Lecture 1

Contents

Proposition or Statement: A proposition is a declarative sentence that is either true or false.

❶ p: Paris is the capital of France	p is a **true** proposition.
❷ p: π is a rational number	p is a **false** proposition.
❸ q: $4 = 7$	q is a **false** proposition.
❹ q: The earth is round	q is a **true** proposition.
❺ q: $3 - x = 5$	It is a proposition but depends on the value of x.
❻ r: $2 + 3 = 5$	r is a **true** proposition.
❼ p: what is your name?	It is not a proposition, why?
❽ q: do your homework	It is not a proposition, why?

Logical Connective:

We can combine these propositions using various operators. There are seven principal logic operators:

INVERTER, AND, OR, NAND, NOR, XOR (exclusive OR) and XNOR (exclusive NOR).

In addition, we will discuss the **conditional** and **biconditional** propositions.

1. Negation "Not" $\bar{p}$

If p is a true proposition, then $\bar{p}$ is a false proposition.

p	$\bar{p}$
T	F
F	T

2. **Conjunction “AND”** $p \wedge q$

it is **true** precisely when both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. Negation of AND "NAND"

it is the complement of AND proposition.

$$\boldsymbol{p}|\boldsymbol{q} \equiv (\overline{p \wedge q}) \equiv \text{invert}\,(p \wedge q)$$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{p}|\boldsymbol{q}$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

4. Disjunction "OR" $p \vee q$

it is true precisely when at least one of p or q is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example-3

If p is $4 \geq 2$ and q is "$5 \leq 2$" then $p \wedge q$ is false and $p \vee q$ is true.

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Example-4

Form the compound propositions $p \wedge q$ and $p \vee q$

① p: 2 is a positive integer q: $\sqrt{2}$ is a rational number

② $p: 2 + 3 \neq 5$ q: Cairo is the capital of Egypt

Solution

① $p \wedge q$: "2 is a positive integer and $\sqrt{2}$ is a rational number" is false

$p \vee q$: "2 is a positive integer or $\sqrt{2}$ is a rational number" is true

5. Negation of OR "NOR"

it is the complement of OR proposition.

$$\overline{p \vee q}$$

p	q	$p \vee q$	$\overline{p \vee q}$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

6. Exclusive OR "XOR" $A \oplus B$

it is true exactly when either A *or* B is **true** and the other is **false.**

$$X = A \oplus B = A\bar{B} + \bar{A}B = (A \wedge \bar{B}) \vee (\bar{A} \wedge B)$$

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
F	F	F

7. Exclusive NOR “XNOR”

It is the complement of the exclusive OR.

$$\overline{(p \oplus q)}$$

p	q	$\overline{p \oplus q}$
T	T	T
T	F	F
F	T	F
F	F	T

<u>8. Conditional Proposition "If p then q"</u>

$p \rightarrow q$

It is called **conditional proposition** or implication.

where, $\boldsymbol{p}$ is called **hypothesis** and $\boldsymbol{q}$ is called **conclusion**

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

9. Biconditional Proposition "p If and only if q"

$p \leftrightarrow q$

It is called bi-conditional proposition or equivalence ["iff"].

$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

NOT

AND

OR

IF ..THEN

IFF

Types of Propositions

Tautology: its truth-values always *true.*

Contradiction: its truth-values always *false.*

Contingency: is neither a tautology nor a contradiction.

