Theory of Algorithms

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Topics

Python

About Python

January 1994 – Python 1.0.0 released.

Guido van Rossum – Designer/Author of Python.

Current versions – 3.5.1 and 2.7.11.

Interpreted – Python implementation must be present at runtime.

Off-side rule – Blocks identified by indentation, as opposed to curly braces.

Popularity – IEEE Spectrum ranks it as the fourth most popular language (July 2015).

Community – Python Enhancement Proposals, notably PEP 8: The Python Style Guide.

spectrum.ieee.org/computing/software/the-2015-top-ten-programming-languages

Guido van Rossum

- Started Python as a hobby.
- Worked for Google, half-time spent on Python.
- Now works at Dropbox.
- Benevolent dictator for life (BDFL).

Conditions

```python
x = int(raw_input("Please enter an integer: "))
if x < 0:
  x = 0
  print 'Negative changed to zero'
elif x == 0:
  print 'Zero'
elif x == 1:
  print 'Single'
else:
  print 'More'
```

docs.python.org/2/tutorial

Loops

```python
# A for loop.
a = ['Mary', 'had', 'a', 'little', 'lamb']
for i in range(len(a)):
  print(i, a[i])

# A while loop.
a, b = 0, 1
while b < 1000:
  print(b)
  a, b = b, a+b
```

docs.python.org/3/tutorial

Functions

```python
# write Fibonacci series up to n
def fib(n):
    """Print a Fibonacci series up to n."""
    a, b = 0, 1
    while a < n:
        print(a)
        a, b = b, a+b
```

docs.python.org/3/tutorial

CPython

Reference implementation – Many different Python implementations exist.

Version 3 – Broke backwards compatibility (somewhat).

Unladen Swallow – Google attempt to fix some Python problems.

Modules – Lots of great Python modules available.

www.python.org

Lists

Lists in Python are usually written as comma-separated values between square brackets.

Types – elements of a list don't have to have the same types.

Slicing is possible, where we take a sublist of the list.

Assignment to slices is possible.

len() is a built-in function that returns the length of a list.

range() is a built-in function that returns a list of numbers. Note: it returns an *iterator*.

```
letters = ['a', 'b', 'c']
letters[1:] = ['c', 'd']
range(10) # [0,1,2,3,4,5,6,7,8,9]
```

docs.python.org/3/tutorial

Strings

Strings are a lot like lists in Python.

Assignment to slices is not allowed, however.

```
words = "This is a sentence."
words[8]          # a
words[5:7]        # is
words[:7]         # This is
words[10:]        # sentence.
words[17:9:-1]    # ecnetnes

len(words)        # 19
"One" + "Two"     # OneTwo
```

docs.python.org/3/tutorial

Functions

def is the keyword for defining a function.

Parameters can be given defaults, so that they are optional.

```
def axn(x, a=1, n=2):
  return a*(x**n)     # ax^n

axn(3)        # 9
axn(3, 2)     # 18
axn(3, 2, 3)  # 54
axn(3, n=3)   # 27
```

docs.python.org/3/tutorial

List comprehensions

Comprehensions are quick ways of creating lists from other lists.

```
nos = range(5) # [0, 1, 2, 3, 4]
squares = [i*i for i in nos] # [0, 1, 4, 9, 16]
oddsqs = [i*i for i in nos if i % 2 == 1] # [1, 9]
```

docs.python.org/3/tutorial

map()

map() takes a function and a list.

New list – it returns a new generator, which is the original list with the function applied to each element.

```
map(len, words)
list(map(len, words))
```

docs.python.org/3/tutorial

Lambda functions

lambda functions are short, inline functions.

Nameless – lambda functions need not have a name.

```
lambda x: x + n
```

docs.python.org/3/tutorial

Without generators

```python
# Build and return a list
def firstn(n):
    num, nums = 0, []
    while num < n:
        nums.append(num)
        num += 1
    return nums

sum_of_first_n = sum(firstn(1000000))
```

https://wiki.python.org/moin/Generators

With generators

```python
# yields items instead of returning a list
def firstn(n):
    num = 0
    while num < n:
        yield num
        num += 1

sum_of_first_n = sum(firstn(1000000))
```

https://wiki.python.org/moin/Generators

Permutations

Permutations

Permutations are rearrangements of ordered collections of items.

