Lecture 6

Binomial Theorem

n is **rational** or **negative number**

$$(a+b)^n = a^n\ b^0\ +\ \frac{n}{1!}a^{n-1}\ b^1$$

$$+\frac{n(n-1)}{2!}a^{n-2}\ b^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}\ b^3$$

$$+\cdots$$

$$\left|\frac{b}{a}\right|<1$$

Example

Expand: $(1-x)^{\frac{1}{3}}$ if $|x|<1$

$$a=1, b=-x\,, n=\frac{1}{3}$$

$$(1-x)^{\frac{1}{3}}=1^{\frac{1}{3}}+\frac{(1/3)}{1!}(1)^{\frac{1}{3}-1}(-x)^{1}$$

$$+\frac{(1/3)\left(\frac{1}{3}-1\right)}{2!}(1)^{\frac{1}{3}-2}(-x)^{2}$$

$$+\frac{(1/3)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(1)^{\frac{1}{3}-3}(-x)^{3}$$

$$=1-\frac{1}{3}x-\frac{1}{9}x^{2}-\frac{5}{81}x^{3}$$

Example

Write down the first four terms of the expansion

$$\frac{1}{(1+x)^2} = (1+x)^{-2} \text{ if } \quad |x| < 1$$

$$a = 1, b = x\,, n = -2$$

$$(1-x)^{\frac{1}{3}} = 1^{-2} + \frac{(-2)}{1!}(1)^{-2-1}(x)^1$$

$$+ \frac{(-2)(-2-1)}{2!}(1)^{-2-2}(x)^2$$

$$+ \frac{(-2)(-2-1)(-2-2)}{3!}(1)^{-2-3}(x)^3$$

$$= 1 - 2x + 3x^2 - 4x^3$$

Example

Write down the first four terms of the expansion

$\frac{1}{(1+x)^2}$ if $|\boldsymbol{x}| > \mathbf{1}$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = \left[x\left(1+\frac{1}{x}\right)\right]^{-2} = \boldsymbol{x^{-2}}\left(1+\frac{1}{x}\right)^{-2}$$

$a = 1, b = \frac{1}{x}, n = -2$

$$= \boldsymbol{x^{-2}}\ [1^{-2} + \frac{(-2)}{1!}(1)^{-2-1}\left(\frac{1}{x}\right)^1$$

$$+\frac{(-2)(-2-1)}{2!}(1)^{-2-2}\left(\frac{1}{x}\right)^2$$

$$+\frac{(-2)(-2-1)(-2-2)}{3!}(1)^{-2-3}\left(\frac{1}{x}\right)^3]$$

$$=\boldsymbol{x^{-2}}\ [1 - 2\frac{1}{x} + 3\frac{1}{x^2} - 4\frac{1}{x^3}] = \boldsymbol{x^{-2} - 2x^{-3} + 3x^{-4} - 4x^{-5}}$$

Example 24

Find the value of $\sqrt{50}$ approximated to four digits.

Solution

$$\sqrt{50} = \sqrt{49+1} = (49+1)^{\frac{1}{2}} = \left[49\left(1+\frac{1}{49}\right)\right]^{\frac{1}{2}} = \sqrt{49}\left(1+\frac{1}{49}\right)^{\frac{1}{2}}, \qquad \left|\frac{1}{49}\right| < 1$$

$$= 7 \times \left[1^{\frac{1}{2}} + \left(\frac{1}{2}\right)(1)^{\left(\frac{1}{2}\right)-1}\left(\frac{1}{49}\right)^{1} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(1)^{\left(\frac{1}{2}\right)-2}\left(\frac{1}{49}\right)^{2}\right.$$

$$\left. + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(1)^{\left(\frac{1}{2}\right)-3}\left(\frac{1}{49}\right)^{3}\right]$$

$$= 7 \times [1 + 0.0102 - 0.0005] = 7.071 \qquad \blacksquare$$

Mathematical Induction

Example 1, 2, 3, 11

Ch 4
Sequence and Series

A sequence is a list of numbers in a defined order.

