Lecture 3

Ch-2
Relations & Functions

Contents

- binary relation
- Matrices of relations
- Closure of relations

The Cartesian product

$$X = \{1, 2, 3, 4\}, \ Y = \{a, b, c\}$$

$$X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}$$

$$\mathrm{X}^2 = \mathrm{X} \times \mathrm{X} = \left\{ \begin{matrix} (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), \\ (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) \end{matrix} \right\}$$

Definition

The **binary relation** R from a set A to B is a subset of the **Cartesian product** $A \times B$ ($R \subset A \times B$)

If $(a,b) \in R$, we say that a is **related** to b **by** R, we also write $\boldsymbol{a\ R\ b}$

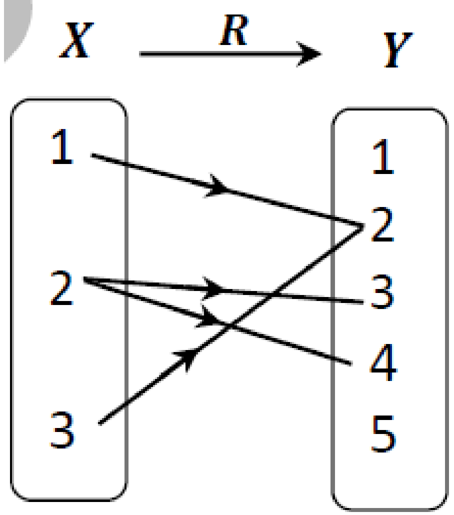
where **A** is the domain [D_R] and **B** is the range (or codomain [C_R]).

If a is not related to b by R we write $a \not R$ b.

Illustrative examples

❶ Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$, then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B and written as: $a\ R\ b$

❷ Let $A = z^+$, the set of all positive integers. If we define the following relation R on A: $a\ R\ b$ iff $\underline{a \text{ divides } b}$, then $4\ R\ 12$, but



$$D_R = \{1, 2, 3\}$$

$$\text{Range} = \{2, 3, 4\}$$

$$C_R = \{1, 2, 3, 4, 5\}$$

<u>NOTE THAT</u>: the **range** is always **contained** in the **codomain** as illustrated above, where the range is the set $\{2, 3, 4\}$ while the codomain of R is $C_R = \{1, 2, 3, 4, 5\}$. ■

Inverse of the relation

If R is a relation from A to B , then the **inverse** of the relation R is R^{-1}.

it is a **relation** from B to A, where $b\, R^{-1} a$

$$R = \{(x, y) | (x \in X) \wedge (y \in Y)\}$$
$$= \{(1,2), (2,3), (2,4), (3,2)\}$$

The ***inverse*** of R is:

$$R^{-1} = \{(y, x) | (x, y) \in R\}$$
$$= \{(2,1), (3,2), (4,2), (2,3)\}$$

Complementary Relation

The complement of R is $\bar{R}$ which is a subset of $A \times B$. It is, of course, a relation from A to B that can be expressed simply in terms of R:

$a\ \bar{R}\ b$ if and only if $a \not{R}\ b$.

Example- 1

If $X = \{1,2,3,4\}$, $Y = \{a,b,c\}$, let

$R = \{(1,a),(1,b),(2,b),(2,c),(3,b),(4,a)\}$ and

$S = \{(1,b),(2,c),(3,b),(4,b)\}$, compute

① $\bar{R}$ ② $R \cap S$ ③ $R \cup S$ ④ R^{-1}

Solution

$$X \times Y = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c),\\ (4,a),(4,b),(4,c)\}$$

Thus, we can find the **complement of** R:

① $\bar{R} = \{(1,c),(2,a),(3,a),(3,c),(4,b),(4,c)\}$

② $R \cap S = \{(1,b),(3,b),(2,c)\}$

③ $R \cup S = \{(1,a),(1,b),(2,b),(2,c),(3,b),(4,a),(4,b)\}$

④ Since $(x,y) \in R^{-1}$ if and only if $(y,x) \in R$, then

$$R^{-1} = \{(a,1),(b,1),(b,2),(c,2),(b,3),(a,4)\}$$ ■

Example- 2

If $X = \{2, 3, 4\}$, $Y = \{3, 4, 5, 6\}$,

① Define the relation R_1 from X to Y by $(x, y) \in R$ where X divide Y.

② Define the relation R_2 from X to Y by $(x, y) \in R$ where $x + y \leq 7$.

③ Find the domain and the range of the relation R_1, R_2.

④ Draw a schematic diagraph of ordered pairs for each relation.

