, which states that **'two systems in thermal equilibrium with a third system separately are in thermal equilibrium with each other'**. R.H. Fowler formulated this law in 1931 long after the first and second Laws of thermodynamics were stated and so numbered.

The Zeroth Law clearly suggests that when two systems *A* and *B*, are in thermal equilibrium, there must be a physical quantity that has the same value for both. This thermodynamic variable whose value is equal for two systems in thermal equilibrium is called temperature (*T*). Thus, if *A* and *B* are separately in equilibrium with *C*, $T_A = T_C$ and $T_B = T_C$. This implies that $T_A = T_B$ i.e. the systems *A* and *B* are also in thermal equilibrium.

We have arrived at the concept of temperature formally via the Zeroth Law. The next question is : how to assign numerical values to temperatures of different bodies ? In other words, how do we construct a scale of temperature ? Thermometry deals with this basic question to which we turn in the next section.



(a)



(b)

Fig. 11.2 *(a) Systems A and B are separated by an adiabatic wall, while each is in contact with a third system C via a conducting wall. (b) The adiabatic wall between A and B is replaced by a conducting wall, while C is insulated from A and B by an adiabatic wall.*

* *Both the variables need not change. It depends on the constraints. For instance, if the gases are in containers of fixed volume, only the pressures of the gases would change to achieve thermal equilibrium.*

11.4 HEAT, INTERNAL ENERGY AND WORK

The Zeroth Law of Thermodynamics led us to the concept of temperature that agrees with our commonsense notion. Temperature is a marker of the 'hotness' of a body. It determines the direction of flow of heat when two bodies are placed in thermal contact. Heat flows from the body at a higher temperature to the one at lower temperature. The flow stops when the temperatures equalise; the two bodies are then in thermal equilibrium. We saw in some detail how to construct temperature scales to assign temperatures to different bodies. We now describe the concepts of heat and other relevant quantities like internal energy and work.

The concept of internal energy of a system is not difficult to understand. We know that every bulk system consists of a large number of molecules. Internal energy is simply the sum of the kinetic energies and potential energies of these molecules. We remarked earlier that in thermodynamics, the kinetic energy of the system, as a whole, is not relevant. Internal energy is thus, the sum of molecular kinetic and potential energies in the frame of reference relative to which the centre of mass of the system is at rest. Thus, it includes only the (disordered) energy associated with the random motion of molecules of the system. We denote the internal energy of a system by U.

Though we have invoked the molecular picture to understand the meaning of internal energy, as far as thermodynamics is concerned, U is simply a macroscopic variable of the system. The important thing about internal energy is that it depends only on the state of the system, not on how that state was achieved. Internal energy U of a system is an example of a thermodynamic 'state variable' – its value depends only on the given state of the system, not on history i.e. not on the 'path' taken to arrive at that state. Thus, the internal energy of a given mass of gas depends on its state described by specific values of pressure, volume and temperature. It does not depend on how this state of the gas came about. Pressure, volume, temperature, and internal energy are thermodynamic state variables of the system (gas) (see section 11.7). If we neglect the small intermolecular forces in a gas, the internal energy of a gas is just the sum of kinetic energies associated with various random motions of its molecules. We will see in the next chapter that in a gas this motion is not only translational (i.e. motion from one point to another in the volume of the container); it also includes rotational and vibrational motion of the molecules (Fig. 11.3).

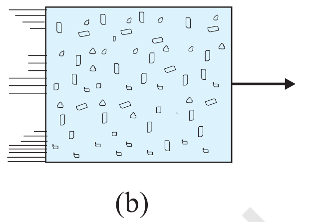


Fig. 11.3 *(a) Internal energy U of a gas is the sum of the kinetic and potential energies of its molecules when the box is at rest. Kinetic energy due to various types of motion (translational, rotational, vibrational) is to be included in U. (b) If the same box is moving as a whole with some velocity, the kinetic energy of the box is not to be included in U.*

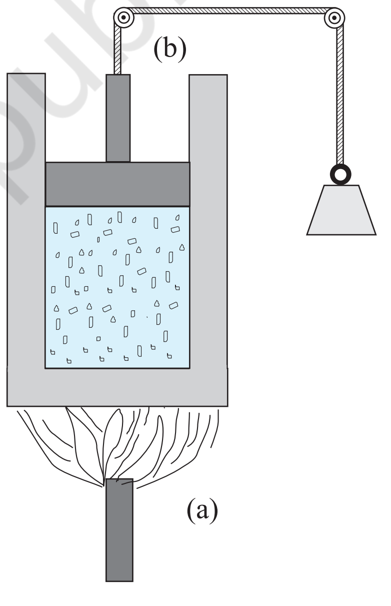


Fig. 11.4 *Heat and work are two distinct modes of energy transfer to a system that results in change in its internal energy. (a) Heat is energy transfer due to temperature difference between the system and the surroundings. (b) Work is energy transfer brought about by means (e.g. moving the piston by raising or lowering some weight connected to it) that do not involve such a temperature difference.*

What are the ways of changing internal energy of a system ? Consider again, for simplicity, the system to be a certain mass of gas contained in a cylinder with a movable piston as shown in Fig. 11.4. Experience shows there are two ways of changing the state of the gas (and hence its internal energy). One way is to put the cylinder in contact with a body at a higher temperature than that of the gas. The temperature difference will cause a flow of energy (heat) from the hotter body to the gas, thus increasing the internal energy of the gas. The other way is to push the piston down i.e. to do work on the system, which again results in increasing the internal energy of the gas. Of course, both these things could happen in the reverse direction. With surroundings at a lower temperature, heat would flow from the gas to the surroundings. Likewise, the gas could push the piston up and do work on the surroundings. In short, heat and work are two different modes of altering the state of a thermodynamic system and changing its internal energy.

The notion of heat should be carefully distinguished from the notion of internal energy. Heat is certainly energy, but it is the energy in transit. This is not just a play of words. The distinction is of basic significance. The state of a thermodynamic system is characterised by its internal energy, not heat. A statement like **'a gas in a given state has a certain amount of heat'** is as meaningless as the statement that **'a gas in a given state has a certain amount of work'**. In contrast, **'a gas in a given state has a certain amount of internal energy'** is a perfectly meaningful statement. Similarly, the statements **'a certain amount of heat is supplied to the system' or 'a certain amount of work was done by the system'** are perfectly meaningful.

To summarise, heat and work in thermodynamics are not state variables. They are modes of energy transfer to a system resulting in change in its internal energy, which, as already mentioned, is a state variable.

In ordinary language, we often confuse heat with internal energy. The distinction between them is sometimes ignored in elementary physics books. For proper understanding of thermodynamics, however, the distinction is crucial.

11.5 FIRST LAW OF THERMODYNAMICS

We have seen that the internal energy U of a system can change through two modes of energy transfer : heat and work. Let

ΔQ = Heat supplied *to* the system *by* the surroundings

ΔW = Work done *by* the system *on* the surroundings

ΔU = Change in internal energy of the system

The general principle of conservation of energy then implies that

$$\Delta Q = \Delta U + \Delta W \tag{11.1}$$

i.e. the energy (ΔQ) supplied to the system goes in partly to increase the internal energy of the system (ΔU) and the rest in work on the environment (ΔW). Equation (11.1) is known as the **First Law of Thermodynamics**. It is simply the general law of conservation of energy applied to any system in which the energy transfer from or to the surroundings is taken into account.

Let us put Eq. (11.1) in the alternative form

$$\Delta Q - \Delta W = \Delta U \tag{11.2}$$

Now, the system may go from an initial state to the final state in a number of ways. For example, to change the state of a gas from (P_1, V_1) to (P_2, V_2), we can first change the volume of the gas from V_1 to V_2, keeping its pressure constant i.e. we can first go the state (P_1, V_2) and then change the pressure of the gas from P_1 to P_2, keeping volume constant, to take the gas to (P_2, V_2). Alternatively, we can first keep the volume constant and then keep the pressure constant. Since U is a state variable, ΔU depends only on the initial and final states and not on the path taken by the gas to go from one to the other. However, ΔQ and ΔW will, in general, depend on the path taken to go from the initial to final states. From the First Law of Thermodynamics, Eq. (11.2), it is clear that the combination $\Delta Q - \Delta W$, is however, path independent. This shows that if a system is taken through a process in which $\Delta U = 0$ (for example, isothermal expansion of an ideal gas, see section 11.8),

$$\Delta Q = \Delta W$$

i.e., heat supplied to the system is used up entirely by the system in doing work on the environment.

If the system is a gas in a cylinder with a movable piston, the gas in moving the piston does work. Since force is pressure times area, and area times displacement is volume, work done by the system against a constant pressure P is

$$\Delta W = P\Delta V$$

where ΔV is the change in volume of the gas. Thus, for this case, Eq. (11.1) gives

$$\Delta Q = \Delta U + P\Delta V \qquad (11.3)$$

As an application of Eq. (11.3), consider the change in internal energy for 1 g of water when we go from its liquid to vapour phase. The measured latent heat of water is 2256 J/g. i.e., for 1 g of water ΔQ = 2256 J. At atmospheric pressure, 1 g of water has a volume 1 cm^3 in liquid phase and 1671 cm^3 in vapour phase.

Therefore,

$$\Delta W = P(V_g - V_l) = 1.013\times10^5 \times (1671 \times 10^{-6}) = 169.2\ \text{J}$$

Equation (11.3) then gives

$$\Delta U = 2256 - 169.2 = 2086.8\ \text{J}$$

We see that most of the heat goes to increase the internal energy of water in transition from the liquid to the vapour phase.

11.6 SPECIFIC HEAT CAPACITY

Suppose an amount of heat ΔQ supplied to a substance changes its temperature from T to $T + \Delta T$. We define heat capacity of a substance (see Chapter 10) to be

$$S = \frac{\Delta Q}{\Delta T} \qquad (11.4)$$

We expect ΔQ and, therefore, heat capacity S to be proportional to the mass of the substance. Further, it could also depend on the temperature, i.e., a different amount of heat may be needed for a unit rise in temperature at different temperatures. To define a constant characteristic of the substance and independent of its amount, we divide S by the mass of the substance m in kg :

$$s = \frac{S}{m} = \left(\frac{1}{m}\right)\frac{\Delta Q}{\Delta T} \qquad (11.5)$$

s is known as the specific heat capacity of the substance. It depends on the nature of the substance and its temperature. The unit of specific heat capacity is J kg^{-1} K^{-1}.

If the amount of substance is specified in terms of moles μ (instead of mass m in kg), we can define heat capacity per mole of the substance by

$$C = \frac{S}{\mu} = \frac{1}{\mu}\frac{\Delta Q}{\Delta T} \qquad (11.6)$$

C is known as molar specific heat capacity of the substance. Like s, C is independent of the amount of substance. C depends on the nature of the substance, its temperature and the conditions under which heat is supplied. The unit of C is J mol^{-1} K^{-1}. As we shall see later (in connection with specific heat capacity of gases), additional conditions may be needed to define C or s. The idea in defining C is that simple predictions can be made in regard to molar specific heat capacities.

Table 11.1 lists measured specific and molar heat capacities of solids at atmospheric pressure and ordinary room temperature.

We will see in Chapter 12 that predictions of specific heats of gases generally agree with experiment. We can use the same law of equipartition of energy that we use there to predict molar specific heat capacities of solids (See Section 12.5 and 12.6). Consider a solid of N atoms, each vibrating about its mean position. An oscillator in one dimension has average energy of $2 \times ½\ k_BT = k_BT$. In three dimensions, the average energy is $3\ k_BT$. For a mole of a solid, the total energy is

$$U = 3\ k_BT \times N_A = 3\ RT \quad (\because k_BT \times N_A = R)$$

Now, at constant pressure, $\Delta Q = \Delta U + P\,\Delta V \cong \Delta U$, since for a solid ΔV is negligible. Therefore,

$$C = \frac{\Delta Q}{\Delta T} = \frac{\Delta U}{\Delta T} = 3R \qquad (11.7)$$

Table 11.1 **Specific and molar heat capacities of some solids at room temperature and atmospheric pressure**

Substance	Specific heat (J kg^{-1} K^{-1})	Molar specific heat (J mol^{-1} K^{-1})
Aluminium	900.0	24.4
Carbon	506.5	6.1
Copper	386.4	24.5
Lead	127.7	26.5
Silver	236.1	25.5
Tungsten	134.4	24.9

As Table 11.1 shows, the experimentally measured values which generally agrees with predicted value 3R at ordinary temperatures. (Carbon is an exception.) The agreement is known to break down at low temperatures.

Specific heat capacity of water

The old unit of heat was calorie. One calorie was earlier defined to be the amount of heat required to raise the temperature of 1g of water by 1 C. With more precise measurements, it was found that the specific heat of water varies slightly with temperature. Figure 11.5 shows this variation in the temperature range 0 to 100 °C.

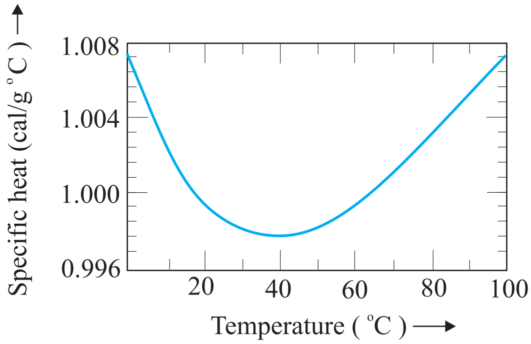


Fig. 11.5 *Variation of specific heat capacity of water with temperature.*

For a precise definition of calorie, it was, therefore, necessary to specify the unit temperature interval. One calorie is defined to be the amount of heat required to raise the temperature of 1g of water from 14.5 °C to 15.5 °C. Since heat is just a form of energy, it is preferable to use the unit joule, J. In SI units, the specific heat capacity of water is 4186 J kg^{-1} K^{-1} i.e. 4.186 J g^{-1} K^{-1}. The so called mechanical equivalent of heat defined as the amount of work needed to produce 1 cal of heat is in fact just a conversion factor between two different units of energy : calorie to joule. Since in SI units, we use the unit joule for heat, work or any other form of energy, the term mechanical equivalent is now superfluous and need not be used.

As already remarked, the specific heat capacity depends on the process or the conditions under which heat capacity transfer takes place. For gases, for example, we can define two specific heats : **specific heat capacity at constant volume and specific heat capacity at constant pressure**. For an ideal gas, we have a simple relation.

$$C_p - C_v = R \tag{11.8}$$

where C_p and C_v are molar specific heat capacities of an ideal gas at constant pressure and volume respectively and R is the universal gas constant. To prove the relation, we begin with Eq. (11.3) for 1 mole of the gas :

$$\Delta Q = \Delta U + P\Delta V$$

If ΔQ is absorbed at constant volume, $\Delta V = 0$

$$C_v = \left(\frac{\Delta Q}{\Delta T}\right)_v = \left(\frac{\Delta U}{\Delta T}\right)_v = \left(\frac{\Delta U}{\Delta T}\right) \tag{11.9}$$

where the subscript v is dropped in the last step, since U of an ideal gas depends only on temperature. (The subscript denotes the quantity kept fixed.) If, on the other hand, ΔQ is absorbed at constant pressure,

$$C_p = \left(\frac{\Delta Q}{\Delta T}\right)_p = \left(\frac{\Delta U}{\Delta T}\right)_p + P\left(\frac{\Delta V}{\Delta T}\right)_p \tag{11.10}$$

The subscript p can be dropped from the first term since U of an ideal gas depends only on T. Now, for a mole of an ideal gas

$$PV = RT$$

which gives

$$P\left(\frac{\Delta V}{\Delta T}\right)_p = R \tag{11.11}$$

Equations (11.9) to (11.11) give the desired relation, Eq. (11.8).

11.7 THERMODYNAMIC STATE VARIABLES AND EQUATION OF STATE

Every **equilibrium state** of a thermodynamic system is completely described by specific values of some macroscopic variables, also called state variables. For example, an equilibrium state of a gas is completely specified by the values of pressure, volume, temperature, and mass (and composition if there is a mixture of gases). A thermodynamic system is not always in equilibrium. For example, a gas allowed to expand freely against vacuum is not an equilibrium state [Fig. 11.6(a)]. During the rapid expansion, pressure of the gas may not be uniform throughout. Similarly, a mixture of gases undergoing an explosive chemical reaction (e.g. a mixture of petrol vapour and air when ignited by a spark) is not an equilibrium state; again its temperature and pressure are not uniform [Fig. 11.6(b)]. Eventually, the gas attains a uniform temperature and pressure and comes to thermal and mechanical equilibrium with its surroundings.

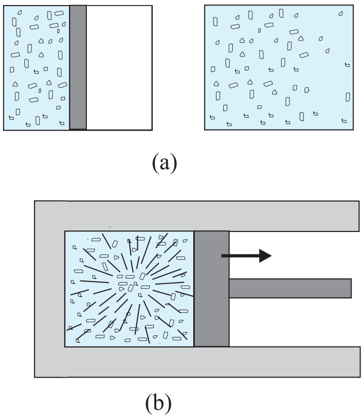


Fig. 11.6 *(a) The partition in the box is suddenly removed leading to free expansion of the gas. (b) A mixture of gases undergoing an explosive chemical reaction. In both situations, the gas is not in equilibrium and cannot be described by state variables.*

In short, thermodynamic state variables describe equilibrium states of systems. The various state variables are not necessarily independent. The connection between the state variables is called the equation of state. For example, for an ideal gas, the equation of state is the ideal gas relation

$$P\,V = \mu\,R\,T$$

For a fixed amount of the gas i.e. given μ, there are thus, only two independent variables, say P and V or T and V. The pressure-volume curve for a fixed temperature is called an **isotherm**. Real gases may have more complicated equations of state.

The thermodynamic state variables are of two kinds: **extensive** and **intensive**. Extensive variables indicate the 'size' of the system. Intensive variables such as pressure and temperature do not. To decide which variable is extensive and which intensive, think of a relevant system in equilibrium, and imagine that it is divided into two equal parts. The variables that remain unchanged for each part are intensive. The variables whose values get halved in each part are extensive. It is easily seen, for example, that internal energy U, volume V, total mass M are extensive variables. Pressure P, temperature T, and density ρ are intensive variables. It is a good practice to check the consistency of thermodynamic equations using this classification of variables. For example, in the equation

$$\Delta Q = \Delta U + P\,\Delta V$$

quantities on both sides are extensive*. (The product of an intensive variable like P and an extensive quantity ΔV is extensive.)

11.8 THERMODYNAMIC PROCESSES

11.8.1 Quasi-static process

Consider a gas in thermal and mechanical equilibrium with its surroundings. The pressure of the gas in that case equals the external pressure and its temperature is the same as that of its surroundings. Suppose that the external pressure is suddenly reduced (say by lifting the weight on the movable piston in the container). The piston will accelerate outward. During the process, the gas passes through states that are not equilibrium states. The non-equilibrium states do not have well-defined pressure and temperature. In the same way, if a finite temperature difference exists between the gas and its surroundings, there will be a rapid exchange of heat during which the gas will pass through non-equilibrium states. In due course, the gas will settle to an equilibrium state with well-defined temperature and pressure equal to those of the surroundings. The free expansion of a gas in vacuum and a mixture of gases undergoing an explosive chemical reaction, mentioned in section 11.7 are also examples where the system goes through non-equilibrium states.

Non-equilibrium states of a system are difficult to deal with. It is, therefore, convenient to imagine an idealised process in which at every stage the system is an equilibrium state. Such a

* *As emphasised earlier, Q is not a state variable. However, ΔQ is clearly proportional to the total mass of system and hence is extensive.*

process is, in principle, infinitely slow, hence the name quasi-static (meaning nearly static). The system changes its variables (P, T, V) so slowly that it remains in thermal and mechanical equilibrium with its surroundings throughout. In a quasi-static process, at every stage, the difference in the pressure of the system and the external pressure is infinitesimally small. The same is true of the temperature difference between the system and its surroundings (Fig.11.7). To take a gas from the state (P, T) to another state (P', T') via a quasi-static process, we change the external pressure by a very small amount, allow the system to equalise its pressure with that of the surroundings and continue the process infinitely slowly until the system achieves the pressure P'. Similarly, to change the temperature, we introduce an infinitesimal temperature difference between the system and the surrounding reservoirs and by choosing reservoirs of progressively different temperatures T to T', the system achieves the temperature T'.

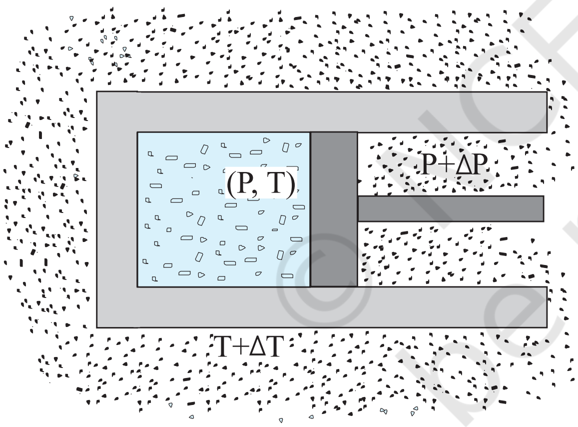


Fig. 11.7 *In a quasi-static process, the temperature of the surrounding reservoir and the external pressure differ only infinitesimally from the temperature and pressure of the system.*

A quasi-static process is obviously a hypothetical construct. In practice, processes that are sufficiently slow and do not involve accelerated motion of the piston, large temperature gradient, etc., are reasonably approximation to an ideal quasi-static process. We shall from now on deal with quasi-static processes only, except when stated otherwise.

A process in which the temperature of the system is kept fixed throughout is called an **isothermal process**. The expansion of a gas in a metallic cylinder placed in a large reservoir of fixed temperature is an example of an isothermal process. (Heat transferred from the reservoir to the system does not materially affect the temperature of the reservoir, because of its very large heat capacity.) In **isobaric processes** the pressure is constant while in **isochoric processes** the volume is constant. Finally, if the system is insulated from the surroundings and no heat flows between the system and the surroundings, the process is **adiabatic**. The definitions of these special processes are summarised in Table. 11.2

Table 11.2 **Some special thermodynamic processes**

Type of processes	Feature
Isothermal	Temperature constant
Isobaric	Pressure constant
Isochoric	Volume constant
Adiabatic	No heat flow between the system and the surroundings ($\Delta Q = 0$)

We now consider these processes in some detail :

11.8.2 Isothermal process

For an isothermal process (T fixed), the ideal gas equation gives

$$PV = \text{constant}$$

i.e., pressure of a given mass of gas varies inversely as its volume. This is nothing but Boyle's Law.

Suppose an ideal gas goes isothermally (at temperature T) from its initial state (P_1, V_1) to the final state (P_2, V_2). At any intermediate stage with pressure P and volume change from V to $V + \Delta V$ (ΔV small)

$$\Delta W = P \Delta V$$

Taking ($\Delta V \to 0$) and summing the quantity ΔW over the entire process,

$$W = \int_{V_1}^{V_2} P \, \mathrm{d}V$$

$$= \mu RT \int_{V_1}^{V_2} \frac{\mathrm{d}V}{V} = \mu RT \ \ln \frac{V_2}{V_1} \qquad (11.12)$$

where in the second step we have made use of the ideal gas equation $PV = \mu\, RT$ and taken the constants out of the integral. For an ideal gas, internal energy depends only on temperature. Thus, there is no change in the internal energy of an ideal gas in an isothermal process. The First Law of Thermodynamics then implies that heat supplied to the gas equals the work done by the gas : $Q = W$. Note from Eq. (11.12) that for $V_2 > V_1$, $W > 0$; and for $V_2 < V_1$, $W < 0$. That is, in an isothermal expansion, the gas absorbs heat and does work while in an isothermal compression, work is done on the gas by the environment and heat is released.

11.8.3 Adiabatic process

In an adiabatic process, the system is insulated from the surroundings and heat absorbed or released is zero. From Eq. (11.1), we see that work done by the gas results in decrease in its internal energy (and hence its temperature for an ideal gas). We quote without proof (the result that you will learn in higher courses) that for an adiabatic process of an ideal gas.

$$P\,V^{\gamma} = \text{const} \qquad (11.13)$$

where γ is the ratio of specific heats (ordinary or molar) at constant pressure and at constant volume.

$$\gamma = \frac{C_p}{C_v}$$

Thus if an ideal gas undergoes a change in its state adiabatically from (P_1, V_1) to (P_2, V_2) :

$$P_1\, V_1^{\gamma} = P_2\, V_2^{\gamma} \qquad (11.14)$$

Figure11.8 shows the P-V curves of an ideal gas for two adiabatic processes connecting two isotherms.

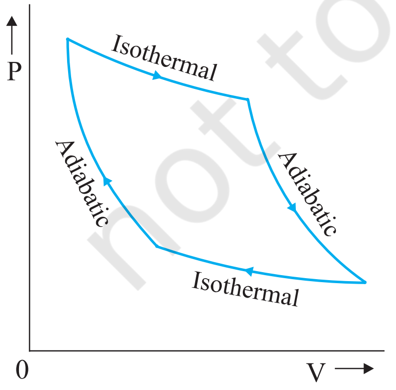


Fig. 11.8 *P-V curves for isothermal and adiabatic processes of an ideal gas.*

We can calculate, as before, the work done in an adiabatic change of an ideal gas from the state (P_1, V_1, T_1) to the state (P_2, V_2, T_2).

$$W = \int_{V_1}^{V_2} P\, \mathrm{d}V$$

$$= constant \times \int_{V_1}^{V_2} \frac{\mathrm{d}V}{V^{\gamma}} = constant \times \left. \frac{V^{-\gamma+1}}{1-\gamma} \right|_{V_1}^{V_2}$$

$$\frac{constant}{(1-\gamma)} \times \left[\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}} \right] \qquad (11.15)$$

From Eq. (11.14), the constant is $P_1 V_1^{\gamma}$ or $P_2 V_2^{\gamma}$

$$W = \frac{1}{1-\gamma} \left[\frac{P_2 V_2^{\gamma}}{V_2^{\gamma-1}} - \frac{P_1 V_1^{\gamma}}{V_1^{\gamma-1}} \right]$$

$$= \frac{1}{1-\gamma} \left[P_2 V_2 - P_1 V_1 \right] = \frac{\mu R (T_1 - T_2)}{\gamma - 1} \qquad (11.16)$$

As expected, if work is done by the gas in an adiabatic process ($W > 0$), from Eq. (11.16), $T_2 < T_1$. On the other hand, if work is done on the gas ($W < 0$), we get $T_2 > T_1$ i.e., the temperature of the gas rises.

11.8.4 Isochoric process

In an isochoric process, V is constant. No work is done on or by the gas. From Eq. (11.1), the heat absorbed by the gas goes entirely to change its internal energy and its temperature. The change in temperature for a given amount of heat is determined by the specific heat of the gas at constant volume.

11.8.5 Isobaric process

In an isobaric process, P is fixed. Work done by the gas is

$$W = P\,(V_2 - V_1) = \mu\, R\,(T_2 - T_1) \qquad (11.17)$$

Since temperature changes, so does internal energy. The heat absorbed goes partly to increase internal energy and partly to do work. The change in temperature for a given amount of heat is determined by the specific heat of the gas at constant pressure.

11.8.6 Cyclic process

In a cyclic process, the system returns to its initial state. Since internal energy is a state variable, $\Delta U = 0$ for a cyclic process. From

Eq. (11.1), the total heat absorbed equals the work done by the system.

11.9 SECOND LAW OF THERMODYNAMICS

The First Law of Thermodynamics is the principle of conservation of energy. Common experience shows that there are many conceivable processes that are perfectly allowed by the First Law and yet are never observed. For example, nobody has ever seen a book lying on a table jumping to a height by itself. But such a thing would be possible if the principle of conservation of energy were the only restriction. The table could cool spontaneously, converting some of its internal energy into an equal amount of mechanical energy of the book, which would then hop to a height with potential energy equal to the mechanical energy it acquired. But this never happens. Clearly, some additional basic principle of nature forbids the above, even though it satisfies the energy conservation principle. This principle, which disallows many phenomena consistent with the First Law of Thermodynamics is known as the Second Law of Thermodynamics.

The Second Law of Thermodynamics gives a fundamental limitation to the efficiency of a heat engine and the co-efficient of performance of a refrigerator. In simple terms, it says that efficiency of a heat engine can never be unity. For a refrigerator, the Second Law says that the co-efficient of performance can never be infinite. The following two statements, one due to Kelvin and Planck denying the possibility of a perfect heat engine, and another due to Clausius denying the possibility of a perfect refrigerator or heat pump, are a concise summary of these observations.

Kelvin-Planck statement

No process is possible whose sole result is the absorption of heat from a reservoir and the complete conversion of the heat into work.

Clausius statement

No process is possible whose sole result is the transfer of heat from a colder object to a hotter object.

It can be proved that the two statements above are completely equivalent.

11.10 REVERSIBLE AND IRREVERSIBLE PROCESSES

Imagine some process in which a thermodynamic system goes from an initial state *i* to a final state *f*. During the process the system absorbs heat Q from the surroundings and performs work W on it. Can we reverse this process and bring both the system and surroundings to their initial states with no other effect anywhere ? Experience suggests that for most processes in nature this is not possible. The spontaneous processes of nature are irreversible. Several examples can be cited. The base of a vessel on an oven is hotter than its other parts. When the vessel is removed, heat is transferred from the base to the other parts, bringing the vessel to a uniform temperature (which in due course cools to the temperature of the surroundings). The process cannot be reversed; a part of the vessel will not get cooler spontaneously and warm up the base. It will violate the Second Law of Thermodynamics, if it did. The free expansion of a gas is irreversible. The combustion reaction of a mixture of petrol and air ignited by a spark cannot be reversed. Cooking gas leaking from a gas cylinder in the kitchen diffuses to the entire room. The diffusion process will not spontaneously reverse and bring the gas back to the cylinder. The stirring of a liquid in thermal contact with a reservoir will convert the work done into heat, increasing the internal energy of the reservoir. The process cannot be reversed exactly; otherwise it would amount to conversion of heat entirely into work, violating the Second Law of Thermodynamics. Irreversibility is a rule rather an exception in nature.

