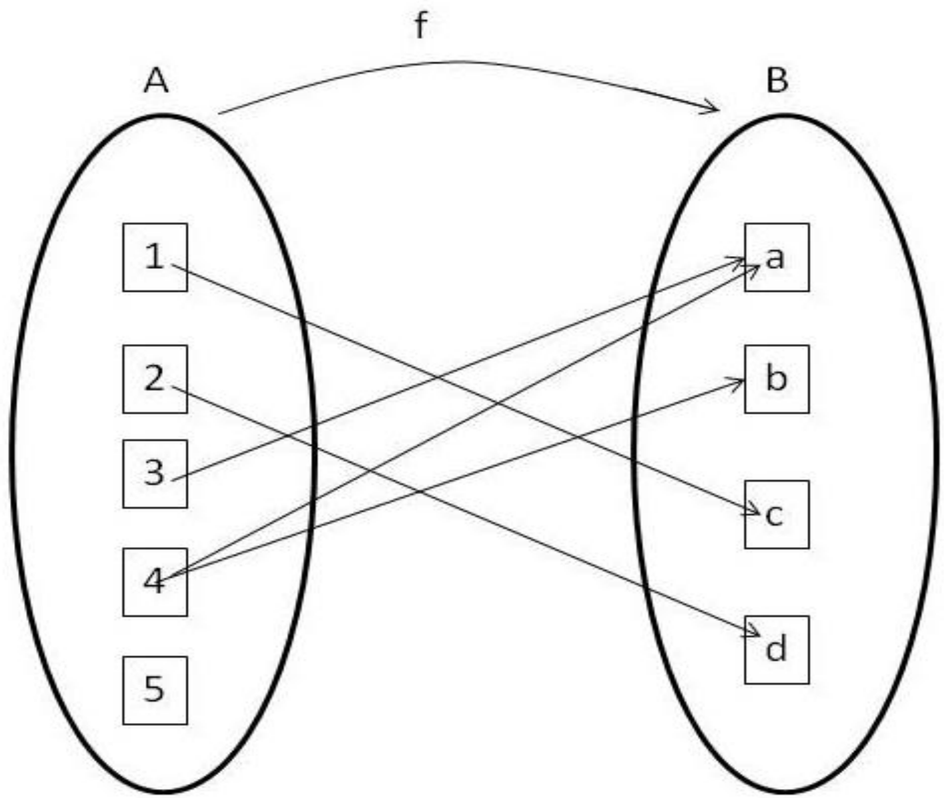
Functions

A **relation** between two sets X and Y is a set of ordered pairs, each pairs of the form (x, y) where x is a member of X and y is a member of Y .