Example: "abcd" is a rearrangement of "bacd".

Think of having a four boxes, where we have to place one of the items in each box.

What are all the different ways of doing this?

What are all the different ways of associating items with boxes?

Abstraction

We can consider permutations in abstraction. For instance, if we have four items to rearrange, we can label the first item 1, the second 2, and so on. Then we can represent the various permutations, in terms of the numbers associated with the items. The permutations "abcd" and "bacd" could be represented by:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

In this way we can consider permutations in their own right.

Counting permutations

With four items, how many distinct permutations are there?
Consider having four placeholders where we can place the items:

$$\boxed{?}\ \boxed{?}\ \boxed{?}\ \boxed{?}$$

When we place an item in the first box, we have four choices, then we are left with only three choice for the second, two choices for the third, and one choice (i.e. not a choice at all) for the last.

So for there are $4 \times 3 \times 2 \times 1 = 4!$ choices in total.

Counting anagrams

Repitition

What happens when we consider two of our items to be the same? For instance, what if we are looking for all distinct rearrangements of "aacd" as opposed to "abcd"? In that case we need to account for the rearrangements of those items by dividing by the factorial of the number of times each item is repeated: $\frac{4!}{2!}$.

If more than one item is repeated a number of times we just keep dividing by the factorials of the numbers of repititons. The distinct number of rearrangements of "aaabbcd" is $\frac{7!}{3!2!}$.

Exercise

Calculate the number of distinct rearragements of the word "Mississippi".

Heap's algorithm

Heap published an algorithm in November 1963 for generating permutations.

Published in The Computer Journal.

Read the article in the link below – it's an easy read.

Pairs of items are interchanged to generate each new permutation.

Induction is used to show the algorithm works.

comjnl.oxfordjournals.org/content/6/3/293.full.pdf

Heap's algorithm description

- Suppose we know how to permute $(n-1)$ items.
- That is, we know all of the different ways of slotting $(n-1)$ items in $(n-1)$ boxes.
- Let's add another, n^{th} item, and another box, an n^{th} box.
- First, place the n^{th} item in the n^{th} box.
- Permute all the the other items, which you know how to do.
- Then swap another item with the n^{th} item.
- Again permute the items in boxes 1 to $(n-1)$.
- Swap the item in the n^{th} box with another, different item.
- Repeat until all items have been in box n.

comjnl.oxfordjournals.org/content/6/3/293.full.pdf

Steinhaus-Johnson-Trotter algorithm

Johnson published another algorithm in 1963 for generating permutations.

Attributed to three people: Steinhaus, Johnson and Trotter.

See the article in the link below – it's a trickier read.

Pairwise – the algorithm can be done pair-wise, like Heap's.

Induction is used to show the algorithm works.

www.ams.org/journals/mcom/1963-17-083/S0025-5718-1963-0159764-2/S0025-5718-1963-0159764-2.pdf

Steinhaus-Johnson-Trotter algorithm description

- Start with two items, 1 and 2, and generate their list of permutations 12 and 21.
- Use the two-item list to generate the three item list in the following way:
  - Place 3 at the right of the first element in the two-item list.
  - Then move 3 one place to the left continuosly until its on the left.
  - Then place 3 at the left of the next element in the two-item list.
  - move 3 one place to the right continuously until it reaches the right.
- Use the three item list to generate the four item list in the same way, and so on.

comjnl.oxfordjournals.org/content/6/3/293.full.pdf

Conundrum – Naive method

```
for permutation in permutations(letters):
  checkIfWord(permutation)
```

Conundrum – Quick method preparation

```
worddict = {}

for word in dictionaryOfWords:
  sortword = sorted(list(word))
  hashword = hash(sortword)
  allWords = worddict.get(hashword, set())
  allWords.update({word})
  worddict[hashword] = allWords
```

Conundrum – Quick method checking

```
word = "conundrum"
sortword = sorted(list(word))
hashword = hash(sortword)

worddict.get(hashword, None)
```

Timing Algorithms

timeit command line

```
$ python -m timeit '"-".join(str(n) for n in range(100))'
10000 loops, best of 3: 30.2 usec per loop
$ python -m timeit '"-".join([str(n) for n in range(100)])
10000 loops, best of 3: 27.5 usec per loop
$ python -m timeit '"-".join(map(str, range(100)))'
10000 loops, best of 3: 23.2 usec per loop
```

docs.python.org/3/library/timeit.html

timeit module

```python
import timeit

def test():
    """Stupid test function"""
    L = [i for i in range(100)]

if __name__ == '__main__':
    import timeit
    print(timeit.timeit("test()",
            setup="from __main__ import test"))
```

docs.python.org/3/library/timeit.html

Functional Programming

John McCarthy

www-formal.stanford.edu/jmc/

- John McCarthy while at MIT – late 1950s.
- Created Lisp.
- Lisp generally considered first functional programming language (not really though).
- Lots of dialects exist today, such as Scheme and Common Lisp.