Irreversibility arises mainly from two causes: one, many processes (like a free expansion, or an explosive chemical reaction) take the system to non-equilibrium states; two, most processes involve friction, viscosity and other dissipative effects (e.g., a moving body coming to a stop and losing its mechanical energy as heat to the floor and the body; a rotating blade in a liquid coming to a stop due to viscosity and losing its mechanical energy with corresponding gain in the internal energy of the liquid). Since dissipative effects are present everywhere and can be minimised but not fully eliminated, most processes that we deal with are irreversible.

A thermodynamic process (state $i \rightarrow$ state f) is reversible if the process can be turned back such that both the system and the surroundings return to their original states, with no other change anywhere else in the universe. From the preceding discussion, a reversible process is an idealised notion. A process is reversible only if it is quasi-static (system in equilibrium with the surroundings at every stage) and there are no dissipative effects. For example, a quasi-static isothermal expansion of an ideal gas in a cylinder fitted with a frictionless movable piston is a reversible process.

Why is reversibility such a basic concept in thermodynamics ? As we have seen, one of the concerns of thermodynamics is the efficiency with which heat can be converted into work. The Second Law of Thermodynamics rules out the possibility of a perfect heat engine with 100% efficiency. But what is the highest efficiency possible for a heat engine working between two reservoirs at temperatures T_1 and T_2? It turns out that a heat engine based on idealised reversible processes achieves the highest efficiency possible. All other engines involving irreversibility in any way (as would be the case for practical engines) have lower than this limiting efficiency.

11.11 CARNOT ENGINE

Suppose we have a hot reservoir at temperature T_1 and a cold reservoir at temperature T_2. What is the maximum efficiency possible for a heat engine operating between the two reservoirs and what cycle of processes should be adopted to achieve the maximum efficiency ? Sadi Carnot, a French engineer, first considered this question in 1824. Interestingly, Carnot arrived at the correct answer, even though the basic concepts of heat and thermodynamics had yet to be firmly established.

We expect the ideal engine operating between two temperatures to be a reversible engine. Irreversibility is associated with dissipative effects, as remarked in the preceding section, and lowers efficiency. A process is reversible if it is quasi-static and non-dissipative. We have seen that a process is not quasi-static if it involves finite temperature difference between the system and the reservoir. This implies that in a reversible heat engine operating between two temperatures, heat should be absorbed (from the hot reservoir) isothermally and released (to the cold reservoir) isothermally. We thus have identified two steps of the reversible heat engine : isothermal process at temperature T_1 absorbing heat Q_1 from the hot reservoir, and another isothermal process at temperature T_2 releasing heat Q_2 to the cold reservoir. To complete a cycle, we need to take the system from temperature T_1 to T_2 and then back from temperature T_2 to T_1. Which processes should we employ for this purpose that are reversible? A little reflection shows that we can only adopt reversible adiabatic processes for these purposes, which involve no heat flow from any reservoir. If we employ any other process that is not adiabatic, say an isochoric process, to take the system from one temperature to another, we shall need a series of reservoirs in the temperature range T_2 to T_1 to ensure that at each stage the process is quasi-static. (Remember again that for a process to be quasi-static and reversible, there should be no finite temperature difference between the system and the reservoir.) But we are considering a reversible engine that operates between only two temperatures. Thus adiabatic processes must bring about the temperature change in the system from T_1 to T_2 and T_2 to T_1 in this engine.

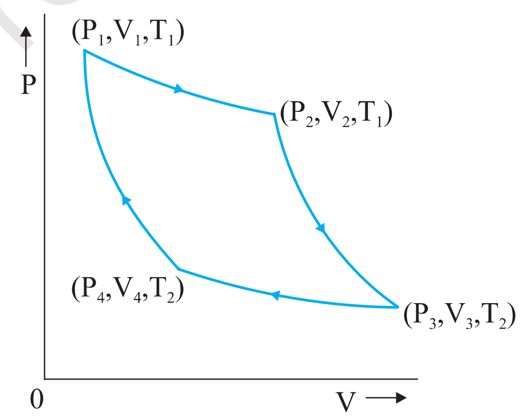


Fig. 11.9 *Carnot cycle for a heat engine with an ideal gas as the working substance.*

A reversible heat engine operating between two temperatures is called a Carnot engine. We have just argued that such an engine must have the following sequence of steps constituting one cycle, called the Carnot cycle, shown in Fig. 11.9. We have taken the working substance of the Carnot engine to be an ideal gas.

(a) *Step* 1 $\rightarrow$ 2 Isothermal expansion of the gas taking its state from (P_1, V_1, T_1) to (P_2, V_2, T_1).

The heat absorbed by the gas (Q_1) from the reservoir at temperature T_1 is given by Eq. (11.12). This is also the work done ($W_{1\rightarrow 2}$) by the gas on the environment.

$$W_{1\rightarrow 2} = Q_1 = \mu R T_1 \ \ln\left(\frac{V_2}{V_1}\right) \qquad (11.18)$$

(b) *Step* 2 $\rightarrow$ 3 Adiabatic expansion of the gas from (P_2, V_2, T_1) to (P_3, V_3, T_2) Work done by the gas, using Eq. (11.16), is

$$W_{2\rightarrow 3} = \frac{\mu R\left(T_1 - T_2\right)}{\gamma - 1} \qquad (11.19)$$

(c) *Step* 3 $\rightarrow$ 4 Isothermal compression of the gas from (P_3, V_3, T_2) to (P_4, V_4, T_2).

Heat released (Q_2) by the gas to the reservoir at temperature T_2 is given by Eq. (11.12). This is also the work done ($W_{3\rightarrow 4}$) on the gas by the environment.

$$W_{3\rightarrow 4} = Q_2 = \mu R T_2 \ln\left(\frac{V_3}{V_4}\right) \qquad (11.20)$$

(d) *Step* 4 $\rightarrow$ 1 Adiabatic compression of the gas from (P_4, V_4, T_2) to (P_1, V_1, T_1).

Work done on the gas, [using Eq.(11.16), is

$$W_{4\rightarrow 1} = \mu R\left(\frac{T_1 - T_2}{\gamma - 1}\right) \qquad (11.21)$$

From Eqs. (11.18) to (11.21) total work done by the gas in one complete cycle is

$$W = W_{1\rightarrow 2} + W_{2\rightarrow 3} - W_{3\rightarrow 4} - W_{4\rightarrow 1}$$

$$= \mu R T_1 \ \ln\left(\frac{V_2}{V_1}\right) - \mu R T_2 \ \ln\left(\frac{V_3}{V_4}\right) \qquad (11.22)$$

The efficiency η of the Carnot engine is

$$\eta = \frac{W}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

$$= 1 - \left(\frac{T_2}{T_1}\right)\frac{\ln\left(\frac{V_3}{V_4}\right)}{\ln\left(\frac{V_2}{V_1}\right)} \qquad (11.23)$$

Now since step 2 $\rightarrow$ 3 is an adiabatic process,

$$T_1 V_2^{\gamma-1} = T_2 V_3^{\gamma-1}$$

i.e. $$\frac{V_2}{V_3} = \left(\frac{T_2}{T_1}\right)^{1/(\gamma-1)} \qquad (11.24)$$

Similarly, since step 4 $\rightarrow$ 1 is an adiabatic process

$$T_2 V_4^{\gamma-1} = T_1 V_1^{\gamma-1}$$

i.e. $$\frac{V_1}{V_4} = \left(\frac{T_2}{T_1}\right)^{1/\gamma-1} \qquad (11.25)$$

From Eqs. (11.24) and (11.25),

$$\frac{V_3}{V_4} = \frac{V_2}{V_1} \qquad (11.26)$$

Using Eq. (11.26) in Eq. (11.23), we get

$$\eta = 1 - \frac{T_2}{T_1} \quad \text{(Carnot engine)} \qquad (11.27)$$

We have already seen that a Carnot engine is a reversible engine. Indeed it is the only reversible engine possible that works between two reservoirs at different temperatures. Each step of the Carnot cycle given in Fig. 11.9 can be reversed. This will amount to taking heat Q_2 from the cold reservoir at T_2, doing work W on the system, and transferring heat Q_1 to the hot reservoir. This will be a reversible refrigerator.

We next establish the important result (sometimes called Carnot's theorem) that (a) working between two given temperatures T_1 and T_2 of the hot and cold reservoirs respectively, no engine can have efficiency more than that of the Carnot engine and (b) the efficiency of the Carnot engine is independent of the nature of the working substance.

To prove the result (a), imagine a reversible (Carnot) engine R and an irreversible engine I working between the same source (hot reservoir) and sink (cold reservoir). Let us couple the engines, I and R, in such a way so that I acts like a heat engine and R acts as a refrigerator. Let I absorb heat Q_1 from the source, deliver work W' and release the heat Q_1- W to the sink. We arrange so that R returns the same heat Q_1 to the source, taking heat Q_2 from the sink and requiring work $W = Q_1 - Q_2$ to be done on it. Now suppose $\eta_R < \eta_I$ i.e. if R were to act as an engine it would give less work output

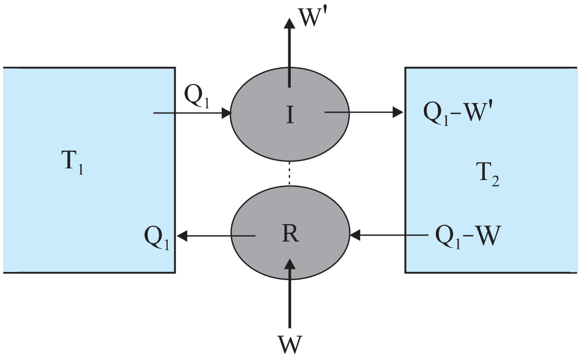


Fig. 11.10 *An irreversible engine (I) coupled to a reversible refrigerator (R). If W ′ > W, this would amount to extraction of heat W′ – W from the sink and its full conversion to work, in contradiction with the Second Law of Thermodynamics.*

than that of I i.e. $W < W'$ for a given Q_1. With R acting like a refrigerator, this would mean $Q_2 = Q_1 - W > Q_1 - W'$. Thus, on the whole, the coupled I-R system extracts heat $(Q_1 - W) - (Q_1 - W') = (W' - W)$ from the cold reservoir and delivers the same amount of work in one cycle, without any change in the source or anywhere else. This is clearly against the Kelvin-Planck statement of the Second Law of Thermodynamics. Hence the assertion $\eta_I > \eta_R$ is wrong. No engine can have efficiency greater than that of the Carnot engine. A similar argument can be constructed to show that a reversible engine with one particular substance cannot be more efficient than the one using another substance. The maximum efficiency of a Carnot engine given by Eq. (11.27) is independent of the nature of the system performing the Carnot cycle of operations. Thus we are justified in using an ideal gas as a system in the calculation of efficiency η of a Carnot engine. The ideal gas has a simple equation of state, which allows us to readily calculate η, but the final result for η, [Eq. (11.27)], is true for any Carnot engine.

This final remark shows that in a Carnot cycle,

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \qquad (11.28)$$

is a universal relation independent of the nature of the system. Here Q_1 and Q_2 are respectively, the heat absorbed and released isothermally (from the hot and to the cold reservoirs) in a Carnot engine. Equation (11.28), can, therefore, be used as a relation to define a truly universal thermodynamic temperature scale that is independent of any particular properties of the system used in the Carnot cycle. Of course, for an ideal gas as a working substance, this universal temperature is the same as the ideal gas temperature introduced in section 11.9.

SUMMARY

1. The zeroth law of thermodynamics states that '*two systems in thermal equilibrium with a third system separately are in thermal equilibrium with each other*'. The Zeroth Law leads to the concept of temperature.
2. Internal energy of a system is the sum of kinetic energies and potential energies of the molecular constituents of the system. It does not include the over-all kinetic energy of the system. Heat and work are two modes of energy transfer to the system. Heat is the energy transfer arising due to temperature difference between the system and the surroundings. Work is energy transfer brought about by other means, such as moving the piston of a cylinder containing the gas, by raising or lowering some weight connected to it.

3. The first law of thermodynamics is the general law of conservation of energy applied to any system in which energy transfer from or to the surroundings (through heat and work) is taken into account. It states that

$$\Delta Q = \Delta U + \Delta W$$

where ΔQ is the heat supplied to the system, ΔW is the work done by the system and ΔU is the change in internal energy of the system.

4. The specific heat capacity of a substance is defined by

$$s = \frac{1}{m}\frac{\Delta Q}{\Delta T}$$

where m is the mass of the substance and ΔQ is the heat required to change its temperature by ΔT. The molar specific heat capacity of a substance is defined by

$$C = \frac{1}{\mu}\frac{\Delta Q}{\Delta T}$$

where μ is the number of moles of the substance. For a solid, the law of equipartition of energy gives

$$C = 3R$$

which generally agrees with experiment at ordinary temperatures.

Calorie is the old unit of heat. 1 calorie is the amount of heat required to raise the temperature of 1 g of water from 14.5 °C to 15.5 °C. 1 cal = 4.186 J.

5. For an ideal gas, the molar specific heat capacities at constant pressure and volume satisfy the relation

$$C_p - C_v = R$$

where R is the universal gas constant.

6. Equilibrium states of a thermodynamic system are described by state variables. The value of a state variable depends only on the particular state, not on the path used to arrive at that state. Examples of state variables are pressure (P), volume (V), temperature (T), and mass (m). Heat and work are not state variables. An Equation of State (like the ideal gas equation $PV = \mu RT$) is a relation connecting different state variables.

7. A quasi-static process is an infinitely slow process such that the system remains in thermal and mechanical equilibrium with the surroundings throughout. In a quasi-static process, the pressure and temperature of the environment can differ from those of the system only infinitesimally.

8. In an isothermal expansion of an ideal gas from volume V_1 to V_2 at temperature T the heat absorbed (Q) equals the work done (W) by the gas, each given by

$$Q = W = \mu R T \ln\left(\frac{V_2}{V_1}\right)$$

9. In an adiabatic process of an ideal gas

$$PV^{\gamma} = \text{constant}$$

where

$$\gamma = \frac{C_p}{C_v}$$

Work done by an ideal gas in an adiabatic change of state from (P_1, V_1, T_1) to (P_2, V_2, T_2) is

$$W = \frac{\mu R(T_1 - T_2)}{\gamma - 1}$$

10. The second law of thermodynamics disallows some processes consistent with the First Law of Thermodynamics. It states

Kelvin-Planck statement

No process is possible whose sole result is the absorption of heat from a reservoir and complete conversion of the heat into work.

Clausius statement

No process is possible whose sole result is the transfer of heat from a colder object to a hotter object.

Put simply, the Second Law implies that no heat engine can have efficiency η equal to 1 or no refrigerator can have co-efficient of performance α equal to infinity.

11. A process is reversible if it can be reversed such that both the system and the surroundings return to their original states, with no other change anywhere else in the universe. Spontaneous processes of nature are irreversible. The idealised reversible process is a quasi-static process with no dissipative factors such as friction, viscosity, etc.

12. Carnot engine is a reversible engine operating between two temperatures T_1 (source) and T_2 (sink). The Carnot cycle consists of two isothermal processes connected by two adiabatic processes. The efficiency of a Carnot engine is given by

$$\eta = 1 - \frac{T_2}{T_1} \qquad \text{(Carnot engine)}$$

No engine operating between two temperatures can have efficiency greater than that of the Carnot engine.

13. If Q > 0, heat is added to the system
If Q < 0, heat is removed to the system
If W > 0, Work is done by the system
If W < 0, Work is done on the system

Quantity	Symbol	Dimensions	Unit	Remark
Co-efficienty of volume expansion	α_v	$[K^{-1}]$	K^{-1}	$\alpha_v = 3\,\alpha_1$
Heat supplied to a system	ΔQ	$[ML^2T^{-2}]$	J	Q is not a state variable
Specific heat capacity	s	$[L^2T^{-2}K^{-1}]$	$J\,kg^{-1}\,K^{-1}$	
Thermal Conductivity	K	$[MLT^{-3}K^{-1}]$	$J\,s^{-1}\,K^{-1}$	$H = -KA\dfrac{dt}{dx}$

POINTS TO PONDER

1. Temperature of a body is related to its average internal energy, not to the kinetic energy of motion of its centre of mass. A bullet fired from a gun is not at a higher temperature because of its high speed.
2. Equilibrium in thermodynamics refers to the situation when macroscopic variables describing the thermodynamic state of a system do not depend on time. Equilibrium of a system in mechanics means the net external force and torque on the system are zero.

3. In a state of thermodynamic equilibrium, the microscopic constituents of a system are not in equilibrium (in the sense of mechanics).
4. Heat capacity, in general, depends on the process the system goes through when heat is supplied.
5. In isothermal quasi-static processes, heat is absorbed or given out by the system even though at every stage the gas has the same temperature as that of the surrounding reservoir. This is possible because of the infinitesimal difference in temperature between the system and the reservoir.

EXERCISES

11.1 A geyser heats water flowing at the rate of 3.0 litres per minute from 27 °C to 77 °C. If the geyser operates on a gas burner, what is the rate of consumption of the fuel if its heat of combustion is 4.0×10^4 J/g ?

11.2 What amount of heat must be supplied to 2.0×10^{-2} kg of nitrogen (at room temperature) to raise its temperature by 45 °C at constant pressure ? (Molecular mass of N_2 = 28; R = 8.3 J mol^{-1} K^{-1}.)

11.3 Explain why

(a) Two bodies at different temperatures T_1 and T_2 if brought in thermal contact do not necessarily settle to the mean temperature $(T_1 + T_2)/2$.

(b) The coolant in a chemical or a nuclear plant (i.e., the liquid used to prevent the different parts of a plant from getting too hot) should have high specific heat.

(c) Air pressure in a car tyre increases during driving.

(d) The climate of a harbour town is more temperate than that of a town in a desert at the same latitude.

11.4 A cylinder with a movable piston contains 3 moles of hydrogen at standard temperature and pressure. The walls of the cylinder are made of a heat insulator, and the piston is insulated by having a pile of sand on it. By what factor does the pressure of the gas increase if the gas is compressed to half its original volume ?

11.5 In changing the state of a gas adiabatically from an equilibrium state A to another equilibrium state B, an amount of work equal to 22.3 J is done on the system. If the gas is taken from state A to B via a process in which the net heat absorbed by the system is 9.35 cal, how much is the net work done by the system in the latter case ? (Take 1 cal = 4.19 J)

11.6 Two cylinders A and B of equal capacity are connected to each other via a stopcock. A contains a gas at standard temperature and pressure. B is completely evacuated. The entire system is thermally insulated. The stopcock is suddenly opened. Answer the following :

(a) What is the final pressure of the gas in A and B ?

(b) What is the change in internal energy of the gas ?

(c) What is the change in the temperature of the gas ?

(d) Do the intermediate states of the system (before settling to the final equilibrium state) lie on its P-V-T surface ?

11.7 An electric heater supplies heat to a system at a rate of 100W. If system performs work at a rate of 75 joules per second. At what rate is the internal energy increasing?

11.8 A thermodynamic system is taken from an original state to an intermediate state by the linear process shown in Fig. (11.13)

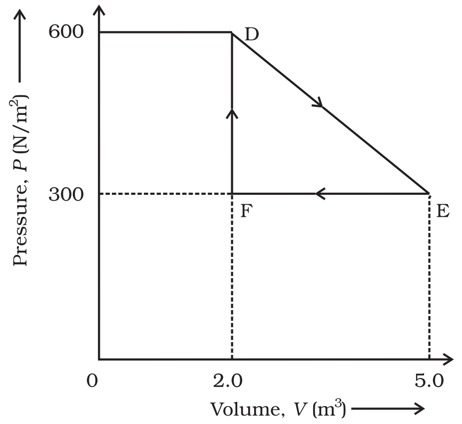


Fig. 11.11

Its volume is then reduced to the original value from E to F by an isobaric process. Calculate the total work done by the gas from D to E to F

CHAPTER TWELVE

KINETIC THEORY

12.1 INTRODUCTION

Boyle discovered the law named after him in 1661. Boyle, Newton and several others tried to explain the behaviour of gases by considering that gases are made up of tiny atomic particles. The actual atomic theory got established more than 150 years later. Kinetic theory explains the behaviour of gases based on the idea that the gas consists of rapidly moving atoms or molecules. This is possible as the inter-atomic forces, which are short range forces that are important for solids and liquids, can be neglected for gases. The kinetic theory was developed in the nineteenth century by Maxwell, Boltzmann and others. It has been remarkably successful. It gives a molecular interpretation of pressure and temperature of a gas, and is consistent with gas laws and Avogadro's hypothesis. It correctly explains specific heat capacities of many gases. It also relates measurable properties of gases such as viscosity, conduction and diffusion with molecular parameters, yielding estimates of molecular sizes and masses. This chapter gives an introduction to kinetic theory.

12.2 MOLECULAR NATURE OF MATTER

Richard Feynman, one of the great physicists of 20th century considers the discovery that "Matter is made up of atoms" to be a very significant one. Humanity may suffer annihilation (due to nuclear catastrophe) or extinction (due to environmental disasters) if we do not act wisely. If that happens, and all of scientific knowledge were to be destroyed then Feynman would like the 'Atomic Hypothesis' to be communicated to the next generation of creatures in the universe. Atomic Hypothesis: All things are made of atoms - little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.

Speculation that matter may not be continuous, existed in many places and cultures. Kanada in India and Democritus

Atomic Hypothesis in Ancient India and Greece

Though John Dalton is credited with the introduction of atomic viewpoint in modern science, scholars in ancient India and Greece conjectured long before the existence of atoms and molecules. In the Vaiseshika school of thought in India founded by Kanada (Sixth century B.C.) the atomic picture was developed in considerable detail. Atoms were thought to be eternal, indivisible, infinitesimal and ultimate parts of matter. It was argued that if matter could be subdivided without an end, there would be no difference between a mustard seed and the Meru mountain. The four kinds of atoms (**Paramanu** — Sanskrit word for the smallest particle) postulated were Bhoomi (Earth), Ap (water), Tejas (fire) and Vayu (air) that have characteristic mass and other attributes, were propounded. Akasa (space) was thought to have no atomic structure and was continuous and inert. Atoms combine to form different molecules (e.g. two atoms combine to form a diatomic molecule dvyanuka, three atoms form a tryanuka or a triatomic molecule), their properties depending upon the nature and ratio of the constituent atoms. The size of the atoms was also estimated, by conjecture or by methods that are not known to us. The estimates vary. In Lalitavistara, a famous biography of the Buddha written mainly in the second century B.C., the estimate is close to the modern estimate of atomic size, of the order of 10^{-10} m.

In ancient Greece, Democritus (Fourth century B.C.) is best known for his atomic hypothesis. The word 'atom' means 'indivisible' in Greek. According to him, atoms differ from each other physically, in shape, size and other properties and this resulted in the different properties of the substances formed by their combination. The atoms of water were smooth and round and unable to 'hook' on to each other, which is why liquid /water flows easily. The atoms of earth were rough and jagged, so they held together to form hard substances. The atoms of fire were thorny which is why it caused painful burns. These fascinating ideas, despite their ingenuity, could not evolve much further, perhaps because they were intuitive conjectures and speculations not tested and modified by quantitative experiments - the hallmark of modern science.

in Greece had suggested that matter may consist of indivisible constituents. The scientific 'Atomic Theory' is usually credited to John Dalton. He proposed the atomic theory to explain the laws of definite and multiple proportions obeyed by elements when they combine into compounds. The first law says that any given compound has, a fixed proportion by mass of its constituents. The second law says that when two elements form more than one compound, for a fixed mass of one element, the masses of the other elements are in ratio of small integers.

To explain the laws Dalton suggested, about 200 years ago, that the smallest constituents of an element are atoms. Atoms of one element are identical but differ from those of other elements. A small number of atoms of each element combine to form a molecule of the compound. Gay Lussac's law, also given in early 19^{th} century, states: When gases combine chemically to yield another gas, their volumes are in the ratios of small integers. Avogadro's law (or hypothesis) says: Equal volumes of all gases at equal temperature and pressure have the same number of molecules. Avogadro's law, when combined with Dalton's theory explains Gay Lussac's law. Since the elements are often in the form of molecules, Dalton's atomic theory can also be referred to as the molecular theory of matter. The theory is now well accepted by scientists. However even at the end of the nineteenth century there were famous scientists who did not believe in atomic theory !

From many observations, in recent times we now know that molecules (made up of one or more atoms) constitute matter. Electron microscopes and scanning tunnelling microscopes enable us to even see them. The size of an atom is about an angstrom (10^{-10} m). In solids, which are tightly packed, atoms are spaced about a few angstroms (2 Å) apart. In liquids the separation between atoms is also about the same. In liquids the atoms are not as rigidly fixed as in solids, and can move around. This enables a liquid to flow. In gases the interatomic distances are in tens of angstroms. The average distance a molecule can travel without colliding is called the **mean free path**. The mean free path, in gases, is of the order of thousands of angstroms. The atoms are much freer in gases and can travel long distances without colliding. If they are not enclosed, gases disperse away. In solids and liquids the closeness makes the interatomic force important. The force has a long range attraction and a short range repulsion. The atoms attract when they are at a few angstroms but repel when they come closer. The static appearance of a gas is misleading. The gas is full of activity and the equilibrium is a dynamic one. In dynamic equilibrium, molecules collide and change their speeds during the collision. Only the average properties are constant.

Atomic theory is not the end of our quest, but the beginning. We now know that atoms are not indivisible or elementary. They consist of a nucleus and electrons. The nucleus itself is made up of protons and neutrons. The protons and neutrons are again made up of quarks. Even quarks may not be the end of the story. There may be string like elementary entities. Nature always has surprises for us, but the search for truth is often enjoyable and the discoveries beautiful. In this chapter, we shall limit ourselves to understanding the behaviour of gases (and a little bit of solids), as a collection of moving molecules in incessant motion.

12.3 BEHAVIOUR OF GASES

Properties of gases are easier to understand than those of solids and liquids. This is mainly because in a gas, molecules are far from each other and their mutual interactions are negligible except when two molecules collide. Gases at low pressures and high temperatures much above that at which they liquefy (or solidify) approximately satisfy a simple relation between their pressure, temperature and volume given by (see Chapter 10)

$$PV = KT \tag{12.1}$$

for a given sample of the gas. Here T is the temperature in kelvin or (absolute) scale. K is a constant for the given sample but varies with the volume of the gas. If we now bring in the idea of atoms or molecules, then K is proportional to the number of molecules, (say) N in the sample. We can write $K = N k$. Observation tells us that this k is same for all gases. It is called Boltzmann constant and is denoted by k_B.

$$\text{As } \frac{P_1 V_1}{N_1 T_1} = \frac{P_2 V_2}{N_2 T_2} = \text{constant} = k_B \tag{12.2}$$

if P, V and T are same, then N is also same for all gases. This is Avogadro's hypothesis, that the number of molecules per unit volume is the same for all gases at a fixed temperature and pressure. The number in 22.4 litres of any gas is 6.02×10^{23}. This is known as Avogadro number and is denoted by N_A. The mass of 22.4 litres of any gas is equal to its molecular weight in grams at S.T.P (standard temperature 273 K and pressure 1 atm). This amount of substance is called a mole (see Chapter 1 for a more precise definition). Avogadro had guessed the equality of numbers in equal volumes of gas at a fixed temperature and pressure from chemical reactions. Kinetic theory justifies this hypothesis.

The perfect gas equation can be written as

$$PV = \mu RT \tag{12.3}$$

where μ is the number of moles and $R = N_A k_B$ is a universal constant. The temperature T is absolute temperature. Choosing kelvin scale for absolute temperature, R = 8.314 J mol^{-1}K^{-1}. Here

$$\mu = \frac{M}{M_0} = \frac{N}{N_A} \tag{12.4}$$

where M is the mass of the gas containing N molecules, M_0 is the molar mass and N_A the Avogadro's number. Using Eqs. (12.4) and (12.3) can also be written as

$$PV = k_B NT \quad \text{or} \quad P = k_B nT$$

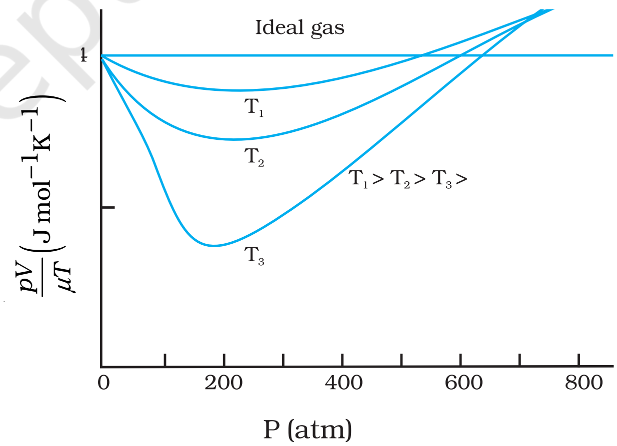


Fig. 12.1 *Real gases approach ideal gas behaviour at low pressures and high temperatures.*

where n is the number density, i.e. number of molecules per unit volume. k_B is the Boltzmann constant introduced above. Its value in SI units is 1.38×10^{-23} J K^{-1}.