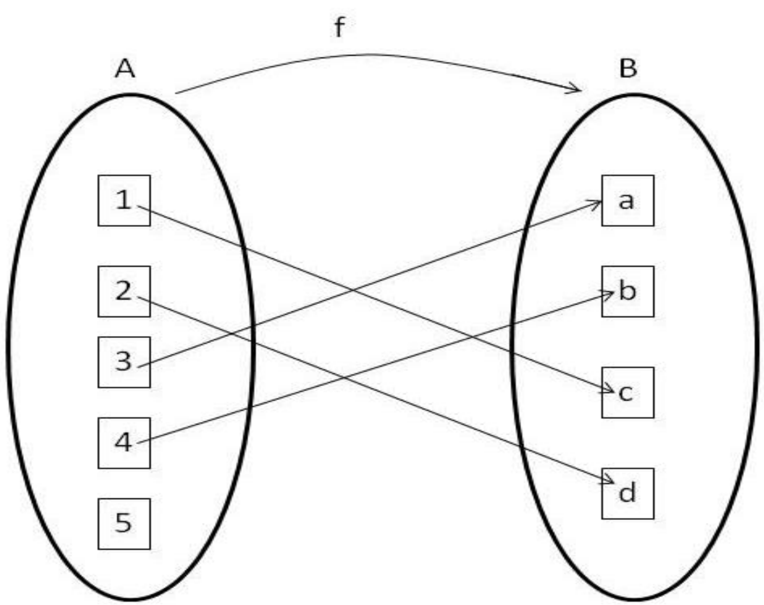


2.1. Basic concepts:

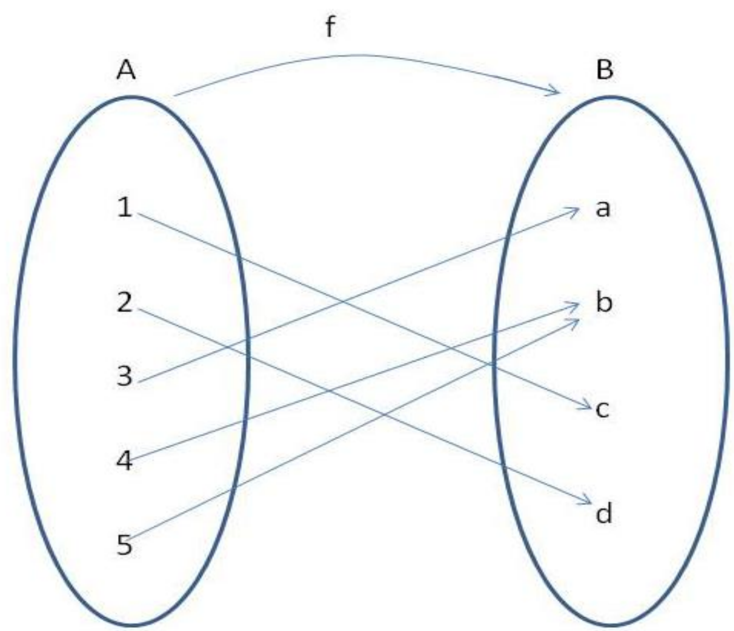
Definition (2. 1. 1): (Real-Valued Function)

Let X and Y are any two sets of real numbers.

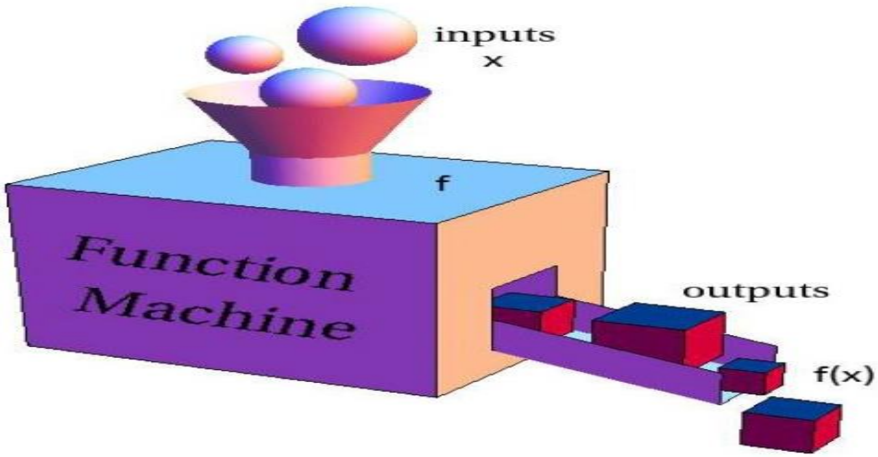
A real valued function f of a real variable x from X to Y is a correspondence that assigns to each number x of X exactly one number y of Y.



A relation but not a function



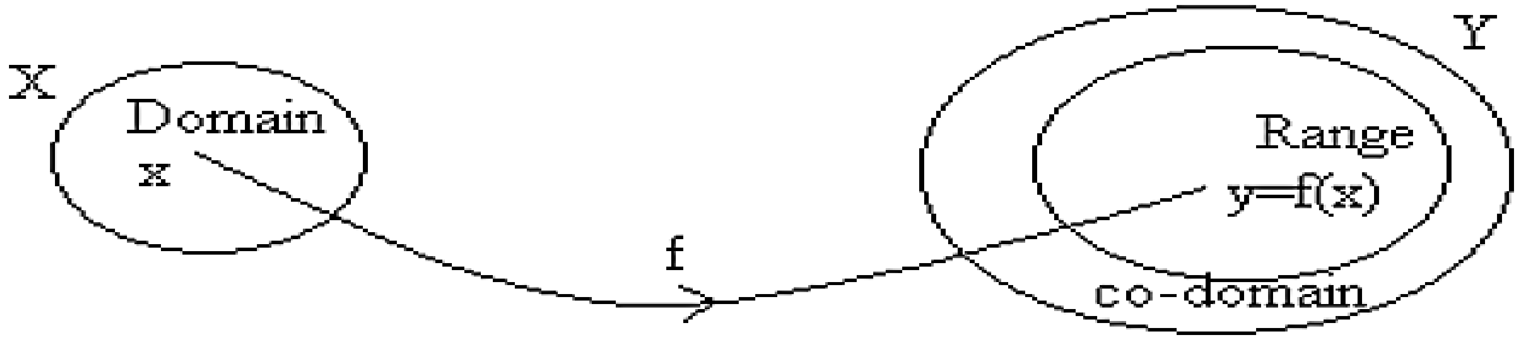
Relation (f) as a function between two sets (A&B)