Solution

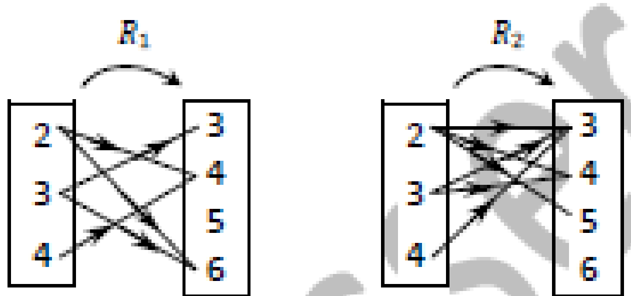
① $R_1 = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$.

② $R_2 = \{(2,3), (2,4), (2,5), (3, 3), (3,4), (4, 3)\}$.

③ $D_{R_1} = \{2, 3, 4\}$, $\quad \text{Range}_{R_1} = \{3, 4, 6\}$,

$D_{R_2} = \{2, 3, 4\}$, $\quad \text{Range}_{R_2} = \{3, 4, 5\}$ ■ 1, 2, 3

④



■ 4

Example- 3

❶ Find the relation '**less than**' on the set $X = \{1, 2, 3, 4\}$.

❷ Draw the graphical representation of R.

❸ Write down the **relation matrix** (Boolean' matrix) for the relation R.

Solution

❶ $R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$ ■1

❷

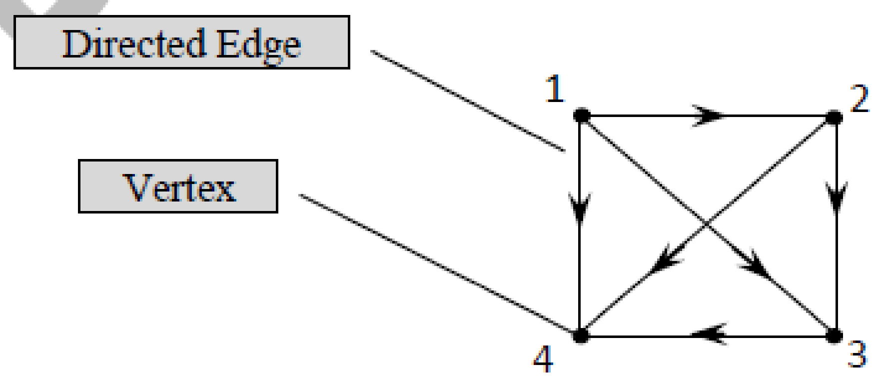
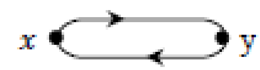
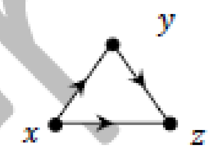
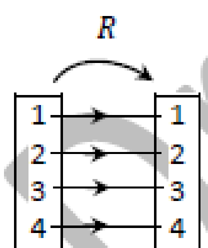


Fig. (2.2) ■2

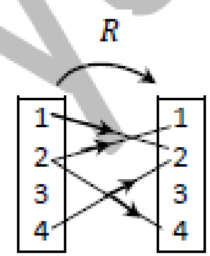
2.3 Properties of relations on a set A

	Property	**Definition**	**Diagraph**
①	**Reflexive**	$x\ R\ x$ for every $x \in A$	
②	**Symmetric**	$x\ R\ y \Rightarrow y\ R\ x$ $\forall\ x, y \in A$	
③	**Transitive**	$x\ R\ y$, and $y\ R\ z\ \Rightarrow x\ R\ z$ $\forall\ x, y, z \in A$	
④	**Irreflexive**	x is not a relation with x	
⑤	**Equivalence**	Reflexive, Symmetric and Transverse	

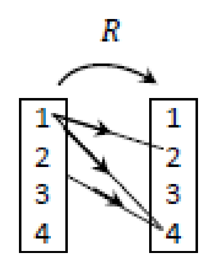
Reflexive



Symmetric



Transitive



Example- 4

Let R be the relation on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y \ \forall\ x, y \in X$.

① **Define the relation** R from X to X.

② Find the **domain** and the **range** of the relation R.

③ Draw its **diagraph** of the relation R.

④ Show that if this relation is **reflexive**, **symmetric** or **transitive**.

Solution

① $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

② $D = \{1, 2, 3, 4\}$, and Range $= \{1, 2, 3, 4\}$ are equal.