Another useful form of Eq. (12.3) is

$$P = \frac{\rho RT}{M_0} \tag{12.5}$$

where ρ is the mass density of the gas.

A gas that satisfies Eq. (12.3) exactly at all pressures and temperatures is defined to be an **ideal gas**. An ideal gas is a simple theoretical model of a gas. No real gas is truly ideal. Fig. 12.1 shows departures from ideal gas behaviour for a real gas at three different temperatures. Notice that all curves approach the ideal gas behaviour for low pressures and high temperatures.

At low pressures or high temperatures the molecules are far apart and molecular interactions are negligible. Without interactions the gas behaves like an ideal one.

If we fix μ and T in Eq. (12.3), we get

$$PV = \text{constant} \tag{12.6}$$

i.e., keeping temperature constant, pressure of a given mass of gas varies inversely with volume. This is the famous **Boyle's law**. Fig. 12.2 shows comparison between experimental *P-V* curves and the theoretical curves predicted by Boyle's law. Once again you see that the agreement is good at high temperatures and low pressures. Next, if you fix P, Eq. (12.1) shows that $V \propto T$ i.e., for a fixed pressure, the volume of a gas is proportional to its absolute temperature T **(Charles' law)**. See Fig. 12.3.

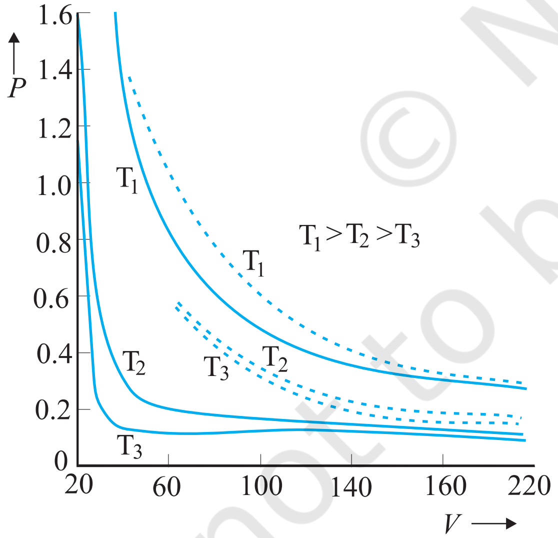


Fig.12.2 *Experimental P-V curves (solid lines) for steam at three temperatures compared with Boyle's law (dotted lines). P is in units of 22 atm and V in units of 0.09 litres.*

Finally, consider a mixture of non-interacting ideal gases: μ_1 moles of gas 1, μ_2 moles of gas 2, etc. in a vessel of volume V at temperature T and pressure P. It is then found that the equation of state of the mixture is :

$$PV = (\mu_1 + \mu_2 + \ldots)\, RT \tag{12.7}$$

$$\text{i.e. } P = \mu_1 \frac{RT}{V} + \mu_2 \frac{RT}{V} + \ldots \tag{12.8}$$

$$= P_1 + P_2 + \ldots \tag{12.9}$$

Clearly $P_1 = \mu_1 RT/V$ is the pressure that gas 1 would exert at the same conditions of volume and temperature if no other gases were present. This is called the partial pressure of the gas. Thus, the total pressure of a mixture of ideal gases is the sum of partial pressures. This is Dalton's law of partial pressures.

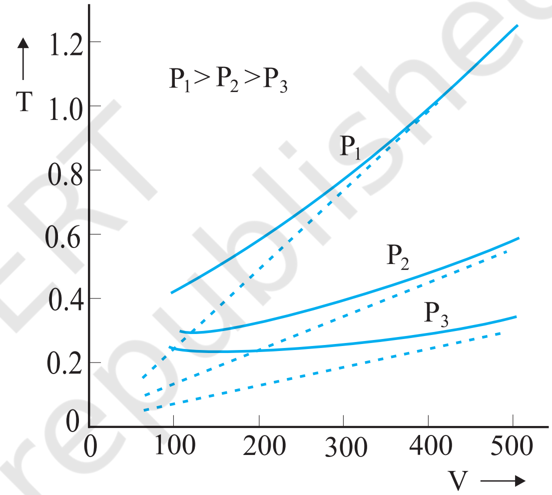


Fig. 12.3 *Experimental T-V curves (solid lines) for CO_2 at three pressures compared with Charles' law (dotted lines). T is in units of 300 K and V in units of 0.13 litres.*

We next consider some examples which give us information about the volume occupied by the molecules and the volume of a single molecule.

Example 12.1 The density of water is 1000 kg m^{-3}. The density of water vapour at 100 °C and 1 atm pressure is 0.6 kg m^{-3}. The volume of a molecule multiplied by the total number gives ,what is called, molecular volume. Estimate the ratio (or fraction) of the molecular volume to the total volume occupied by the water vapour under the above conditions of temperature and pressure.

Answer For a given mass of water molecules, the density is less if volume is large. So the volume of the vapour is $1000/0.6 = 1/(6 \times 10^{-4})$ times larger. If densities of bulk water and water molecules are same, then the fraction of molecular volume to the total volume in liquid state is 1. As volume in vapour state has increased, the fractional volume is less by the same amount, i.e. 6×10^{-4}. ◀

▶ **Example 12.2** Estimate the volume of a water molecule using the data in Example 12.1.

Answer In the liquid (or solid) phase, the molecules of water are quite closely packed. The density of water molecule may therefore, be regarded as roughly equal to the density of bulk water = 1000 kg m^{-3}. To estimate the volume of a water molecule, we need to know the mass of a single water molecule. We know that 1 mole of water has a mass approximately equal to

$(2 + 16)$g = 18 g = 0.018 kg.

Since 1 mole contains about 6×10^{23} molecules (Avogadro's number), the mass of a molecule of water is $(0.018)/(6 \times 10^{23})$ kg = 3×10^{-26} kg. Therefore, a rough estimate of the volume of a water molecule is as follows :

Volume of a water molecule

$= (3 \times 10^{-26}\ \text{kg})/(1000\ \text{kg m}^{-3})$

$= 3 \times 10^{-29}\ \text{m}^3$

$= (4/3)\ \pi\ (\text{Radius})^3$

Hence, Radius $\approx 2 \times 10^{-10}$ m = 2 Å ◀

▶ **Example 12.3** What is the average distance between atoms (interatomic distance) in water? Use the data given in Examples 12.1 and 12.2.

Answer : A given mass of water in vapour state has 1.67×10^3 times the volume of the same mass of water in liquid state (Ex. 12.1). This is also the increase in the amount of volume available for each molecule of water. When volume increases by 10^3 times the radius increases by $V^{1/3}$ or 10 times, i.e., 10 × 2 Å = 20 Å. So the average distance is 2 × 20 = 40 Å. ◀

▶ **Example 12.4** A vessel contains two non-reactive gases : neon (monatomic) and oxygen (diatomic). The ratio of their partial pressures is 3:2. Estimate the ratio of (i) number of molecules and (ii) mass density of neon and oxygen in the vessel. Atomic mass of Ne = 20.2 u, molecular mass of O_2 = 32.0 u.

Answer Partial pressure of a gas in a mixture is the pressure it would have for the same volume and temperature if it alone occupied the vessel. (The total pressure of a mixture of non-reactive gases is the sum of partial pressures due to its constituent gases.) Each gas (assumed ideal) obeys the gas law. Since V and T are common to the two gases, we have $P_1V = \mu_1 RT$ and $P_2V = \mu_2 RT$, i.e. $(P_1/P_2) = (\mu_1 / \mu_2)$. Here 1 and 2 refer to neon and oxygen respectively. Since (P_1/P_2) = (3/2) (given), $(\mu_1/ \mu_2) = 3/2$.

(i) By definition $\mu_1 = (N_1/N_A)$ and $\mu_2 = (N_2/N_A)$ where N_1 and N_2 are the number of molecules of 1 and 2, and N_A is the Avogadro's number. Therefore, $(N_1/N_2) = (\mu_1 / \mu_2) = 3/2$.

(ii) We can also write $\mu_1 = (m_1/M_1)$ and $\mu_2 = (m_2/M_2)$ where m_1 and m_2 are the masses of 1 and 2; and M_1 and M_2 are their molecular masses. (Both m_1 and M_1; as well as m_2 and M_2 should be expressed in the same units). If ρ_1 and ρ_2 are the mass densities of 1 and 2 respectively, we have

$$\frac{\rho_1}{\rho_2} = \frac{m_1/V}{m_2/V} = \frac{m_1}{m_2} = \frac{\mu_1}{\mu_2} \times \left(\frac{M_1}{M_2}\right)$$

$$= \frac{3}{2} \times \frac{20.2}{32.0} = 0.947$$

◀

12.4 KINETIC THEORY OF AN IDEAL GAS

Kinetic theory of gases is based on the molecular picture of matter. A given amount of gas is a collection of a large number of molecules (typically of the order of Avogadro's number) that are in incessant random motion. At ordinary pressure and temperature, the average distance between molecules is a factor of 10 or more than the typical size of a molecule (2 Å). Thus, interaction between molecules is negligible and we can assume that they move freely in straight lines according to Newton's first law. However, occasionally, they come close to each other, experience intermolecular forces and their velocities change. These interactions are called collisions. The molecules collide incessantly against each other or with the walls and change their velocities. The collisions are considered to be elastic. We can derive an expression for the pressure of a gas based on the kinetic theory.

We begin with the idea that molecules of a gas are in incessant random motion, colliding against one another and with the walls of the container. All collisions between molecules among themselves or between molecules and the walls are elastic. This implies that total kinetic energy is conserved. The total momentum is conserved as usual.

12.4.1 Pressure of an Ideal Gas

Consider a gas enclosed in a cube of side l. Take the axes to be parallel to the sides of the cube, as shown in Fig. 12.4. A molecule with velocity (v_x, v_y, v_z) hits the planar wall parallel to *yz*-plane of area A (= l^2). Since the collision is elastic, the molecule rebounds with the same velocity; its y and z components of velocity do not change in the collision but the x-component reverses sign. That is, the velocity after collision is ($-v_x$, v_y, v_z) . The change in momentum of the molecule is: $-mv_x - (mv_x) = -2mv_x$. By the principle of conservation of momentum, the momentum imparted to the wall in the collision $= 2mv_x$.

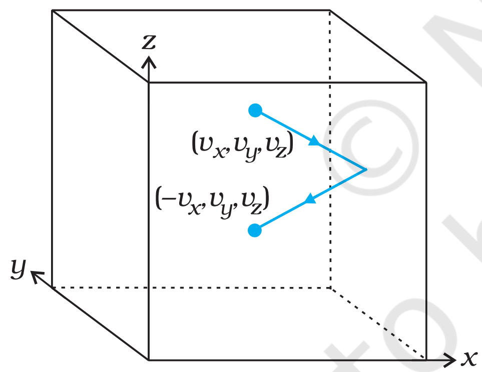


Fig. 12.4 *Elastic collision of a gas molecule with the wall of the container.*

To calculate the force (and pressure) on the wall, we need to calculate momentum imparted to the wall per unit time. In a small time interval Δt, a molecule with x-component of velocity v_x will hit the wall if it is within the distance $v_x \Delta t$ from the wall. That is, all molecules within the volume $Av_x \Delta t$ only can hit the wall in time Δt. But, on the average, half of these are moving towards the wall and the other half away from the wall. Thus, the number of molecules with velocity (v_x, v_y, v_z) hitting the wall in time Δt is $\frac{1}{2}A\, v_x\, \Delta t\, n,$ where n is the number of molecules per unit volume. The total momentum transferred to the wall by these molecules in time Δt is :

$$Q = (2mv_x)\, (\tfrac{1}{2}\, n\, A\, v_x\, \Delta t) \qquad (12.10)$$

The force on the wall is the rate of momentum transfer $Q/\Delta t$ and pressure is force per unit area :

$$P = Q\,/(A\,\Delta t) = n\, m\, v_x^2 \qquad (12.11)$$

Actually, all molecules in a gas do not have the same velocity; there is a distribution in velocities. The above equation, therefore, stands for pressure due to the group of molecules with speed v_x in the x-direction and n stands for the number density of that group of molecules. The total pressure is obtained by summing over the contribution due to all groups:

$$P = n\, m\, \overline{v_x^2} \qquad (12.12)$$

where $\overline{v_x^2}$ is the average of v_x^2 . Now the gas is isotropic, i.e. there is no preferred direction of velocity of the molecules in the vessel. Therefore, by symmetry,

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$

$$= (1/3)\, [\, \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}\,] = (1/3)\, \overline{v^2} \qquad (12.13)$$

where v is the speed and $\overline{v^2}$ denotes the mean of the squared speed. Thus

$$P = (1/3)\, n\, m\, \overline{v^2} \qquad (12.14)$$

Some remarks on this derivation. First, though we choose the container to be a cube, the shape of the vessel really is immaterial. For a vessel of arbitrary shape, we can always choose a small infinitesimal (planar) area and carry through the steps above. Notice that both A and Δt do not appear in the final result. By Pascal's law, given in Ch. 9, pressure in one portion of the gas in equilibrium is the same as anywhere else. Second, we have ignored any collisions in the derivation. Though this assumption is difficult to justify rigorously, we can qualitatively see that it will not lead to erroneous results. The number of molecules hitting the wall in time Δt was found to be $\frac{1}{2}\, n\, Av_x \Delta t$. Now the collisions are random and the gas is in a steady state. Thus, if a molecule with velocity (v_x, v_y, v_z) acquires a different velocity due to collision with some molecule, there will always be some other molecule with a different initial velocity which after a collision acquires the velocity (v_x, v_y, v_z). If this were not so, the distribution of velocities would not remain steady. In any case we are finding $\overline{v_x^2}$. Thus, on the whole, molecular collisions (if they are not too frequent and the time spent in a collision is negligible compared to time between collisions) will not affect the calculation above.

12.4.2 Kinetic Interpretation of Temperature

Equation (13.14) can be written as

$$PV = (1/3)\, nV m\, \overline{v^2} \qquad (12.15a)$$

$$PV = (2/3)\, N \times ½\, m\, \overline{v^2} \qquad (12.15b)$$

where N (= nV) is the number of molecules in the sample.

The quantity in the bracket is the average translational kinetic energy of the molecules in the gas. Since the internal energy E of an ideal gas is purely kinetic*,

$$E = N\ (1/2)\, m\, \overline{v^2} \qquad (12.16)$$

Equation (12.15) then gives :

$$PV = (2/3)\, E \qquad (12.17)$$

We are now ready for a kinetic interpretation of temperature. Combining Eq. (12.17) with the ideal gas Eq. (12.3), we get

$$E = (3/2)\ k_B\, NT \qquad (12.18)$$

or $E/N = ½\ m\, \overline{v^2} = (3/2)\, k_B T$ (12.19)

i.e., the average kinetic energy of a molecule is proportional to the absolute temperature of the gas; it is independent of pressure, volume or the nature of the ideal gas. This is a fundamental result relating temperature, a macroscopic measurable parameter of a gas (a thermodynamic variable as it is called) to a molecular quantity, namely the average kinetic energy of a molecule. The two domains are connected by the Boltzmann constant. We note in passing that Eq. (12.18) tells us that internal energy of an ideal gas depends only on temperature, not on pressure or volume. With this interpretation of temperature, kinetic theory of an ideal gas is completely consistent with the ideal gas equation and the various gas laws based on it.

For a mixture of non-reactive ideal gases, the total pressure gets contribution from each gas in the mixture. Equation (12.14) becomes

$$P = (1/3)\, [n_1 m_1 \overline{v_1^2} + n_2 m_2\, \overline{v_2^2} + \ldots\,] \qquad (12.20)$$

In equilibrium, the average kinetic energy of the molecules of different gases will be equal. That is,

$$½\ m_1\, \overline{v_1^2} = ½\, m_2\, \overline{v_2^2} = (3/2)\, k_B\, T$$

so that

$$P = (n_1 + n_2 + \ldots\,)\, k_B\, T \qquad (12.21)$$

which is Dalton's law of partial pressures.

From Eq. (12.19), we can get an idea of the typical speed of molecules in a gas. At a temperature T = 300 K, the mean square speed of a molecule in nitrogen gas is :

$$m = \frac{M_{N_2}}{N_A} = \frac{28}{6.02 \times 10^{26}} = 4.65 \times 10^{-26}\ \text{kg}.$$

$$\overline{v^2} = 3\, k_B\, T\, /\, m = (516)^2\ \text{m}^2\text{s}^{-2}$$

The square root of $\overline{v^2}$ is known as root mean square (rms) speed and is denoted by v_{rms},

(We can also write $\overline{v^2}$ as $< v^2 >$.)

$$v_{rms} = 516\ \text{m s}^{-1}$$

The speed is of the order of the speed of sound in air. It follows from Eq. (12.19) that at the same temperature, lighter molecules have greater rms speed.

Example 12.5 A flask contains argon and chlorine in the ratio of 2:1 by mass. The temperature of the mixture is 27 C. Obtain the ratio of (i) average kinetic energy per molecule, and (ii) root mean square speed v_{rms} of the molecules of the two gases. Atomic mass of argon = 39.9 u; Molecular mass of chlorine = 70.9 u.

Answer The important point to remember is that the average kinetic energy (per molecule) of any (ideal) gas (be it monatomic like argon, diatomic like chlorine or polyatomic) is always equal to $(3/2)\ k_B T$. It depends only on temperature, and is independent of the nature of the gas.

(i) Since argon and chlorine both have the same temperature in the flask, the ratio of average kinetic energy (per molecule) of the two gases is 1:1.

(ii) Now $½\ m\, v_{rms}{}^2$ = average kinetic energy per molecule = $(3/2)$) $k_B T$ where m is the mass

* *E denotes the translational part of the internal energy U that may include energies due to other degrees of freedom also. See section 12.5.*

of a molecule of the gas. Therefore,

$$\frac{\left(\mathbf{v}_{rms}^2\right)_{\mathrm{Ar}}}{\left(\mathbf{v}_{rms}^2\right)_{\mathrm{Cl}}} = \frac{(m)_{\mathrm{Cl}}}{(m)_{\mathrm{Ar}}} = \frac{(M)_{\mathrm{Cl}}}{(M)_{\mathrm{Ar}}} = \frac{70.9}{39.9} = 1.77$$

where M denotes the molecular mass of the gas. (For argon, a molecule is just an atom of argon.) Taking square root of both sides,

$$\frac{\left(\mathbf{v}_{rms}\right)_{\mathrm{Ar}}}{\left(\mathbf{v}_{rms}\right)_{\mathrm{Cl}}} = 1.33$$

You should note that the composition of the mixture by mass is quite irrelevant to the above calculation. Any other proportion by mass of argon and chlorine would give the same answers to (i) and (ii), provided the temperature remains unaltered. ◀

▶ ***Example 12.6*** Uranium has two isotopes of masses 235 and 238 units. If both are present in Uranium hexafluoride gas which would have the larger average speed ? If atomic mass of fluorine is 19 units, estimate the percentage difference in speeds at any temperature.

Answer At a fixed temperature the average energy $= \frac{1}{2}\, m \langle v^2 \rangle$ is constant. So smaller the mass of the molecule, faster will be the speed. The ratio of speeds is inversely proportional to the square root of the ratio of the masses. The masses are 349 and 352 units. So

$$v_{349} / v_{352} = (352/349)^{1/2} = 1.0044 .$$

$$\text{Hence difference } \frac{\Delta V}{V} = 0.44\ \%.$$

[^{235}U is the isotope needed for nuclear fission. To separate it from the more abundant isotope ^{238}U, the mixture is surrounded by a porous cylinder. The porous cylinder must be thick and narrow, so that the molecule wanders through individually, colliding with the walls of the long pore. The faster molecule will leak out more than the slower one and so there is more of the lighter molecule (enrichment) outside the porous cylinder (Fig. 12.5). The method is not very efficient and has to be repeated several times for sufficient enrichment.]. ◀

When gases diffuse, their rate of diffusion is inversely proportional to square root of the masses (see Exercise 12.12). Can you guess the explanation from the above answer?

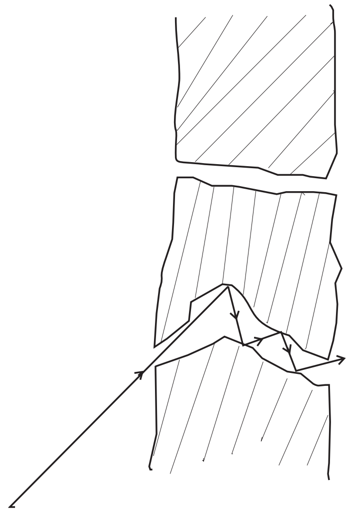


Fig. 12.5 *Molecules going through a porous wall.*

▶ ***Example 12.7*** (a) When a molecule (or an elastic ball) hits a (massive) wall, it rebounds with the same speed. When a ball hits a massive bat held firmly, the same thing happens. However, when the bat is moving towards the ball, the ball rebounds with a different speed. Does the ball move faster or slower? (Ch.5 will refresh your memory on elastic collisions.)

(b) When gas in a cylinder is compressed by pushing in a piston, its temperature rises. Guess at an explanation of this in terms of kinetic theory using (a) above.

(c) What happens when a compressed gas pushes a piston out and expands. What would you observe ?

(d) Sachin Tendulkar used a heavy cricket bat while playing. Did it help him in anyway ?

Answer (a) Let the speed of the ball be u relative to the wicket behind the bat. If the bat is moving towards the ball with a speed V relative to the wicket, then the relative speed of the ball to bat is $V + u$ towards the bat. When the ball rebounds (after hitting the massive bat) its speed, relative to bat, is $V + u$ moving away from the bat. So relative to the wicket the speed of the rebounding ball is $V + (V + u) = 2V + u$, moving away from the wicket. So the ball speeds up after the collision with the bat. The rebound speed will be less than u if the bat is not massive. For a molecule this would imply an increase in temperature.

You should be able to answer (b) (c) and (d) based on the answer to (a).
(Hint: Note the correspondence, piston→ bat, cylinder → wicket, molecule → ball.)

12.5 LAW OF EQUIPARTITION OF ENERGY

The kinetic energy of a single molecule is

$$\varepsilon_t = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 \qquad (12.22)$$

For a gas in thermal equilibrium at temperature T the average value of energy denoted by $< \varepsilon_t >$ is

$$\langle \varepsilon_t \rangle = \left\langle \frac{1}{2}mv_x^2 \right\rangle + \left\langle \frac{1}{2}mv_y^2 \right\rangle + \left\langle \frac{1}{2}mv_z^2 \right\rangle = \frac{3}{2}k_BT \qquad (12.23)$$

Since there is no preferred direction, Eq. (12.23) implies

$$\left\langle \frac{1}{2}\ mv_x^2 \right\rangle = \frac{1}{2}\ k_BT\ , \left\langle \frac{1}{2}\ mv_y^2 \right\rangle = \frac{1}{2}\ k_BT\ ,$$

$$\left\langle \frac{1}{2}\ mv_z^2 \right\rangle = \frac{1}{2}\ k_BT \qquad (12.24)$$

A molecule free to move in space needs three coordinates to specify its location. If it is constrained to move in a plane it needs two; and if constrained to move along a line, it needs just one coordinate to locate it. This can also be expressed in another way. We say that it has one degree of freedom for motion in a line, two for motion in a plane and three for motion in space. Motion of a body as a whole from one point to another is called translation. Thus, a molecule free to move in space has three translational degrees of freedom. Each translational degree of freedom contributes a term that contains square of some variable of motion, e.g., ½ mv_x^2 and similar terms in v_y and v_z. In, Eq. (12.24) we see that in thermal equilibrium, the average of each such term is ½ k_BT.

Molecules of a monatomic gas like argon have only translational degrees of freedom. But what about a diatomic gas such as O_2 or N_2? A molecule of O_2 has three translational degrees of freedom. But in addition it can also rotate about its centre of mass. Figure 12.6 shows the two independent axes of rotation 1 and 2, normal to the axis joining the two oxygen atoms about which the molecule can rotate*. The molecule thus has two rotational degrees of freedom, each of which contributes a term to the total energy consisting of translational energy ε_t and rotational energy ε_r.

$$\varepsilon_t + \varepsilon_r = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 \qquad (12.25)$$

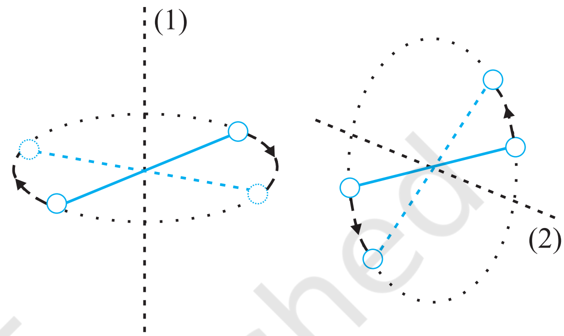


Fig. 12.6 *The two independent axes of rotation of a diatomic molecule*

where ω_1 and ω_2 are the angular speeds about the axes 1 and 2 and I_1, I_2 are the corresponding moments of inertia. Note that each rotational degree of freedom contributes a term to the energy that contains square of a rotational variable of motion.

We have assumed above that the O_2 molecule is a 'rigid rotator', i.e., the molecule does not vibrate. This assumption, though found to be true (at moderate temperatures) for O_2, is not always valid. Molecules, like CO, even at moderate temperatures have a mode of vibration, i.e., its atoms oscillate along the interatomic axis like a one-dimensional oscillator, and contribute a vibrational energy term ε_v to the total energy:

$$\varepsilon_v = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}ky^2$$

$$\varepsilon = \varepsilon_t + \varepsilon_r + \varepsilon_v \qquad (12.26)$$

where k is the force constant of the oscillator and y the vibrational co-ordinate.

Once again the vibrational energy terms in Eq. (12.26) contain squared terms of vibrational variables of motion y and dy/dt.

* *Rotation along the line joining the atoms has very small moment of inertia and does not come into play for quantum mechanical reasons. See end of section 12.6.*

At this point, notice an important feature in Eq.(12.26). While each translational and rotational degree of freedom has contributed only one 'squared term' in Eq.(12.26), one vibrational mode contributes two 'squared terms' : kinetic and potential energies.

Each quadratic term occurring in the expression for energy is a mode of absorption of energy by the molecule. We have seen that in thermal equilibrium at absolute temperature T, for each translational mode of motion, the average energy is ½ $k_B T$. The most elegant principle of classical statistical mechanics (first proved by Maxwell) states that this is so for each mode of energy: translational, rotational and vibrational. That is, in equilibrium, the total energy is equally distributed in all possible energy modes, with each mode having an average energy equal to ½ $k_B T$. This is known as the **law of equipartition of energy**. Accordingly, each translational and rotational degree of freedom of a molecule contributes ½ $k_B T$ to the energy, while each vibrational frequency contributes $2 \times ½\, k_B T = k_B T$, since a vibrational mode has both kinetic and potential energy modes.

The proof of the law of equipartition of energy is beyond the scope of this book. Here, we shall apply the law to predict the specific heats of gases theoretically. Later, we shall also discuss briefly, the application to specific heat of solids.

12.6 SPECIFIC HEAT CAPACITY

12.6.1 Monatomic Gases

The molecule of a monatomic gas has only three translational degrees of freedom. Thus, the average energy of a molecule at temperature T is $(3/2)k_B T$. The total internal energy of a mole of such a gas is

$$U = \frac{3}{2} k_B T \times N_A = \frac{3}{2} RT \tag{12.27}$$

The molar specific heat at constant volume, C_v, is

$$C_v \text{ (monatomic gas)} = \frac{dU}{dT} = \frac{3}{2} RT \tag{12.28}$$

For an ideal gas,

$$C_p - C_v = R \tag{12.29}$$

where C_p is the molar specific heat at constant pressure. Thus,

$$C_p = \frac{5}{2} R \tag{12.30}$$

The ratio of specific heats $\gamma = \dfrac{C_p}{C_v} = \dfrac{5}{3}$ (12.31)

12.6.2 Diatomic Gases

As explained earlier, a diatomic molecule treated as a rigid rotator, like a dumbbell, has 5 degrees of freedom: 3 translational and 2 rotational. Using the law of equipartition of energy, the total internal energy of a mole of such a gas is

$$U = \frac{5}{2} k_B T \times N_A = \frac{5}{2} RT \tag{12.32}$$

The molar specific heats are then given by

$$C_v \text{ (rigid diatomic)} = \frac{5}{2} R,\ C_p = \frac{7}{2} R \tag{12.33}$$

$$\gamma \text{ (rigid diatomic)} = \frac{7}{5} \tag{12.34}$$

If the diatomic molecule is not rigid but has in addition a vibrational mode

$$U = \left(\frac{5}{2} k_B T + k_B T\right) N_A = \frac{7}{2} RT$$

$$C_v = \frac{7}{2} R,\ C_p = \frac{9}{2} R,\ \gamma = \frac{9}{7} R \tag{12.35}$$

12.6.3 Polyatomic Gases

In general a polyatomic molecule has 3 translational, 3 rotational degrees of freedom and a certain number (f) of vibrational modes. According to the law of equipartition of energy, it is easily seen that one mole of such a gas has

$$U = \left(\frac{3}{2} k_B T + \frac{3}{2} k_B T + f k_B T\right) N_A$$

i.e., $C_v = (3 + f)\, R$, $C_p = (4 + f)\, R$,

$$\gamma = \frac{(4 + f)}{(3 + f)} \tag{12.36}$$

Note that $C_p - C_v = R$ is true for any ideal gas, whether mono, di or polyatomic.

