Lecture 5

Method of Enumeration

Example

How many ways for a number of two digits can be formed by using {1,2,3,8}

❶ With Repeating

Number of Units	Number of Tens
4	4

No. of ways= **4 × 4 = 16**

❷ Without Repeating

Number of Units	Number of Tens
4	3

No. of ways= 4 × 3 = 12

Method of Enumeration

Example

How many ways for a number of three digits can be formed by using {2,3,5,7,8}

❶ with Repeating		
Number of Units	Number of Tens	Number of Hundreds
5	5	5
No. of ways= $5 \times 5 \times 5 = 125$		

❷ without Repeating		
Number of Units	Number of Tens	Number of Hundreds
5	4	3
No. of ways= $5 \times 4 \times 3 = 60$		

Factorial

$$n!$$

$$n! = n \times (n-1) \times (n-2) \times \cdots (3) \times (2) \times (1)$$

$$n! = \prod_{j=1}^{n} j$$

$$5! = 5 \times 4 \times 3 \times (2) \times (1) = 120$$

$$5 \text{ SHIFT } x^{-1} = 120$$

$$n! = n(n-1)! = n(n-1)(n-2)!$$

$$7! = 7 \times 6! = 7 \times 6 \times 5!$$

$$0! = 1$$

$$1! = 1$$

Permutations

$$^nP_r = P(n,r)$$

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$
$$= \frac{n!}{(n-r)!} \qquad n \geq r$$

$$P(n,n) = n!$$

$$P(5,3)=5\times4\times3=60$$

$$P(4,2)=4\times3=12$$

$$P(3,3)=3!=3\times2\times1=6$$

$$5 \ \text{SHIFT} \ \times 3 = 120$$

Arrangement

Order

Schedule

Example

How many permutations for 5 books to be stored in 5 places of bookshelf?

The required number

$P(5,5)=5!=5\times4\times3\times2\times1=120$

Example

How many permutations for 3 books to be stored in 5 places of bookshelf?

The required number

$P(5,3) = 5 \times 4 \times 3 = 60$

Example

❑ How many different ways can three of the letters of the word **BYTES** be chosen and written in a row?

The required number $P(5,3)=5\times4\times3=60$

❑ If the first position must be B, then how many different ways can this are done?

The required number $= 1\times4\times3=12$

Combination

$$^nC_r = c(n, r) = \binom{n}{r}$$

$$\binom{n}{r} = \frac{P(n, r)}{r!}$$

$$\prod_{k=1}^{r} \frac{n - k + 1}{k} = \frac{n!}{r!\,(n - r)!}$$

$$C(5, 2) = \frac{5 \times 4}{2 \times 1} = 10$$

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

$$6 \ \text{SHIFT} \ \div \ 3 = 20$$

Group Committee Sample

Example

How many ways in which two persons can be selected at random from a group containing 4 persons?

The required number $= \binom{4}{2} = \frac{4 \times 3}{2 \times 1} = 6$

Example

From 6 men and 5 women. How many ways in which a committee can conformed of 3 men and 2 women?

The required number $= \binom{6}{3} \times \binom{5}{2} = 200$

Example

Department consists of 10 men and 5 women. How many ways to conform a committee consists of 4 persons provided that at least 2 men are selected?

$$\text{No. of all possible cases} = \binom{10}{2}\binom{5}{2} + \binom{10}{3}\binom{5}{1} + \binom{10}{4}\binom{5}{0}$$
$$= 450 + 600 + 210 = 1260$$

Example

How many different ways can be selected at least one person from a group consists of 4 persons?

The required number $= \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 15$

Permutations with repeated elements

In general, if we have a set of n objects has $n1$ identical objects of type 1, $n2$ identical objects of type 2, ..., and nr identical objects of type r, Then the number of ordering of this set is:

$$P(n\ ;\ n_1, n_2, .., n_r) = \frac{n!}{n_1!\ n_2! \ldots n_r!}$$

Permutations with repeated elements

Example

How many numbers can be formed from the group $\{1, 3, 3, 5, 6, 5, 5\}$

The required number is $= \dfrac{7!}{1! \times 2! \times 3! \times 1!} = 740$

Definition

The number of permutations of n objects **around a circle** is $(n-1)!$

Example

How many different ways to arrange 4 persons around a circular table?

