# Lecture 2

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## 1.1. Classification of Numbers

**Natural Numbers** $\mathcal{N} = \{ 1, 2, \dots \}$

**Whole Numbers** $W = \{0, 1, 2, \dots\} = \mathcal{Z}^{+} \cup \{0\}$

**Odd Numbers** $\mathbb{O} = \{1, 3, 5, \dots .\}$

**Even Numbers** $\mathbb{E} = \{0, 2, 4, \dots .\}$

**Prime Numbers** $\mathbb{P} = \{2, 3, 5, 7, 11, \dots .\}$

**Integer's Numbers** $\mathcal{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

**Positive Integer** $\mathcal{Z}^{+} = \{1, 2, \dots\}$,

**Negative Integer** $\mathcal{Z}^{-} = \{-1, -2, \dots\}$

**Rational Numbers** $\mathbb{Q} = \{a/b : a \text{ and } b \text{ are integres}, b \neq 0\} \equiv$

All terminating or repeating decimals, , ex. $\frac{3}{7} = 0.75$ *or* $\frac{27}{110} = 0.2\overline{45} = 0.2454545 \dots \}$

**Irrational Numbers** $\mathbb{Q}'$ = *All nonterminating or nonrepeating decimals,* $\pi$ *and* $e$

**Real Numbers** $\mathcal{R} = ]-\infty, \infty[ = \mathbb{Q} \cup \mathbb{Q}'$

**Complex[1] Numbers** $\mathcal{C} = \{x + i\, y : x, y \in \mathcal{R},\ i = \sqrt{-1}\}$

**predicate** or **propositional function :** It is a proposition containing variables.

**Example- 21**

| | |
|---|---|
| ① | If $p(x)$ is the predicate "$x^2 < 6$", $x \in \mathcal{N}$. Determine the truth values of $p(1)$ and $p(3)$. |
| ② | If $q(x)$ is the predicate "$x^2 \geq 0$", $x \in \mathcal{R}$. Determine the truth-values of $p(-1), p(0)$ and $p(3)$ |

**Answer:**

① $p(1)$ : $1 < 4$, represents a true proposition.

$p(3)$ : $9 < 4$, represents a false proposition.

② $q(-1)$: $1 \geq 0$, represents a true proposition

$q(0)$ : $0 \geq 0$, represents a true proposition

$q(3)$ : $9 \geq 0$, represents a true proposition ■

## Illustrative Example- 22

| ① | ② |
|---|---|
| The predicate:<br>$p(x)$, "$x^2 - 1 = (x-1)(x+1)$"<br>is true *for (**every** or **all**) any* real number $x$.<br>$\forall x,\ x^2 - 1 = (x-1)(x+1)$ | If $x$ is a real number variable, the predicate $p(x)$, "$x^2 \geq 6$" is true *for **some*** real number $x$.<br>$\exists x,\ x^2 \geq 6$ |

## **Universal quantifier:**

We will use the notation $(\forall x)$ to denote that the predicate will be true **for all** values of $x$.

"$\forall$" called "universal quantifier" **:** ***for all x, for every x , for each x***

## Existential quantifier:

We will use the notation $(\exists x)$ to denote that the predicate will be true **for some** values of $x$.

"∃" called existential quantifier. "***for some x***"

## <u>Examples- 23</u>

❶ If "$P(x)$: $x + 2 = 7$" and the domain is the set of integers $\mathbb{Z}$, <u>*then* $\forall x\, P(x)$ *is false*</u>. But if we say: "<u>$\exists x\, P(x)$</u>" then the sentence will be true.

❷ If $Q(x)$ is "$(x+1)^2 = x^2 + 2x + 1$" and the domain is the set of integers $\mathbb{Z}$, <u>*then* $\forall x\, Q(x)$ *is true*</u>.

## Generalized De- Morgan Laws for Logic

❶ If $\exists x, P(x)$ is false then there is no value of $x$ for which $p(x)$ is true

$$\text{i.e. } \neg[\exists x,\ P(x)] \equiv \forall x\ \neg P(x)$$

❷ If $\forall x, P(x)$ is false, then for some $x$, $P(x)$ must be false

$$\text{i.e. } \neg[\forall x,\ P(x)] \equiv \exists x\ \neg P(x)$$

❸ In general $\neg[\exists x\ \forall y,\ P(x,y)] \equiv \forall x\ \exists y\ \neg P(x,y)$

**Example- 24**

Write formally the statement "for all real number there is a greater real number". Write the negation of that statement.