Table 12.1 summarises the theoretical predictions for specific heats of gases ignoring any vibrational modes of motion. The values are in good agreement with experimental values of specific heats of several gases given in Table 12.2. Of course, there are discrepancies between predicted and actual values of specific heats of several other gases (not shown in the table), such as Cl_2, C_2H_6 and many other polyatomic gases. Usually, the experimental values for specific heats of these gases are greater than the predicted values as given in Table12.1 suggesting that the agreement can be improved by including vibrational modes of motion in the calculation. The law of equipartition of energy is, thus, well verified experimentally at ordinary temperatures.

Table 12.1 **Predicted values of specific heat capacities of gases (ignoring vibrational modes)**

Nature of Gas	C_v (J mol^{-1} K^{-1})	C_p (J mol^{-1} K^{-1})	$C_p - C_v$ (J mol^{-1} K^{-1})	γ
Monatomic	12.5	20.8	8.31	1.67
Diatomic	20.8	29.1	8.31	1.40
Triatomic	24.93	33.24	8.31	1.33

Table12.2 **Measured values of specific heat capacities of some gases**

Nature of gas	Gas	C_v (J mol^{-1} K^{-1})	C_p (J mol^{-1} K^{-1})	$C_p - C_v$ (J mol^{-1} K^{-1})	γ
Monatomic	He	12.5	20.8	8.30	1.66
Monatomic	Ne	12.7	20.8	8.12	1.64
Monatomic	Ar	12.5	20.8	8.30	1.67
Diatomic	H_2	20.4	28.8	8.45	1.41
Diatomic	O_2	21.0	29.3	8.32	1.40
Diatomic	N_2	20.8	29.1	8.32	1.40
Triatomic	H_2O	27.0	35.4	8.35	1.31
Polyatomic	CH_4	27.1	35.4	8.36	1.31

Example 12.8 A cylinder of fixed capacity 44.8 litres contains helium gas at standard temperature and pressure. What is the amount of heat needed to raise the temperature of the gas in the cylinder by 15.0 °C ? (R = 8.31 J mo1^{-1} K^{-1}).

Answer Using the gas law $PV = \mu RT$, you can easily show that 1 mol of any (ideal) gas at standard temperature (273 K) and pressure (1 atm = 1.01×10^5 Pa) occupies a volume of 22.4 litres. This universal volume is called molar volume. Thus the cylinder in this example contains 2 mol of helium. Further, since helium is monatomic, its predicted (and observed) molar specific heat at constant volume, $C_v = (3/2)\ R$, and molar specific heat at constant pressure, $C_p = (3/2)\ R + R = (5/2)\ R$. Since the volume of the cylinder is fixed, the heat required is determined by C_v. Therefore,

Heat required = no. of moles × molar specific heat rise in temperature

$= 2 \times 1.5\ R \times 15.0 = 45\ R$

$= 45 \times 8.31 = 374$ J.

12.6.4 Specific Heat Capacity of Solids

We can use the law of equipartition of energy to determine specific heats of solids. Consider a solid of N atoms, each vibrating about its mean position. An oscillation in one dimension has average energy of $2 \times ½\ k_BT = k_BT$. In three dimensions, the average energy is $3\ k_BT$. For a mole of solid, $N = N_A$, and the total energy is

$U = 3\ k_BT \times N_A = 3\ RT$

Now at constant pressure $\Delta Q = \Delta U + P\Delta V = \Delta U$, since for a solid ΔV is negligible. Hence,

$$C = \frac{\Delta Q}{\Delta T} = \frac{\Delta U}{\Delta T} = 3R \qquad (12.37)$$

Table 12.3 **Specific Heat Capacity of some solids at room temperature and atmospheric pressure**

Substance	Specific heat (J kg^{-1} K^{-1})	Molar specific heat (J mol^{-1} K^{-1})
Aluminium	900.0	24.4
Carbon	506.5	6.1
Copper	386.4	24.5
Lead	127.7	26.5
Silver	236.1	25.5
Tungsten	134.4	24.9

As Table 12.3 shows the prediction generally agrees with experimental values at ordinary temperature (Carbon is an exception).

12.7 MEAN FREE PATH

Molecules in a gas have rather large speeds of the order of the speed of sound. Yet a gas leaking from a cylinder in a kitchen takes considerable time to diffuse to the other corners of the room. The top of a cloud of smoke holds together for hours. This happens because molecules in a gas have a finite though small size, so they are bound to undergo collisions. As a result, they cannot move straight unhindered; their paths keep getting incessantly deflected.

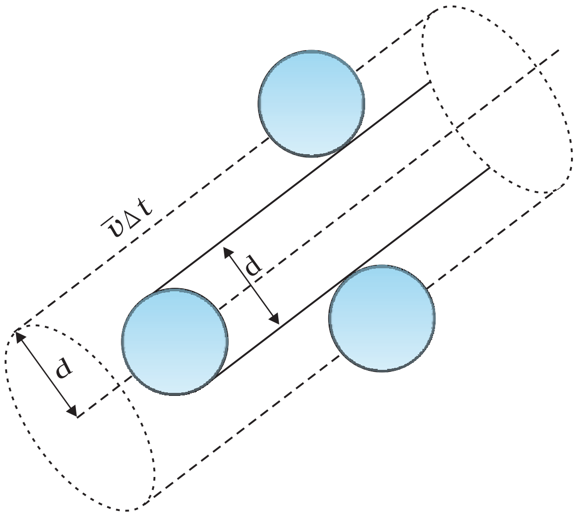


Fig. 12.7 *The volume swept by a molecule in time Δt in which any molecule will collide with it.*

Suppose the molecules of a gas are spheres of diameter d. Focus on a single molecule with the average speed $\langle v\rangle$. It will suffer collision with any molecule that comes within a distance d between the centres. In time Δt, it sweeps a volume $\pi d^2 \langle v\rangle \Delta t$ wherein any other molecule will collide with it (see Fig. 12.7). If n is the number of molecules per unit volume, the molecule suffers $n\pi d^2 \langle v\rangle \Delta t$ collisions in time Δt. Thus the rate of collisions is $n\pi d^2 \langle v\rangle$ or the time between two successive collisions is on the average,

$$\tau = 1/(n\pi \langle v\rangle d^2) \qquad (12.38)$$

The average distance between two successive collisions, called the mean free path l, is :

$$l = \langle v\rangle \tau = 1/(n\pi d^2) \qquad (12.39)$$

In this derivation, we imagined the other molecules to be at rest. But actually all molecules are moving and the collision rate is determined by the average relative velocity of the molecules. Thus we need to replace $\langle v\rangle$ by $\langle v\rangle_r$ in Eq. (12.38). A more exact treatment gives

$$l = 1/\left(\sqrt{2}\ n\pi d^2\right) \qquad (12.40)$$

Let us estimate l and τ for air molecules with average speeds $\langle v\rangle = (485 \text{m/s})$. At STP

$$n = \frac{\left(0.02 \times 10^{23}\right)}{\left(22.4 \times 10^{-3}\right)}$$

$= 2.7 \times 10^{25}\ \text{m}^{-3.}$

Taking, $d = 2 \times 10^{-10}$ m,

$\tau = 6.1 \times 10^{-10}$ s

and $l = 2.9 \times 10^{-7}\ \text{m} \approx 1500\ d$ (12.41)

As expected, the mean free path given by Eq. (12.40) depends inversely on the number density and the size of the molecules. In a highly evacuated tube n is rather small and the mean free path can be as large as the length of the tube.

▶ ***Example 12.9*** Estimate the mean free path for a water molecule in water vapour at 373 K. Use information from Exercises 12.1 and Eq. (12.41) above.

Answer The d for water vapour is same as that of air. The number density is inversely proportional to absolute temperature.

$$\text{So } n = 2.7 \times 10^{25} \times \frac{273}{373} = 2 \times 10^{25}\ \text{m}^{-3}$$

Hence, mean free path $l = 4 \times 10^{-7}$ m ◀

Note that the mean free path is 100 times the interatomic distance ~ 40 Å = 4×10^{-9} m calculated earlier. It is this large value of mean free path that leads to the typical gaseous behaviour. Gases can not be confined without a container.

Using, the kinetic theory of gases, the bulk measurable properties like viscosity, heat conductivity and diffusion can be related to the microscopic parameters like molecular size. It is through such relations that the molecular sizes were first estimated.

SUMMARY

1. The ideal gas equation connecting pressure (P), volume (V) and absolute temperature (T) is

$$PV = \mu RT \quad = k_B NT$$

where μ is the number of moles and N is the number of molecules. R and k_B are universal constants.

$$R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1}, \qquad k_B = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

Real gases satisfy the ideal gas equation only approximately, more so at low pressures and high temperatures.

2. Kinetic theory of an ideal gas gives the relation

$$P = \frac{1}{3} n m \overline{v^2}$$

where n is number density of molecules, m the mass of the molecule and $\overline{v^2}$ is the mean of squared speed. Combined with the ideal gas equation it yields a kinetic interpretation of temperature.

$$\frac{1}{2} m \overline{v^2} = \frac{3}{2} k_B T, \quad v_{rms} = \left(\overline{v^2}\right)^{1/2} = \sqrt{\frac{3k_B T}{m}}$$

This tells us that the temperature of a gas is a measure of the average kinetic energy of a molecule, *independent of the nature of the gas or molecule*. In a mixture of gases at a fixed temperature the heavier molecule has the lower average speed.

3. The translational kinetic energy

$$E = \frac{3}{2} k_B NT.$$

This leads to a relation

$$PV = \frac{2}{3} E$$

4. The law of equipartition of energy states that if a system is in equilibrium at absolute temperature T, the total energy is distributed equally in different energy modes of absorption, the energy in each mode being equal to ½ $k_B T$. Each translational and rotational degree of freedom corresponds to one energy mode of absorption and has energy ½ $k_B T$. Each vibrational frequency has two modes of energy (kinetic and potential) with corresponding energy equal to

$2 \times$ ½ $k_B T = k_B T$.

5. Using the law of equipartition of energy, the molar specific heats of gases can be determined and the values are in agreement with the experimental values of specific heats of several gases. The agreement can be improved by including vibrational modes of motion.

6. The mean free path l is the average distance covered by a molecule between two successive collisions :

$$l = \frac{1}{\sqrt{2}\, n \pi d^2}$$

where n is the number density and d the diameter of the molecule.

POINTS TO PONDER

1. Pressure of a fluid is not only exerted on the wall. Pressure exists everywhere in a fluid. Any layer of gas inside the volume of a container is in equilibrium because the pressure is the same on both sides of the layer.

2. We should not have an exaggerated idea of the intermolecular distance in a gas. At ordinary pressures and temperatures, this is only 10 times or so the interatomic distance in solids and liquids. What is different is the mean free path which in a gas is 100 times the interatomic distance and 1000 times the size of the molecule.

3. The law of equipartition of energy is stated thus: the energy for each degree of freedom in thermal equilibrium is ½ $k_B T$. Each quadratic term in the total energy expression of a molecule is to be counted as a degree of freedom. Thus, each vibrational mode gives 2 (not 1) degrees of freedom (kinetic and potential energy modes), corresponding to the energy 2 × ½ $k_B T = k_B T$.

4. Molecules of air in a room do not all fall and settle on the ground (due to gravity) because of their high speeds and incessant collisions. In equilibrium, there is a very slight increase in density at lower heights (like in the atmosphere). The effect is small since the potential energy (*mgh*) for ordinary heights is much less than the average kinetic energy ½ mv^2 of the molecules.

5. $< v^2 >$ is not always equal to $(< v >)^2$. The average of a squared quantity is not necessarily the square of the average. Can you find examples for this statement.

EXERCISES

12.1 Estimate the fraction of molecular volume to the actual volume occupied by oxygen gas at STP. Take the diameter of an oxygen molecule to be 3 Å.

12.2 Molar volume is the volume occupied by 1 mol of any (ideal) gas at standard temperature and pressure (STP : 1 atmospheric pressure, 0 °C). Show that it is 22.4 litres.

12.3 Figure 12.8 shows plot of PV/T versus P for 1.00×10^{-3} kg of oxygen gas at two different temperatures.

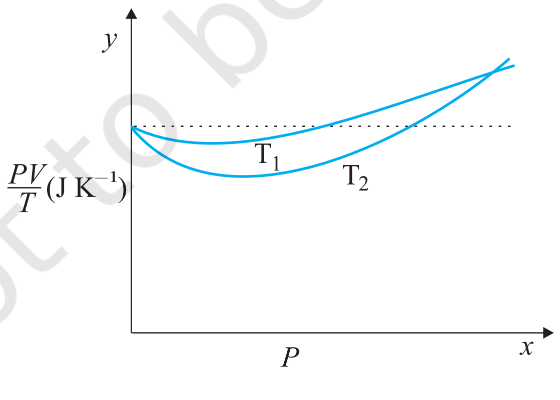


Fig. 12.8

(a) What does the dotted plot signify?
(b) Which is true: $T_1 > T_2$ or $T_1 < T_2$?
(c) What is the value of PV/T where the curves meet on the y-axis?

(d) If we obtained similar plots for 1.00×10^{-3} kg of hydrogen, would we get the same value of PV/T at the point where the curves meet on the y-axis? If not, what mass of hydrogen yields the same value of PV/T (for low pressure high temperature region of the plot) ? (Molecular mass of H_2 = 2.02 u, of O_2 = 32.0 u, R = 8.31 J mol^{-1} K^{-1}.)

12.4 An oxygen cylinder of volume 30 litre has an initial gauge pressure of 15 atm and a temperature of 27 °C. After some oxygen is withdrawn from the cylinder, the gauge pressure drops to 11 atm and its temperature drops to 17 °C. Estimate the mass of oxygen taken out of the cylinder (R = 8.31 J mol^{-1} K^{-1}, molecular mass of O_2 = 32 u).

12.5 An air bubble of volume 1.0 cm^3 rises from the bottom of a lake 40 m deep at a temperature of 12 °C. To what volume does it grow when it reaches the surface, which is at a temperature of 35 °C ?

12.6 Estimate the total number of air molecules (inclusive of oxygen, nitrogen, water vapour and other constituents) in a room of capacity 25.0 m^3 at a temperature of 27 °C and 1 atm pressure.

12.7 Estimate the average thermal energy of a helium atom at (i) room temperature (27 °C), (ii) the temperature on the surface of the Sun (6000 K), (iii) the temperature of 10 million kelvin (the typical core temperature in the case of a star).

12.8 Three vessels of equal capacity have gases at the same temperature and pressure. The first vessel contains neon (monatomic), the second contains chlorine (diatomic), and the third contains uranium hexafluoride (polyatomic). Do the vessels contain equal number of respective molecules ? Is the root mean square speed of molecules the same in the three cases? If not, in which case is v_{rms} the largest ?

12.9 At what temperature is the root mean square speed of an atom in an argon gas cylinder equal to the rms speed of a helium gas atom at – 20 °C ? (atomic mass of Ar = 39.9 u, of He = 4.0 u).

12.10 Estimate the mean free path and collision frequency of a nitrogen molecule in a cylinder containing nitrogen at 2.0 atm and temperature 17 ^{0}C. Take the radius of a nitrogen molecule to be roughly 1.0 Å. Compare the collision time with the time the molecule moves freely between two successive collisions (Molecular mass of N_2 = 28.0 u).

Chapter Thirteen

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Oscillations

13.1 INTRODUCTION

In our daily life we come across various kinds of motions. You have already learnt about some of them, e.g., rectilinear motion and motion of a projectile. Both these motions are non-repetitive. We have also learnt about uniform circular motion and orbital motion of planets in the solar system. In these cases, the motion is repeated after a certain interval of time, that is, it is periodic. In your childhood, you must have enjoyed rocking in a cradle or swinging on a swing. Both these motions are repetitive in nature but different from the periodic motion of a planet. Here, the object moves to and fro about a mean position. The pendulum of a wall clock executes a similar motion. Examples of such periodic to and fro motion abound: a boat tossing up and down in a river, the piston in a steam engine going back and forth, etc. Such a motion is termed as oscillatory motion. In this chapter we study this motion.

The study of oscillatory motion is basic to physics; its concepts are required for the understanding of many physical phenomena. In musical instruments, like the sitar, the guitar or the violin, we come across vibrating strings that produce pleasing sounds. The membranes in drums and diaphragms in telephone and speaker systems vibrate to and fro about their mean positions. The vibrations of air molecules make the propagation of sound possible. In a solid, the atoms vibrate about their equilibrium positions, the average energy of vibrations being proportional to temperature. AC power supply give voltage that oscillates alternately going positive and negative about the mean value (zero).

The description of a periodic motion, in general, and oscillatory motion, in particular, requires some fundamental concepts, like period, frequency, displacement, amplitude and phase. These concepts are developed in the next section.

13.2 PERIODIC AND OSCILLATORY MOTIONS

Fig. 13.1 shows some periodic motions. Suppose an insect climbs up a ramp and falls down, it comes back to the initial point and repeats the process identically. If you draw a graph of its height above the ground versus time, it would look something like Fig. 13.1 (a). If a child climbs up a step, comes down, and repeats the process identically, its height above the ground would look like that in Fig. 13.1 (b). When you play the game of bouncing a ball off the ground, between your palm and the ground, its height versus time graph would look like the one in Fig. 13.1 (c). Note that both the curved parts in Fig. 13.1 (c) are sections of a parabola given by the Newton's equation of motion (see section 2.6),

$h = ut + \frac{1}{2}gt^2$ for downward motion, and

$h = ut - \frac{1}{2}gt^2$ for upward motion,

with different values of u in each case. These are examples of periodic motion. Thus, a motion that repeats itself at regular intervals of time is called **periodic motion.**

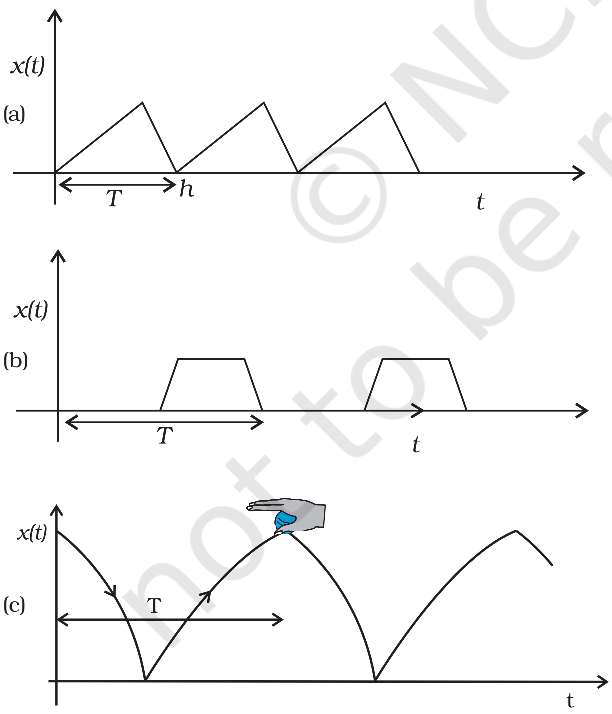


Fig. 13.1 Examples of periodic motion. The period T is shown in each case.

Very often, the body undergoing periodic motion has an equilibrium position somewhere inside its path. When the body is at this position no net external force acts on it. Therefore, if it is left there at rest, it remains there forever. If the body is given a small displacement from the position, a force comes into play which tries to bring the body back to the equilibrium point, giving rise to **oscillations** or **vibrations**. For example, a ball placed in a bowl will be in equilibrium at the bottom. If displaced a little from the point, it will perform oscillations in the bowl. Every oscillatory motion is periodic, but every periodic motion need not be oscillatory. Circular motion is a periodic motion, but it is not oscillatory.

There is no significant difference between oscillations and vibrations. It seems that when the frequency is small, we call it oscillation (like, the oscillation of a branch of a tree), while when the frequency is high, we call it vibration (like, the vibration of a string of a musical instrument).

Simple harmonic motion is the simplest form of oscillatory motion. This motion arises when the force on the oscillating body is directly proportional to its displacement from the mean position, which is also the equilibrium position. Further, at any point in its oscillation, this force is directed towards the mean position.

In practice, oscillating bodies eventually come to rest at their equilibrium positions because of the damping due to friction and other dissipative causes. However, they can be forced to remain oscillating by means of some external periodic agency. We discuss the phenomena of damped and forced oscillations later in the chapter.

Any material medium can be pictured as a collection of a large number of coupled oscillators. The collective oscillations of the constituents of a medium manifest themselves as waves. Examples of waves include water waves, seismic waves, electromagnetic waves. We shall study the wave phenomenon in the next chapter.

13.2.1 Period and frequency

We have seen that any motion that repeats itself at regular intervals of time is called **periodic motion**. **The smallest interval of time after which the motion is repeated is called its period**. Let us denote the period by the symbol T. Its SI unit is second. For periodic motions, which are either too fast or too slow on the scale of seconds, other convenient units of time are used. The period of vibrations of a quartz crystal is expressed in units of microseconds (10^{-6} s) abbreviated as μs. On the other hand, the orbital period of the planet Mercury is 88 earth days. The Halley's comet appears after every 76 years.

The reciprocal of T gives the number of repetitions that occur per unit time. This quantity is called the **frequency of the periodic motion**. It is represented by the symbol ν. The relation between ν and T is

$$\nu = 1/T \qquad (13.1)$$

The unit of ν is thus s^{-1}. After the discoverer of radio waves, Heinrich Rudolph Hertz (1857–1894), a special name has been given to the unit of frequency. It is called hertz (abbreviated as Hz). Thus,

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1 \text{ s}^{-1} \qquad (13.2)$$

Note, that the frequency, ν, is not necessarily an integer.

Example 13.1 On an average, a human heart is found to beat 75 times in a minute. Calculate its frequency and period.

Answer The beat frequency of heart = 75/(1 min)
= 75/(60 s)
= 1.25 s^{-1}
= 1.25 Hz

The time period T = 1/(1.25 s^{-1})
= 0.8 s

13.2.2 Displacement

In section 3.2, we defined displacement of a particle as the change in its position vector. In this chapter, we use the term displacement in a more general sense. It refers to change with time of any physical property under consideration. For example, in case of rectilinear motion of a steel ball on a surface, the distance from the starting point as a function of time is its position displacement. The choice of origin is a matter of convenience. Consider a block attached to a spring, the other end of the spring is fixed to a rigid wall [see Fig. 13.2(a)]. Generally, it is convenient to measure displacement of the body from its equilibrium position. For an oscillating simple pendulum, the angle from the vertical as a function of time may be regarded as a displacement variable [see Fig. 13.2(b)]. The term displacement is not always to be referred

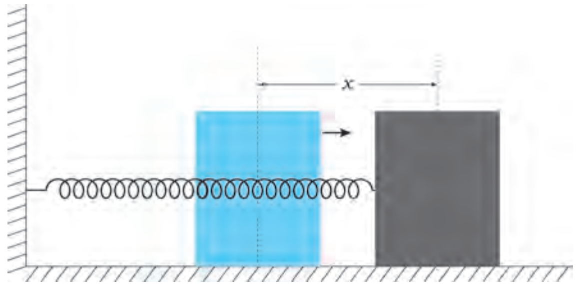


Fig. 13.2(a) A block attached to a spring, the other end of which is fixed to a rigid wall. The block moves on a frictionless surface. The motion of the block can be described in terms of its distance or displacement x from the equilibrium position.

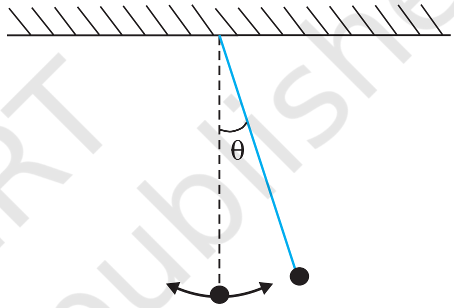


Fig. 13.2(b) An oscillating simple pendulum; its motion can be described in terms of angular displacement θ from the vertical.

in the context of position only. There can be many other kinds of displacement variables. The voltage across a capacitor, changing with time in an AC circuit, is also a displacement variable. In the same way, pressure variations in time in the propagation of sound wave, the changing electric and magnetic fields in a light wave are examples of displacement in different contexts. The displacement variable may take both positive and negative values. In experiments on oscillations, the displacement is measured for different times.

The displacement can be represented by a mathematical function of time. In case of periodic motion, this function is periodic in time. One of the simplest periodic functions is given by

$$f(t) = A \cos \omega t \qquad (13.3a)$$

If the argument of this function, ωt, is increased by an integral multiple of 2π radians, the value of the function remains the same. The function $f(t)$ is then periodic and its period, T, is given by

$$T = \frac{2\pi}{\omega} \tag{13.3b}$$

Thus, the function $f(t)$ is periodic with period T,

$$f(t) = f(t+T)$$

The same result is obviously correct if we consider a sine function, $f(t) = A \sin \omega t$. Further, a linear combination of sine and cosine functions like,

$$f(t) = A \sin \omega t + B \cos \omega t \tag{13.3c}$$

is also a periodic function with the same period T. Taking,

$$A = D \cos \phi \text{ and } B = D \sin \phi$$

Eq. (13.3c) can be written as,

$$f(t) = D \sin (\omega t + \phi)\ , \tag{13.3d}$$

Here D and ϕ are constant given by

$$D = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

The great importance of periodic sine and cosine functions is due to a remarkable result proved by the French mathematician, Jean Baptiste Joseph Fourier (1768–1830): **Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients**.

> ***Example 13.2*** Which of the following functions of time represent (a) periodic and (b) non-periodic motion? Give the period for each case of periodic motion [ω is any positive constant].
> (i) $\sin \omega t + \cos \omega t$
> (ii) $\sin \omega t + \cos 2\,\omega t + \sin 4\,\omega t$
> (iii) $e^{-\omega t}$
> (iv) $\log (\omega t)$

Answer

(i) $\sin \omega t + \cos \omega t$ is a periodic function, it can also be written as $\sqrt{2}\ \sin (\omega t + \pi/4)$.

Now $\sqrt{2}\ \sin (\omega t + \pi/4) = \sqrt{2}\ \sin (\omega t + \pi/4 + 2\pi)$

$= \sqrt{2}\ \sin [\omega (t + 2\pi/\omega) + \pi/4]$

The periodic time of the function is $2\pi/\omega$.

(ii) This is an example of a periodic motion. It can be noted that each term represents a periodic function with a different angular frequency. Since period is the least interval of time after which a function repeats its value, $\sin \omega t$ has a period $T_0 = 2\pi/\omega$; $\cos 2\,\omega t$ has a period $\pi/\omega = T_0/2$; and $\sin 4\,\omega t$ has a period $2\pi/4\omega = T_0/4$. The period of the first term is a multiple of the periods of the last two terms. Therefore, the smallest interval of time after which the sum of the three terms repeats is T_0, and thus, the sum is a periodic function with a period $2\pi/\omega$.

(iii) The function $e^{-\omega t}$ is not periodic, it decreases monotonically with increasing time and tends to zero as $t \to \infty$ and thus, never repeats its value.

(iv) The function $\log(\omega t)$ increases monotonically with time t. It, therefore, never repeats its value and is a non-periodic function. It may be noted that as $t \to \infty$, $\log(\omega t)$ diverges to ∞. It, therefore, cannot represent any kind of physical displacement. ◀

13.3 SIMPLE HARMONIC MOTION

Consider a particle oscillating back and forth about the origin of an x-axis between the limits $+A$ and $-A$ as shown in Fig. 13.3. This oscillatory motion is said to be simple harmonic if the displacement x of the particle from the origin varies with time as :

$$x(t) = A \cos (\omega t + \phi) \tag{13.4}$$

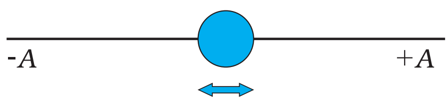


Fig. 13.3 *A particle vibrating back and forth about the origin of x-axis, between the limits +A and –A.*

where A, ω and ϕ are constants.

Thus, simple harmonic motion (SHM) is not any periodic motion but one in which displacement is a sinusoidal function of time. Fig. 13.4 shows the positions of a particle executing SHM at discrete value of time, each interval of time being $T/4$, where T is the period of motion. Fig. 13.5 plots the graph of x versus t, which gives the values of displacement as a continuous function of time. The quantities A,

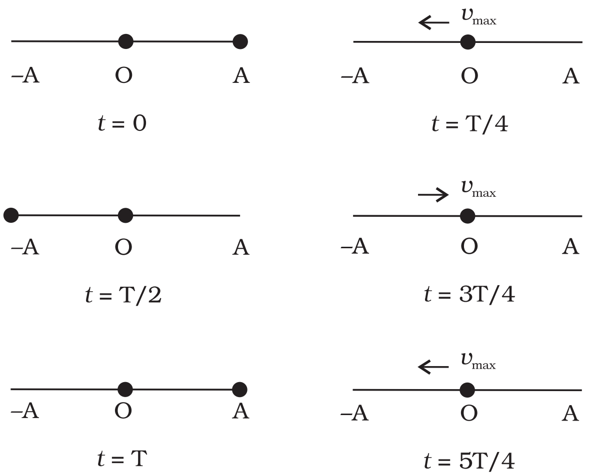


Fig. 13.4 *The location of the particle in SHM at the discrete values t = 0, T/4, T/2, 3T/4, T, 5T/4. The time after which motion repeats itself is T. T will remain fixed, no matter what location you choose as the initial (t = 0) location. The speed is maximum for zero displacement (at x = 0) and zero at the extremes of motion.*

ω and ϕ which characterize a given SHM have standard names, as summarised in Fig. 13.6. Let us understand these quantities.