Wasteful for loops

```python
# How do you parallelise this?
inds = list(range(10))
total = 0
for i in inds:
  total = total + (i * 7)

def byseven(i):
  return i * 7

inds = list(range(10))
sevens = map(byseven, inds)
total = sum(sevens)
```

State

Imperative programming is a programming paradigm where statements are used to change the *state*.

State is the name given to the current data/values related to an executing process, including internal stuff like the call stack.

Processes begin with an initial state and (possibly) have (a number of) halt states.

Statements change the state.

Functions in imperative programming languages might return different values for the same input at different times, because of the state.

Functional programming languages (try to) not depend on state.

Side effects

Functions are said to have side effects if they modify the state (on top of returning a value).

Static and global variables are often good examples of side effects in action.

Functional programming tries to avoid side effects.

It's tricky to avoid them – such as when we need user input.

Basic operators

```
> (+ 3 4)
7
> (* 3 2)
6
> (- 5 3)
2
> (/ 6 3)
2
```

More arguments

```
> (+ 3 4 5)
(+ 3 4 5)
> (- 3 4 5)
-6
> (* 2 3 4)
24
> (/ 6 3 3)
2/3
> (/ 6 3 3 3)
2/9
```

Functions and values

```scheme
; Define a value called foo with value 3.
>(define foo 3)

; Define a function f.
>(define (f x)
  (+ (* 3 x) 12))

>(define (g x)
  (* 3 (+ x 4)))

>(g 2)
18
```

www.artificialworlds.net/presentations/scheme-01-intro/scheme-01-intro.html

Conditionals

```
> (if (< 1 2) '(y e s) '(n o))
(y e s)

>(define (abs x)
  (if (< x 0)
  (- x)
  x))
```

www.artificialworlds.net/presentations/scheme-01-intro/scheme-01-intro.html

Lists

```
>(list 1 2 3)
(1 2 3)

>(list 'a 'b 'c)
(a b c)

> (length (list 1 2 3))
3
```

www.artificialworlds.net/presentations/scheme-01-intro/scheme-01-intro.html

car and cdr

```
> (car (list 1 2 3))
1
> (cdr (list 1 2 3))
(2 3)
> (define l (list 1 2 3))
> (car l)
1
> (cdr l)
(2 3)
> (car (cdr l))
2
> (cadr l)
2
```

www.artificialworlds.net/presentations/scheme-01-intro/scheme-01-intro.html

Recursion

```
> (define (sum lv)
    (if (null? lv)
      0
      (+ (car lv) (sum (cdr lv)))))
> (sum (list 1 2 3))
6
> (define (derange n)
    (if (= 0 n)
      '()
      (cons n (derange (- n 1)))))
> (derange 12)
(12 11 10 9 8 7 6 5 4 3 2 1)
```

www.artificialworlds.net/presentations/scheme-01-intro/scheme-01-intro.html

Looping recursively

```
> (let loop ((i 5))
(print "i is " i ".\n")
(if (> i 0) (loop (- i 1))))
i is 5.
i is 4.
i is 3.
i is 2.
i is 1.
i is 0.
```

gambitscheme.org/wiki/images/a/a7/A_Tour_of_Scheme_in_Gambit.pdf

Data and code

```
> (define (swap3-1-2 x)
  (list (cadr x) (car x) (caddr x)))

> (swap3-1-2 (list 1 2 3))
(2 1 3)

> (define four-over-two (list 4 '/ 2))

> four-over-two
(4 / 2)

> (eval (swap3-1-2 four-over-two))
```

