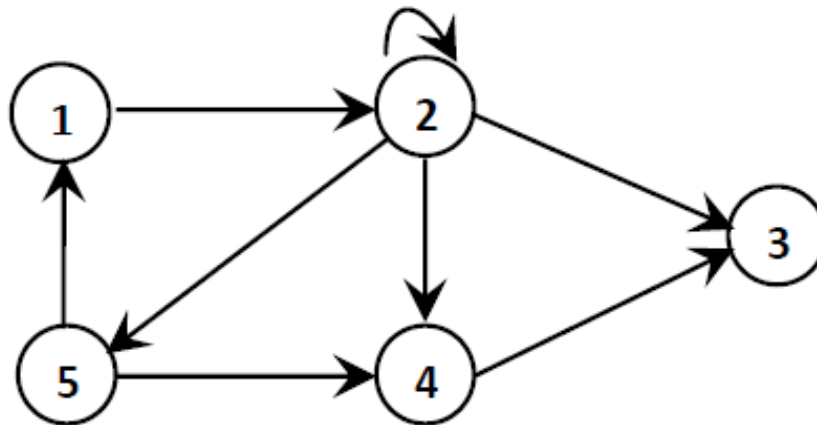
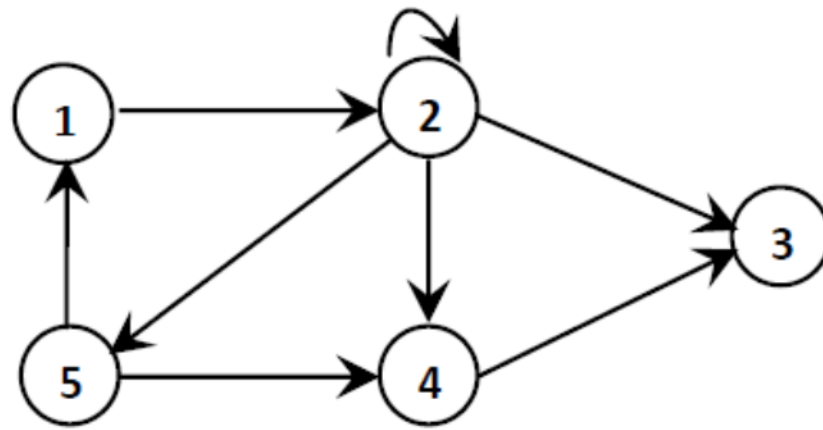


Lecture 4

Paths in Relations and Diagraphs

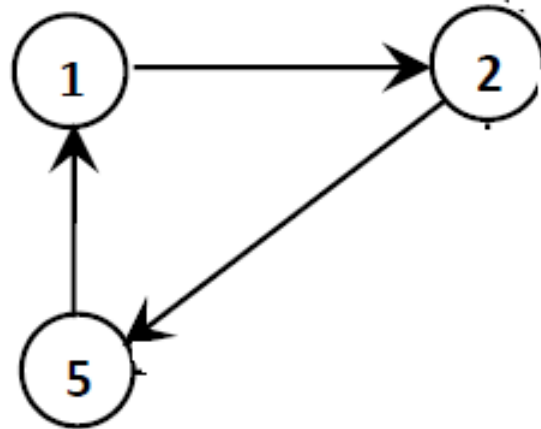
- ❑ A **path** is succession of edges.
- ❑ The **length** of a path is "the number of edges" in the path





- The path $\pi_1: 1, 2, 5, 4, 3$ is a path of length 4
- The path $\pi_2: 1, 2, 5, 1$ is a path of length 3 (three edges).
- The path $\pi_3: 2, 2$ is a path of length 1 from vertex 2 to itself.

A path that begins and ends at the same vertex is called a **cycle**.



