Lecture 7

Sigma Notation Σ

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the kth term.

The index k starts at $k = 1$.

$$\sum_{k=1}^{k=6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$\sum_{k=1}^{\boldsymbol{n}} k = \frac{\boldsymbol{n}(\boldsymbol{n}+1)}{2} = \frac{n^2+n}{2}$$

$$\sum_{k=1}^{\boldsymbol{n}} k^2 = \frac{\boldsymbol{n}(\boldsymbol{n}+1)(2\boldsymbol{n}+1)}{6}$$

$$\sum_{k=1}^{\boldsymbol{n}} k^3 = \frac{\boldsymbol{n}^2(\boldsymbol{n}+1)^2}{4}$$

$$\sum_{k=1}^{\boldsymbol{n}} \frac{1}{n} = \boldsymbol{n}\frac{1}{n} = 1$$

Example

If $f(x) = 1 - x^2$

Find $\lim_{n \to \infty} [\ \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \left(\frac{1}{n}\right)]$

Solution

$$f\left(\frac{k}{n}\right) = 1 - \left(\frac{k}{n}\right)^2$$

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left\{\left(1 - \left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\right)\right\}$$

$$= \sum_{k=1}^{n}\left\{\frac{1}{n} - \frac{k^2}{n^3}\right\}$$

$$= \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= \sum_{k=1}^{n} \frac{1}{n} - \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{\boldsymbol{n}} \frac{1}{n} = \boldsymbol{n} \frac{1}{n} = 1, \qquad \sum_{k=1}^{\boldsymbol{n}} k^2 = \frac{\boldsymbol{n}(\boldsymbol{n}+1)(2\boldsymbol{n}+1)}{6}$$

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) = 1 - \frac{n(n+1)(2n+1)}{6n^3}$$

$$= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[1-\frac{2n^3+3n^2+n}{6n^3}\right]$$

$$= 1-\frac{2}{6}=\frac{2}{3}$$

Example

Prove that

$$\sum_{1}^{m} \frac{1}{n(n+1)} = \frac{m}{m+1}$$

From partial fractions

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Solution

$$\sum_{1}^{m} \frac{1}{n(n+1)} = \sum_{1}^{m} \frac{1}{n} - \sum_{1}^{m} \frac{1}{n+1}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m+1}\right)$$

$$= 1 - \frac{1}{m+1} = \frac{m}{m+1}$$

Changing Index in the sigma notation

<u>Example</u>

Change the lower index of the sum

$$\sum_{k=1}^{k=n} a_{k-1} \; a_{n-k}$$

to start with $j = 3$

old $\quad \mathrm{k} = 1, 2, 3 \;\ldots$

new $\quad \mathrm{j} = 3, 4, 5 \;\ldots$

$$\mathrm{j} = \mathrm{k} + 2 \quad \rightarrow \quad k = j - 2$$

$$\sum_{k=1}^{k=n} a_{k-1}\, a_{n-k} = \sum_{j-2=1}^{j-2=n} a_{j-2-1}\, a_{n-j+2} = \sum_{j=3}^{j=n+2} a_{j-3}\, a_{n-j+2}$$

Example

Change the lower index of the sum

$$\sum_{k=5}^{k=n} a_{n+k} \ \ b_{n-k}$$

to start with $j = 0$

old $k = 5, 6, 7 \;\ldots$

new $j = 0, 1, 2 \;\ldots$

$k = j + 5$

$$\sum_{k=5}^{k=n} a_{n+k}\, b_{n-k} = \sum_{j+5=5}^{j+5=n} a_{n+j+5}\, b_{n-j-5} = \sum_{j=0}^{j=n-5} a_{n+j+5}\, b_{n-j-5}$$

Series

A series is the sum of a sequence.