	CONTINGENCY	CONTRADICTION	TAUTOLOGY
$\boldsymbol{p}$	$\overline{\boldsymbol{p}}$	$\boldsymbol{p} \wedge \overline{\boldsymbol{p}}$	$\boldsymbol{p} \vee \overline{\boldsymbol{p}}$
T	F	$\boldsymbol{F}$	$\boldsymbol{T}$
F	T	$\boldsymbol{F}$	$\boldsymbol{T}$

Statement forms associated with the implication

Contrapositive of: $p \to q$	is:	$\bar{q} \to \bar{p}$
Converse of $p \to q$	is:	$q \to p$
Inverse of $p \to q$	is:	$\bar{p} \to \bar{q}$

$\equiv$

				Statement	Converse	Contrapositive	Inverse
p	q	$\bar{q}$	$\bar{p}$	$p \to q$	$q \to p$	$\bar{q} \to \bar{p}$	$\bar{p} \to \bar{q}$
T	T	F	F	T	T	T	T
T	F	T	F	F	T	F	T
F	T	F	T	T	F	T	F
F	F	T	T	T	T	T	T

Example-17

Start with the statement, "$\forall\, n \in \mathcal{N}$, if n is prime (which is true), then n is not even (which is false)." Form the contrapositive, converse, and inverse of this statement. Which statements are true?

Solution

Let p: n is prime (T), $\quad \bar{p} \equiv n$ is not prime (F)

Let q: n is not even (F), $\quad \bar{q} \equiv n$ is even (T)

The statement $\quad p \to q \equiv T \to F \equiv F \quad$ is false

The contrapositive of $\quad p \to q$ is $\bar{q} \to \bar{p} \equiv T \to F \equiv F \quad$ is false

The converse of $\quad p \to q$ is $q \to p \equiv F \to T \equiv T \quad$ is true

The inverse of $\quad p \to q$ is $\bar{p} \to \bar{q} \equiv F \to T \equiv T \quad$ is true ■

List of Identities

- Commutative Rule

$$p \land q \equiv q \land p$$

$$p \lor q \equiv q \lor p$$

- Associatively Rule

$$p \land (q \land r) \equiv (p \land q) \land r$$

$$p \lor (q \lor r) \equiv (p \lor q) \lor r$$

- Distribution Rule

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

$$p \equiv p \vee p$$

$$p \equiv p \wedge p$$

$$p \vee T \equiv T$$

$$p \wedge T \equiv p$$

$$p \vee F \equiv p$$

$$p \wedge F \equiv F$$

$$p \vee \bar{p} \equiv T$$

$$p \wedge \bar{p} \equiv F$$

- De- Morgan Rule

$$\overline{p \wedge q} \equiv \bar{p} \vee \bar{q}$$

$$\overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$$

- Double Negation

$$\overline{(\overline{p})} \equiv p$$

- Implication Rule

$$p \rightarrow q \equiv \bar{p} \vee q$$

- Equivalence Rule

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

- Contra positively Rule

$$p \rightarrow q \equiv \bar{q} \rightarrow \bar{p}$$

- Exportation Rule

$$(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$$

Example-5

Assuming that p is a **true**, q is **false**, and r is **true** propositions. Find the truth-value of each proposition:

① $(p \wedge q) \to r$ ② $(p \vee q) \to \bar{r}$

③ $p \wedge (q \to r)$ ④ $p \to (q \to r)$

Solution

① $(p \wedge q) \to r \equiv (T \wedge F) \to T \equiv F \to T \equiv T$

② $(p \vee q) \to \bar{r} \equiv (T \vee F) \to F \equiv T \to F \equiv F$

Example-7

① Assuming that p is a <u>true</u>, q is <u>false</u>, and r is <u>true</u> propositions. Find the truth-value of the propositions: $(p \wedge q) \rightarrow r$ and $(p \vee q) \rightarrow \bar{r}$

② Use the truth table to prove that: $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$. ■-1

Solution

① $(p \wedge q) \rightarrow r \equiv (T \wedge F) \rightarrow T \equiv F \rightarrow T \equiv T$

$(p \vee q) \rightarrow \bar{r} \equiv (T \vee F) \rightarrow F \equiv T \rightarrow F \equiv F$

②

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example-8

Compute the truth table of the statement $(p \to q) \leftrightarrow (\bar{q} \to \bar{p})$