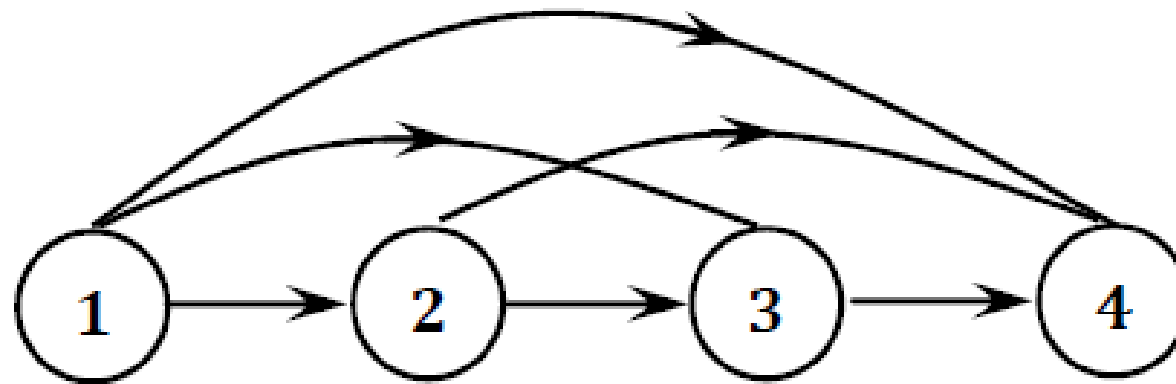
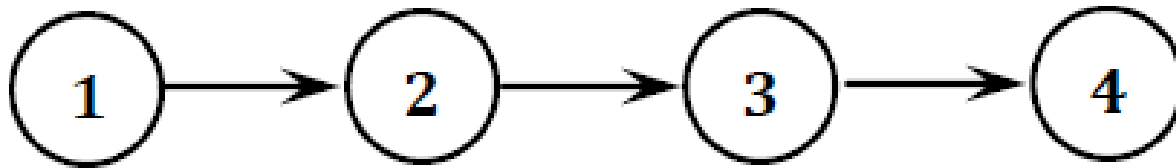
$$xR^n y$$

n is positive integer

We define a relation R^n as follows:

$xR^n y$ means that there is a path of length n from x to y in R .