③

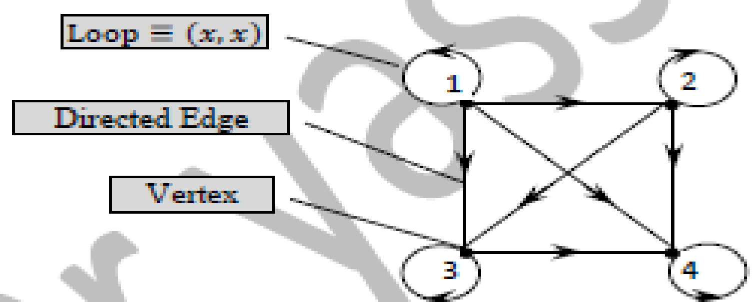


Fig. (2.3)

④ The relation is <u>reflexive</u> because there exist a loop at every vertex, defined by the ordered pairs: $\{(1,1), (2,2), (3,3), (4,4)\}$ in R. The relation R is not <u>symmetric</u>[1] because $(1,2) \in R$ but $(2,1) \notin R$. The relation R is <u>transitive</u> because for all $x, y, z \in X$, when (x, y) and $(y, z) \in R$ we have found $(x, z) \in R$. ∎

Example- 5

Classify the following relations, which are defined on the set of integers z:

❶ "is less than or equal"

❷ "is divisible by"

Solution

- Since $x \leq x$ is always true, so the relation is reflexive (and not irreflexive).
- if $x \leq y$, then it is never the case that $y \leq x$ except when $x = y$, so the relation is antisymmetric (and not symmetric). Therefore, the relations "$\leq$" and "$\geq$" are antisymmetric relations.

❶

- if $x \leq y$ and $y \leq z$, then $x \leq z$, so the relation is transitive.

- This relation is reflexive since n is divisible by n, $\forall n \in z$.
- This relation is transitive[2], for example 2 divides 4 and 4 divides 8, this implies that 2 divides 8.

❷ - This relation is neither symmetric nor antisymmetric. To see that it is not symmetric, note that 2 divides 4 but 4 not divides 2. To see that it is not antisymmetric, note that –2 is divisible by 2 and 2 is divisible by –2, but $2 \neq -2$.

2.4. Composition of Relations

Let R_1 be a relation from X to Y and R_2 from Y to Z. Then the composition of R_1 and R_2, denoted by $R_2 \circ R_1$, is the relation from X to Z, and equals

$$R_2 \circ R_1 = \{(x,z) | (x,y) \in R_1 \text{ and } (y,z) \in R_2 \text{ for some } y \in Y\}$$

Example- 6

Let $R_1 = \{ (1,2), (1,6), (2,4), (3,4), (3,6), (3,8)\}$ and

$$R_2 = \{(2,\mathrm{u}), (4,\mathrm{s}), (4,\mathrm{t}), (6,\mathrm{t}), (8,\mathrm{u})\}. \text{ Find } \mathrm{R}_2 \circ \mathrm{R}_1$$

Solution

$$R_2 \circ R_1 = \{(1,u), (1,t), (2,s), (2,t), (3,s), (3,t), (3,u)\} \quad \blacksquare$$

Matrices of relations $\boldsymbol{M_R}$

$$R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

$$\begin{array}{c c} & \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The matrix of the relation R, where $R = \{(1,b),(1,d),(2,c),(3,c),(3,b),(4,a)\}$ from $X = \{1,2,3,4\}$ to $Y = \{a,b,c,d\}$ relative to the ordering $1,2,3,4$ and $\{a,b,c,d\}$ is:

$$\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The matrix of the relation: $R = \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$ on $\{a,b,c,d\}$, relative to the ordering a,b,c,d is:

$$\begin{array}{c} \\ \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example- 11

If $R_1 = \{(1,x),(1,y),(2,x),(3,x)\}$ and

$R_2 = \{(x,b),(y,b),(y,a),(y,c)\}$,

with ordering 1, 2, 3; x, y; and a, b, c

① Find the matrix A_1 of the relation R_1 (relative to the given ordering).

② Find the matrix A_2 of the relation R_2 (relative to the given ordering).

③ Find the matrix of the product $A_1 \times A_2$.

④ Use the product $A_1 \times A_2$ to find the composition $R_2 \circ R_1$.

⑤ Find $R_2 \circ R_1$ as a set of ordered pairs.

Solution

① $A_1 = M_{R_1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array}$

② $A_2 = M_{R_2} = \begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$

③ $A_1 \times A_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$

④ $R_2 \circ R_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$

⑤ $R_2 \circ R_1 = \{(1,a),(1,b),(1,c),(2,b),(3,b)\}$ ■

Note that, we have changed each non zero entry in $A_1 \times A_2$ to be 1 to get $R_2 \circ R_1$.

2.5.1. Operations on the Matrix Representations of R

If R and S are relations on set A, then

❶ $M_{R \cap S} = M_R \wedge M_S$

❷ $M_{R \cup S} = M_R \vee M_S$

❸ $M_{R^{-1}} = (M_R)^T$

Example- 12

Let $A = \{1, 2, 3\}$, let R and S be relations on A. Suppose that the matrices of R and S are $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

① $M_{\bar{R}}$ ② $M_{R \cap S}$ ③ $M_{R \cup S}$ ④ $M_{R^{-1}}$

Solution

① $M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

② $M_{R \cap S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

③ $M_{R \cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

④ $M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ ■

- To get $R_2 \circ R_1$ we have changed each non zero entry in $A_1 \times A_2$ to be 1.