www.artificialworlds.net/presentations/scheme-01-intro/scheme-01-intro.html

More on functions

```
; Printing stuff to terminal.
> (print "Ay" "-" "yo.\n")
; Proper way to define a function.
> (define foo (lambda (bar) (print "Bar is " bar ".\n")))
; Shorthand.
> (define (foo bar) (print "Bar is " bar ".\n"))

; Local variables.
> (define (foo bar) (let ((thing "Bar"))
    (print thing " is " bar ".\n")))
> (foo "open")
Bar is open.
```

gambitscheme.org/wiki/images/a/a7/A_Tour_of_Scheme_in_Gambit.pdf

Function example

```scheme
> (define l
    (let
      ((d 4) (e 5))
      (lambda (a b c) (list a b c d e))
    )
  )
> (l 1 2 3)
(1 2 3 4 5)
```

gambitscheme.org/wiki/images/a/a7/A_Tour_of_Scheme_in_Gambit.pdf

cons

```
> (cons 1 '())
(1)
> (cons 1 (cons 2 null))
(1 2)
> (cons 1 (cons 2 (cons 3 null)))
(1 2 3)
> (define mylist (cons 1 (cons 2 (cons 3 null))))
> mylist
(1 2 3)
> (car mylist)
1
> (cdr mylist)
(2 3)
```

www.artificialworlds.net/presentations/scheme-02-basics/scheme-02-basics.html

More on lists

```
> (list "a" "b" "c")
("a" "b" "c")
> (list a b c)
reference to undefined identifier: a
> (list 'a 'b 'c)
(a b c)

> (equal?
(list 1 2 3)
(cons 1 (cons 2 (cons 3 '()))))
#t
```

www.artificialworlds.net/presentations/scheme-02-basics/scheme-02-basics.html

Quoting

```
> (list a b c)
*** ERROR IN (console)@1.7--Unbound variable: a
> (quote (a b c))
(a b c)
> (quote a b c)
*** ERROR IN (console)@2.1--Ill-formed special form: quote
> '(a b c)
(a b c)
> (define forty-two '(* 6 9))
> forty-two
(* 6 9)
> (eval forty-two)
54
```

www.artificialworlds.net/presentations/scheme-05-quotation/scheme-05-quotation.html

Null list

```
> ()
missing procedure expression
> (list)
()
> '()
()
> null
()
> 'null
null
```

www.artificialworlds.net/presentations/scheme-05-quotation/scheme-05-quotation.html

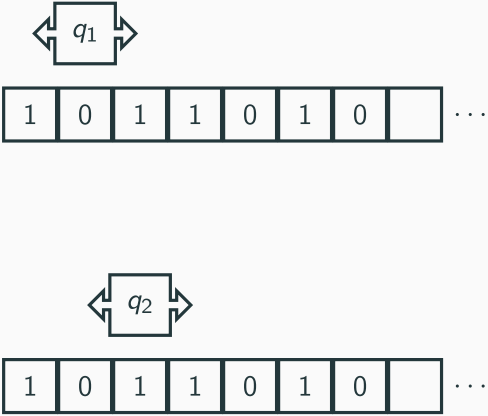
Closures

```
> (define (container value)
(lambda ()
(string-append "This container contains " value ".")))
> (define apple (container "an apple"))
> (define pie (container "a pie"))
> (apple)
"This container contains an apple."
> (apple)
"This container contains an apple."
> (pie)
"This container contains a pie."
```

gambitscheme.org/wiki/images/a/a7/A_Tour_of_Scheme_in_Gambit.pdf

Turing Machines

Visualisation



Symbols

Q Set of states (finite).

G Tape alphabet (finite).

B Blank symbol, element of G.

S Input alphabet, subset of $G \setminus \{B\}$.

δ Transition function.

q_0 Initial state, $\in Q$.

q_a Accept state, $\in Q$.

q_r Reject state, $\in Q$.

M Turing Machine: $[Q, G, B, S, \delta, q_0, q_a, q_r]$.

State Table

State	Input	Write	Move	Next
	B	B		Accept
0	0	0	L	0
	1	1	L	1
	B	B		Fail
1	0	0	L	1
	1	1	L	0

$$\delta(q_i, g_n) \to (q_j, g_m, L/R)$$

Turing's Second Example

A slightly more difficult example

We can construct a machine to compute the sequence

$$001011011101111011111\ldots$$

The machine is to be capable of five m-configurations, viz. o, q, p, f, b and of printing a, x, 0, 1. The first three symbols on the tape will be aaO; the other figures follow on alternate squares. On the intermediate squares we never print anything but x. These letters serve to keep the place for us and are erased when we have finished with them. We also arrange that in the sequence of figures on alternate squares there shall be no blanks.