$$R^{\infty}$$



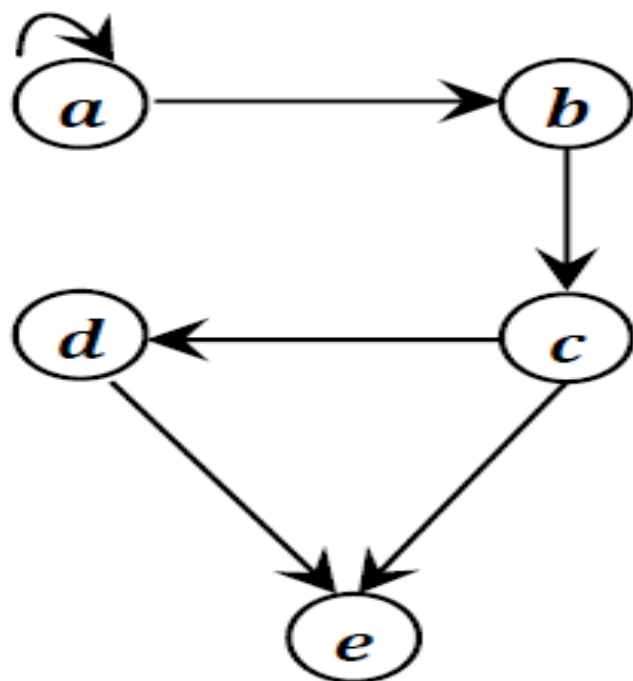
Let $A = \{a, b, c, d, e\}$ and

$$R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$$

Compute

❶ R^2

❷ R^∞



$a R^2 a$ since $a R a$ and $a R a$

$a R^2 b$ since $a R a$ and $a R b$

$a R^2 c$ since $a R b$ and $b R c$

$b R^2 e$ since $b R c$ and $c R e$

$b R^2 d$ since $b R c$ and $c R d$

$c R^2 e$ since $c R d$ and $d R e$

Hence $R^2 = \{(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)\}$ ■

$$R^\infty = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e)\}$$

Transitive closure

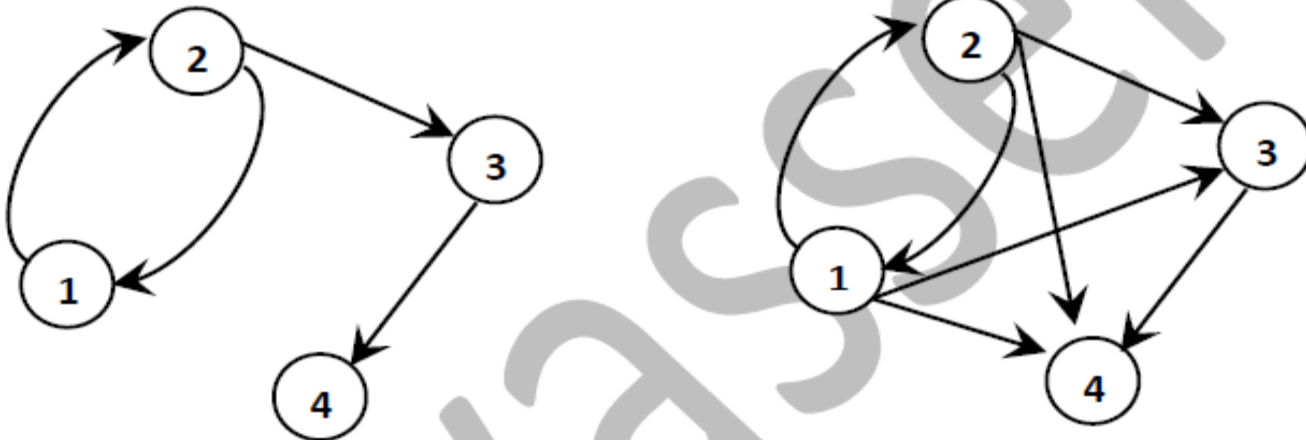
Transitive Closure $R_1 = R^\infty$

$(c,d) \dots (d,c) \rightarrow (c,c)$
 $(d,c) \dots (c,d) \rightarrow (d,d)$



Example

Let $A=\{1,2,3,4\}$ and $R=\{(1,2),(2,3),(3,4),(2,1)\}$, find the transitive closure of R .

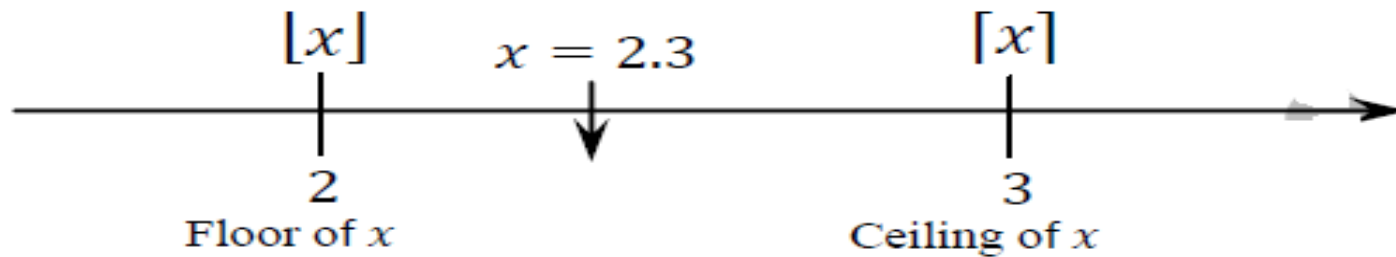


$$R^\infty = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4), (3,3)\}$$

Special Types of Functions

Floor and Ceiling Functions

Let x be a real number



The ***floor*** of a real number x , denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x

$$\lfloor 2 \rfloor = 2,$$

$$\lfloor \pi \rfloor = 3,$$

$$\lfloor -2.5 \rfloor = -3$$

The *ceiling* of a real number x , denoted by $\lceil x \rceil$, is the least integer greater than or equal to x

$$\lceil 2 \rceil = 2,$$

$$\lceil \pi \rceil = 4,$$

$$\lceil -2.5 \rceil = -2$$

$$\lfloor 2.3 \rfloor = 2$$

$$\lfloor 0.3 \rfloor = 0$$

$$\lfloor -2.3 \rfloor = -3$$

$$\lfloor 2.3 \rfloor = 3$$

$$\lfloor 0.3 \rfloor = 1$$

$$\lfloor -2.3 \rfloor = -2$$

$$\lfloor \sqrt{5} \rfloor = 3$$

$$\lfloor -0.2 \rfloor = 0$$

$$\lfloor -4 \rfloor = -4$$

$$\lfloor 5 \rfloor = 5$$

$$\lfloor 5 \rfloor = 5$$

$$\lfloor 0 \rfloor = 0$$

□ $[x] = \lfloor x \rfloor, \forall x \in \mathbb{Z}$ (integer number)

□ $[x] = \lfloor x \rfloor + 1, \forall x \notin \mathbb{Z}$ and x is rational (\mathbb{Q}).