- $1, 3, 5, 7, \ldots.$

$a_n = 2n - 1$

- $-1, -3, -5, -7, \ldots.$

$a_n = 1 - 2n$

- $0, 3, 8, 15, \ldots.$

$a_n = n^2 - 1$

- $-2, 4, -6, 8, \ldots.$

$$a_n = (-1)^n 2n$$

- $\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \ldots.$

$$a_n = (-1)^{n+1} \frac{n}{n+1}$$

- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$

$$a_n = \frac{1}{2^n}$$

Types of Sequences

Finite sequence $\{a_n\}_1^n$

$$\{a_n\}_1^n = \{a_1, a_2, a_3, \ldots, a_n\}$$

Infinite sequence $\{a_n\}_1^\infty$

$$\{a_n\}_1^\infty = \{a_1, a_2, a_3, \ldots\}$$

Arithmetic Sequences

$$(\boldsymbol{a}, \boldsymbol{a}+\boldsymbol{d}, \boldsymbol{a}+\boldsymbol{2d}\ , \boldsymbol{a}+\boldsymbol{3d}+\cdots.., \boldsymbol{a}+(\boldsymbol{n}-\boldsymbol{1})\boldsymbol{d})$$

- d is called "common difference"

$d = a_{n+1} - a_n$

- The n-th term $a_n = a + (n-1)d$

- The n-th partial sum s_n

$$s_n = \frac{n}{2}(a + a_n) = \frac{n}{2}\ [2a + (n-1)d]$$

Geometric Sequences

$$(\boldsymbol{a}, \boldsymbol{ar}, \boldsymbol{ar^2}, \boldsymbol{ar^3}, \ldots, \boldsymbol{ar^{n-1}})$$

- r is called "common ratio"

$$r = \frac{a_{n+1}}{a_n}$$

- The n-th term $a_n = a\, \boldsymbol{r^{n-1}}$

- The n-th partial sum s_n

$$s_n = \frac{a(r^n - 1)}{r - 1} \qquad r \neq 1$$

Increasing and Decreasing Sequences

A sequence $\{a_n\}_1^\infty$ is called:

❶ Strictly increasing if $a_1 < a_2 < a_3 <..< a_n < \cdots$

❷ Increasing if $a_1 \le a_2 \le a_3 \le..\le a_n \le \cdots$

❸ Strictly decreasing if $a_1 > a_2 > a_3 >..> a_n > \cdots$

❹ Decreasing if $a_1 \ge a_2 \ge a_3 \ge..\ge a_n \ge \cdots$

Example

$$\left\{\frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \ldots\right\}$$

show that the sequence is strictly increasing

$$a_n = \frac{n}{n+1}, \quad a_{n+1} = \frac{n+1}{n+2}$$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+2)(n+1)} > \mathbf{0}$$

$$a_{n+1} \div a_n = \frac{n+1}{n+2} \times \frac{n+1}{n} = \frac{(n+1)^2}{(n+2)(n)} = \frac{n^2+2n+1}{n^2+2n}$$

$$= 1 + \frac{1}{n^2+2n} > \mathbf{1}$$

Recursively Sequences

- The k^{th} order linear **non-homogenous** recurrence relation (RR) is:

$$c_0x_n + c_1x_{n-1} + c_2x_{n-2} + \cdots + c_kx_{n-k} = b_n$$

- The k^{th} order linear **homogenous** RR is:

$$c_0x_n + c_1x_{n-1} + c_2x_{n-2} + \cdots + c_kx_{n-k} = 0$$

if $b_n = 0$, ***homogenous.***

$b_n \neq 0$ ***non- homogenous***

Order

$$c_0 x_n + \cdots + c_k x_{n-k} = 0 \quad k^{th} \text{ order}$$

$$c_0 x_n + c_1 \boldsymbol{x_{n-1}} = 0 \quad 1^{th} \text{ order}$$

$$c_0 x_n + \cdots + c_2 \boldsymbol{x_{n-2}} = 0 \quad 2^{th} \text{ order}$$

$$c_0 x_n + \cdots + c_3 \boldsymbol{x_{n-3}} = 0 \quad 3^{th} \text{ order}$$

$$c_0 x_n + \cdots + c_4 \boldsymbol{x_{n-4}} = 0 \quad 4^{th} \text{ order}$$

homogenous and ***non- homogenous***

homogenous

$$c_0 x_n + \cdots + c_k x_{n-k} = 0$$

non- homogenous

$$c_0 x_n + \cdots + c_k x_{n-k} = 5\mathrm{n}$$

Linear

- Linear

All coefficients are constants

$$c_0, c_1, c_2 \ldots, c_k \text{ are } (4, -2, 8, \ldots)$$

- Not linear

$3n\, x_n$ ✗

$2 \; x_n \; x_{n-1}$ ✗

$2 \; {x_n}^2$ ✗

Examples

$S_n = 2S_{n-1}$	LHRR
$f(n) = f(n-1) + f(n-2)$	LHRR
$a_n = 3a_{n-1}a_{n-2}$	Not **LHRR** because the term $a_{n-1}a_{n-2}$
$a_n - a_{n-1} = 2n$	Not L**H**RR because $2n$ must be zero
$a_n = 3\,n\,a_{n-1}$	Not **L**HRR because $3n$ is not constant coefficient
$a_n = a_{n-1} + a_{n-2}^2$	Not **LHRR** because the exponent of a_{n-2}