Definition : Function

A function is a rule for transforming an object into another object see figure 2.1. The object you start with is called the input (x) , and comes from some set called the domain. What you get back is called the output (y) ; it comes from some set called the codomain.

**** The variable x is the independent variable, and the variable y is the dependent variable.**



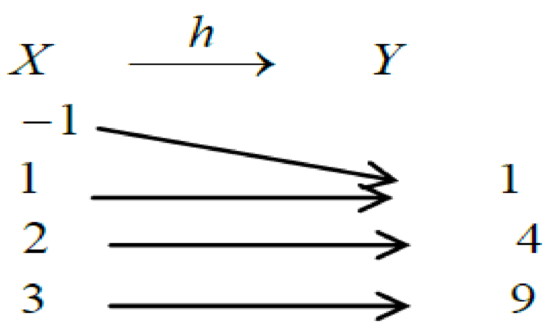
Example(2. 1. 2) : Let $X=\{-1,1,2,3\}$ and $Y=\{1,4,9\}$.

Define $h:X\to Y$ by ; $h(x)=x^2$, $\forall x\in X$.

Is h a real valued function? If it is , find its range?

Solution:

Since $h(x)=x^2$, $\forall x\in X$, so we have



$h(1)=h(-1)=1$

$h(2)=4$

$h(3)=9$

Since X ,Y are subsets of $\mathbb{R}$ and each number in X has a relation with only one number in Y, then h is a real valued function. We note that $Y=R_h=\{1,4,9\}$

Domain & Rang of Root functions

- Determined D_f & R_f for the following functions

1): $f(x) = 2 + \sqrt{x-1}$

$** \; The \; domain \; of \; f \; must \; satisfies \; x - 1 \geq 0,$

$$x \geq 1,$$

$$D_f = [1, \infty).$$

To find R_f:

$$x \geq 1$$

$$x - 1 \geq 0$$

$$\sqrt{x-1} \geq 0$$

$$2 + \sqrt{x-1} \geq 2$$

$$f(x) \geq 2$$

$$R_f = [2, \infty)$$

- 2) Find $\boldsymbol{R_f}$ $f(x) = \frac{1}{x-1}$, $\forall x \in [2,5)$.

<u>Solution:</u>

$Domain\ of\ f = [2,5)$.

$$2 \leq x < 5$$
$$1 \leq x - 1 < 4$$
$$1 \geq \frac{1}{x-1} > \frac{1}{4}$$
$$\frac{1}{4} < \frac{1}{x-1} \leq 1$$