**Solution:**

x and y are in the domain of real numbers R. The statement is:

$\forall x\ \exists y\ (x < y)$.

The negation of the statement is: $\neg[\forall x\ \exists y\ (x < y)] \equiv \exists x\ \forall y\ (x \geq y)$ ■

- A proof is a clear explanation, accepted by the mathematical community, for the truth of a proposition

- **Types of proofs**

- Direct proof
- Indirect proof **or** proof by contrapositive
- Proof by contradiction

1. **Direct proof :** $P$(hypothesis)$\rightarrow Q$(conclusion).

**❶ Example- (25-a) "direct proof"**

| |
|---|
| ① Prove that if $x > 2$ and $y > 3$ then $x + y > 5$ |

**Proof**

Let $x > 2$ and $y > 3$, and adding the two inequalities to get $x + y > 5$

## ❶ Example- (25-b) "direct proof"

② Prove that for all real numbers: $d,\ d_1,\ d_2$ and $x$

"If $d = \min\{d_1, d_2\}$ and $x \leq d$ then $x \leq d_1$ and $x \leq d_2$"

**Proof**

From the definition of *min*, it follows that:

if $x \leq d$ and $d \leq d_1$ then $x \leq d_1$.

Also, if $x \leq d$ and $d \leq d_2$ then $x \leq d_2$.

Then $x \leq d_1$ and $x \leq d_2$. ∎

**❶ Example- (25-c) "direct proof"**

③ If $n$ is an odd positive integer, then $n^2$ is odd as well.

**Proof**

If $n$ is an odd positive integer, then it can be written as $n = 2k + 1$ for some integer $k \geq 0$, then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2(2k^2 + 2k)$ is even, and "even plus one is odd", we can conclude that $n^2$ is odd. ■

## 2. **indirect proof** or **proof by contrapositive:**

Instead of proving $P \rightarrow Q$, we prove $\neg Q \rightarrow \neg P$.

**Examples:**

Prove that, if $x^2$ is even, then $x$ is even.

Here: $p \equiv$ "$x^2$ is even number" and $q \equiv$ "$x$ is also even".

Let us start with $\bar{q}$ by assuming that $x$ is an odd, so $x = 2k + 1$ for some integer $k$. This leads to $x^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, then $x^2$ is odd number. This completes the proof.

## ❷ Example- (26-b) "indirect proof" or "proof by contrapositive"

Prove that for $n \geq 2$: if $2^n - 1$ is prime, then $n$ is odd.

**Proof**

$p$: $2^n - 1$ is prime, $\quad \bar{p}$: $2^n - 1$ is not prime (or it can be factored)

$q$: $n$ is odd, $\quad \bar{q}$: $n$ is even

By using indirect proof, we start with $\bar{q}$: $n$ is even and hence

$$n = 2k, \qquad k \geq 1$$

$$2^n - 1 = 2^{2k} - 1 = (2^k - 1)(2^k + 1)$$

Therefore, $2^n - 1$ is factored and hence it is not prime ($\equiv \bar{p}$). ■

## 3. Proof by contradiction

We want to reach to a proposition of the form

$P \to \neg Q.$

# Examples

① Prove by contradiction that

if $x+y>5$ then either $x>2$ or $y>3$

- Suppose that the conclusion "$x>2$ **or** $y>3$" is false
- so $x\leq 2$ **and** $y\leq 3$ is true
- Adding the two inequalities gives $x+y\leq 5$
- This contradicts with the hypothesis $x+y>5$
- hence we conclude that the assumption $x\leq 2$ or $y\leq 3$ cannot be right.
- So, $x>2$ or $y>3$ must be true.

② Prove by contradiction that $\sqrt{2}$ is not a rational number, i.e. there are no integers $a$, $b$ such that $\sqrt{2} = \frac{a}{b}$

**Proof**

Assume that $\sqrt{2}$ is a rational number, that can be written as a fraction of two integers $a$ and $b$, then there exist $\sqrt{2} = \frac{a}{b}$, $a$ and $b$ are integers and the fraction is written in least term (has no common factor).