The required is: $(4-1)!=6$

Binomial Theorem

$(a + b)^0$	1
$(a + b)^1$	$a + b$
$(a + b)^2$	$a^2 + 2\,a\,b + b^2$
$(a + b)^3$	$a^3 + 3\,a^2 b + 3\,a\,b^2 + b^3$
$(a + b)^4$	$a^4 + 4\,a^3 b + 6\,a^2 b^2 + 4\,a^1 b^3 + b^4$
$(a + b)^5$	$a^5 + 5\,a^4\,b + 10\,a^3 b^2 + 10\,a^2\,b^3 + 5\,a^1 b^4 + b^5$

$$(a + b)^n = \binom{n}{0} a^n \, b^0 \; + \; \binom{n}{1} a^{n-1} \, b^1$$

$$+ \binom{n}{2} a^{n-2} \, b^2 + \; \ldots \; + \binom{n}{n} a^0 b^n$$

$$(a + b)^n = \sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} \, b^k$$

n is a **positive integer**

- The first term always a^n, and the exponent n **decrease by one** for each successive term.

- The exponent of b start from **zero** at the **first term** and then **increase by one** for each successive term to reach to the last term b^n.

- The degree of each term is n (**sum** of the two exponents of a and b).

- The expansion of $(a + b)^n$ has $n+1$ terms.

Example

Expand: $(2x^2 + 3)^4$

$$= \binom{4}{0} (2x^2)^4\, 3^0\; +\; \binom{4}{1} (2x^2)^3\, 3^1$$

$$+ \binom{4}{2} (2x^2)^2\, 3^2\; +\; \binom{4}{3} (2x^2)^1\, 3^3$$

$$+ \binom{4}{4} (2x^2)^0\, 3^4$$

$$= 16\, x^8 + 96\, x^6 + 216\, x^4 + 216\, x^2 + 81$$

Example

Find the term involve: x^3 in $(1+2x)^5$

$$(1+2x)^5 = \sum_{k=0}^{k=5} \binom{5}{k} (1)^{5-k} (2x)^k$$

To get x^3, you must put $k=3$

$$\binom{5}{3} (1)^{5-3} (2x)^3 = 10 \times 8 \times x^3 = 80\, x^3$$

Example

Find the term involve: y^8 in $(2\,x + y^2)^6$

$$(2x + y^2)^6 = \sum_{k=0}^{k=6} \binom{6}{k} (2x)^{6-k} (y^2)^k$$

To get y^8, you must put $k=4$

$$\binom{6}{4} (2\,x)^{6-4} (y^2)^4 = 15 \times 4 \times x^2\, y^8 = 60\, x^2 x^2\, y^8$$

Example

Find the **coefficient** of the term involve x^2y^4 in $(2\,x + y)^6$

$$(2x + y)^6 = \sum_{k=0}^{k=6} \binom{6}{k} (2x)^{6-k} (y)^k$$

To get x^2y^4, you must put $k=4$

$$\binom{6}{4} (2\,x)^2 (y)^4 = 15 \times 4 \times x^2\,y^4 = \mathbf{60}\ x^2y^4$$

$(a + b)^n$

❑ The term $\quad P_{r+1} = \binom{n}{r} (a)^{n-r} \, b^r$

❑The ratio between the two successive terms

$$\frac{P_{r+1}}{P_r} = \frac{n-r+1}{r} \times \frac{b}{a}$$

Example

Find the fourth term in: $(2x^3 - 3y^2)^5$

$$P_{r+1} = \binom{n}{r}(a)^{n-r}\,b^r$$

$$P_{3+1} = \binom{5}{3}(2x^3)^2\,(-3y^2)^3$$

$$= 10 \times 4 \times -27 \times x^6\ y^6 = \text{-}1080\,x^6\ y^6$$

Results

$$(1 + x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots + \binom{n}{n} x^n$$

$$(1 - x)^n = 1 - \binom{n}{1} x + \binom{n}{2} x^2 - \binom{n}{3} x^3 \ldots + \binom{n}{n} x^n$$

n is a **positive integer**

Example

Approximate $(0.98)^8$ by evaluating the first three terms of $(1 - 0.02)^8$.

$$(1 - 0.02)^8 = 1 - \binom{8}{1}(0.02)^1 + \binom{8}{2}(0.02)^2$$

$$= 1 - 0.16 + 0.0112 = 0.8512$$

Multinomial Theorem

$$(a_1 + a_2 + a_3 + \cdots + a_k)^n = \sum_{k=0}^{n} \frac{n!}{k_1! \, k_2! \ldots k_n!} a_1^{k_1} a_2^{k_2} a_3^{k_3} \ldots a_n^{k_n}$$

The number of terms is: $\binom{n + k - 1}{k - 1}$

Example

Find the **coefficient** of the term involve
$x^2 y^3 z^5$ in $(x + y + z)^{10}$

$$(x + y + z)^{10} = \sum_{k=0}^{k=10} \frac{10!}{k_1! \times k_2! \times k_3!} (x)^{k_1} (y)^{k_2} (z)^{k_3}$$