The amplitutde A of SHM is the magnitude of maximum displacement of the particle. [Note, A can be taken to be positive without any loss of generality]. As the cosine function of time varies from +1 to –1, the displacement varies between the extremes A and $-A$. Two simple harmonic motions may have same ω and ϕ but different amplitudes A and B, as shown in Fig. 13.7 (a).

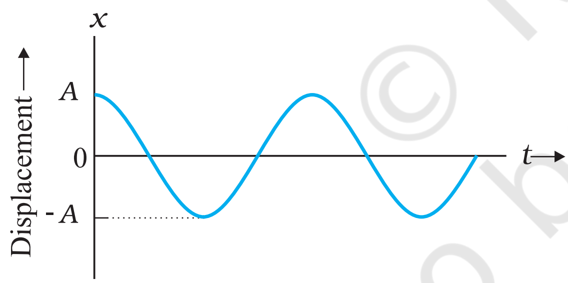


Fig. 13.5 *Displacement as a continuous function of time for simple harmonic motion.*

$x(t)$	:	displacement x as a function of time t
A	:	amplitude
ω	:	angular frequency
$\omega t + \phi$	:	phase (time-dependent)
ϕ	:	phase constant

Fig. 13.6 *The meaning of standard symbols in Eq. (13.4)*

While the amplitude A is fixed for a given SHM, the state of motion (position and velocity) of the particle at any time t is determined by the argument $(\omega t + \phi)$ in the cosine function. This time-dependent quantity, $(\omega t + \phi)$ is called the *phase* of the motion. The value of plase at $t = 0$ is ϕ and is called the *phase constant* (or *phase angle*). If the amplitude is known, ϕ can be determined from the displacement at $t = 0$. Two simple harmonic motions may have the same A and ω but different phase angle ϕ, as shown in Fig. 13.7 (b).

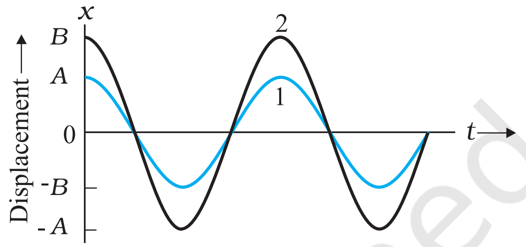


Fig. 13.7 (a) *A plot of displacement as a function of time as obtained from Eq. (14.4) with ϕ = 0. The curves 1 and 2 are for two different amplitudes A and B.*

Finally, the quantity ω can be seen to be related to the period of motion T. Taking, for simplicity, $\phi = 0$ in Eq. (13.4), we have

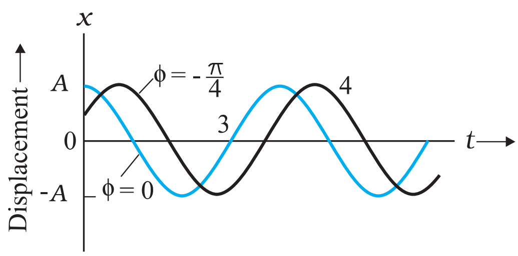


Fig. 13.7 (b) *A plot obtained from Eq. (13.4). The curves 3 and 4 are for ϕ = 0 and $-\pi/4$ respectively. The amplitude A is same for both the plots.*

$$x(t) = A \cos \omega t \qquad (13.5)$$

Since the motion has a period T, $x(t)$ is equal to $x(t + T)$. That is,

$$A \cos \omega t = A \cos \omega (t + T) \qquad (13.6)$$

Now the cosine function is periodic with period 2π, i.e., it first repeats itself when the argument changes by 2π. Therefore,

$$\omega(t + T) = \omega t + 2\pi$$

that is $\omega = 2\pi/ T$ (13.7)

ω is called the angular frequency of SHM. Its S.I. unit is radians per second. Since the frequency of oscillations is simply $1/T$, ω is 2π times the frequency of oscillation. Two simple harmonic motions may have the same A and ϕ, but different ω, as seen in Fig. 13.8. In this plot the curve (b) has half the period and twice the frequency of the curve (a).

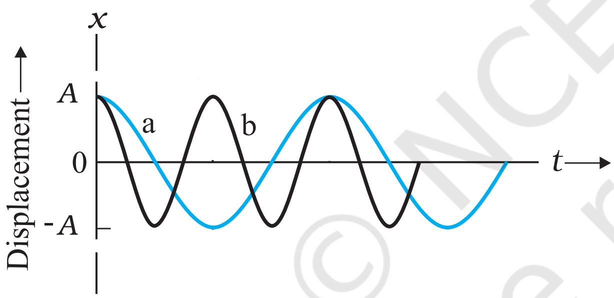


Fig. 13.8 *Plots of Eq. (13.4) for ϕ = 0 for two different periods.*

> **Example 13.3** Which of the following functions of time represent (a) simple harmonic motion and (b) periodic but not simple harmonic? Give the period for each case.
> (1) $\sin \omega t - \cos \omega t$
> (2) $\sin^2 \omega t$

Answer

(a) $\sin \omega t - \cos \omega t$

$$= \sin \omega t - \sin (\pi/2 - \omega t)$$
$$= 2 \cos (\pi/4) \sin (\omega t - \pi/4)$$
$$= \sqrt{2} \sin (\omega t - \pi/4)$$

This function represents a simple harmonic motion having a period $T = 2\pi/\omega$ and a phase angle $(-\pi/4)$ or $(7\pi/4)$

(b) $\sin^2 \omega t$

$$= \tfrac{1}{2} - \tfrac{1}{2} \cos 2 \omega t$$

The function is periodic having a period $T = \pi/\omega$. It also represents a harmonic motion with the point of equilibrium occurring at ½ instead of zero.

13.4 SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

In this section, we show that the projection of uniform circular motion on a diameter of the circle follows simple harmonic motion. A simple experiment (Fig. 13.9) helps us visualise this connection. Tie a ball to the end of a string and make it move in a horizontal plane about a fixed point with a constant angular speed. The ball would then perform a uniform circular motion in the horizontal plane. Observe the ball sideways or from the front, fixing your attention in the plane of motion. The ball will appear to execute to and fro motion along a horizontal line with the point of rotation as the midpoint. You could alternatively observe the shadow of the ball on a wall which is perpendicular to the plane of the circle. In this process what we are observing is the motion of the ball on a diameter of the circle normal to the direction of viewing.

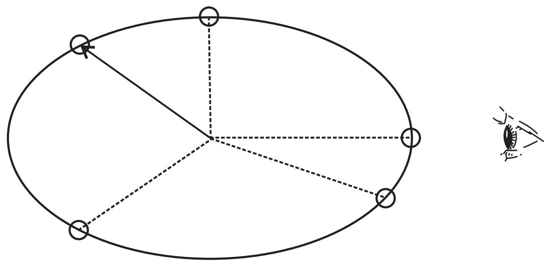


Fig. 13.9 *Circular motion of a ball in a plane viewed edge-on is SHM.*

Fig. 13.10 describes the same situation mathematically. Suppose a particle P is moving uniformly on a circle of radius A with angular speed ω. The sense of rotation is anticlockwise. The initial position vector of the particle, i.e., the vector $\overline{\mathbf{OP}}$ at $t = 0$ makes an angle of ϕ with the positive direction of x-axis. In time t, it will cover a further angle ωt and its position vector

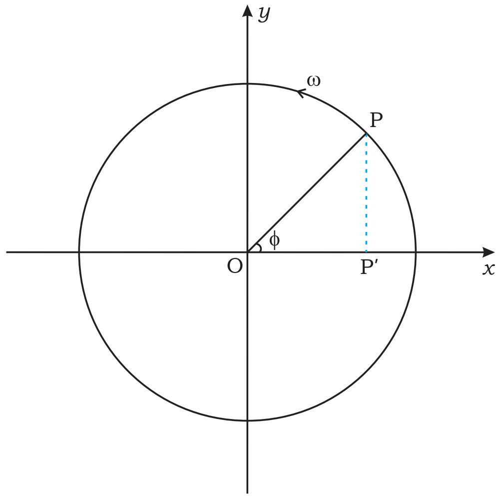


Fig. 13.10

will make an angle of $\omega t + \phi$ with the +ve x-axis. Next, consider the projection of the position vector OP on the x-axis. This will be OP′. The position of P′ on the x-axis, as the particle P moves on the circle, is given by

$$x(t) = A \cos (\omega t + \phi)$$

which is the defining equation of SHM. This shows that if P moves uniformly on a circle, its projection P′ on a diameter of the circle executes SHM. The particle P and the circle on which it moves are sometimes referred to as the *reference particle* and the *reference circle*, respectively.

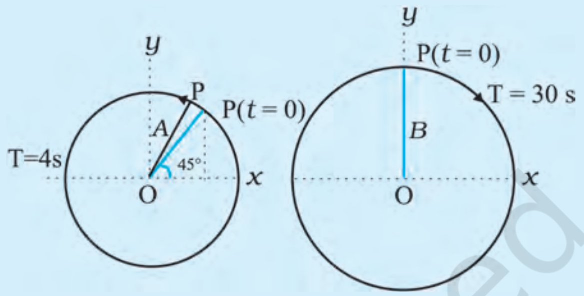
We can take projection of the motion of P on any diameter, say the y-axis. In that case, the displacement $y(t)$ of P′ on the y-axis is given by

$$y = A \sin (\omega t + \phi)$$

which is also an SHM of the same amplitude as that of the projection on x-axis, but differing by a phase of $\pi/2$.

In spite of this connection between circular motion and SHM, the force acting on a particle in linear simple harmonic motion is very different from the centripetal force needed to keep a particle in uniform circular motion.

Example 13.4 The figure given below depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated in the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.



Answer

(a) At $t = 0$, OP makes an angle of $45° = \pi/4$ rad with the (positive direction of) x-axis. After time t, it covers an angle $\frac{2\pi}{T}t$ in the anticlockwise sense, and makes an angle of $\frac{2\pi}{T}t + \frac{\pi}{4}$ with the x-axis.

The projection of OP on the x-axis at time t is given by,

$$x(t) = A \cos\left(\frac{2\pi}{T}t + \frac{\pi}{4}\right)$$

For $T = 4$ s,

$$x(t) = A \cos\left(\frac{2\pi}{4}t + \frac{\pi}{4}\right)$$

which is a SHM of amplitude A, period 4 s, and an initial phase* $= \frac{\pi}{4}$.

* *The natural unit of angle is radian, defined through the ratio of arc to radius. Angle is a dimensionless quantity. Therefore it is not always necessary to mention the unit 'radian' when we use π, its multiples or submultiples. The conversion between radian and degree is not similar to that between metre and centimetre or mile. If the argument of a trigonometric function is stated without units, it is understood that the unit is radian. On the other hand, if degree is to be used as the unit of angle, then it must be shown explicitly. For example, sin(15°) means sine of 15 degree, but sin(15) means sine of 15 radians. Hereafter, we will often drop 'rad' as the unit, and it should be understood that whenever angle is mentioned as a numerical value, without units, it is to be taken as radians.*

(b) In this case at $t = 0$, OP makes an angle of $90^o = \dfrac{\pi}{2}$ with the x-axis. After a time t, it covers an angle of $\dfrac{2\pi}{T}t$ in the clockwise sense and makes an angle of $\left(\dfrac{\pi}{2} - \dfrac{2\pi}{T}t\right)$ with the x-axis. The projection of OP on the x-axis at time t is given by

$$x(t) = B\cos\left(\frac{\pi}{2} - \frac{2\pi}{T}t\right)$$

$$= B\sin\left(\frac{2\pi}{T}t\right)$$

For $T = 30$ s,

$$x(t) = B\sin\left(\frac{\pi}{15}t\right)$$

Writing this as $x(t) = B\cos\left(\dfrac{\pi}{15}t - \dfrac{\pi}{2}\right)$, and comparing with Eq. (13.4). We find that this represents a SHM of amplitude B, period 30 s, and an initial phase of $-\dfrac{\pi}{2}$. ◀

13.5 VELOCITY AND ACCELERATION IN SIMPLE HARMONIC MOTION

The speed of a particle v in uniform circular motion is its angular speed ω times the radius of the circle A.

$$v = \omega A \tag{13.8}$$

The direction of velocity $\bar{\mathbf{v}}$ at a time t is along the tangent to the circle at the point where the particle is located at that instant. From the geometry of Fig. 13.11, it is clear that the velocity of the projection particle P′ at time t is

$$v(t) = -\omega A \sin(\omega t + \phi) \tag{13.9}$$

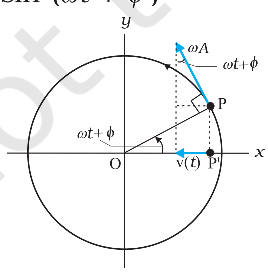


Fig. 13.11 *The velocity, v (t), of the particle P′ is the projection of the velocity $\bar{\mathbf{v}}$ of the reference particle, P.*

where the negative sign shows that $v(t)$ has a direction opposite to the positive direction of x-axis. Eq. (13.9) gives the instantaneous velocity of a particle executing SHM, where displacement is given by Eq. (13.4). We can, of course, obtain this equation without using geometrical argument, directly by differentiating (Eq. 13.4) with respect of t:

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) \tag{13.10}$$

The method of reference circle can be similarly used for obtaining instantaneous acceleration of a particle undergoing SHM. We know that the centripetal acceleration of a particle P in uniform circular motion has a magnitude v^2/A or $\omega^2 A$, and it is directed towards the centre i.e., the direction is along PO. The instantaneous acceleration of the projection particle P′ is then (See Fig. 13.12)

$$a(t) = -\omega^2 A\cos(\omega t + \phi)$$

$$= -\omega^2 x(t) \tag{13.11}$$

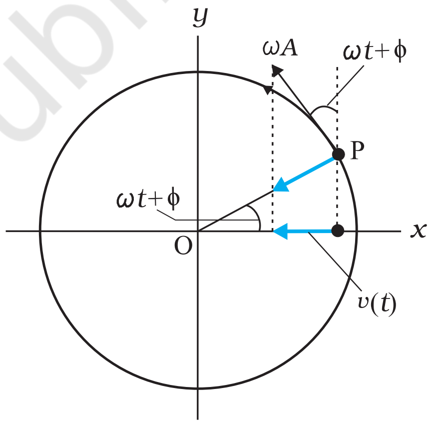


Fig. 13.12 *The acceleration, a(t), of the particle P′ is the projection of the acceleration **a** of the reference particle P.*

Eq. (13.11) gives the acceleration of a particle in SHM. The same equation can again be obtained directly by differentiating velocity $v(t)$ given by Eq. (13.9) with respect to time:

$$a(t) = \frac{\mathrm{d}}{\mathrm{d}t}v(t) \tag{13.12}$$

We note from Eq. (13.11) the important property that acceleration of a particle in SHM is proportional to displacement. For $x(t) > 0$, $a(t) < 0$ and for $x(t) < 0$, $a(t) > 0$. Thus, whatever the value of x between $-A$ and A, the acceleration $a(t)$ is always directed towards the centre.

For simplicity, let us put $\phi = 0$ and write the expression for $x(t)$, $v(t)$ and $a(t)$

$x(t) = A \cos \omega t$, $v(t) = -\omega A \sin \omega t$, $a(t) = -\omega^2 A \cos \omega t$

The corresponding plots are shown in Fig. 13.13. All quantities vary sinusoidally with time; only their maxima differ and the different plots differ in phase. x varies between $-A$ to A; $v(t)$ varies from $-\omega A$ to ωA and $a(t)$ from $-\omega^2 A$ to $\omega^2 A$. With respect to displacement plot, velocity plot has a phase difference of $\pi/2$ and acceleration plot has a phase difference of π.

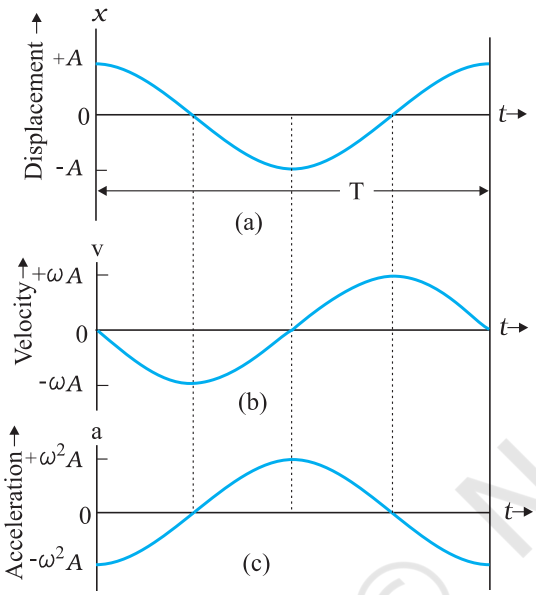


Fig. 13.13 *Displacement, velocity and acceleration of a particle in simple harmonic motion have the same period T, but they differ in phase*

> ***Example 13.5*** A body oscillates with SHM according to the equation (in SI units),
>
> $x = 5 \cos [2\pi\, t + \pi/4]$.
>
> At $t = 1.5$ s, calculate the (a) displacement, (b) speed and (c) acceleration of the body.

Answer The angular frequency ω of the body $= 2\pi\ \mathrm{s^{-1}}$ and its time period $T = 1$ s.

At $t = 1.5$ s

(a) displacement $= (5.0\ \mathrm{m}) \cos [(2\pi\ \mathrm{s^{-1}}) \times 1.5\ \mathrm{s} + \pi/4]$

$= (5.0\ \mathrm{m}) \cos [(3\pi + \pi/4)]$

$= -5.0 \times 0.707\ \mathrm{m}$

$= -3.535\ \mathrm{m}$

(b) Using Eq. (13.9), the speed of the body

$= -(5.0\ \mathrm{m})(2\pi\ \mathrm{s^{-1}}) \sin [(2\pi\ \mathrm{s^{-1}}) \times 1.5\ \mathrm{s} + \pi/4]$

$= -(5.0\ \mathrm{m})(2\pi\ \mathrm{s^{-1}}) \sin [(3\pi + \pi/4)]$

$= 10\pi \times 0.707\ \mathrm{m\ s^{-1}}$

$= 22\ \mathrm{m\ s^{-1}}$

(c) Using Eq.(13.10), the acceleration of the body

$= -(2\pi\ \mathrm{s^{-1}})^2 \times$ displacement

$= -(2\pi\ \mathrm{s^{-1}})^2 \times (-3.535\ \mathrm{m})$

$= 140\ \mathrm{m\ s^{-2}}$ ◀

13.6 FORCE LAW FOR SIMPLE HARMONIC MOTION

Using Newton's second law of motion, and the expression for acceleration of a particle undergoing SHM (Eq. 13.11), the force acting on a particle of mass m in SHM is

$$F(t) = ma$$

$$= -m\omega^2 x(t)$$

i.e., $$F(t) = -k\, x(t) \qquad (13.13)$$

where $$k = m\omega^2 \qquad (13.14a)$$

or $$\omega = \sqrt{\frac{k}{m}} \qquad (13.14b)$$

Like acceleration, force is always directed towards the mean position—hence it is sometimes called the restoring force in SHM. To summarise the discussion so far, simple harmonic motion can be defined in two equivalent ways, either by Eq. (13.4) for displacement or by Eq. (13.13) that gives its force law. Going from Eq. (13.4) to Eq. (13.13) required us to differentiate two times. Likewise, by integrating the force law Eq. (13.13) two times, we can get back Eq. (13.4).

Note that the force in Eq. (13.13) is linearly proportional to $x(t)$. A particle oscillating under such a force is, therefore, calling a linear harmonic oscillator. In the real world, the force may contain small additional terms proportional to x^2, x^3, etc. These then are called non-linear oscillators.

> ***Example 13.6*** Two identical springs of spring constant k are attached to a block of mass m and to fixed supports as shown in Fig. 13.14. Show that when the mass is displaced from its equilibrium position on either side, it executes a simple harmonic motion. Find the period of oscillations.

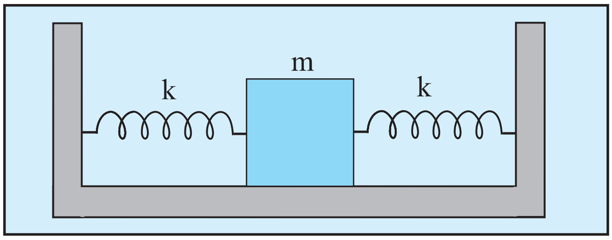


Fig. 13.14

Answer Let the mass be displaced by a small distance x to the right side of the equilibrium position, as shown in Fig. 13.15. Under this situation the spring on the left side gets

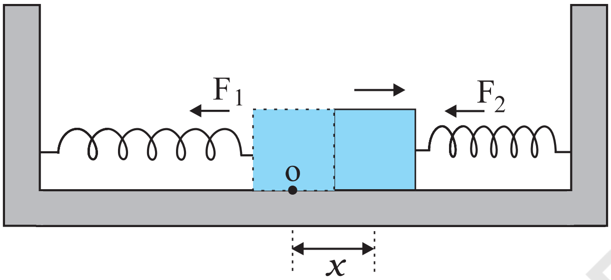


Fig. 13.15

elongated by a length equal to x and that on the right side gets compressed by the same length. The forces acting on the mass are then,

$F_1 = -k\,x$ (force exerted by the spring on the left side, trying to pull the mass towards the mean position)

$F_2 = -k\,x$ (force exerted by the spring on the right side, trying to push the mass towards the mean position)

The net force, F, acting on the mass is then given by,

$$F = -2kx$$

Hence the force acting on the mass is proportional to the displacement and is directed towards the mean position; therefore, the motion executed by the mass is simple harmonic. The time period of oscillations is,

$$T = 2\pi\sqrt{\frac{m}{2k}}$$

13.7 ENERGY IN SIMPLE HARMONIC MOTION

Both kinetic and potential energies of a particle in SHM vary between zero and their maximum values.

In section 13.5 we have seen that the velocity of a particle executing SHM, is a periodic function of time. It is zero at the extreme positions of displacement. Therefore, the kinetic energy (K) of such a particle, which is defined as

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\,\omega^2 A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}k\,A^2 \sin^2(\omega t + \phi) \qquad (13.15)$$

is also a periodic function of time, being zero when the displacement is maximum and maximum when the particle is at the mean position. Note, since the sign of v is immaterial in K, the period of K is $T/2$.

What is the potential energy (U) of a particle executing simple harmonic motion? In Chapter 6, we have seen that the concept of potential energy is possible only for conservative forces. The spring force $F = -kx$ is a conservative force, with associated potential energy

$$U = \frac{1}{2}k\,x^2 \qquad (13.16)$$

Hence the potential energy of a particle executing simple harmonic motion is,

$$U(x) = \frac{1}{2}k\,x^2$$

$$= \frac{1}{2}k\,A^2 \cos^2(\omega t + \phi) \qquad (13.17)$$

Thus, the potential energy of a particle executing simple harmonic motion is also periodic, with period $T/2$, being zero at the mean position and maximum at the extreme displacements.

It follows from Eqs. (13.15) and (13.17) that the total energy, E, of the system is,

$$E = U + K$$

$$= \frac{1}{2} k\,A^2 \cos^2(\omega t + \phi) + \frac{1}{2} k\,A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} k\,A^2 \left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right]$$

Using the familiar trigonometric identity, the value of the expression in the brackets is unity. Thus,

$$E = \frac{1}{2} k\,A^2 \qquad (13.18)$$

The total mechanical energy of a harmonic oscillator is thus independent of time as expected for motion under any conservative force. The time and displacement dependence of the potential and kinetic energies of a linear simple harmonic oscillator are shown in Fig. 13.16.

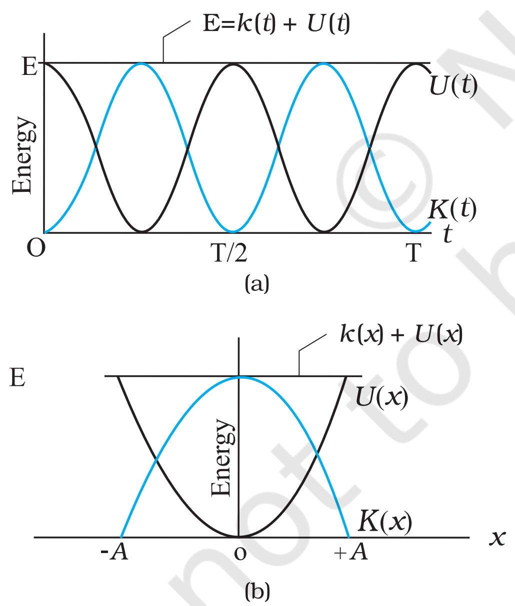


Fig. 13.16 *Kinetic energy, potential energy and total energy as a function of time [shown in (a)] and displacement [shown in (b)] of a particle in SHM. The kinetic energy and potential energy both repeat after a period T/2. The total energy remains constant at all t or x.*

Observe that both kinetic energy and potential energy in SHM are seen to be always positive in Fig. 13.16. Kinetic energy can, of course, be never negative, since it is proportional to the square of speed. Potential energy is positive by choice of the undermined constant in potential energy. Both kinetic energy and potential energy peak twice during each period of SHM. For $x = 0$, the energy is kinetic; at the extremes $x = \pm A$, it is all potential energy. In the course of motion between these limits, kinetic energy increases at the expense of potential energy or vice-versa.

Example 13.7 A block whose mass is 1 kg is fastened to a spring. The spring has a spring constant of 50 N m^{-1}. The block is pulled to a distance x = 10 cm from its equilibrium position at $x = 0$ on a frictionless surface from rest at t = 0. Calculate the kinetic, potential and total energies of the block when it is 5 cm away from the mean position.

Answer The block executes SHM, its angular frequency, as given by Eq. (13.14b), is

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{50 \text{ N m}^{-1}}{1 \text{kg}}}$$

$$= 7.07 \text{ rad s}^{-1}$$

Its displacement at any time t is then given by,

$$x(t) = 0.1 \cos(7.07t)$$

Therefore, when the particle is 5 cm away from the mean position, we have

$$0.05 = 0.1 \cos(7.07t)$$

Or $\cos(7.07t) = 0.5$ and hence

$$\sin(7.07t) = \frac{\sqrt{3}}{2} = 0.866$$

Then, the velocity of the block at $x = 5$ cm is

$$= 0.1 \times 7.07 \quad 0.866 \text{ m s}^{-1}$$

$$= 0.61 \text{ m s}^{-1}$$

Hence the K.E. of the block,

$$= \frac{1}{2} m v^2$$

$$= ½[1\text{kg} \times (0.6123 \text{ m s}^{-1})^2]$$

$$= 0.19 \text{ J}$$

The P.E. of the block,

$$= \frac{1}{2} k x^2$$

$$= ½(50 \text{ N m}^{-1} \times 0.05 \text{ m} \times 0.05 \text{ m})$$

$$= 0.0625 \text{ J}$$

The total energy of the block at $x = 5$ cm,

$$= \text{K.E.} + \text{P.E.}$$

$$= 0.25 \text{ J}$$

we also know that at maximum displacement, K.E. is zero and hence the total energy of the system is equal to the P.E. Therefore, the total energy of the system,

$$= ½(50 \text{ N m}^{-1} \times 0.1 \text{ m} \times 0.1 \text{ m})$$

$$= 0.25 \text{ J}$$

which is same as the sum of the two energies at a displacement of 5 cm. This is in conformity with the principle of conservation of energy. ◀

13.8 The Simple Pendulum

It is said that Galileo measured the periods of a swinging chandelier in a church by his pulse beats. He observed that the motion of the chandelier was periodic. The system is a kind of pendulum. You can also make your own pendulum by tying a piece of stone to a long unstretchable thread, approximately 100 cm long. Suspend your pendulum from a suitable support so that it is free to oscillate. Displace the stone to one side by a small distance and let it go. The stone executes a to and fro motion, it is periodic with a period of about two seconds.

We shall show that this periodic motion is simple harmonic for small displacements from

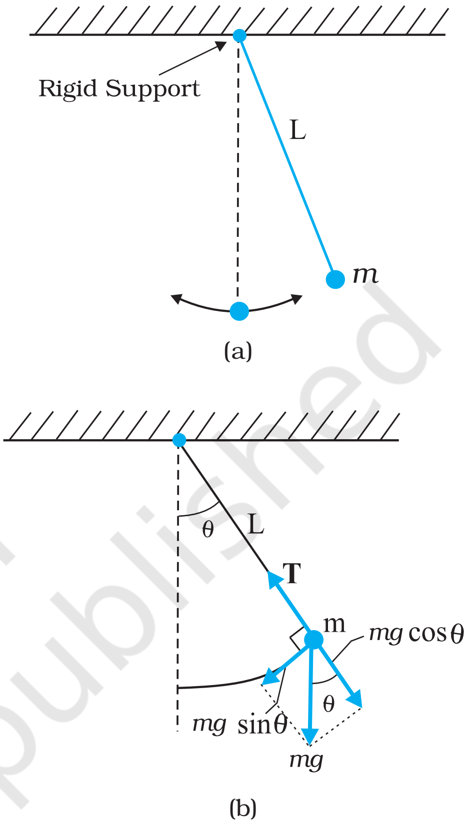


Fig. 13.17 *(a) A bob oscillating about its mean position. (b) The radial force T-mg cosθ provides centripetal force but no torque about the support. The tangential force mg sinθ provides the restoring torque.*

the mean position. Consider simple pendulum — a small bob of mass m tied to an inextensible massless string of length L. The other end of the string is fixed to a rigid support. The bob oscillates in a plane about the vertical line through the support. Fig. 13.17(a) shows this system. Fig. 13.17(b) is a kind of 'free-body' diagram of the simple pendulum showing the forces acting on the bob.