Squaring both sides to get $2 = \frac{a^2}{b^2} \rightarrow a^2 = 2b^2$ then $a^2$ is an even number. This implies that $a$ itself [9] is an even number which can be written as $a = 2l$. Return to the previous relation and substitute to get: $2 = \frac{4l^2}{b^2}$, then $b^2 = 2l^2$ which represent an even number and again this implies that b itself is an even number which can be written as $b = 2m$.

So, $\sqrt{2} = \frac{2l}{2m}$ which contradict to the original hypothesis that $\frac{a}{b}$ was in least term. Thus $\sqrt{2}$ is not a rational number. ■

③ Use "the proof by contradiction" to prove that $\sqrt{3}$ is irrational number.

**Proof**

Say $\sqrt{3}$ is a rational number, that can be written as a fraction of two integers $a$ and $b$, then there exist $\sqrt{3} = \frac{a}{b}$, $a$ and $b$ are integers and the fraction is written in least term (**has no common factor**). Squaring both sides to get $3 = \frac{a^2}{b^2} \to a^2 = 3b^2$. This leads to that $a^2$ must be divisible by 3 and so $a$ itself must be divisible by 3.

Since $a$ is divisible 3, then $a = 3\ell$ and hence $3b^2 = 9\ell^2$ or $b^2 = 3\ell^2$ which means that $b^2$ is divisible by 3 and hence $b$ is also divisible by 3, i.e., $b = 3m$. Therefore, $\sqrt{3} = \frac{a}{b} = \frac{3\ell}{3m}$ which has a common factor and now we have a contradiction. So, $\sqrt{3}$ is irrational number. ■

**Arguments:** it is a sequence of propositions $p1, p2, .., pn$ called hypotheses followed by a proposition called conclusion.

**An argument is usually written as:**

$$\begin{array}{c} p_1 \\ p_2 \\ . \\ . \\ p_n \\ \hline \therefore q \end{array}$$

Or: $p_1, p_2, \ldots, p_n / \therefore q$ , which means that if $(p_1 \wedge p_2 \wedge \ldots\ldots \wedge p_n) \rightarrow q$

The argument is called **valid** or **invalid**.

**Example- 28**

Determine whether the following arguments are ***valid*** or ***invalid***

① $\begin{array}{c} p \to q \\ p \\ \hline \therefore q \end{array}$

② $\begin{array}{c} p \to (q \to r) \\ q \to (p \to r) \\ \hline \therefore (p \vee q) \to r \end{array}$

**Proof:**

① By using the truth table to check the truth values of $p \to q$, $p$ and $q$

| $p$ | $q$ | $p \to q$ | $(p \to q) \wedge \mathrm{p}$ | $[(p \to q) \wedge \mathrm{p}] \to q$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $\boldsymbol{T}$ |
| $T$ | $F$ | $F$ | $F$ | $\boldsymbol{T}$ |
| $F$ | $T$ | $T$ | $F$ | $\boldsymbol{T}$ |
| $F$ | $F$ | $T$ | $F$ | $\boldsymbol{T}$ |

The last column is a *tutorial*, so the argument is ***valid***. ■

② Let $u = p \to (q \to r)$, $v = q \to (p \to r)$, and $w = (p \vee q) \to r$

Now, we want to prove that $(u \wedge v) \to w$ represent a tutorial

| $p$ | $q$ | $r$ | $p \vee q$ | $q \to r$ | $p \to r$ | $u$ | $v$ | $w$ | $(u \wedge v)$ | $(u \wedge v) \to w$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |

The last column $(u \wedge v) \to w$ does not represent a tutorial, then the argument is *invalid*. ■

## 1.6. Boolean Matrix Operations

A Boolean matrix (or a bit matrix) is an $m \times n$ matrix whose entries are either **zeros** or **ones**. Here, we will define three operations on Boolean matrices that have useful applications in Chapter V.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ Boolean matrices, then define:

**1-** $A \vee B = [C_{ij}]$, where $C_{ij} = \begin{cases} 1 & \text{if } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{otherwise} \end{cases}$

**2-** $A \wedge B = [C_{ij}]$, where $C_{ij} = \begin{cases} 1 & \text{if } a_{ij} \text{ and } b_{ij} = 1 \text{ are both } 1 \\ 0 & \text{otherwise} \end{cases}$

**3-** $A \times B = [C_{ij}]$, where

## Example- 29

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, compute

① $A \vee B$ ② $A \wedge B$ ③ $A \times C$

## Solution

① $A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ ② $A \wedge B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ③ $A \times C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$