Example 4

If $a_1 = 1$, and $a_n = n\,a_{n-1}$, find a_2, a_3 and a_4

Solution

$$\boldsymbol{a_n = n\,a_{n-1}}$$

$$\boldsymbol{a_1} = 1$$

$$\boldsymbol{a_2} = 2\ a_1 = 2 \times 1 = 2$$

$$\boldsymbol{a_3} = 3\,a_2 = 3 \times 2 = 6$$

$$\boldsymbol{a_4} = 4\ a_3 = 4 \times 6 = 24$$

Example 6

Let $(f_1, f_2, f_3, \ldots)$ denote the Fibonacci sequence,

$f_n = f_{n-1} + f_{n-2}$ for $n > 2$ and $f(1) = f(2) = 1$

Find the first five terms of Fibonacci sequence.

Solution

$$f_1 = 1, \quad f_2 = 1, \qquad f_3 = f_1 + f_2 = 1 + 1 = 2$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3, \quad f_5 = f_4 + f_3 = 3 + 2 = 5$$

Solving Recurrence Relations

Iteration Method

Example

Solve the recurrence relation $a_n = a_{n-1} + 3$,subject to the initial condition $a_1 = 2$

$$a_n = a_{n-1} + 3$$

$$= (a_{n-2} + 3) + 3 = a_{n-2} + 6 = a_{n-2} + 2 * 3$$

$$= (a_{n-3} + 3) + 2 * 3 = a_{n-3} + 9 = a_{n-3} + 3 * 3$$
$$= (a_{n-4} + 3) + 3 * 3 = a_{n-4} + 12 = a_{n-4} + 4 * 3$$
$$\vdots$$
$$= a_{n-k} + k * 3$$

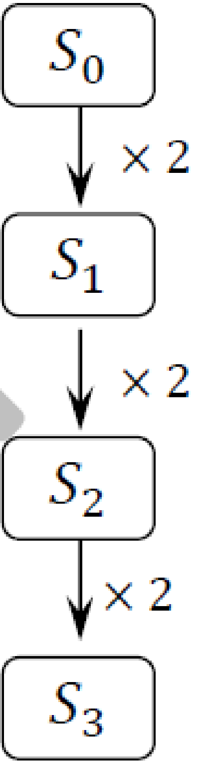
Put $k = n - 1$

$$a_n = a_{n-(n-1)} + 3(n-1) = a_1 + 3(n-1) = 2 + 3(n-1) = 3n - 1$$

Example 8

Solve the recurrence relation $S_n = 2S_{n-1}$ subject to the initial condition $S_0 = 1$

Solution



$$
\begin{aligned}
S_n &= 2S_{n-1} \\
&= 2(2\,S_{n-2}) = 2 * 2\,S_{n-2} = 2^2 S_{n-2} \\
&= 2 * 2\,(2\,S_{n-3}) = 2 * 2 * 2\,S_{n-3} = 2^3 S_{n-3} = \cdots = \\
&= 2^n\,S_{n-n} = 2^n\,S_0 = 2^n \qquad \twoheadrightarrow \qquad \boldsymbol{S_n = 2^n}\,,
\end{aligned}
$$

We have proceeded to n terms to produce S_0 directly. ■

Example

Solve the recurrence relation $P_n = a + sP_{n-1}$ of the economic model where, a and s are parameters depend on the model

Solution

$$
\begin{aligned}
P_n &= a + s\,P_{n-1} \\
&= a + s\,(a + s\,P_{n-2}) = a + as + s^2 P_{n-2} \\
&= a + as + s^2(a + sP_{n-3}) = a + as + as^2 + s^3 P_{n-3} \\
&\ \vdots \qquad\qquad\qquad \vdots \\
&= \Big(\overbrace{a + as + as^2 + as^3 + \cdots + as^{n-1}}^{\text{Sum of } n - \text{terms of a geometric series}}\Big) + s^n P_{n-n} \\
&= \frac{a(s^n - 1)}{s-1} + s^n P_0 \qquad \blacksquare
\end{aligned}
$$