Let θ be the angle made by the string with the vertical. When the bob is at the mean position, $\theta = 0$

There are only two forces acting on the bob; the tension T along the string and the vertical force due to gravity (=mg). The force mg can be resolved into the component $mg\cos\theta$ along the string and $mg\sin\theta$ perpendicular to it. Since the motion of the bob is along a circle of length L and centre at the support point, the bob has a radial acceleration ($\omega^2 L$) and also a tangental acceleration; the latter arises since motion along the arc of the circle is not uniform. The radial acceleration is provided by the net radial force $T - mg\cos\theta$, while the tangential acceleration is provided by $mg\sin\theta$. It is more convenient to work with torque about the support since the radial force gives zero torque. Torque τ about the support is entirely provided by the tangental component of force

$$\tau = -L(mg\sin\theta) \qquad (13.19)$$

This is the restoring torque that tends to reduce angular displacement — hence the negative sign. By Newton's law of rotational motion,

$$\tau = I\alpha \qquad (13.20)$$

where I is the moment of inertia of the system about the support and α is the angular acceleration. Thus,

$$I\alpha = -mg\sin\theta \;\; L \qquad (13.21)$$

Or,

$$\alpha = -\frac{mgL}{I}\sin\theta \qquad (13.22)$$

We can simplify Eq. (13.22) if we assume that the displacement θ is small. We know that $\sin\theta$ can be expressed as,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \pm \ldots \qquad (13.23)$$

where θ is in radians.

Now if θ is small, $\sin\theta$ can be approximated by θ and Eq. (13.22) can then be written as,

$$\alpha = -\frac{mgL}{I}\theta \qquad (13.24)$$

In Table 13.1, we have listed the angle θ in degrees, its equivalent in radians, and the value of the function $\sin\theta$. From this table it can be seen that for θ as large as 20 degrees, $\sin\theta$ is nearly the same as θ **expressed in radians**.

Table 13.1 sin θ as ma function of angle θ

θ (degrees)	θ (radians)	$\sin\theta$
0	0	0
5	0.087	0.087
10	0.174	0.174
15	0.262	0.259
20	0.349	0.342

Equation (13.24) is mathematically, identical to Eq. (13.11) except that the variable is angular displacement. Hence we have proved that for small q, the motion of the bob is simple harmonic. From Eqs. (13.24) and (13.11),

$$\omega = \sqrt{\frac{mgL}{I}}$$

and

$$T = 2\pi\sqrt{\frac{I}{mgL}} \qquad (13.25)$$

Now since the string of the simple pendulum is massless, the moment of inertia I is simply mL^2. Eq. (13.25) then gives the well-known formula for time period of a simple pendulum.

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad (13.26)$$

Example 13.8 What is the length of a simple pendulum, which ticks seconds ?

Answer From Eq. (13.26), the time period of a simple pendulum is given by,

$$T = 2\pi\sqrt{\frac{L}{g}}$$

From this relation one gets,

$$L = \frac{gT^2}{4\pi^2}$$

The time period of a simple pendulum, which ticks seconds, is 2 s. Therefore, for $g = 9.8\ \mathrm{m\ s^{-2}}$ and $T = 2$ s, L is

$$= \frac{9.8(\mathrm{m\ s^{-2}}) \times 4(\mathrm{s^2})}{4\pi^2}$$

$$= 1\ \mathrm{m}$$

SUMMARY

1. The motion that repeats itself is called *periodlic motion.*
2. The *period T* is the time reequired for one complete oscillation, or cycle. It is related to the frequency v by,

$$T = \frac{1}{v}$$

The *frequency v* of periodic or oscillatory motion is the number of oscillations per unit time. In the SI, it is measured in hertz :

1 hertz = 1 Hz = 1 oscillation per second = $1 s^{-1}$

3. In *simple harmonic motion* (SHM), the displacement $x(t)$ of a particle from its equilibrium position is given by,

$$x(t) = A \cos(\omega t + \phi) \quad \text{(displacement)},$$

in which A is the *amplitude* of the displacement, the quantity $(\omega t + \phi)$ is the phase of the motion, and ϕ is the *phase constant*. The *angular frequency* ω is related to the period and frequency of the motion by,

$$\omega = \frac{2\pi}{T} = 2\pi v \quad \text{(angular frequency)}.$$

4. Simple harmonic motion can also be viewed as the projection of uniform circular motion on the diameter of the circle in which the latter motion occurs.
5. The particle velocity and acceleration during SHM as functions of time are given by,

$$v(t) = -\omega A \sin(\omega t + \phi) \quad \text{(velocity)},$$

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

$$= -\omega^2 x(t) \quad \text{(acceleration)},$$

Thus we see that both velocity and acceleration of a body executing simple harmonic motion are periodic functions, having the velocity *amplitude* $v_m = \omega A$ and *acceleration amplitude* $a_m = \omega^2 A$, respectively.

6. The force acting in a simple harmonic motion is proportional to the displacement and is always directed towards the centre of motion.
7. A particle executing simple harmonic motion has, at any time, kinetic energy $K = ½\, mv^2$ and potential energy $U = ½\, kx^2$. If no friction is present the mechanical energy of the system, $E = K + U$ always remains constant even though K and U change with time.
8. A particle of mass m oscillating under the influence of Hooke's law restoring force given by $F = -k\,x$ exhibits simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency)}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period)}$$

Such a system is also called a linear oscillator.

9. The motion of a simple pendulum swinging through small angles is approximately simple harmonic. The period of oscillation is given by,

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Physical quantity	Symbol	Dimensions	Unit	Remarks
Period	T	[T]	s	The least time for motion to repeat itself
Frequency	ν (or f)	[T^{-1}]	s^{-1}	$\nu = \frac{1}{T}$
Angular frequency	ω	[T^{-1}]	s^{-1}	$\omega = 2\,\pi\,\nu$
Phase constant	ϕ	Dimensionless	rad	Initial value of phase of displacement in SHM
Force constant	k	[MT^{-2}]	N m^{-1}	Simple harmonic motion $F = -k\,x$

POINTS TO PONDER

1. The period T is the *least time* after which motion repeats itself. Thus, motion repeats itself after nT where n is an integer.
2. Every periodic motion is not simple harmonic motion. Only that periodic motion governed by the force law $F = -k\,x$ is simple harmonic.
3. Circular motion can arise due to an inverse-square law force (as in planetary motion) as well as due to simple harmonic force in two dimensions equal to: $-m\omega^2 r$. In the latter case, the phases of motion, in two perpendicular directions (x and y) must differ by $\pi/2$. Thus, for example, a particle subject to a force $-m\omega^2 r$ with initial position (0, A) and velocity (ωA, 0) will move uniformly in a circle of radius A.
4. For linear simple harmonic motion with a given ω, two initial conditions are necessary and sufficient to determine the motion completely. The initial conditions may be (i) initial position and initial velocity or (ii) amplitude and phase or (iii) energy and phase.
5. From point 4 above, given amplitude or energy, phase of motion is determined by the initial position or initial velocity.
6. A combination of two simple harmonic motions with arbitrary amplitudes and phases is not necessarily periodic. It is periodic only if frequency of one motion is an integral multiple of the other's frequency. However, a periodic motion can always be expressed as a sum of infinite number of harmonic motions with appropriate amplitudes.
7. The period of SHM does not depend on amplitude or energy or the phase constant. Contrast this with the periods of planetary orbits under gravitation (Kepler's third law).
8. The motion of a simple pendulum is simple harmonic for small angular displacement.
9. For motion of a particle to be simple harmonic, its displacement x must be expressible in either of the following forms :

$$x = A\cos\omega t + B\sin\omega t$$

$$x = A\cos(\omega t + \alpha),\quad x = B\sin(\omega t + \beta)$$

The three forms are completely equivalent (any one can be expressed in terms of any other two forms).

Thus, damped simple harmonic motion is not strictly simple harmonic. It is approximately so only for time intervals much less than $2m/b$ where b is the damping constant.

Exercises

13.1 Which of the following examples represent periodic motion?
- (a) A swimmer completing one (return) trip from one bank of a river to the other and back.
- (b) A freely suspended bar magnet displaced from its N-S direction and released.
- (c) A hydrogen molecule rotating about its centre of mass.
- (d) An arrow released from a bow.

13.2 Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?
- (a) the rotation of earth about its axis.
- (b) motion of an oscillating mercury column in a U-tube.
- (c) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.
- (d) general vibrations of a polyatomic molecule about its equilibrium position.

13.3 Fig. 13.18 depicts four *x-t* plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion) ?

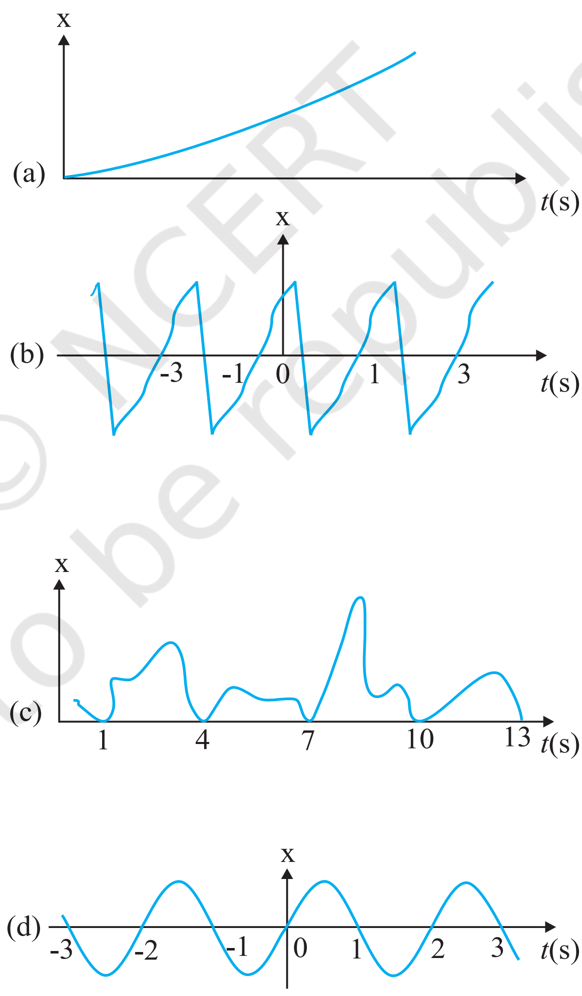


Fig. 18.18

13.4 Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant):

(a) $\sin \omega t - \cos \omega t$
(b) $\sin^3 \omega t$
(c) $3 \cos (\pi/4 - 2\omega t)$
(d) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$
(e) $\exp (-\omega^2 t^2)$
(f) $1 + \omega t + \omega^2 t^2$

13.5 A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is

(a) at the end A,
(b) at the end B,
(c) at the mid-point of AB going towards A,
(d) at 2 cm away from B going towards A,
(e) at 3 cm away from A going towards B, and
(f) at 4 cm away from B going towards A.

13.6 Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?

(a) $a = 0.7x$
(b) $a = -200x^2$
(c) $a = -10x$
(d) $a = 100x^3$

13.7 The motion of a particle executing simple harmonic motion is described by the displacement function,

$$x(t) = A \cos (\omega t + \phi).$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle ? The angular frequency of the particle is π s^{-1}. If instead of the cosine function, we choose the sine function to describe the SHM : $x = B \sin (\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.

13.8 A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body ?

13.9 A spring having with a spring constant 1200 N m^{-1} is mounted on a horizontal table as shown in Fig. 13.19. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

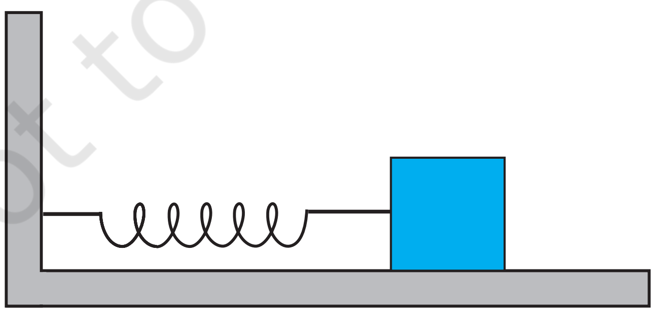


Fig. 13.19

Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.

13.10 In Exercise 13.9, let us take the position of mass when the spring is unstreched as $x = 0$, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch ($t = 0$), the mass is

(a) at the mean position,

(b) at the maximum stretched position, and

(c) at the maximum compressed position.

In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

13.11 Figures 13.20 correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e. clockwise or anti-clockwise) are indicated on each figure.

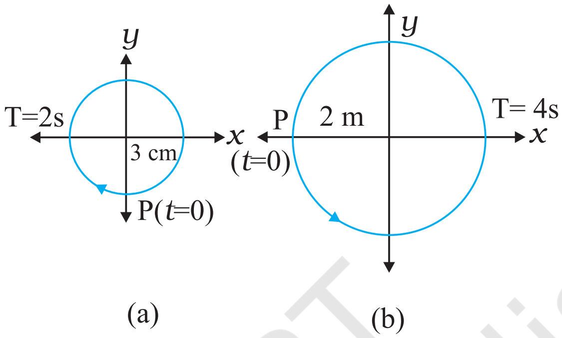


Fig. 13.20

Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.

13.12 Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t =0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

(a) $x = -2 \sin (3t + \pi/3)$

(b) $x = \cos (\pi/6 - t)$

(c) $x = 3 \sin (2\pi t + \pi/4)$

(d) $x = 2 \cos \pi t$

13.13 Figure 13.21(a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force **F** applied at the free end stretches the spring. Figure 13.21 (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. 13.21(b) is stretched by the same force **F**.

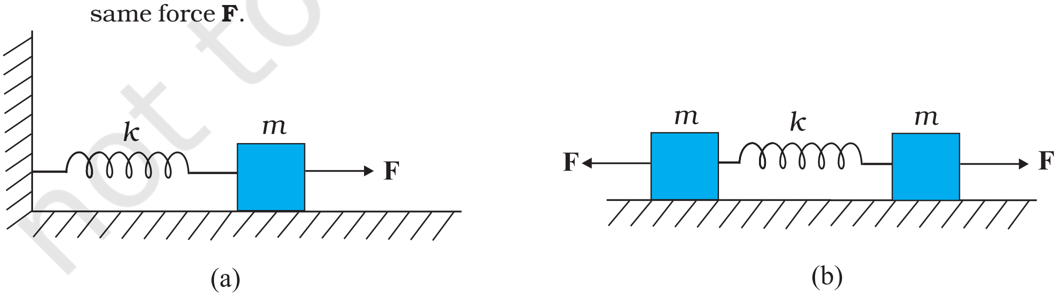


Fig. 13.21

(a) What is the maximum extension of the spring in the two cases ?

(b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case ?

13.14 The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed ?

13.15 The acceleration due to gravity on the surface of moon is 1.7 m s^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s ? (g on the surface of earth is 9.8 m s^{-2})

13.16 A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period ?

13.17 A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_l. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}$$

where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).

13.18 One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

Chapter Fourteen

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Waves

14.1 INTRODUCTION

In the previous Chapter, we studied the motion of objects oscillating in isolation. What happens in a system, which is a collection of such objects? A material medium provides such an example. Here, elastic forces bind the constituents to each other and, therefore, the motion of one affects that of the other. If you drop a little pebble in a pond of still water, the water surface gets disturbed. The disturbance does not remain confined to one place, but propagates outward along a circle. If you continue dropping pebbles in the pond, you see circles rapidly moving outward from the point where the water surface is disturbed. It gives a feeling as if the water is moving outward from the point of disturbance. If you put some cork pieces on the disturbed surface, it is seen that the cork pieces move up and down but do not move away from the centre of disturbance. This shows that the water mass does not flow outward with the circles, but rather a moving disturbance is created. Similarly, when we speak, the sound moves outward from us, without any flow of air from one part of the medium to another. The disturbances produced in air are much less obvious and only our ears or a microphone can detect them. These patterns, which move without the actual physical transfer or flow of matter as a whole, are called **waves**. In this Chapter, we will study such waves.

Waves transport energy and the pattern of disturbance has information that propagate from one point to another. All our communications essentially depend on transmission of signals through waves. Speech means production of sound waves in air and hearing amounts to their detection. Often, communication involves different kinds of waves. For example, sound waves may be first converted into an electric current signal which in turn may generate an electromagnetic wave that may be transmitted by an optical cable or via a satellite. Detection of the original signal will usually involve these steps in reverse order.

Not all waves require a medium for their propagation. We know that light waves can travel through vacuum. The light emitted by stars, which are hundreds of light years away, reaches us through inter-stellar space, which is practically a vacuum.

The most familiar type of waves such as waves on a string, water waves, sound waves, seismic waves, etc. is the so-called mechanical waves. These waves require a medium for propagation, they cannot propagate through vacuum. They involve oscillations of constituent particles and depend on the elastic properties of the medium. The electromagnetic waves that you will learn in Class XII are a different type of wave. Electromagnetic waves do not necessarily require a medium - they can travel through vacuum. Light, radiowaves, X-rays, are all electromagnetic waves. In vacuum, all electromagnetic waves have the same speed c, whose value is :

$$c = 299,792,458 \text{ ms}^{-1}. \tag{14.1}$$

A third kind of wave is the so-called Matter waves. They are associated with constituents of matter : electrons, protons, neutrons, atoms and molecules. They arise in quantum mechanical description of nature that you will learn in your later studies. Though conceptually more abstract than mechanical or electro-magnetic waves, they have already found applications in several devices basic to modern technology; matter waves associated with electrons are employed in electron microscopes.

In this chapter we will study mechanical waves, which require a material medium for their propagation.

The aesthetic influence of waves on art and literature is seen from very early times; yet the first scientific analysis of wave motion dates back to the seventeenth century. Some of the famous scientists associated with the physics of wave motion are Christiaan Huygens (1629-1695), Robert Hooke and Isaac Newton. The understanding of physics of waves followed the physics of oscillations of masses tied to springs and physics of the simple pendulum. Waves in elastic media are intimately connected with harmonic oscillations. (Stretched strings, coiled springs, air, etc., are examples of elastic media). We shall illustrate this connection through simple examples.

Consider a collection of springs connected to one another as shown in Fig. 14.1. If the spring at one end is pulled suddenly and released, the disturbance travels to the other end. What has happened? The first spring is disturbed from its equilibrium length. Since the second spring is connected to the first, it is also stretched or compressed, and so on. The disturbance moves from one end to the other; but each spring only executes small oscillations about its equilibrium position. As a practical example of this situation, consider a stationary train at a railway station. Different bogies of the train are coupled to each other through a spring coupling. When an engine is attached at one end, it gives a push to the bogie next to it; this push is transmitted from one bogie to another without the entire train being bodily displaced.

Fig. 14.1 *A collection of springs connected to each other. The end A is pulled suddenly generating a disturbance, which then propagates to the other end.*

Now let us consider the propagation of sound waves in air. As the wave passes through air, it compresses or expands a small region of air. This causes a change in the density of that region, say $\delta\rho$, this change induces a change in pressure, δp, in that region. Pressure is force per unit area, so there is a **restoring force proportional** to the disturbance, just like in a spring. In this case, the quantity similar to extension or compression of the spring is the change in density. If a region is compressed, the molecules in that region are packed together, and they tend to move out to the adjoining region, thereby increasing the density or creating compression in the adjoining region. Consequently, the air in the first region undergoes rarefaction. If a region is comparatively rarefied the surrounding air will rush in making the rarefaction move to the adjoining region. Thus, the compression or rarefaction moves from one region to another, making the propagation of a disturbance possible in air.

In solids, similar arguments can be made. In a crystalline solid, atoms or group of atoms are arranged in a periodic lattice. In these, each atom or group of atoms is in equilibrium, due to forces from the surrounding atoms. Displacing one atom, keeping the others fixed, leads to restoring forces, exactly as in a spring. So we can think of atoms in a lattice as end points, with springs between pairs of them.

In the subsequent sections of this chapter we are going to discuss various characteristic properties of waves.

14.2 TRANSVERSE AND LONGITUDINAL WAVES

We have seen that motion of mechanical waves involves oscillations of constituents of the medium. If the constituents of the medium oscillate perpendicular to the direction of wave propagation, we call the wave a transverse wave. If they oscillate along the direction of wave propagation, we call the wave a longitudinal wave.

Fig.14.2 shows the propagation of a single pulse along a string, resulting from a single up and down jerk. If the string is very long compared

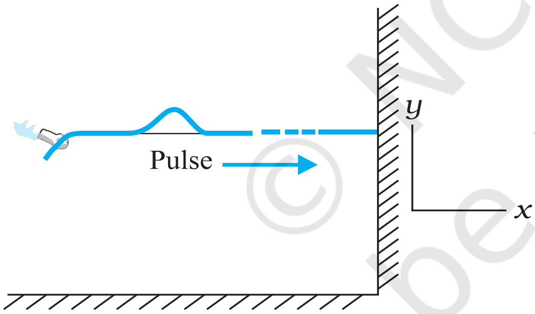


Fig. 14.2 When a pulse travels along the length of a stretched string (x-direction), the elements of the string oscillate up and down (y-direction)

to the size of the pulse, the pulse will damp out before it reaches the other end and reflection from that end may be ignored. Fig. 14.3 shows a similar situation, but this time the external agent gives a continuous periodic sinusoidal up and down jerk to one end of the string. The resulting disturbance on the string is then a sinusoidal wave. In either case the elements of the string oscillate about their equilibrium mean

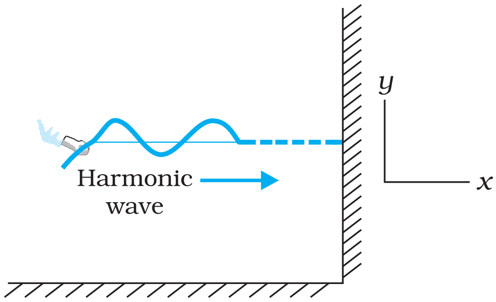


Fig. 14.3 A harmonic (sinusoidal) wave travelling along a stretched string is an example of a transverse wave. An element of the string in the region of the wave oscillates about its equilibrium position perpendicular to the direction of wave propagation.

position as the pulse or wave passes through them. The oscillations are normal to the direction of wave motion along the string, so this is an example of transverse wave.

We can look at a wave in two ways. We can fix an instant of time and picture the wave in space. This will give us the shape of the wave as a whole in space at a given instant. Another way is to fix a location i.e. fix our attention on a particular element of string and see its oscillatory motion in time.

Fig. 14.4 describes the situation for longitudinal waves in the most familiar example of the propagation of sound waves. A long pipe filled with air has a piston at one end. A single sudden push forward and pull back of the piston will generate a pulse of condensations (higher density) and rarefactions (lower density) in the medium (air). If the push-pull of the piston is continuous and periodic (sinusoidal), a

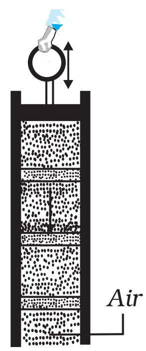


Fig. 14.4 Longitudinal waves (sound) generated in a pipe filled with air by moving the piston up and down. A volume element of air oscillates in the direction parallel to the direction of wave propagation.

sinusoidal wave will be generated propagating in air along the length of the pipe. This is clearly an example of longitudinal waves.

The waves considered above, transverse or longitudinal, are travelling or progressive waves since they travel from one part of the medium to another. The material medium as a whole does not move, as already noted. A stream, for example, constitutes motion of water as a whole. In a water wave, it is the disturbance that moves, not water as a whole. Likewise a wind (motion of air as a whole) should not be confused with a sound wave which is a propagation of disturbance (in pressure density) in air, without the motion of air medium as a whole.

In transverse waves, the particle motion is normal to the direction of propagation of the wave. Therefore, as the wave propagates, each element of the medium undergoes a shearing strain. Transverse waves can, therefore, be propagated only in those media, which can sustain shearing stress, such as solids and not in fluids. Fluids, as well as, solids can sustain compressive strain; therefore, longitudinal waves can be propagated in all elastic media. For example, in medium like steel, both transverse and longitudinal waves can propagate, while air can sustain only longitudinal waves. The waves on the surface of water are of two kinds: **capillary waves** and **gravity waves**. The former are ripples of fairly short wavelength—not more than a few centimetre—and the restoring force that produces them is the surface tension of water. Gravity waves have wavelengths typically ranging from several metres to several hundred meters. The restoring force that produces these waves is the pull of gravity, which tends to keep the water surface at its lowest level. The oscillations of the particles in these waves are not confined to the surface only, but extend with diminishing amplitude to the very bottom. The particle motion in water waves involves a complicated motion—they not only move up and down but also back and forth. The waves in an ocean are the combination of both longitudinal and transverse waves.

It is found that, generally, transverse and longitudinal waves travel with different speed in the same medium.

Example 14.1 Given below are some examples of wave motion. State in each case if the wave motion is transverse, longitudinal or a combination of both:

(a) Motion of a kink in a longitudinal spring produced by displacing one end of the spring sideways.
(b) Waves produced in a cylinder containing a liquid by moving its piston back and forth.
(c) Waves produced by a motorboat sailing in water.
(d) Ultrasonic waves in air produced by a vibrating quartz crystal.

Answer

(a) Transverse and longitudinal
(b) Longitudinal
(c) Transverse and longitudinal
(d) Longitudinal

14.3 DISPLACEMENT RELATION IN A PROGRESSIVE WAVE

For mathematical description of a travelling wave, we need a function of both position x and time t. Such a function at every instant should give the shape of the wave at that instant. Also, at every given location, it should describe the motion of the constituent of the medium at that location. If we wish to describe a sinusoidal travelling wave (such as the one shown in Fig. 14.3) the corresponding function must also be sinusoidal. For convenience, we shall take the wave to be transverse so that if the position of the constituents of the medium is denoted by x, the displacement from the equilibrium position may be denoted by y. A sinusoidal travelling wave is then described by:

$$y(x,t) = a\sin(kx - \omega t + \phi) \qquad (14.2)$$

The term ϕ in the argument of sine function means equivalently that we are considering a linear combination of sine and cosine functions:

$$y(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t) \qquad (14.3)$$

From Equations (14.2) and (14.3),

$$a = \sqrt{A^2 + B^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

To understand why Equation (14.2) represents a sinusoidal travelling wave, take a fixed instant, say $t = t_0$. Then, the argument of the sine function in Equation (14.2) is simply

kx + constant. Thus, the shape of the wave (at any fixed instant) as a function of x is a sine wave. Similarly, take a fixed location, say $x = x_0$. Then, the argument of the sine function in Equation (14.2) is constant $-\omega t$. The displacement y, at a fixed location, thus, varies sinusoidally with time. That is, the constituents of the medium at different positions execute simple harmonic motion. Finally, as t increases, x must increase in the positive direction to keep $kx - \omega t + \phi$ constant. Thus, Eq. (14.2) represents a sinusiodal (harmonic) wave travelling along the positive direction of the x-axis. On the other hand, a function

$$y(x,t) = a\sin(kx + \omega t + \phi) \qquad (14.4)$$

represents a wave travelling in the negative direction of x-axis. Fig. (14.5) gives the names of the various physical quantities appearing in Eq. (14.2) that we now interpret.

$y(x,t)$	:	displacement as a function of position x and time t
a	:	amplitude of a wave
ω	:	angular frequency of the wave
k	:	angular wave number
$kx-\omega t+\phi$	:	initial phase angle (a+x = 0, t = 0)

***Fig. 14.5** The meaning of standard symbols in Eq. (14.2)*

Fig. 14.6 shows the plots of Eq. (14.2) for different values of time differing by equal intervals of time. In a wave, the crest is the point of maximum positive displacement, the trough is the point of maximum negative displacement. To see how a wave travels, we can fix attention on a crest and see how it progresses with time. In the figure, this is shown by a cross ( ) on the crest. In the same manner, we can see the motion of a particular constituent of the medium at a fixed location, say at the origin of the x-axis. This is shown by a solid dot (•). The plots of Fig. 14.6 show that with time, the solid dot (•) at the origin moves periodically, i.e., the particle at the origin oscillates about its mean position as the wave progresses. This is true for any other location also. We also see that during the time the solid dot (•) has completed one full oscillation, the crest has moved further by a certain distance.

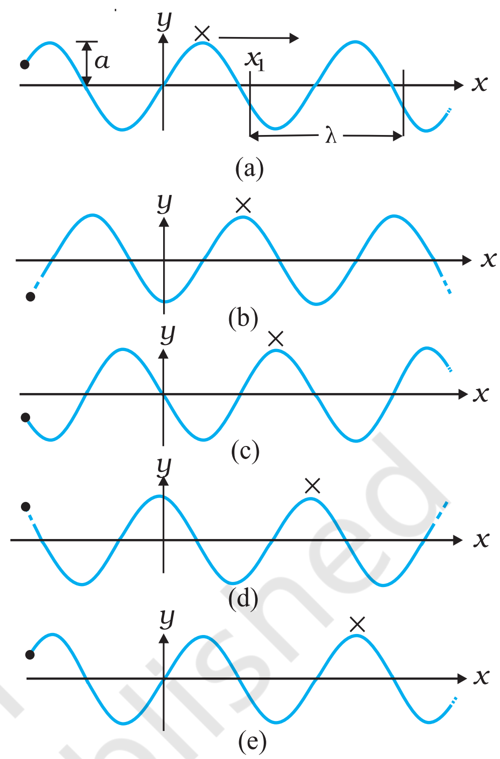


***Fig. 14.6** A harmonic wave progressing along the positive direction of x-axis at different times.*

Using the plots of Fig. 14.6, we now define the various quantities of Eq. (14.2).