$Range\ \ of\ f = (\frac{1}{4}, 1]$

Section 2: Composite Functions

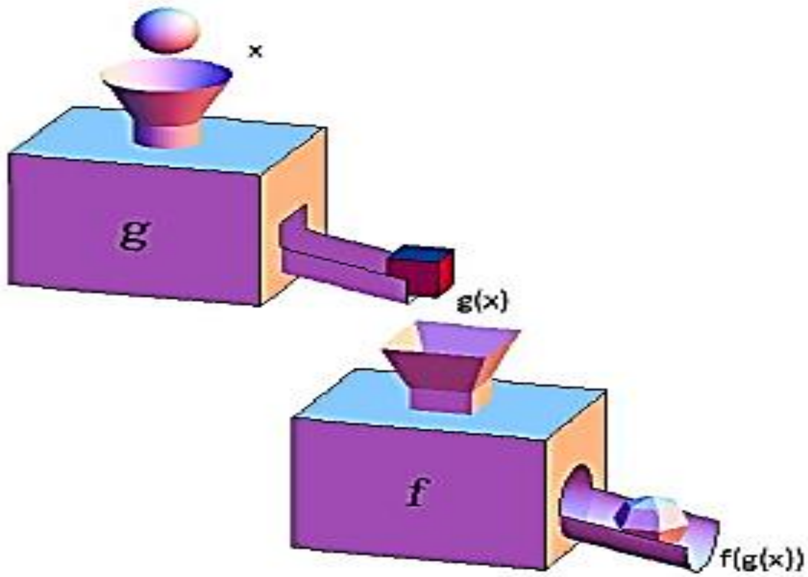
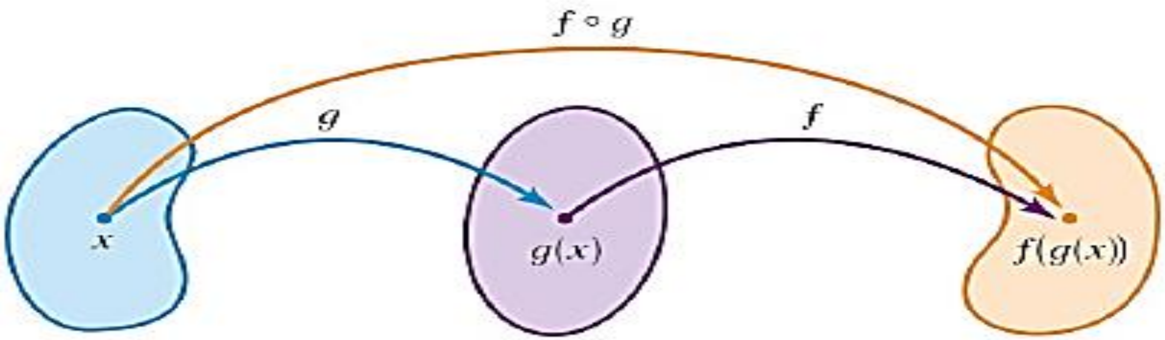
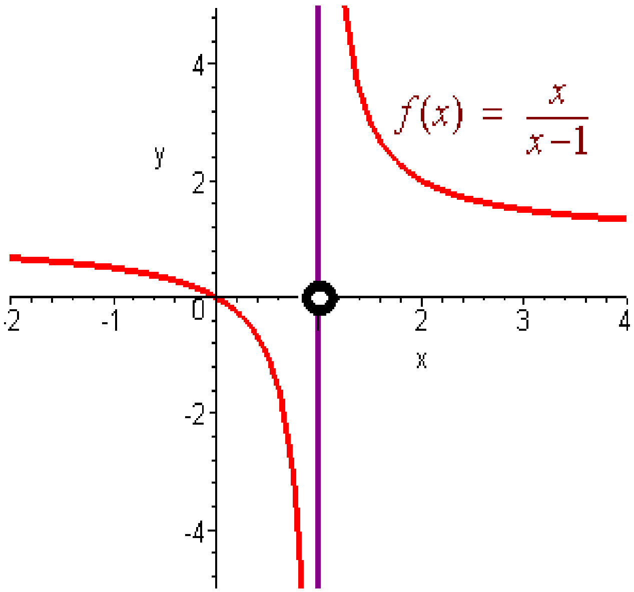


Figure 2.33: Composition of Functions



- Example : Let $f(x) = \frac{x}{x-1}$, $g(x) = x^{10}$ *and* $h(x) = x + 3$
- Find : 1) Domains and Ranges?
- 2) $(f0g)(x), (g0f)(x), (f0g0h)(x)$
- Solution:
- $D_f = R - \{1\}$, $D_g = R$, $D_h = R$
- $R_f = R - \{0\}$, $R_g = R^+$, $R_h = R$



- $(f0g)(x) = f(g(x)) = \frac{x^{10}}{x^{10}-1}$
- $(g0f)(x) = g(f(x)) = (\frac{x}{x-1})^{10}$
- $(f0g0h)(x) = f(g0h)(x) = f(g(h))$
- $(g0h)(x) = g(h(x)) = (x+3)^{10}$
- $(f0g0h)(x) = f(g0h)(x) = f(g(h)) = \frac{(x+3)^{10}}{(x+3)^{10} - 1}$