Turing's Second Example: Table

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$		$P\text{ə}, R, P\text{ə}, R, P0, R, R, P0, L, L$	$\mathfrak{o}$
$\mathfrak{o}$	1	R, Px, L, L, L	$\mathfrak{o}$
	0		$\mathfrak{q}$
$\mathfrak{q}$	Any (0 or 1)	R, R	$\mathfrak{q}$
	None	$P1, L$	$\mathfrak{p}$
$\mathfrak{p}$	x	E, R	$\mathfrak{q}$
	ə	R	$\mathfrak{f}$
	None	L, L	$\mathfrak{p}$
$\mathfrak{f}$	Any	R, R	$\mathfrak{f}$
	None	$P0, L, L$	$\mathfrak{o}$

Turing's Second Example: JavaScript 1

```
// The contents of the tape.
var tape = []
s// The current position of the machine on the tape.
var pos = 0
// The current state;
var state = b;
```

Turing's Second Example: JavaScript 2

```javascript
// Writes a symbol to the current cell on the tape.
function write(sym) {
  tape[pos] = sym;
}

// Returns true iff the current cell contains sym.
function read(sym) {
  return sym == tape[pos] ? true : false;
}
```

Turing's Second Example: JavaScript 3

```
// Erases the symbol in the current cell of the tape.
function erase() {
  delete tape[pos];
}

// Returns true iff the current cell is blank.
// Returns true iff the current cell is blank.
function blank() {
  return typeof(tape[pos])
    == 'undefined' ? true : false;
}
```

Turing's Second Example: JavaScript 4

```
function b() {
  write('e');
  pos++;
  write('e');
  pos++;
  write('0');
  pos++;
  pos++;
  write('0');
  pos--;
  pos--;
  state = o;
}
```

Languages

Alphabet Finite set of symbols, denoted Σ.

Word Sequence of symbols, w from Σ.

Language Set of words, denoted L.

Length Of a word, denoted $|w|$.

Empty word Unique word of length 0, denoted λ.

Kleene star

Words $w_1 w_2$ is the concatenation of words w_1 and w_2.

Languages $L_1 L_2$ is the language resulting from the concatenation of all words in L_1 and all words in L_2, in that order.

Powers $L^0 = \{\lambda\}$, $L^1 = L$ and $L^{n+1} = L^n L$ for all $n > 1$.

Kleene Star

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

Note that treating the alphabet Σ as a language in itself, we get that Σ^* is the set of all words over Σ.

Example

$$
\begin{array}{rl}
\Sigma & \{0, 1\} \\
L & \{00, 01, 10, 11\} \\
w_1 & 01 \\
w_3 & 11 \\
w_1 w_3 & 0111 \\
\Sigma^* & \{\lambda, 0, 1, 00, 01, 10, 11, 001, 010, \ldots\} \\
L^* & \{\lambda, 00, 01, 10, 11, 0000, 0001, \ldots\} \\
L^+ & \{00, 01, 10, 11, 0000, 0001, \ldots\}
\end{array}
$$

Complexity Classes

Polynomial time

Definition

An algorithm is said to be solvable in *polynomial time* if the number of steps required to complete the algorithm for a given input is $O(n^k)$ for some nonnegative integer k, where n is the complexity of the input.

P complexity class

The P complexity class is the set of problems for which there exists, for each such problem, at least one algorithm to solve that problem in polynomial time.

mathworld.wolfram.com/P-Problem.html

Non-deterministic polynomial time

Definition

A problem is in the *NP* complexity class if it is solvable by a non-deterministic Turing Machine in polynomial time. A non-deterministic Turing Machine is one which may not have a unique action to take for some or all states and inputs.

P is a subset of NP

The P complexity class is a subset of P because all polynomial time solvabled problems can be modelled using Nondeterministic Turing Machinses.

mathworld.wolfram.com/P-Problem.html

Decision problems

Decision problems are problems where the answer is 0 or 1.

Restricting ourselves to decision problems is convenient and fair.

$f : \{0,1\}^n \to \{0,1\}$ is useful notation for considering decision problems.