Solution

p	q	$p \to q$	$\bar{q}$	$\bar{p}$	$\bar{q} \to \bar{p}$	$(\boldsymbol{p} \to \boldsymbol{q}) \leftrightarrow (\bar{\boldsymbol{q}} \to \bar{\boldsymbol{p}})$
T	T	T	F	F	T	$\boldsymbol{T}$
T	F	F	T	F	F	$\boldsymbol{T}$
F	T	T	F	T	T	$\boldsymbol{T}$
F	F	T	T	T	T	$\boldsymbol{T}$

■

Example-16

Is the statement $(p \wedge \bar{q}) \wedge (\bar{p} \vee q) \wedge r$ forming a tautology, contradiction or contingency?

Solution

p	q	r	$\bar{p}$	$\bar{q}$	$p \wedge \bar{q}$	$\bar{p} \vee q$	$(p \wedge \bar{q}) \wedge (\bar{p} \vee q)$	$(p \wedge \bar{q}) \wedge (\bar{p} \vee q) \wedge r$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	F	T	F	F
T	F	T	F	T	T	F	F	F
T	F	F	F	T	T	F	F	F
F	T	T	T	F	F	T	F	F
F	T	F	T	F	F	T	F	F
F	F	T	T	T	F	T	F	F
F	F	F	T	T	F	T	F	F

Thus, the statement forms a **contradiction**. ■

Example- 18: Check the following logical equivalences:

❶ $\overline{p \to q} \equiv p \land \bar{q}$

Solution:

❶ L.H.S: $p \to q \equiv \bar{p} \lor q$

$$\overline{p \to q} \equiv \overline{\bar{p} \lor q} \quad \text{Demorgan's Rule}$$

$$\overline{p \to q} \equiv \bar{\bar{p}} \land \bar{q}$$

$$\overline{p \to q} \equiv p \land \bar{q} = R.H.S$$

❷ $p \vee (p \wedge q) \equiv p$

❷ L. H. S: $p \vee (p \wedge q)$, and by using $p = p \wedge T$, we get:

$$(\boldsymbol{p} \wedge T) \vee (\boldsymbol{p} \wedge q) \equiv \boldsymbol{p} \wedge (T \vee q) = p \wedge T = p = \text{R. H. S}$$

❸ $(p \vee q) \wedge \bar{p} \equiv \bar{p} \wedge q$

❸ L. H. S: $(p \vee q) \wedge \bar{p} \equiv (\bar{p} \wedge p) \vee (\bar{p} \wedge q)$

$$\equiv F \vee (\bar{p} \wedge q)$$

$$\equiv \bar{p} \wedge q \ = \text{R. H. S}$$

Example- 19

- Simplify the following logical predicate

$$(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})$$

- Verify your answer using truth table

Solution

$(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})$

$\equiv (p \wedge q) \vee (p \vee \bar{p}) \wedge \bar{q}$

$\equiv (p \wedge q) \vee (T \wedge \bar{q})$

$\equiv (p \wedge q) \vee \bar{q}$

$\equiv \bar{q} \vee (p \wedge q)$

$\equiv (\bar{q} \vee p) \wedge (\bar{q} \vee q)$

$\equiv (\bar{q} \vee p) \wedge T$

$\equiv (\bar{q} \vee p)$ ■

Example- 20

Show that $\bar{p} \wedge (p \vee q) \rightarrow q$ is equal to true, then verify your answer using truth table.

Solution

$\bar{p} \wedge (p \vee q) \rightarrow q \equiv \overline{\bar{p} \wedge (p \vee q)} \vee q$ *where* $p \rightarrow q \equiv \bar{p} \vee q$

$\equiv \bar{\bar{p}} \vee (\overline{p \vee q}) \vee q$ Demorgan rule

$\equiv p \vee (\overline{p \vee q}) \vee q$

$\equiv (p \vee q) \vee (\overline{p \vee q}) \equiv T$