Inverse of Function

6. The Inverse Functions :

Definition (2. 2. 16) :

A function f from X to Y is said to be one –to- one (or 1-1;injective) if to every y –value in the range there exists exactly one x-value in the domain .

i.e. $f : X \to Y$ is one – to –one if

$$\forall\ x_1, x_2 \in A :\ f(x_1) = f(x_2) \ \Rightarrow\ x_1 = x_2$$

Which is equivalent to the statement :

$$\forall x_1, x_2 \in A\ :\ x_1 \neq x_2 \ \Rightarrow\ f(x_1) \ \neq\ (x_2)$$

A function f from X to Y is said to be onto (or subjective) if the range consists of all values of Y, i.e. if $R_f = Y$. $f : X \to Y$ is said to be 1-1 correspondence (objective) If its one - to - one and onto .

Then $$bijective \Leftrightarrow injective + surjective$$

Examples (2. 2. 17) :

The function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x$, $\forall\, x\in\mathbb{R}$ is one - to - one and onto .

(ii) The function $f:[0,1]\to\mathbb{R}$, $f(x)=x^2$, is one - to - one , but not onto .

(iv) The function $f:[0,1]\to[0,1]$, $f(x)=x^2$, is one - to - one and onto

<u>How one can finds the inverse of a function :</u>

1. Determine (by the previous theorem) whether the function given by $y = f(x)$ has an inverse.

2. Solve for x as a function of y .

3. Interchange x and y . The resulting equation $y = f^{-1}(x)$.

4. Define the domain of f^{-1} to be the range of f .

5. Verify that

$$(f o f^{-1})(x) = f(f^{-1}(x)) = x \quad ; \quad (f^{-1} o f)(x) = f^{-1}(f(x)) = x$$

Example (2. 2. 23) :

Find the inverse of $f(x) = \sqrt{2x - 3}$

i.e. $f : X \to Y$ is one – to – one if

$$\forall \; x_1, x_2 \in A : \; f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$$

$$\sqrt{2x_1 - 3} = \sqrt{2x_2 - 3}$$

$$2x_1 - 3 = 2x_2 - 3$$

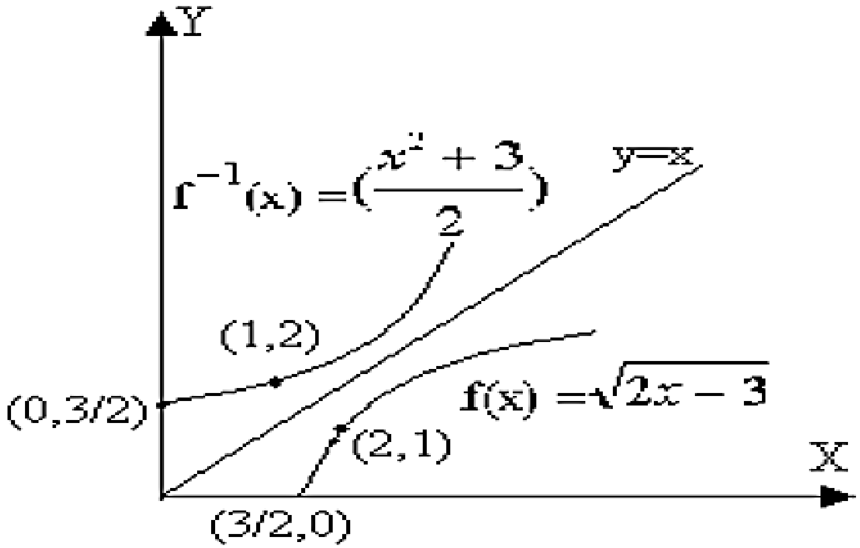
$x_1 = x_2$, then f is 1-1 , has inverse