14.3.1 Amplitude and Phase

In Eq. (14.2), since the sine function varies between 1 and –1, the displacement $y(x,t)$ varies between a and $-a$. We can take a to be a positive constant, without any loss of generality. Then, a represents the maximum displacement of the constituents of the medium from their equilibrium position. Note that the displacement y may be positive or negative, but a is positive. It is called the **amplitude** of the wave.

The quantity ($kx - \omega t + \phi$) appearing as the argument of the sine function in Eq. (14.2) is called the phase of the wave. Given the amplitude a, the phase determines the displacement of the wave at any position and at any instant. Clearly ϕ is the phase at x = 0 and t = 0. Hence, ϕ is called the initial phase angle. By suitable choice of origin on the x-axis and the intial time, it is possible to have ϕ = 0. Thus there is no loss of generality in dropping ϕ, i.e., in taking Eq. (14.2) with ϕ = 0.

14.3.2 Wavelength and Angular Wave Number

The minimum distance between two points having the same phase is called the wavelength of the wave, usually denoted by λ. For simplicity, we can choose points of the same phase to be crests or troughs. The wavelength is then the distance between two consecutive crests or troughs in a wave. Taking $\phi = 0$ in Eq. (14.2), the displacement at $t = 0$ is given by

$$y(x,0) = a \sin kx \tag{14.5}$$

Since the sine function repeats its value after every 2π change in angle,

$$\sin kx = \sin(kx + 2n\pi) = \sin k\left(x + \frac{2n\pi}{k}\right)$$

That is the displacements at points x and at

$$x + \frac{2n\pi}{k}$$

are the same, where n=1,2,3,... The least distance between points with the same displacement (at any given instant of time) is obtained by taking $n = 1$. λ is then given by

$$\lambda = \frac{2\pi}{k} \quad \text{or} \quad k = \frac{2\pi}{\lambda} \tag{14.6}$$

k is the angular wave number or propagation constant; its SI unit is radian per metre or

rad m^{-1} *

14.3.3 Period, Angular Frequency and Frequency

Fig. 14.7 shows again a sinusoidal plot. It describes not the shape of the wave at a certain instant but the displacement of an element (at any fixed location) of the medium as a function of time. We may for, simplicity, take Eq. (14.2) with $\phi = 0$ and monitor the motion of the element say at $x = 0$. We then get

$$y(0,t) = a \sin(-\omega t)$$
$$= -a \sin \omega t$$

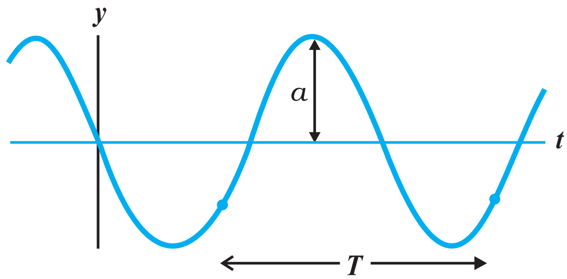


Fig. 14.7 *An element of a string at a fixed location oscillates in time with amplitude a and period T, as the wave passes over it.*

Now, the period of oscillation of the wave is the time it takes for an element to complete one full oscillation. That is

$$-a \sin \omega t = -a \sin \omega(t + T)$$
$$= -a \sin(\omega t + \omega T)$$

Since sine function repeats after every 2π,

$$\omega T = 2\pi \quad \text{or} \quad \omega = \frac{2\pi}{T} \tag{14.7}$$

ω is called the angular frequency of the wave. Its SI unit is rad s^{-1}. The frequency ν is the number of oscillations per second. Therefore,

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \tag{14.8}$$

ν is usually measured in hertz.

In the discussion above, reference has always been made to a wave travelling along a string or a transverse wave. In a longitudinal wave, the displacement of an element of the medium is parallel to the direction of propagation of the wave. In Eq. (14.2), the displacement function for a longitudinal wave is written as,

$$s(x, t) = a \sin(kx - \omega t + \phi) \tag{14.9}$$

where $s(x, t)$ is the displacement of an element of the medium in the direction of propagation of the wave at position x and time t. In Eq. (14.9), a is the displacement amplitude; other quantities have the same meaning as in case of a transverse wave except that the displacement function $y(x, t)$ is to be replaced by the function $s(x, t)$.

* *Here again, 'radian' could be dropped and the units could be written merely as* m^{-1}. *Thus, k represents* 2π *times the number of waves (or the total phase difference) that can be accommodated per unit length, with SI units* m^{-1}.

> **Example 14.2** A wave travelling along a string is described by,
>
> $y(x, t) = 0.005 \sin (80.0\, x - 3.0\, t)$,
>
> in which the numerical constants are in SI units (0.005 m, 80.0 rad m^{-1}, and 3.0 rad s^{-1}). Calculate (a) the amplitude, (b) the wavelength, and (c) the period and frequency of the wave. Also, calculate the displacement y of the wave at a distance $x = 30.0$ cm and time $t = 20$ s ?

Answer On comparing this displacement equation with Eq. (14.2),

$$y\,(x,\, t) = a \sin (kx - \omega t),$$

we find

(a) the amplitude of the wave is 0.005 m = 5 mm.

(b) the angular wave number k and angular frequency ω are

$$k = 80.0 \ \text{m}^{-1} \text{ and } \omega = 3.0 \text{ s}^{-1}$$

We, then, relate the wavelength λ to k through Eq. (14.6),

$$\lambda = 2\pi / k$$

$$= \frac{2\pi}{80.0 \text{ m}^{-1}}$$

$$= 7.85 \text{ cm}$$

(c) Now, we relate T to ω by the relation

$$T = 2\pi / \omega$$

$$= \frac{2\pi}{3.0 \text{ s}^{-1}}$$

$$= 2.09 \text{ s}$$

and frequency, $\nu = 1/T = 0.48$ Hz

The displacement y at $x = 30.0$ cm and time $t = 20$ s is given by

$$y = (0.005 \text{ m}) \sin (80.0 \times 0.3 - 3.0 \times 20)$$
$$= (0.005 \text{ m}) \sin (-36 + 12\pi)$$
$$= (0.005 \text{ m}) \sin (1.699)$$
$$= (0.005 \text{ m}) \sin (97^0) \simeq 5 \text{ mm}$$

14.4 THE SPEED OF A TRAVELLING WAVE

To determine the speed of propagation of a travelling wave, we can fix our attention on any particular point on the wave (characterised by some value of the phase) and see how that point moves in time. It is convenient to look at the motion of the crest of the wave. Fig. 14.8 gives the shape of the wave at two instants of time, which differ by a small time internal Δt. The entire wave pattern is seen to shift to the right (positive direction of x-axis) by a distance Δx. In particular, the crest shown by a dot (●) moves a

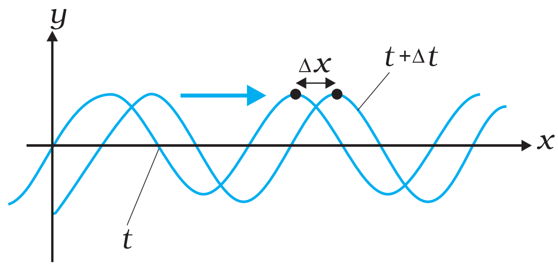


Fig. 14.8 *Progression of a harmonic wave from time t to $t + \Delta t$. where Δt is a small interval. The wave pattern as a whole shifts to the right. The crest of the wave (or a point with any fixed phase) moves right by the distance Δx in time Δt.*

distance Δx in time Δt. The speed of the wave is then $\Delta x/\Delta t$. We can put the dot (●) on a point with any other phase. It will move with the same speed v (otherwise the wave pattern will not remain fixed). The motion of a fixed phase point on the wave is given by

$$kx - \omega t = \text{constant} \tag{14.10}$$

Thus, as time t changes, the position x of the fixed phase point must change so that the phase remains constant. Thus,

$$kx - \omega t = k(x+\Delta x) - \omega(t+\Delta t)$$

or $k\,\Delta x - \omega\,\Delta t = 0$

Taking Δx, Δt vanishingly small, this gives

$$\frac{dx}{dt} = \frac{\omega}{k} = v \tag{14.11}$$

Relating ω to T and k to λ, we get

$$v = \frac{2\pi\nu}{2\pi/\lambda} = \lambda\nu = \frac{\lambda}{T} \tag{14.12}$$

Eq. (14.12), a general relation for all progressive waves, shows that in the time required for one full oscillation by any constituent of the medium, the wave pattern travels a distance equal to the wavelength of the wave. It should be noted that the speed of a mechanical wave is determined by the inertial (linear mass density for strings, mass density in general) and elastic properties (Young's modulus for linear media/ shear modulus, bulk modulus) of the medium. The medium determines the speed; Eq. (14.12) then relates wavelength to frequency for the given speed. Of course, as remarked earlier, the medium can support both transverse and longitudinal waves, which will have different speeds in the same medium. Later in this chapter, we shall obtain specific expressions for the speed of mechanical waves in some media.

14.4.1 Speed of a Transverse Wave on Stretched String

The speed of a mechanical wave is determined by the restoring force setup in the medium when it is disturbed and the inertial properties (mass density) of the medium. The speed is expected to be directly related to the former and inversely to the latter. For waves on a string, the restoring force is provided by the tension T in the string. The inertial property will in this case be linear mass density μ, which is mass m of the string divided by its length L. Using Newton's Laws of Motion, an exact formula for the wave speed on a string can be derived, but this derivation is outside the scope of this book. We shall, therefore, use dimensional analysis. We already know that dimensional analysis alone can never yield the exact formula. The overall dimensionless constant is always left undetermined by dimensional analysis.

The dimension of μ is $[ML^{-1}]$ and that of T is like force, namely $[MLT^{-2}]$. We need to combine these dimensions to get the dimension of speed v $[LT^{-1}]$. Simple inspection shows that the quantity T/μ has the relevant dimension

$$\frac{[MLT^{-2}]}{[ML]} = [L^2T^{-2}]$$

Thus if T and μ are assumed to be the only relevant physical quantities,

$$v = C\sqrt{\frac{T}{\mu}} \qquad (14.13)$$

where C is the undetermined constant of dimensional analysis. In the exact formula, it turms out, C=1. The speed of transverse waves on a stretched string is given by

$$v = \sqrt{\frac{T}{\mu}} \qquad (14.14)$$

Note the important point that the speed v depends only on the properties of the medium T and μ (T is a property of the stretched string arising due to an external force). It does not depend on wavelength or frequency of the wave itself. In higher studies, you will come across waves whose speed is not independent of frequency of the wave. Of the two parameters λ and ν the source of disturbance determines the frequency of the wave generated. Given the speed of the wave in the medium and the frequency Eq. (14.12) then fixes the wavelength

$$\lambda = \frac{v}{\nu} \qquad (14.15)$$

Example 14.3 A steel wire 0.72 m long has a mass of 5.0×10^{-3} kg. If the wire is under a tension of 60 N, what is the speed of transverse waves on the wire ?

Answer Mass per unit length of the wire,

$$\mu = \frac{5.0\times10^{-3}\text{ kg}}{0.72\text{ m}}$$

$$= 6.9\times10^{-3}\text{ kg m}^{-1}$$

Tension, T = 60 N

The speed of wave on the wire is given by

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{60\text{ N}}{6.9\times10^{-3}\text{kg m}^{-1}}} = 93\text{ m s}^{-1}$$

14.4.2 Speed of a Longitudinal Wave (Speed of Sound)

In a longitudinal wave, the constituents of the medium oscillate forward and backward in the direction of propagation of the wave. We have already seen that the sound waves travel in the form of compressions and rarefactions of small volume elements of air. The elastic property that determines the stress under compressional strain is the bulk modulus of the medium defined by (see Chapter 8)

$$B = -\frac{\Delta P}{\Delta V/V} \qquad (14.16)$$

Here, the change in pressure ΔP produces a volumetric strain $\frac{\Delta V}{V}$. B has the same dimension as pressure and given in SI units in terms of pascal *(Pa)*. The inertial property relevant for the propagation of wave is the mass density ρ, with dimensions $[ML^{-3}]$. Simple inspection reveals that quantity B/ρ has the relevant dimension:

$$\frac{[ML^{-2}T^{-2}]}{[ML^{-3}]} = [L^2T^{-2}] \tag{14.17}$$

Thus, if B and ρ are considered to be the only relevant physical quantities,

$$v = C\sqrt{\frac{B}{\rho}} \tag{14.18}$$

where, as before, C is the undetermined constant from dimensional analysis. The exact derivation shows that C=1. Thus, the general formula for longitudinal waves in a medium is:

$$v = \sqrt{\frac{B}{\rho}} \tag{14.19}$$

For a linear medium, like a solid bar, the lateral expansion of the bar is negligible and we may consider it to be only under longitudinal strain. In that case, the relevant modulus of elasticity is Young's modulus, which has the same dimension as the Bulk modulus. Dimensional analysis for this case is the same as before and yields a relation like Eq. (14.18), with an undetermined C, which the exact derivation shows to be unity. Thus, the speed of longitudinal waves in a solid bar is given by

$$v = \sqrt{\frac{Y}{\rho}} \tag{14.20}$$

where Y is the Young's modulus of the material of the bar. Table 14.1 gives the speed of sound in some media.

Table 14.1 Speed of Sound in some Media

Medium	Speed (m s^{-1})
Gases	
Air (0 °C)	331
Air (20 °C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0 °C)	1402
Water (20 °C)	1482
Seawater	1522
Solids	
Aluminium	6420
Copper	3560
Steel	5941
Granite	6000
Vulcanised Rubber	54

Liquids and solids generally have higher speed of sound than gases. [Note for solids, the speed being referred to is the speed of longitudinal waves in the solid]. This happens because they are much more difficult to compress than gases and so have much higher values of bulk modulus. Now, see Eq. (14.19). Solids and liquids have higher mass densities (ρ) than gases. But the corresponding increase in both the modulus (B) of solids and liquids is much higher. This is the reason why the sound waves travel faster in solids and liquids.

We can estimate the speed of sound in a gas in the ideal gas approximation. For an ideal gas, the pressure P, volume V and temperature T are related by (see Chapter 10).

$$PV = Nk_BT \tag{14.21}$$

where N is the number of molecules in volume V, k_B is the Boltzmann constant and T the temperature of the gas (in Kelvin). Therefore, for an isothermal change it follows from Eq.(14.21) that

$$V\Delta P + P\Delta V = 0$$

or $$-\frac{\Delta P}{\Delta V/V} = P$$

Hence, substituting in Eq. (14.16), we have

$$B = P$$

Therefore, from Eq. (14.19) the speed of a longitudinal wave in an ideal gas is given by,

$$v = \sqrt{\frac{P}{\rho}} \tag{14.22}$$

This relation was first given by Newton and is known as Newton's formula.

Example 14.4 Estimate the speed of sound in air at standard temperature and pressure. The mass of 1 mole of air is 29.0×10^{-3} kg.

Answer We know that 1 mole of any gas occupies 22.4 litres at STP. Therefore, density of air at STP is:

ρ_o = (mass of one mole of air)/ (volume of one mole of air at STP)

$$= \frac{29.0 \times 10^{-3}\ \text{kg}}{22.4 \times 10^{-3}\ \text{m}^3}$$

$$= 1.29\ \text{kg m}^{-3}$$

According to Newton's formula for the speed of sound in a medium, we get for the speed of sound in air at STP,

$$v = \left[\frac{1.01 \times 10^5 \text{ N m}^{-2}}{1.29 \text{ kg m}^{-3}}\right]^{1/2} = 280 \text{ m s}^{-1} \quad (14.23)$$

The result shown in Eq.(14.23) is about 15% smaller as compared to the experimental value of 331 m s^{-1} as given in Table 14.1. Where did we go wrong? If we examine the basic assumption made by Newton that the pressure variations in a medium during propagation of sound are isothermal, we find that this is not correct. It was pointed out by Laplace that the pressure variations in the propagation of sound waves are so fast that there is little time for the heat flow to maintain constant temperature. These variations, therefore, are adiabatic and not isothermal. For adiabatic processes the ideal gas satisfies the relation (see Section 11.8),

$$PV^{\gamma} = \text{constant}$$

i.e. $\Delta(PV^{\gamma}) = 0$

or $P\gamma V^{\gamma-1} \Delta V + V^{\gamma} \Delta P = 0$

where γ is the ratio of two specific heats, C_p/C_v.

Thus, for an ideal gas the adiabatic bulk modulus is given by,

$$B_{ad} = -\frac{\Delta P}{\Delta V/V} = \gamma P$$

The speed of sound is, therefore, from Eq. (14.19), given by,

$$v = \sqrt{\frac{\gamma P}{\rho}} \quad (14.24)$$

This modification of Newton's formula is referred to as the **Laplace correction**. For air $\gamma = 7/5$. Now using Eq. (14.24) to estimate the speed of sound in air at STP, we get a value 331.3 m s^{-1}, which agrees with the measured speed.

14.5 THE PRINCIPLE OF SUPERPOSITION OF WAVES

What happens when two wave pulses travelling in opposite directions cross each other (Fig. 14.9)? It turns out that wave pulses continue to retain their identities after they have crossed. However, during the time they overlap, the wave pattern is different from either of the pulses. Figure 14.9 shows the situation when two pulses of equal and opposite shapes move towards each other. When the pulses overlap, the resultant displacement is the algebraic sum of the displacement due to each pulse. This is known as the principle of superposition of waves. According to this principle, each pulse moves as if others are not present. The constituents of the medium, therefore, suffer displacments due to both and since the displacements can be positive and negative, the net displacement is an algebraic sum of the two. Fig. 14.9 gives graphs of the wave shape at different times. Note the dramatic effect in the graph (c); the displacements due to the two pulses have exactly cancelled each other and there is zero displacement throughout.

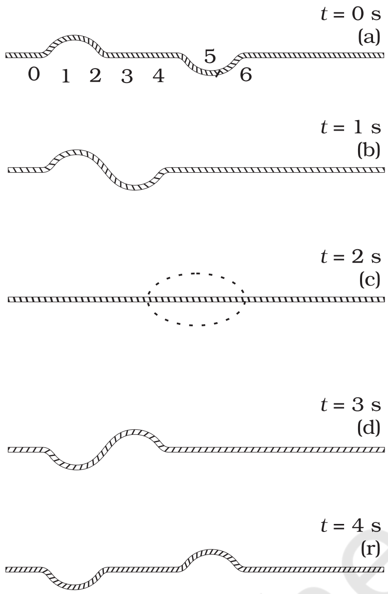


Fig. 14.9 Two pulses having equal and opposite displacements moving in opposite directions. The overlapping pulses add up to zero displacement in curve (c).

To put the principle of superposition mathematically, let $y_1(x,t)$ and $y_2(x,t)$ be the displacements due to two wave disturbances in the medium. If the waves arrive in a region simultaneously, and therefore, overlap, the net displacement $y(x,t)$ is given by

$$y(x, t) = y_1(x, t) + y_2(x, t) \quad (14.25)$$

If we have two or more waves moving in the medium the resultant waveform is the sum of wave functions of individual waves. That is, if the wave functions of the moving waves are

$$y_1 = f_1(x - vt),$$

$$y_2 = f_2(x - vt),$$

....

....

$$y_n = f_n(x - vt)$$

then the wave function describing the disturbance in the medium is

$$y = f_1(x - vt) + f_2(x - vt) + \ldots + f_n(x - vt)$$

$$= \sum_{i=1}^{n} f_i\,(x - vt) \qquad (14.26)$$

The principle of superposition is basic to the phenomenon of interference.

For simplicity, consider two harmonic travelling waves on a stretched string, both with the same ω (angular frequency) and k (wave number), and, therefore, the same wavelength λ. Their wave speed will be identical. Let us further assume that their amplitudes are equal and they are both travelling in the positive direction of x-axis. The waves only differ in their initial phase. According to Eq. (14.2), the two waves are described by the functions:

$$y_1(x, t) = a \sin (kx - \omega t) \qquad (14.27)$$

and $y_2(x, t) = a \sin (kx - \omega t + \phi)$ (14.28)

The net displacement is then, by the principle of superposition, given by

$$y\ (x, t) = a \sin (kx - \omega t) + a \sin (kx - \omega t + \phi) \qquad (14.29)$$

$$= a\left[2\sin\left[\frac{(kx-\omega t)+(kx-\omega t+\phi)}{2}\right]\cos\frac{\phi}{2}\right] \qquad (14.30)$$

where we have used the familiar trignometric identity for $\sin A + \sin B$. We then have

$$y(x,t) = 2a\ \cos\frac{\phi}{2}\ \ \sin\left(kx - \omega t + \frac{\phi}{2}\right) \qquad (14.31)$$

Eq. (14.31) is also a harmonic travelling wave in the positive direction of x-axis, with the same frequency and wavelength. However, its initial phase angle is $\dfrac{\phi}{2}$. The significant thing is that its amplitude is a function of the phase difference

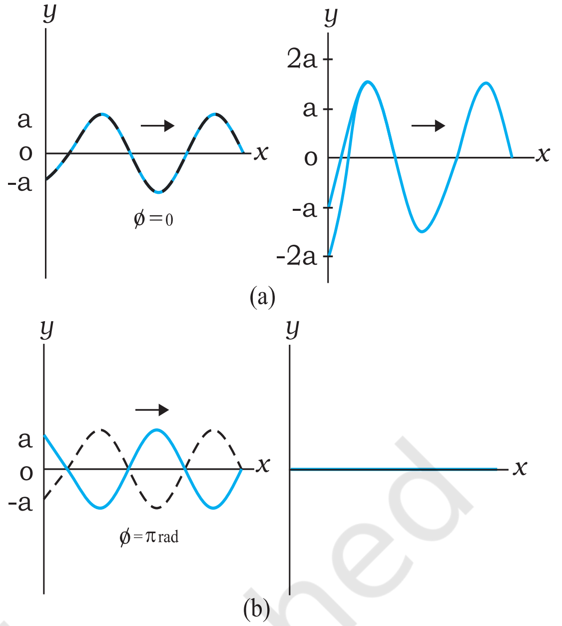


Fig. 14.10 *The resultant of two harmonic waves of equal amplitude and wavelength according to the principle of superposition. The amplitude of the resultant wave depends on the phase difference ϕ, which is zero for (a) and π for (b)*

ϕ between the constituent two waves:

$$A(\phi) = 2a \cos ½\phi \qquad (14.32)$$

For $\phi = 0$, when the waves are in phase,

$$y(x,t) = 2a\ \sin(kx - \omega t) \qquad (14.33)$$

i.e., the resultant wave has amplitude 2a, the largest possible value for A. For $\phi = \pi$, the waves are completely, out of phase and the resultant wave has zero displacement everywhere at all times

$$y\ (x, t) = 0 \qquad (14.34)$$

Eq. (14.33) refers to the so-called constructive interference of the two waves where the amplitudes add up in the resultant wave. Eq. (14.34) is the case of destructive intereference where the amplitudes subtract out in the resultant wave. Fig. 14.10 shows these two cases of interference of waves arising from the principle of superposition.

14.6 REFLECTION OF WAVES

So far we considered waves propagating in an unbounded medium. What happens if a pulse or a wave meets a boundary? If the boundary is rigid, the pulse or wave gets reflected. The phenomenon of echo is an example of reflection by a rigid boundary. If the boundary is not completely rigid or is an interface between two different elastic media, the situation is some what complicated. A part of the incident wave is reflected and a part is transmitted into the second medium. If a wave is incident obliquely on the boundary between two different media the transmitted wave is called the **refracted wave**. The incident and refracted waves obey Snell's law of refraction, and the incident and reflected waves obey the usual laws of reflection.

Fig. 14.11 shows a pulse travelling along a stretched string and being reflected by the boundary. Assuming there is no absorption of energy by the boundary, the reflected wave has the same shape as the incident pulse but it suffers a phase change of π or 180^0 on reflection. This is because the boundary is rigid and the disturbance must have zero displacement at all times at the boundary. By the principle of superposition, this is possible only if the reflected and incident waves differ by a phase of π, so that the resultant displacement is zero. This reasoning is based on boundary condition on a rigid wall. We can arrive at the same conclusion dynamically also. As the pulse arrives at the wall, it exerts a force on the wall. By Newton's Third Law, the wall exerts an equal and opposite force on the string generating a reflected pulse that differs by a phase of π.

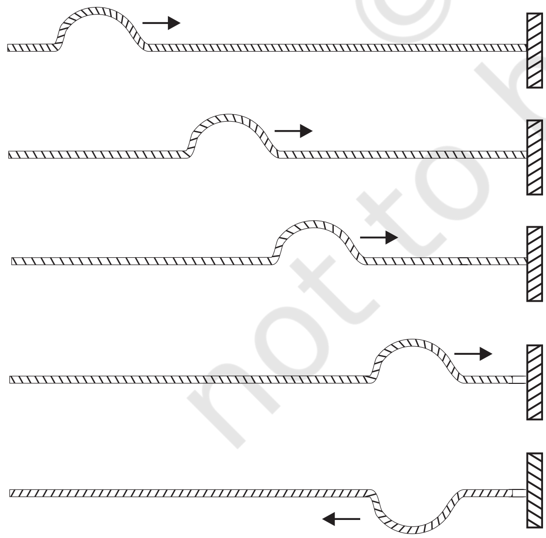


Fig. 14.11 Reflection of a pulse meeting a rigid boundary.

If on the other hand, the boundary point is not rigid but completely free to move (such as in the case of a string tied to a freely moving ring on a rod), the reflected pulse has the same phase and amplitude (assuming no energy dissipation) as the incident pulse. The net maximum displacement at the boundary is then twice the amplitude of each pulse. An example of non- rigid boundary is the open end of an organ pipe.

To summarise, a travelling wave or pulse suffers a phase change of π on reflection at a rigid boundary and no phase change on reflection at an open boundary. To put this mathematically, let the incident travelling wave be

$$y_2(x,t) = a\sin(kx - \omega t)$$

At a rigid boundary, the reflected wave is given by

$$y_r(x,\ t) = a \sin\ (kx - \omega t + \pi).$$
$$= -\ a \sin\ (kx - \omega t) \qquad (14.35)$$

At an open boundary, the reflected wave is given by

$$y_r(x,\ t) = a \sin\ (kx - \omega t + 0).$$
$$= a \sin\ (kx - \omega t) \qquad (14.36)$$

Clearly, at the rigid boundary, $y = y_2 + y_r = 0$ at all times.

14.6.1 Standing Waves and Normal Modes

We considered above reflection at one boundary. But there are familiar situations (a string fixed at either end or an air column in a pipe with either end closed) in which reflection takes place at two or more boundaries. In a string, for example, a wave travelling in one direction will get reflected at one end, which in turn will travel and get reflected from the other end. This will go on until there is a steady wave pattern set up on the string. Such wave patterns are called standing waves or stationary waves. To see this mathematically, consider a wave travelling along the positive direction of x-axis and a reflected wave of the same amplitude and wavelength in the negative direction of x-axis. From Eqs. (14.2) and (14.4), with $\phi = 0$, we get:

$$y_1(x,\ t) = a \sin\ (kx - \omega t)$$

$$y_2(x,\ t) = a \sin\ (kx + \omega t)$$

The resultant wave on the string is, according to the principle of superposition:

$$y\,(x,\ t) = y_1(x,\ t) + y_2(x,\ t)$$

$= a\,[\sin(kx - \omega t) + \sin(kx + \omega t)]$

Using the familiar trignometric identity Sin (A+B) + Sin (A–B) = 2 sin A cosB we get,

$$y(x, t) = 2a \sin kx \cos \omega t \qquad (14.37)$$

Note the important difference in the wave pattern described by Eq. (14.37) from that described by Eq. (14.2) or Eq. (14.4). The terms kx and ωt appear separately, not in the combination $kx - \omega t$. The amplitude of this wave is $2a \sin kx$. Thus, in this wave pattern, the amplitude varies from point-to-point, but each element of the string oscillates with the same angular frequency ω or time period. There is no phase difference between oscillations of different elements of the wave. The string as a whole vibrates in phase with differing amplitudes at different points. The wave pattern is neither moving to the right nor to the left. Hence, they are called standing or stationary waves. The amplitude is fixed at a given location but, as remarked earlier, it is different at different locations. The points at which the amplitude is zero (i.e., where there is no motion at all) are **nodes**; the points at which the amplitude is the largest are called **antinodes.** Fig. 14.12 shows a stationary wave pattern resulting from superposition of two travelling waves in opposite directions.

The most significant feature of stationary waves is that the boundary conditions constrain the possible wavelengths or frequencies of vibration of the system. The system cannot oscillate with any arbitrary frequency (contrast this with a harmonic travelling wave), but is characterised by a set of natural frequencies or **normal modes** of oscillation. Let us determine these normal modes for a stretched string fixed at both ends.