Example 10

Solve the following system of recurrence relations in terms of x_0 and y_0

$$x_{n+1} = 7x_n + 4y_n$$
$$y_{n+1} = -9x_n - 5y_n, \quad \text{for } n \geq 0$$

Solution

Combine the above equations into a single matrix equation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

Or, $\mathbf{x}_{n+1} = A\mathbf{x}_n$, where $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$, and $\mathbf{x}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$

We see that:

$$\mathbf{x}_1 = A\mathbf{x}_0,$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = A^2\mathbf{x}_0,$$

$$\vdots$$

$$\mathbf{x}_n = A^n\mathbf{x}_0,$$

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$$A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$$

So, we can get $\mathbf{x}_n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} (1+6n)x_0 + (4n)y_0 \\ (-9n)x_0 + (1-6n)y_0 \end{bmatrix}$

and hence, $x_n = (1+6n)x_0 + (4n)y_0$ and

$$y_n = (-9n)x_0 + (1-6n)y_0 \qquad \blacksquare$$

Sigma Notation Σ

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the kth term.

The index k starts at $k = 1$.

$$\sum_{k=1}^{k=6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

④ Note the difference between the two notations:

$$\sum_1^\infty a_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n + ..$$

$$\prod_1^\infty a_n = a_1 \times a_2 \times a_3 \times a_4 \times .. \times a_n \times ..$$

Example 15

If $a_n = 2n \;, n \geq 1$, find $\sum_{i=1}^3 a_i$ and $\prod_{i=1}^3 a_i$

Solution

$$\sum_{i=1}^3 a_i = a_1 + a_2 + a_3 = 2 + 4 + 6 = 12$$

$$\prod_{i=1}^3 a_i = a_1 \times a_2 \times a_3 = 2 \times 4 \times 6 = 48 \qquad \blacksquare$$

$$\sum_{i=0}^{n}(a_i \mp b_i) = \sum_{i=0}^{n} a_i \mp \sum_{i=0}^{n} b_i$$

$$\sum_{i=0}^{n} c a_i = c \sum_{i=0}^{n} a_i$$

$$\sum_{i=1}^{\boldsymbol{n}} c = c + c + c + \cdots + c = \boldsymbol{n}\, c$$

$$\sum_{k=1}^{\boldsymbol{n}} (c + k) = \boldsymbol{n}\, c + \sum_{k=1}^{n} k$$

$$\sum_{k=1}^{\boldsymbol{n}} k = \frac{\boldsymbol{n}(\boldsymbol{n}+1)}{2} = \frac{n^2+n}{2}$$

$$\sum_{k=1}^{\boldsymbol{n}} k^2 = \frac{\boldsymbol{n}(\boldsymbol{n}+1)(2\boldsymbol{n}+1)}{6}$$

$$\sum_{k=1}^{\boldsymbol{n}} k^3 = \frac{\boldsymbol{n}^2(\boldsymbol{n}+1)^2}{4}$$

$$\sum_{k=1}^{\boldsymbol{n}} \frac{1}{n} = \boldsymbol{n}\frac{1}{n} = 1$$

$$\sum_{k=1}^{\mathbf{800}}(2k+1) = 2\sum_{k=1}^{800} k + \sum_{k=1}^{800} 1 = 2 \times \frac{\boldsymbol{n}(\boldsymbol{n}+1)}{2} + 1 \times n$$

n=800

$$= 2 \times \frac{800(800+1)}{2} + 1 \times 800 = 641600$$

$$\sum_{\boldsymbol{k=1}}^{\boldsymbol{5}}(3k+2)=3\sum_{k=1}^{5}k+\sum_{k=1}^{5}2=3\times\frac{\boldsymbol{n(n+1)}}{2}+2\times n$$

n=5

$$=3\times\frac{5(5+1)}{2}+2\times 5=55$$

$$\sum_{k=1}^{\mathbf{10}} (k^2 + 3k + 2) = \sum_{k=1}^{10} k^2 + 3\sum_{k=1}^{10} k + \sum_{k=1}^{10} 2$$

$$= \frac{\boldsymbol{n}(\boldsymbol{n}+1)(2\boldsymbol{n}+1)}{6} + 3 \times \frac{\boldsymbol{n}(\boldsymbol{n}+1)}{2} + 2 \times n$$

n=10

$$= \frac{10(10+1)(2 \times 10 + 1)}{6} + 3 \times \frac{10(10+1)}{2} + 2 \times 10 = 570$$