$\sqrt{2x - 3} = y$ let $y = f(x)$

$\therefore 2x - 3 = y^2$ square both sides

$\therefore \quad x = \dfrac{y^2 + 3}{2}$ interchange x and y

$f^{-1}(x) = \dfrac{x^2 + 3}{2}$ replace y by $f^{-1}(x)$



You can verify this result as follows

$$(f o f^{-1})(x) = f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3}$$

$$= \sqrt{x^2} = x \quad ; \quad x \geq 0$$

$$(f^{-1} o f)(x) = f^{-1}(f(x)) = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2}$$

$$= \frac{2x - 3 + 3}{2} = x \quad ; \quad x \geq \frac{3}{2}$$

Floor and Ceiling Functions

- The **floor** and **ceiling** functions map the real numbers onto the integers ($\mathbf{R} \to \mathbf{Z}$).
- The **floor** function assigns to $r \in \mathbf{R}$ the largest $z \in \mathbf{Z}$ with $z \leq r$, denoted by $\lfloor r \rfloor$.
- **Examples:** $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$
- The **ceiling** function assigns to $r \in \mathbf{R}$ the smallest $z \in \mathbf{Z}$ with $z \geq r$, denoted by $\lceil r \rceil$.
- **Examples:** $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Geometric Sequences and Series

A sequence is **geometric** if the ratios of consecutive terms are the same.

2, 8, 32, 128, 512, . . .

geometric sequence

$$\frac{8}{2} = 4$$

$$\frac{32}{8} = 4$$

$$\frac{128}{32} = 4$$

$$\frac{512}{128} = 4$$

The **common ratio**, *r*, is 4.

Example 1.

a. Is the sequence geometric? If so, what is r?

$$2, 4, 8, 16, \ldots 2^n, \ldots$$

$$\frac{4}{2} = 2, \quad \frac{8}{4} = 2, \quad \frac{16}{8} = 2 \qquad r = 2$$

b. Is the sequence geometric? If so, what is r?

$$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots$$

$$\frac{\frac{1}{9}}{-\frac{1}{3}} = -\frac{1}{3} \quad \frac{-\frac{1}{27}}{\frac{1}{9}} = -\frac{1}{3} \quad \frac{\frac{1}{81}}{-\frac{1}{27}} = -\frac{1}{3} \qquad r = -\frac{1}{3}$$

The ***n*th term** of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence.

$$r = \frac{75}{15} = 5$$

$a_1 = 15$

$$15, \quad 75, \quad 375, \quad 1875, \ldots$$

$a_2 =$ $15(5)$ $\quad a_3 =$ $15(5^2)$ $\quad a_4 =$ $15(5^3)$

The nth term is $15(5^{n-1})$.

Arithmetic Sequences and Series

Arithmetic Sequence: ***sequence whose consecutive terms have a common difference.***

3, 5, 7, 9, 11, 13, ... ***Example:***

The terms have a common difference of 2.

The common difference is the number *d*.

To find the common difference you use $a_{n+1} - a_n$

Example: Is the sequence arithmetic?

–45, –30, –15, 0, 15, 30

Yes, ***the common difference is*** **15**

How do you find any term in this sequence?

To find any term in an arithmetic sequence, use the formula

$$a_n = a_1 + (n - 1)d$$

where d is the common difference.

Examples:

Find the 14th term of the arithmetic sequence

4, 7, 10, 13,……

$$a_n = a_1 + (n-1)d$$

$$a_{14} = 4 + (14-1)3$$

$$= 4 + (13)3$$

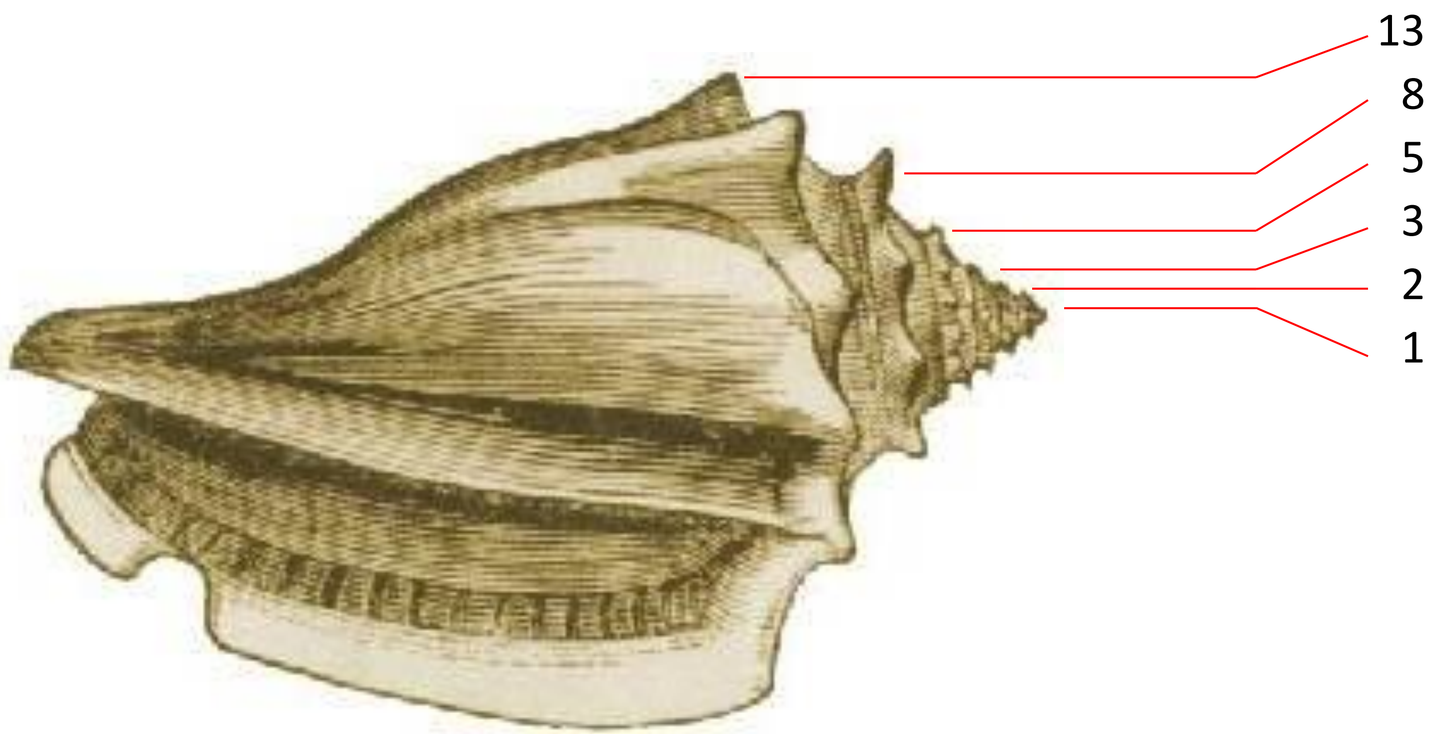
$$= 4 + 39$$

$$= 43$$

Fibonacci sequence

- Sequences can be neither geometric or arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence

Fibonacci sequence in nature



Summations

- A summation:

upper limit

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \textstyle\sum_{j=m}^{n} a_j$$

lower limit

index of summation

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
    sum += a(j);
```

Evaluating sequences

$$\sum_{k=1}^{5}(k+1)$$

- $2 + 3 + 4 + 5 + 6 = 20$

$$\sum_{j=0}^{4}(-2)^{j}$$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$$\sum_{i=1}^{10}3$$

- $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$

$$\sum_{j=0}^{8}\left(2^{j+1} - 2^{j}\right)$$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 511$

Evaluating sequences

$S = \{ 1, 3, 5, 7 \}$

- What is $\Sigma_{j \in S}\, j$
  $1 + 3 + 5 + 7 = 16$
- What is $\Sigma_{j \in S}\, j^2$
  $1^2 + 3^2 + 5^2 + 7^2 = 84$
- What is $\Sigma_{j \in S}\, (1/j)$
  $1/1 + 1/3 + 1/5 + 1/7 = 176/105$
- What is $\Sigma_{j \in S}\, 1$
  $1 + 1 + 1 + 1 = 4$

Summation of a geometric series

- Sum of a geometric series:

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

- Example:

$$\sum_{j=0}^{10} 2^{n} = \frac{2^{10+1}-1}{2-1} = \frac{2048-1}{1} = 2047$$

Double summations

- Like a nested for loop

$$\boxed{\sum_{i=1}^{4}\sum_{j=1}^{3} ij}$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
for ( int j = 1; j <= 3; j++ )
sum += i*j;
```

Double summations

- These have the meaning you'd expect.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3)$$

$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4)$$

$$= 6 \cdot 10 = 60$$