- To get $M_{\bar{R}}$ (complement) from M_R we have changed each zero to be 1 **and** each 1 to be zero in M_R.

- To get $M_{R^{-1}}$ we have changed the rows into columns and columns into rows for the matrix M_R.

Closure

Reflexive closure

If $A = \{a_1, a_2, a_3, \ldots\}$

Then $\Delta = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), \ldots\}$

R is not **reflexive**

$R_1 = R \cup \Delta$

R_1 is called **reflexive closure of** R

symmetric closure

If $R = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots\}$

Then $R^{-1} = \{(b_1, a_1), (b_2, a_2), (b_3, a_3), \ldots\}$

R is not **symmetric**

$$R_1 = R \cup R^{-1}$$

R_1 is called **symmetric closure** of R

Consider the relation that $R = \{(1,1),(1,2),(2,3),(3,1)\}$ defined on the set $X = \{1,2,3\}$. Find the relations R_1 that makes R reflexive and then find R_2 that makes R symmetric.

Solution

$$\begin{aligned} R_1 = R \cup \Delta \quad &= \{(1,1),(1,2),(2,3),(3,1)\} \cup \{(1,1),(2,2),(3,3)\} \\ &= \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,3)\} \end{aligned}$$

$$\begin{aligned} R_2 = R \cup R^{-1} &= \{(1,1),(1,2),(2,3),(3,1)\} \cup \{(2,1),(3,2),(1,3)\} \\ &= \{(1,1),(1,\ ),(2,1),(2,3),(3,2),(3,1),(1,3)\} \quad \blacksquare \end{aligned}$$

Transitive closure

Transitive Closure $R_1 = R^{\infty}$

Example- 15

Graph the relation: $R = \{(1,2),(2,3),(1,4)\}$ and then graph its transitive closure.

Solution

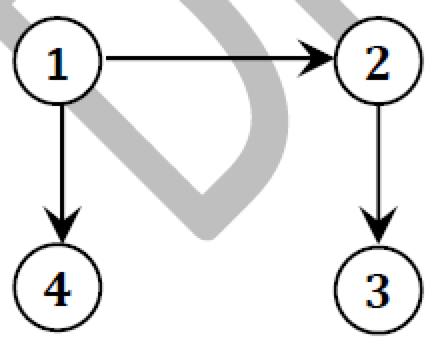


Fig. (2.7**a**)

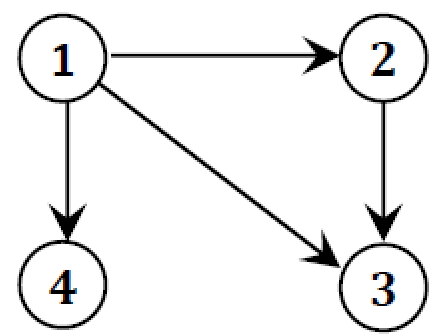


Fig. (2.7**b**)

The **transitive closure** of a relation R is the smallest transitive relation containing R. Therefore, the transitive closure is:

$R_1 = \{(1,2),(2,3),(\mathbf{1},\mathbf{3}),(1,4)\}$. Note that: $R \subset R_1 \subset R^{\infty}$. ■

Example- 16

> Let $A = \{1, 2, 3.4\}$ and $R = \{(1,2), (2,3), (3,4), (2,1)\}$, find the transitive closure of R.

Solution

The transitive closure is the smallest transitive relation containing R,

$$R_1 = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,4)\}$$

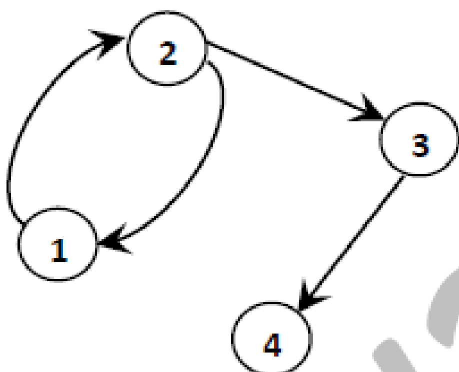


Fig. (2.8a)

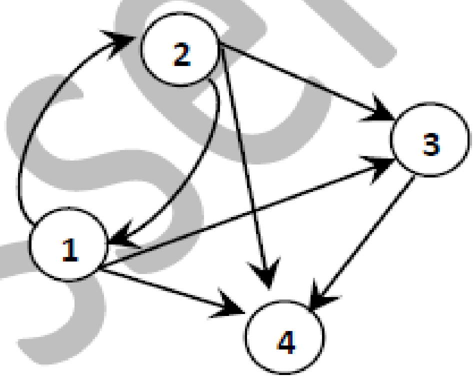


Fig. (2.8b)

Here, $R \subset R_1 \equiv R^{\infty}$. ■