□ The functions $f(x) = \lfloor x \rfloor$ and $f(x) = [x]$ map from \mathcal{R} to \mathcal{Z} is **onto** but **not one to one**

Mod Function

Define the function $f(x,y)=x \bmod y$ as the remainder when x is divided by y

where x is any integer ($x \in \mathbb{Z}$)

y is positive natural number ($y \in \mathbb{N}$)

$$f:(x,y)\rightarrow(x \bmod y)$$

domain $\{(x,y):x\in\mathbb{Z}, \text{ and } y\in\mathbb{Z}^+\}$

codomain $W=\mathbb{N}\cup\{0\}$

range is the interval $[0,y)$.

The mod function **onto** but **not one to one**,

$1 \bmod 12=1$ **and also** $13 \bmod 12=1$.

If $x \bmod y = r$, then $0 \leq r < y$

$7 \bmod 4 = 3$ $\frac{7}{4} = 1 + \frac{3}{4}$	$5 \bmod 1 = 0$ $\frac{5}{1} = 5 + \frac{0}{1}$	$5 \bmod 6 = 5$ $\frac{5}{6} = 0 + \frac{5}{6}$
$25 \bmod 7 = 4$ $\frac{25}{7} = 3 + \frac{4}{7}$	$16 \bmod 4 = 0$ $\frac{16}{4} = 4 + \frac{0}{4}$	$0 \bmod 4 = 0$ $\frac{0}{4} = 0 + \frac{0}{4}$

when x is negative

$$x \bmod y = y - [|x| \bmod y]$$

$$-26 \bmod 7 = 7 - [26 \bmod 7] = 7 - 5 = 2.$$

$-16 \bmod 3 = 2$	$-5 \bmod 1 = 1$	$-5 \bmod 5 = 5$
$3 - (16 \bmod 3) = 2$	$1 - (5 \bmod 1) = 1$	$5 - (5 \bmod 5) = 5$

Theorem

If $x \bmod y = r$, then y divides $x - r$.

Application

What day of the week will it be after 365 days from Wednesday?

Solution

Since after 7 days the **Wednesday** will come again

$$365 \text{ mod } 7 = 1$$

then Thursday is the required day

Barcodes



International Standard Book Number (ISBN)

ISBN 978-0-306-40615-7



Formally, the ISBN-10 check digit calculation is:

$$x_{10} = [(x_1 \times 1) + (x_2 \times 2) + (x_3 \times 3) + \dots + (x_9 \times 9)] \bmod 11]$$

Example-

Find the *check digit* for an ISBN-10 of 0 – 306 – 40615 – x_{10} ?

Solution

$$x_{10} = [(0 \times 1) + (3 \times 2) + (0 \times 3) + (6 \times 4) + (4 \times 5) + (0 \times 6) + (6 \times 7) + (1 \times 8) + (5 \times 9)] \bmod 11 = 145 \bmod 11 = 2 \quad \blacksquare$$

□ Formally, the ISBN-13 check digit calculation is:

$$x_{13} = 10 - [(x_1 + 3x_2 + x_3 + 3x_4 + x_5 + \cdots + 3x_{12}) \bmod 10]$$

Example

Find the check digit for an ISBN-13 of

$$978-0-306-40615-x_{13}$$

$$x_1 + 3x_2 + x_3 + 3x_4 + x_5 + \cdots + 3x_{12} =$$

$$= (9 \times 1) + (7 \times 3) + (8 \times 1) + (0 \times 3) + (3 \times 1) + (0 \times 3) + (6 \times 1)$$

$$+ (4 \times 3) + (0 \times 1) + (6 \times 3) + (1 \times 1) + (5 \times 3) =$$

$$= 9 + 21 + 8 + 0 + 3 + 0 + 6 + 12 + 0 + 18 + 1 + 15 = 93$$

$$x_{13} = 10 - (93 \bmod 10) = 10 - 3 = 7$$



Thus, the check digit is 7, the ISBN 978-0-306-40615-7.