To get $x^2 y^3 z^5$, you must put $k_1 = 2, k_2 = 3, k_3 = 5$

$$\frac{10!}{2! \times 3! \times 5!} x^2 y^3 z^5 = \mathbf{2520}\ x^2 y^3 z^5$$

The number of terms $= \binom{n+k-1}{k-1} = \binom{10+3-1}{3-1} = \binom{12}{2} = 66$

Example

Find the **coefficient** of the term involve

$$w^2 x^3 y^2 z^5 \quad \text{in} \ (\, 2w + x + 3y + z)^{12}$$

$$(2w + x + 3y + z)^{12}$$

$$= \sum_{k=0}^{k=12} \frac{12!}{k_1! \times k_2\,! \times k_3\,! \times k_4} (2w)^{k_1} (x)^{k_2} \, (3y)^{k_3} (z)^{k_4}$$

To get $w^2 x^3 y^2 z^5$, you must put $k_1 = 2, k_2 = 3, k_3 = 2, k_4 = 5$

$$\frac{12!}{2! \times 3! \times 2! \times 5!} (\mathbf{2}w)^2 (x)^3 (\mathbf{3}y)^2 (z)^5$$

$$= \frac{12!}{2! \times 3! \times 2! \times 5!} \times (\mathbf{2})^{\mathbf{2}} \times (\mathbf{3})^{\mathbf{2}} w^2 x^3 y^2 z^5$$

$$= 5987520$$

The number of terms $= \binom{n+k-1}{k-1} = \binom{12+4-1}{4-1} = \binom{15}{3}$
$= 455$

Example
13, 19, 20

Example- 13

Prove that: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof

L. H. S. $= \binom{n}{k} + \binom{n}{k-1} = \dfrac{n!}{k!\,(n-k)!} + \dfrac{n!}{(k-1)!\,(n-k+1)!}$

$= \dfrac{n!}{k\,(k-1)!\,(n-k)!} + \dfrac{n!}{(k-1)!\,(n-k+1)(n-k)!}$

$= \dfrac{n!}{(k-1)!\,(n-k)!}\left[\dfrac{1}{k} + \dfrac{1}{n-k+1}\right]$

$= \dfrac{n!}{(k-1)!\,(n-k)!}\left[\dfrac{n-k+1+k}{k(n-k+1)}\right]$

$= \dfrac{n!}{(k-1)!\,(n-k)!}\left[\dfrac{n+1}{k(n-k+1)}\right]$

$= \dfrac{(n+1)!}{k!\,(n-k+1)!} = \binom{n+1}{k} = $ R. H. S $\blacksquare$

Example 19   Prove that: $\sum_{k=0}^{k=n} \binom{n}{k} (-1)^k = 0$

Proof

$$\sum_{k=0}^{k=n} \binom{n}{k} (-1)^k = \binom{n}{0} (-1)^0 + \binom{n}{1} (-1)^1 + \binom{n}{2} (-1)^2 + \cdots + \binom{n}{n} (-1)^n$$

$$= \binom{n}{0} (-1)^0 (1)^n + \binom{n}{1} (-1)^1 (1)^{n-1} + \binom{n}{2} (-1)^2 (1)^{n-2} + \cdots + \binom{n}{n} (-1)^n (1)^{n-n}$$

$$= (1 - 1)^n = 0 \qquad \blacksquare$$

Example 20

If $y(x) = x^n$, then prove that: $\quad \dfrac{dy}{dx} = \text{Lim}_{h \to 0} \dfrac{(x+h)^n - x^n}{h} = n\,x^{n-1}$

Proof

$$\frac{(x+h)^n - x^n}{h}$$

$$= \frac{x^n + \binom{n}{1} x^{n-1}h^1 + \binom{n}{2} x^{n-2}h^2 + \binom{n}{3} x^{n-3}h^3 + \cdots + h^n - x^n}{h} =$$

$$= \frac{nx^{n-1}h^1 + \binom{n}{2} x^{n-2}h^2 + \binom{n}{3} x^{n-3}h^3 + \cdots + h^n}{h}$$

$$= \frac{h\left[nx^{n-1} + \binom{n}{2} x^{n-2}h^1 + \binom{n}{3} x^{n-3}h^2 + \cdots + h^{n-1}\right]}{h}$$

$$= \text{Lim}_{h \to 0} \frac{(x+h)^n - x^n}{h} = \text{Lim}_{h \to 0} \left[nx^{n-1} + \binom{n}{2} x^{n-2}h^1 + \binom{n}{3} x^{n-3}h^2 + \cdots + h^{n-1}\right]$$

$$\therefore \quad \frac{dy}{dx} = nx^{n-1} \qquad \blacksquare$$