Other problems can be easily (polynomial time) adapted into decision problems.

NP-complete problems

Definition

An problem is *NP*-hard if each problem in *NP* can be reduced to it in polynomial time.

_NP_-hard problems are hard

NP-hard problems are at least as hard to solve as the hardest *NP* problems. Note that *NP*-hard problems don't have to be *NP*.

Definition

An problem is *NP*-complete if it's in *NP* and in *NP*-hard.

Subset sum problem

Problem

Given a set of integers S, is there a non-empty subset whose elements sum to zero?

Example

Does $\{1, 3, 7, -5, -13, 2, 9, -8\}$ have such a subset?

Note

If somebody suggests a soltuion, it is very quick to check it. Being able to quickly verify a solution is a characteristic of NP problems.

Propositional logic

Literals are Boolean variables (can be True or False), and their negations. They are represented by lower case letters like a and x_i.

Clause are expressions based on literals, that evaluate as True or False based on the literals. We use not, and and or on the literals. We'll sometimes call them expressions.

Not is depicted by $\neg$. So "not a" is denoted by $\neg a$.

Or is depicted by $\vee$. So "a or b" is denoted by $a \vee b$.

And is depicted by $\wedge$. So "a and b" is denoted by $a \wedge b$.

Normal forms

CNF A clause is in Conjunctive Normal Form if it is a "conjunction of disjunctions": $(a \vee b) \wedge (\neg a \vee c) \wedge d$.

DNF A cluse is in Disjunctive Normal Form if it is a "disjunction of conjunctions": $(a \wedge b) \vee (\neg a \wedge c) \vee d$.

Converting to CNF and DNF

Every Boolean expression can be converted to CNF, and every Boolean expression can also be converted to DNF.

Four laws

The following four laws can be used to convert expressions to CNF and DNF. The first two are known as De Morgan's laws, and the latter two are called the distributivity laws.

Conversion laws

$$\neg(a \vee b) = \neg a \wedge \neg b$$

$$\neg(a \wedge b) = \neg a \vee \neg b$$

$$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$$

$$c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b)$$

Boolean Satisfiability Problem (SAT)

The problem

We're often interested in knowing if there is any setting of the variables in a Boolean expression that makes the expression true. Another way of asking the question is: is the expression satisfiable? The real question is, is there a polynomial time algorithm that takes as input *any* Boolean expression and outputs true if there is any setting, and false otherwise.

The problem is the prototypical NP-complete problem. Note that it's quick to check the correctness of a solution for a given expression.

CNF SAT

Only using CNF

All Boolean expressions can be converted in CNF. However, it's not always possible to do that in polynomial time. We can though, in polynomial time, create new expressions using some extra(neous) variables that are satisfiable if and only if the original expression is.

k-SAT

k-SAT

k-SAT is like SAT except that all expressions must be in CNF and each clause must be a disjunction of k literals.

2-SAT $(a \vee b) \wedge (\neg c \vee d) \wedge \dots$

3-SAT $(a \vee b \vee c) \wedge (\neg c \vee d \vee \neg a) \wedge \dots$

2-SAT is not NP-complete. There are polynomial time algorithms that solve it.

3-SAT is NP complete.

3-SAT is NP-Complete

Reduction

3-SAT is a special case of SAT, so 3-SAT must be in NP. We can reduce SAT to 3-SAT in polynomial time. First take the expression and convert it to a CNF expression (in polynomial time). Then we just need to convert each clause into a CNF expression with 3 literals per clause.

Suppose we have a clause with 1 literal: a. Convert this to $(a \vee u_1 \vee u_2) \wedge (a \vee u_1 \vee \neg u_2) \wedge (a \vee \neg u_1 \vee u_2) \wedge (a \vee \neg u_1 \vee \neg u_2)$. Suppose we have a clause with 2 literals: $a \vee b$. Convert this to $(a \vee b \vee u_1) \wedge (a \vee b \vee \neg u_1))$.

Now suppose we have a clause with n literals: $a \vee b \vee c \vee \ldots$. Convert this to $(a \vee b \vee u_1) \wedge (c \vee \neg u_1 \vee u_2)) \wedge \ldots \wedge (i \vee \neg u_{n-4} \vee u_{n-3}) \wedge (j \vee k \vee \neg u_{n-3})$.