Hash Function

If the computer memory cell is indexed from 0 to 10, then we have 11 places to store and retrieve numbers.

165			102	15	258	137		558		76
0	1	2	3	4	5	6	7	8	9	10

$$h(n) = n \bmod 11$$

$$h(165)=0,$$

$$h(102)=3,$$

$$h(15)=4,$$

$$h(3)=3$$

$$h(76)=10$$

The Hash function

$$h(x): \mathcal{N}(\text{natural numbers}) \rightarrow \{0, 1, \dots, 10\}$$

It is **not one to one**

$$137 \neq 258 \rightarrow h(137) = h(258) = 5$$

but it is **onto**

Example

Find the value of: $2h(258) - 4(93 \bmod 10)$

Solution

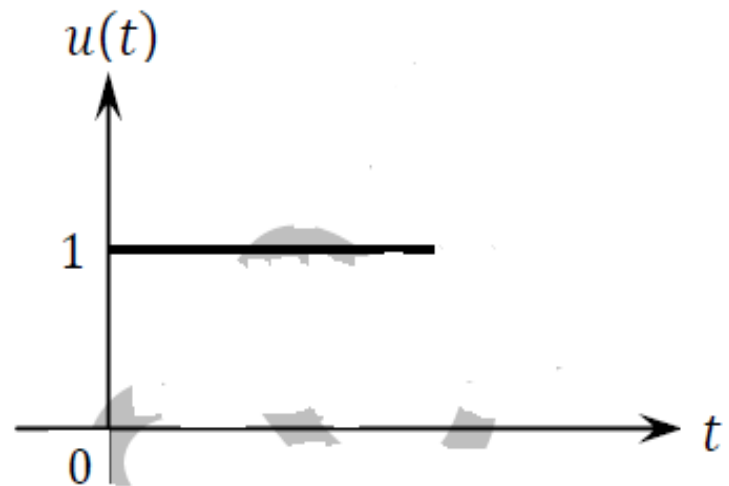
$$2h(258) - 4(93 \bmod 10) = 2(5) - 4(3) = -2$$

Try to

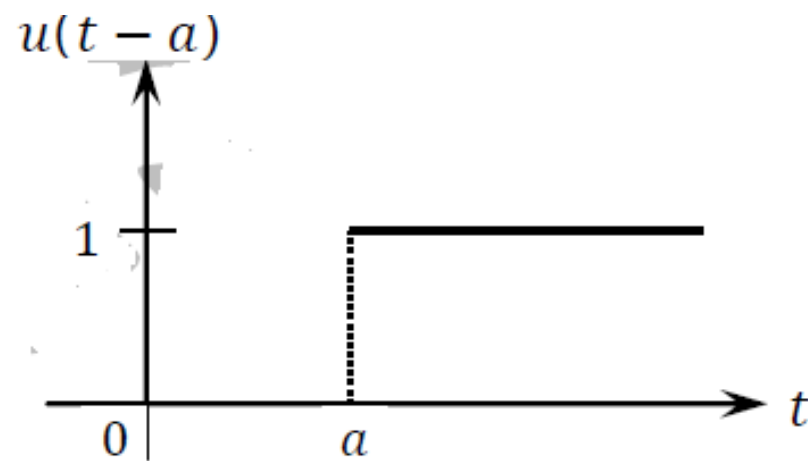
find the value of: $2h(122) + 5 \lceil 2.7 \rceil - 2 \lfloor -2.7 \rfloor - 5 \times (10 \bmod 7)$