First, from Eq. (14.37), the positions of nodes (where the amplitude is zero) are given by

sin $kx = 0$.

which implies

$$kx = n\pi; \quad n = 0, 1, 2, 3, \ldots$$

Since, $k = 2\pi/\lambda$, we get

$$x = \frac{n\lambda}{2}\ ; n = 0, 1, 2, 3, \ldots \qquad (14.38)$$

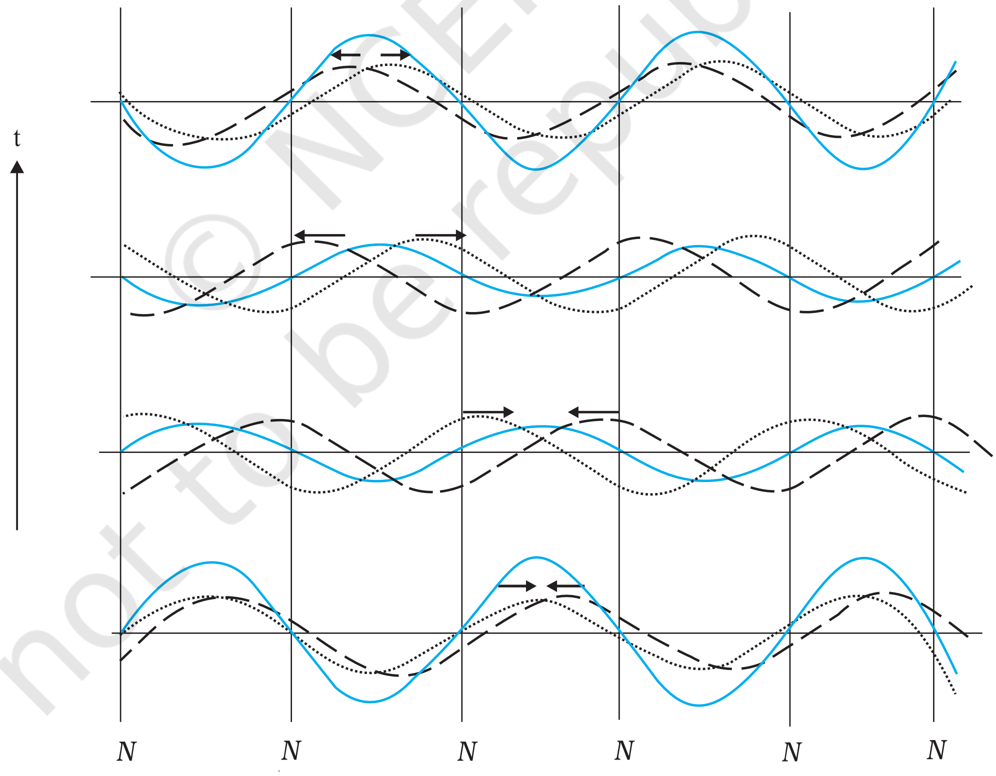


Fig. 14.12 *Stationary waves arising from superposition of two harmonic waves travelling in opposite directions. Note that the positions of zero displacement (nodes) remain fixed at all times.*

Clearly, the distance between any two successive nodes is $\frac{\lambda}{2}$. In the same way, the positions of antinodes (where the amplitude is the largest) are given by the largest value of sin kx:

$$|\sin kx| = 1$$

which implies

$$kx = (n + ½)\ \pi\ ;\ n = 0,\ 1,\ 2,\ 3,\ ...$$

With $k = 2\pi/\lambda$, we get

$$x = (n + ½)\frac{\lambda}{2}\ ;\ n = 0,\ 1,\ 2,\ 3,\ ... \qquad (14.39)$$

Again the distance between any two consecutive antinodes is $\frac{\lambda}{2}$. Eq. (14.38) can be applied to the case of a stretched string of length L fixed at both ends. Taking one end to be at $x = 0$, the boundary conditions are that $x = 0$ and $x = L$ are positions of nodes. The $x = 0$ condition is already satisfied. The $x = L$ node condition requires that the length L is related to λ by

$$L = n\frac{\lambda}{2};\quad n = 1, 2, 3, ... \qquad (14.40)$$

Thus, the possible wavelengths of stationary waves are constrained by the relation

$$\lambda = \frac{2L}{n};\quad n = 1, 2, 3, ... \quad (14.41)$$

with corresponding frequencies

$$\nu = \frac{nv}{2L},\ \text{for}\ \ n = 1, 2, 3, \quad (14.42)$$

We have thus obtained the natural frequencies - the normal modes of oscillation of the system. The lowest possible natural frequency of a system is called its **fundamental mode** or the **first harmonic**. For the stretched string fixed at either end it is given by $\nu = \frac{v}{2L}$, corresponding to $n = 1$ of Eq. (14.42). Here v is the speed of wave determined by the properties of the medium. The $n = 2$ frequency is called the second harmonic; $n = 3$ is the third harmonic and so on. We can label the various harmonics by the symbol ν_n ($n = 1, 2, ...$).

Fig. 14.13 shows the first six harmonics of a stretched string fixed at either end. A string need not vibrate in one of these modes only. Generally, the vibration of a string will be a superposition of different modes; some modes may be more strongly excited and some less. Musical instruments like sitar or violin are based on this principle. Where the string is plucked or bowed, determines which modes are more prominent than others.

Let us next consider normal modes of oscillation of an air column with one end closed

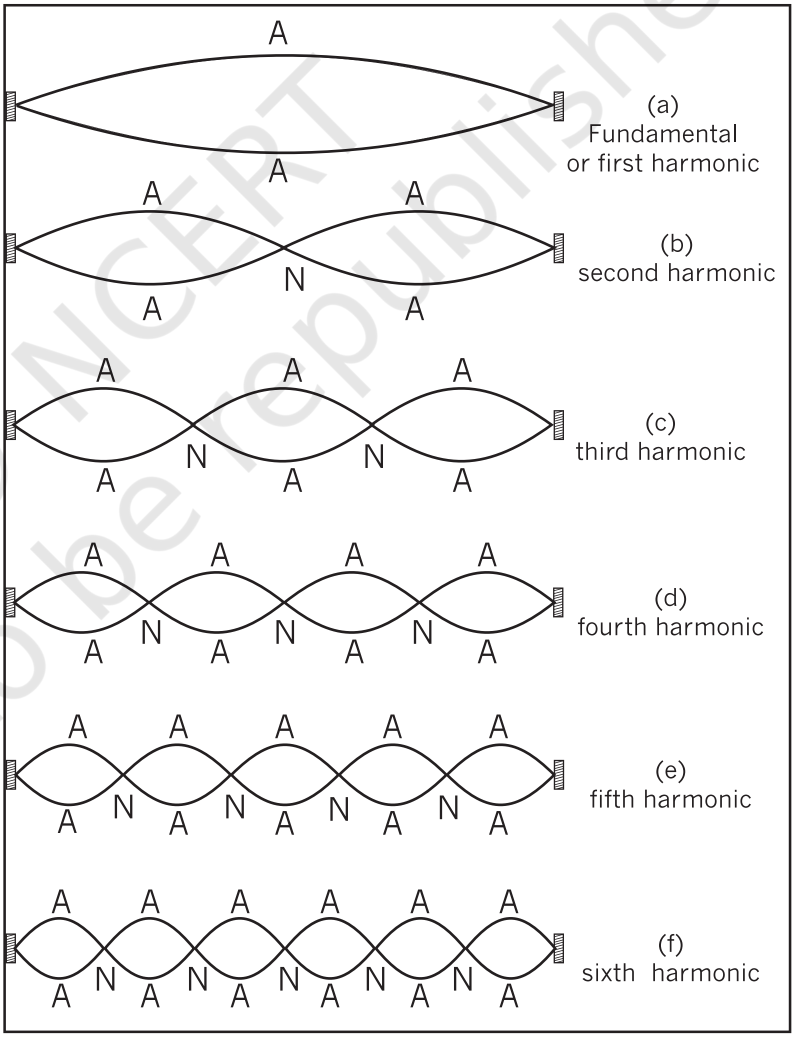


Fig. 14.13 *The first six harmonics of vibrations of a stretched string fixed at both ends.*

and the other open. A glass tube partially filled with water illustrates this system. The end in contact with water is a node, while the open end is an antinode. At the node the pressure changes are the largest, while the displacement is minimum (zero). At the open end - the antinode, it is just the other way - least pressure change and maximum amplitude of displacement. Taking the end in contact with water to be $x = 0$, the node condition (Eq. 14.38) is already satisfied. If the other end $x = L$ is an antinode, Eq. (14.39) gives

$$L = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \text{ for } n = 0, 1, 2, 3, \ldots$$

The possible wavelengths are then restricted by the relation :

$$\lambda = \frac{2L}{(n + 1/2)}, \text{ for } n = 0, 1, 2, 3,\ldots \quad (14.43)$$

The normal modes – the natural frequencies – of the system are

$$\nu = \left(n + \frac{1}{2}\right)\frac{v}{2L};\ n = 0, 1, 2, 3, \ldots \quad (14.44)$$

The fundamental frequency corresponds to $n = 0$, and is given by $\frac{v}{4L}$. The higher frequencies are **odd harmonics**, i.e., odd multiples of the fundamental frequency : $3\frac{v}{4L}$, $5\frac{v}{4L}$, etc.

Fig. 14.14 shows the first six odd harmonics of air column with one end closed and the other open. For a pipe open at both ends, each end is an antinode. It is then easily seen that an open air column at both ends generates all harmonics (See Fig. 14.15).

The systems above, strings and air columns, can also undergo forced oscillations (Chapter 13). If the external frequency is close to one of the natural frequencies, the system shows **resonance**.

Normal modes of a circular membrane rigidly clamped to the circumference as in a tabla are determined by the boundary condition that no point on the circumference of the membrane vibrates. Estimation of the frequencies of normal modes of this system is more complex. This problem involves wave propagation in two dimensions. However, the underlying physics is the same.

Example 14.5 A pipe, 30.0 cm long, is open at both ends. Which harmonic mode of the pipe resonates a 1.1 kHz source? Will resonance with the same source be observed if one end of the pipe is closed ? Take the speed of sound in air as 330 m s^{-1}.

Answer The first harmonic frequency is given by

$$\nu_1 = \frac{v}{\lambda_1} = \frac{v}{2L} \quad \text{(open pipe)}$$

where L is the length of the pipe. The frequency of its nth harmonic is:

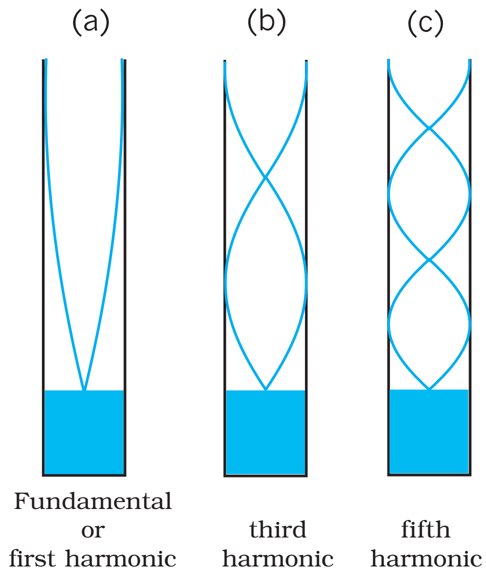
$$\nu_n = \frac{nv}{2L}, \text{ for } n = 1, 2, 3, \ldots \text{ (open pipe)}$$

First few modes of an open pipe are shown in Fig. 14.15.

For $L = 30.0$ cm, $v = 330$ m s^{-1},

$$\nu_n = \frac{n\,330\ (\text{m s}^{-1})}{0.6\ (\text{m})} = 550\ n\ \text{s}^{-1}$$

Clearly, a source of frequency 1.1 kHz will resonate at v_2, i.e. the **second harmonic**.



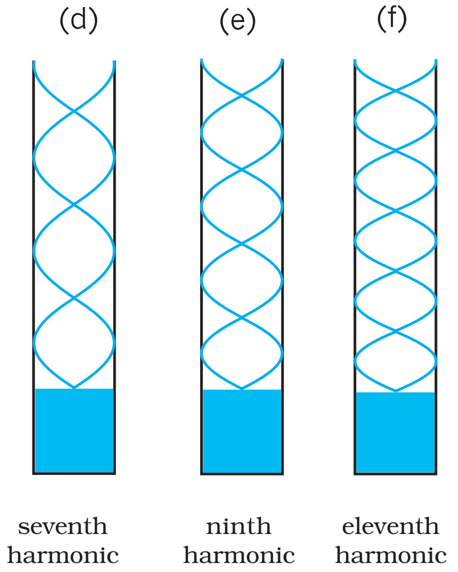


Fig. 14.14 *Normal modes of an air column open at one end and closed at the other end. Only the odd harmonics are seen to be possible.*

Now if one end of the pipe is closed (Fig. 14.15), it follows from Eq. (14.15) that the fundamental frequency is

$$\nu_1 = \frac{v}{\lambda_1} = \frac{v}{4L} \text{ (pipe closed at one end)}$$

and only the odd numbered harmonics are present :

$$\nu_3 = \frac{3v}{4L}, \quad \nu_5 = \frac{5v}{4L}, \text{ and so on.}$$

For L = 30 cm and v = 330 m s^{-1}, the fundamental frequency of the pipe closed at one end is 275 Hz and the source frequency corresponds to its fourth harmonic. Since this harmonic is not a possible mode, no resonance will be observed with the source, the moment one end is closed.

14.7 BEATS

'Beats' is an interesting phenomenon arising from interference of waves. When two harmonic sound waves of close (but not equal) frequencies are heard at the same time, we hear a sound of similar frequency (the average of two close frequencies), but we hear something else also. We hear audibly distinct waxing and waning of the intensity of the sound, with a frequency equal to the difference in the two close frequencies. Artists use this phenomenon often

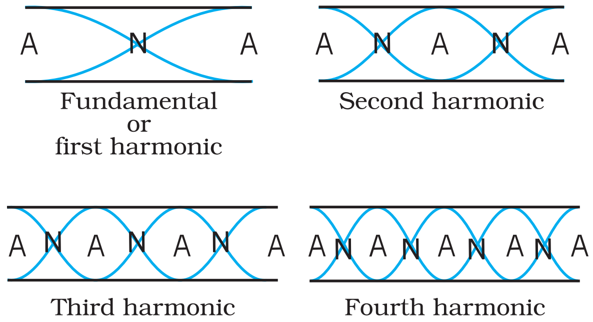


Fig. 14.15 *Standing waves in an open pipe, first four harmonics are depicted.*

while tuning their instruments with each other. They go on tuning until their sensitive ears do not detect any beats.

To see this mathematically, let us consider two harmonic sound waves of nearly equal angular frequency ω_1 and ω_2 and fix the location to be x = 0 for convenience. Eq. (14.2) with a suitable choice of phase ($\phi = \pi/2$ for each) and, assuming equal amplitudes, gives

$$s_1 = a \cos \omega_1 t \quad \text{and} \quad s_2 = a \cos \omega_2 t \qquad (14.45)$$

Here we have replaced the symbol y by s, since we are referring to longitudinal not transverse displacement. Let ω_1 be the (slightly) greater of the two frequencies. The resultant displacement is, by the principle of superposition,

$$s = s_1 + s_2 = a\,(\cos \omega_1 t + \cos \omega_2 t)$$

Using the familiar trignometric identity for cos A + cosB, we get

$$= 2\,a \cos \frac{(\omega_1 - \omega_2)t}{2} \cos \frac{(\omega_1 + \omega_2)t}{2} \qquad (14.46)$$

which may be written as :

$$s = [2\,a \cos \omega_b t] \cos \omega_a t \qquad (14.47)$$

If $|\omega_1 - \omega_2| << \omega_1, \omega_2, \omega_a >> \omega_b$, th where

$$\omega_b = \frac{(\omega_1 - \omega_2)}{2} \text{ and } \omega_a = \frac{(\omega_1 + \omega_2)}{2}$$

Now if we assume $|\omega_1 - \omega_2| << \omega_1$, which means $\omega_a >> \omega_b$, we can interpret Eq. (14.47) as follows. The resultant wave is oscillating with the average angular frequency ω_a; however its amplitude is **not** constant in time, unlike a pure harmonic wave. The amplitude is the largest when the term *cos* $\omega_b t$ takes its limit +1 or –1. In other words, the intensity of the resultant wave waxes and wanes with a frequency which is $2\omega_b = \omega_1 -$

Musical Pillars

Temples often have some pillars portraying human figures playing musical instruments, but seldom do these pillars themselves produce music. At the Nellaiappar temple in Tamil Nadu, gentle taps on a cluster of pillars carved out of a single piece of rock produce the basic notes of Indian classical music, viz. Sa, Re, Ga, Ma, Pa, Dha, Ni, Sa. Vibrations of these pillars depend on elasticity of the stone used, its density and shape.

Musical pillars are categorised into three types: The first is called the **Shruti Pillar**, as it can produce the basic notes — the "swaras". The second type is the **Gana Thoongal**, which generates the basic tunes that make up the "ragas". The third variety is the **Laya Thoongal** pillars that produce "taal" (beats) when tapped. The pillars at the Nellaiappar temple are a combination of the Shruti and Laya types.

Archaeologists date the Nelliappar temple to the 7th century and claim it was built by successive rulers of the Pandyan dynasty.

The musical pillars of Nelliappar and several other temples in southern India like those at Hampi (picture), Kanyakumari, and Thiruvananthapuram are unique to the country and have no parallel in any other part of the world.

ω_2. Since $\omega = 2\pi\nu$, the beat frequency ν_{beat}, is given by

$$\nu_{beat} = \nu_1 - \nu_2 \qquad (14.48)$$

Fig. 14.16 illustrates the phenomenon of beats for two harmonic waves of frequencies 11 Hz and 9 Hz. The amplitude of the resultant wave shows beats at a frequency of 2 Hz.

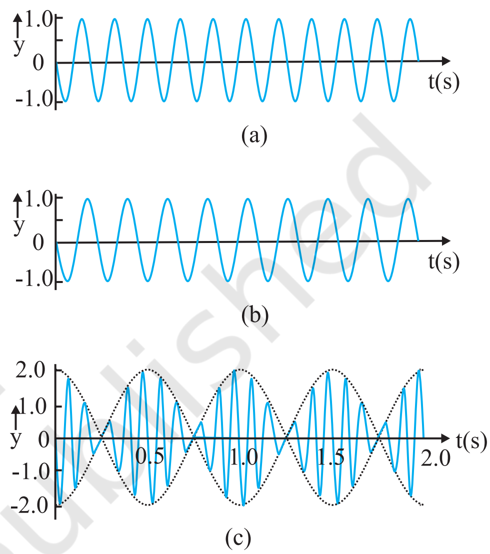


Fig. 14.16 *Superposition of two harmonic waves, one of frequency 11 Hz (a), and the other of frequency 9Hz (b), giving rise to beats of frequency 2 Hz, as shown in (c).*

Example 14.6 Two sitar strings A and B playing the note '*Dha*' are slightly out of tune and produce beats of frequency 5 Hz. The tension of the string B is slightly increased and the beat frequency is found to decrease to 3 Hz. What is the original frequency of B if the frequency of A is 427 Hz ?

Answer Increase in the tension of a string increases its frequency. If the original frequency of B (ν_B) were greater than that of A (ν_A), further increase in ν_B should have resulted in an increase in the beat frequency. But the beat frequency is found to decrease. This shows that $\nu_B < \nu_A$. Since $\nu_A - \nu_B = 5$ Hz, and $\nu_A = 427$ Hz, we get $\nu_B = 422$ Hz.

SUMMARY

1. *Mechanical waves* can exist in material media and are governed by Newton's Laws.
2. *Transverse waves* are waves in which the particles of the medium oscillate perpendicular to the direction of wave propagation.
3. *Longitudinal waves* are waves in which the particles of the medium oscillate along the direction of wave propagation.
4. *Progressive wave* is a wave that moves from one point of medium to another.
5. *The displacement* in a sinusoidal wave propagating in the positive x direction is given by

$$y(x, t) = a \sin(kx - \omega t + \phi)$$

where a *is the* amplitude of the wave, k is the angular wave number, ω *is the* angular frequency, $(kx - \omega t + \phi)$ *is the* phase, *and* ϕ *is the* phase constant or phase angle.

6. *Wavelength* λ of a progressive wave is the distance between two consecutive points of the same phase at a given time. In a stationary wave, it is twice the distance between two consecutive nodes or antinodes.
7. *Period* T of oscillation of a wave is defined as the time any element of the medium takes to move through one complete oscillation. It is related to the *angular frequency* ω through the relation

$$T = \frac{2\pi}{\omega}$$

8. *Frequency* ν of a wave is defined as $1/T$ and is related to angular frequency by

$$\nu = \frac{\omega}{2\pi}$$

9. *Speed* of a progressive wave is given by $v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda\nu$
10. *The speed of a transverse wave* on a stretched string is set by the properties of the string. The speed on a string with tension T and linear mass density μ is

$$v = \sqrt{\frac{T}{\mu}}$$

11. *Sound waves* are longitudinal mechanical waves that can travel through solids, liquids, or gases. The speed v of sound wave in a fluid having *bulk modulus* B and density ρ is

$$v = \sqrt{\frac{B}{\rho}}$$

The speed of longitudinal waves in a metallic bar is

$$v = \sqrt{\frac{Y}{\rho}}$$

For gases, since $B = \gamma P$, the speed of sound is

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

12. When two or more waves traverse simultaneously in the same medium, the displacement of any element of the medium is the algebraic sum of the displacements due to each wave. This is known as the *principle of superposition* of waves

$$y = \sum_{i=1}^{n} f_i(x - vt)$$

13. Two sinusoidal waves on the same string exhibit *interference*, adding or cancelling according to the principle of superposition. If the two are travelling in the same direction and have the same amplitude a and frequency but differ in phase by a *phase constant* ϕ, the result is a single wave with the same frequency ω:

$$y(x, t) = \left[2a\cos\frac{1}{2}\phi\right]\sin\left(kx - \omega t + \frac{1}{2}\phi\right)$$

If $\phi = 0$ or an integral multiple of 2π, the waves are exactly in phase and the interference is constructive; if $\phi = \pi$, they are exactly out of phase and the interference is destructive.

14. A travelling wave, at a rigid boundary or a closed end, is reflected with a phase reversal but the reflection at an open boundary takes place without any phase change.

For an incident wave

$$y_i(x, t) = a\sin(kx - \omega t)$$

the reflected wave at a rigid boundary is

$$y_r(x, t) = -a\sin(kx + \omega t)$$

For reflection at an open boundary

$$y_r(x, t) = a\sin(kx + \omega t)$$

15. The interference of two identical waves moving in opposite directions produces *standing waves.* For a string with fixed ends, the standing wave is given by

$$y(x, t) = [2a\sin kx]\cos\omega t$$

Standing waves are characterised by fixed locations of zero displacement called *nodes* and fixed locations of maximum displacements called *antinodes.* The separation between two consecutive nodes or antinodes is $\lambda/2$.

A stretched string of length L fixed at both the ends vibrates with frequencies given by

$$\nu = \frac{n\,v}{2\,L}, \qquad n = 1, 2, 3, \ldots$$

The set of frequencies given by the above relation are called the *normal modes* of oscillation of the system. The oscillation mode with lowest frequency is called the *fundamental mode* or the *first harmonic.* The *second harmonic* is the oscillation mode with $n = 2$ and so on.

A pipe of length L with one end closed and other end open (such as air columns) vibrates with frequencies given by

$$\nu = \left(n + \tfrac{1}{2}\right)\frac{v}{2L}, \qquad n = 0, 1, 2, 3, \ldots$$

The set of frequencies represented by the above relation are the *normal modes* of oscillation of such a system. The lowest frequency given by $v/4L$ is the fundamental mode or the first harmonic.

16. A string of length L fixed at both ends or an air column closed at one end and open at the other end or open at both the ends, vibrates with certain frequencies called their normal modes. Each of these frequencies is a *resonant frequency* of the system.

17. *Beats* arise when two waves having slightly different frequencies, ν_1 and ν_2 and comparable amplitudes, are superposed. The beat frequency is

$$\nu_{beat} = \nu_1 \sim \nu_2$$

Physical quantity	Symbol	Dimensions	Unit	Remarks
Wavelength	λ	[L]	m	Distance between two consecutive points with the same phase.
Propagation constant	k	$[L^{-1}]$	m^{-1}	$k = \frac{2\pi}{\lambda}$
Wave speed	v	$[LT^{-1}]$	$m\ s^{-1}$	$v = \nu\lambda$
Beat frequency	ν_{beat}	$[T^{-1}]$	s^{-1}	Difference of two close frequencies of superposing waves.

POINTS TO PONDER

1. A wave is not motion of matter as a whole in a medium. A wind is different from the sound wave in air. The former involves motion of air from one place to the other. The latter involves compressions and rarefactions of layers of air.
2. In a wave, energy and *not the matter* is transferred from one point to the other.
3. In a mechanical wave, energy transfer takes place because of the coupling through elastic forces between neighbouring oscillating parts of the medium.
4. Transverse waves can propagate only in medium with shear modulus of elasticity, Longitudinal waves need bulk modulus of elasticity and are therefore, possible in all media, solids, liquids and gases.
5. In a harmonic progressive wave of a given frequency, all particles have the same amplitude but different phases at a given instant of time. In a stationary wave, all particles between two nodes have the same phase at a given instant but have different amplitudes.
6. Relative to an observer at rest in a medium the speed of a mechanical wave in that medium (v) depends only on elastic and other properties (such as mass density) of the medium. It does not depend on the velocity of the source.

EXERCISES

14.1 A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If the transverse jerk is struck at one end of the string, how long does the disturbance take to reach the other end?

14.2 A stone dropped from the top of a tower of height 300 m splashes into the water of a pond near the base of the tower. When is the splash heard at the top given that the speed of sound in air is 340 m s^{-1}? (g = 9.8 m s^{-2})

14.3 A steel wire has a length of 12.0 m and a mass of 2.10 kg. What should be the tension in the wire so that speed of a transverse wave on the wire equals the speed of sound in dry air at 20 °C = 343 m s^{-1}.

14.4 Use the formula $v = \sqrt{\dfrac{\gamma P}{\rho}}$ to explain why the speed of sound in air

(a) is independent of pressure,

(b) increases with temperature,

(c) increases with humidity.

14.5 You have learnt that a travelling wave in one dimension is represented by a function $y = f(x, t)$ where x and t must appear in the combination $x - v\,t$ or $x + v\,t$, i.e. $y = f(x \pm v\,t)$. Is the converse true? Examine if the following functions for y can possibly represent a travelling wave :

(a) $(x - vt)^2$

(b) $\log\,[(x + vt)/x_0]$

(c) $1/(x + vt)$

14.6 A bat emits ultrasonic sound of frequency 1000 kHz in air. If the sound meets a water surface, what is the wavelength of (a) the reflected sound, (b) the transmitted sound? Speed of sound in air is 340 m s^{-1} and in water 1486 m s^{-1}.

14.7 A hospital uses an ultrasonic scanner to locate tumours in a tissue. What is the wavelength of sound in the tissue in which the speed of sound is 1.7 km s^{-1}? The operating frequency of the scanner is 4.2 MHz.

14.8 A transverse harmonic wave on a string is described by

$$y(x, t) = 3.0 \sin (36\ t + 0.018\ x + \pi/4)$$

where x and y are in cm and t in s. The positive direction of x is from left to right.

(a) Is this a travelling wave or a stationary wave ?

If it is travelling, what are the speed and direction of its propagation ?

(b) What are its amplitude and frequency ?

(c) What is the initial phase at the origin ?

(d) What is the least distance between two successive crests in the wave ?

14.9 For the wave described in Exercise 14.8, plot the displacement (y) versus (t) graphs for $x = 0$, 2 and 4 cm. What are the shapes of these graphs? In which aspects does the oscillatory motion in travelling wave differ from one point to another: amplitude, frequency or phase ?

14.10 For the travelling harmonic wave

$$y(x, t) = 2.0 \cos 2\pi\,(10t - 0.0080\ x + 0.35)$$

where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) 4 m,

(b) 0.5 m,

(c) $\lambda/2$,

(d) $3\lambda/4$

14.11 The transverse displacement of a string (clamped at its both ends) is given by

$$y(x, t) = 0.06 \sin\left(\frac{2\pi}{3}x\right) \cos\,(120\ \pi t)$$

where x and y are in m and t in s. The length of the string is 1.5 m and its mass is 3.0×10^{-2} kg.

Answer the following :

(a) Does the function represent a travelling wave or a stationary wave?

(b) Interpret the wave as a superposition of two waves travelling in opposite directions. What is the wavelength, frequency, and speed of each wave ?

(c) Determine the tension in the string.

14.12 (i) For the wave on a string described in Exercise 15.11, do all the points on the string oscillate with the same (a) frequency, (b) phase, (c) amplitude? Explain your answers. (ii) What is the amplitude of a point 0.375 m away from one end?

14.13 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represent (i) a travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2 \cos (3x) \sin (10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3 \sin (5x - 0.5t) + 4 \cos (5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

14.14 A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is 3.5×10^{-2} kg and its linear mass density is 4.0×10^{-2} kg m^{-1}. What is (a) the speed of a transverse wave on the string, and (b) the tension in the string?

14.15 A metre-long tube open at one end, with a movable piston at the other end, shows resonance with a fixed frequency source (a tuning fork of frequency 340 Hz) when the tube length is 25.5 cm or 79.3 cm. Estimate the speed of sound in air at the temperature of the experiment. The edge effects may be neglected.

14.16 A steel rod 100 cm long is clamped at its middle. The fundamental frequency of longitudinal vibrations of the rod are given to be 2.53 kHz. What is the speed of sound in steel?

14.17 A pipe 20 cm long is closed at one end. Which harmonic mode of the pipe is resonantly excited by a 430 Hz source ? Will the same source be in resonance with the pipe if both ends are open? (speed of sound in air is 340 m s^{-1}).

14.18 Two sitar strings A and B playing the note '*Ga*' are slightly out of tune and produce beats of frequency 6 Hz. The tension in the string A is slightly reduced and the beat frequency is found to reduce to 3 Hz. If the original frequency of A is 324 Hz, what is the frequency of B?

14.19 Explain why (or how):

(a) in a sound wave, a displacement node is a pressure antinode and vice versa,

(b) bats can ascertain distances, directions, nature, and sizes of the obstacles without any "eyes",

(c) a violin note and sitar note may have the same frequency, yet we can distinguish between the two notes,

(d) solids can support both longitudinal and transverse waves, but only longitudinal waves can propagate in gases, and

(e) the shape of a pulse gets distorted during propagation in a dispersive medium.