Unit Step Function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

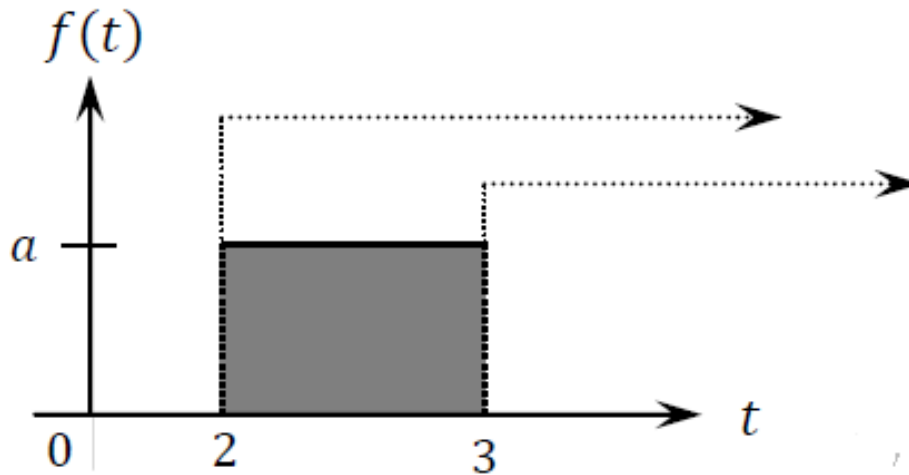


$$u(t - a) = \begin{cases} 0 & t - a < 0 \\ 1 & t - a \geq 0 \end{cases}$$



Example-

Graph the function: $f(t) = a$, $2 < t < 3$; express the function $f(t)$ in terms of unit step functions.



$$f(t) = a[u(t-2) - u(t-3)]$$

Example-

Express the function $g(t)$ in terms of unit step functions;

$$g(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases}$$

$$g(t) = t u(t) - t u(t-1) + u(t-1)$$

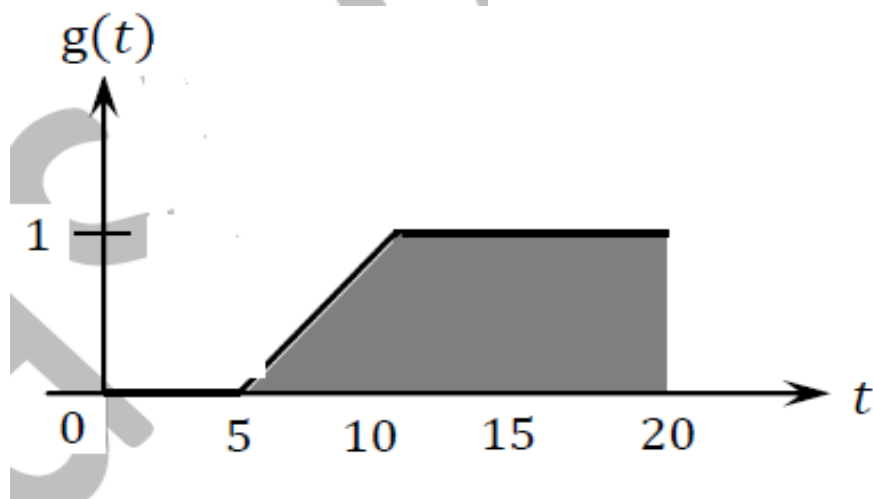
$$u(t) = 1 \quad 0 \leq t < 1$$

$$g(t) = t - (t-1) u(t-1)$$

Example-38

Express the function $g(t)$ in terms of unit step functions;

$$g(t) = \begin{cases} 0 & 0 < t < 5 \\ (t - 5)/5 & 5 \leq t < 10 \\ 1 & t \geq 10 \end{cases}$$



$$g(t) = 0 \times [u(t - 0) - u(t - 5)] + \left(\frac{t-5}{5}\right) \times [u(t - 5) - u(t - 10)] + 1 \times [u(t - 10)]$$

$$g(t) = \left(\frac{t-5}{5}\right) u(t - 5) - \left(\frac{t-5}{5}\right) u(t - 10) + u(t - 10)$$

$$g(t) = \left(\frac{t-5}{5}\right) u(t - 5) - \left(\frac{t-5}{5} - 1\right) u(t - 10)$$

$$g(t) = \left(\frac{t-5}{5}\right) u(t - 5) - \left(\frac{t-10}{5}\right) u(t - 10)$$