- ❑ The infinite sequence

$$\{a_n\}_1^\infty = \{a_1, a_2, a_3, \dots\}$$

- ❑The infinite series

$$a_1 + a_2 + a_3 + \cdots$$

Partial sums

- Infinite series $a_1 + a_2 + a_3 + \cdots$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

- The sequence of partial sum

$$S_n = \{S_1, S_2, S_3, S_4, \dots, S_n, \dots\}$$

The sum *S* of the infinite series

$$\boxed{S = \sum_{1}^{\infty} a_n = \lim_{n \to \infty} S_n}$$

- The infinite sequence $\left(\frac{3}{10}, \frac{3}{10^2}, \frac{3}{10^3}, \ldots, \frac{3}{10^n}, \cdots\right)$
- The infinite series $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots + \frac{3}{10^n} + \cdots$

$$S_1 = a_1 = 0.3$$

$$S_2 = a_1 + a_2 = 0.3 + 0.03 = 0.33$$

$$S_3 = a_1 + a_2 + a_3 = 0.3 + 0.03 + 0.003 = 0.333$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = 0.3 + 0.03 + 0.003 + 0.0003 = 0.3333$$

$$\vdots$$

- The sequence of partial sum

$$S_n = \{S_1, S_2, S_3, S_4, \dots, S_n, \dots\}$$
$$= \{0.3, 0.33, 0.333, 0.3333, \dots.\ \}$$

- The sum $\boldsymbol{S}$ of the infinite series

$$\boldsymbol{S} = 0.33333333 = \frac{1}{3}$$

Converges and diverges

<u>Theorem</u>

- A series Σa_n is **convergent** (or **converges**) if its sequence of partial sums $\{S_n\}$ converges

$$\text{Lim}_{n\to\infty} S_n = S$$

- The limit S is the **sum** of the series

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- The series Σa_n is divergent (or diverges) if $\{S_n\}$ diverges. A divergent series has no sum.

Example

Given the series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}+\cdots,$$

a) Find $S_1,\ S_2,\ S_3, \ldots,\ S_6,$ **b)** Find S_n

c) Show that the series converges and find its sum.

Solution

a)

$$S_1 = a_1 = \frac{1}{1 \times 2} = \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{2}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} = \frac{3}{4}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{4}{5}$$

$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = \frac{5}{6}$$

$$S_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = \frac{6}{7}$$

b)

- The **sequence** of partial sum

$$S_n = \{S_1, S_2, S_3, S_4, \ldots, S_n, \ldots\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7} \ldots . \right\}$$

- The general formula for S_n

$$S_n = \frac{n}{n+1}$$

c)

Converges and its sum

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

The **sum** of the series S=1

Geometric Series

$$\boldsymbol{a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots}$$

- ❑ converges if $|r| < 1$

$$S = \frac{a}{1 - r}$$

- ❑diverges if $|r| > 1$

Example

Prove that the following series converges and find its sum:

$$0.6 + 0.06 + 0.006 + \cdots + \frac{6}{10^n} + \cdots$$

The series is a geometric series $r = \frac{0.06}{0.6} = 0.1$

Since $|r|<1$ $\rightarrow$ geometric series **convergent**

$$S = \frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{2}{3}$$

$$0.6 + 0.06 + 0.006 + \cdots + \frac{6}{10^n} + \cdots = \frac{2}{3}$$

The ratio test for convergent

- Convergent

$$\left|\frac{A_{n+1}}{A_n}\right| < 1;$$

- Divergent

$$\left|\frac{A_{n+1}}{A_n}\right| > 1$$

- Failed

$$\left|\frac{A_{n+1}}{A_n}\right| = 1;$$

Example

Find all values of $\boldsymbol{x}$ for which the following series will be convergent, and find its sum

$$\frac{1}{2}+\frac{(x-3)}{4}+\frac{(x-3)^2}{8}+\cdots+\frac{(x-3)^n}{2^{n+1}}+\cdots$$

Solution

The series to be convergent, it must satisfy the ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\left|\frac{(x-3)}{4} \div \frac{1}{2}\right| = \left|\frac{(x-3)}{2}\right| < 1$$

$$|x-3| < 2 \;\rightarrow\; -2 < x-3 < 2 \;\rightarrow\; 1 < x < 5$$

The series is a geometric series

$$a = 0.5 \qquad\qquad r = \frac{x-3}{2}$$

Thus, the sum is

$$S = \frac{a}{1-r} = \frac{0.5}{1-\frac{x-3}{2}} = \frac{1}{5-x}$$

Taylor Series

Any defined continuous function $f(x)$ can be approximated by **Taylor series** as:

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \frac{(x - x_0)^3}{3!} f'''(x_0) + \cdots$$

Maclaurin series is a **special case** of Taylor series

If $x_0 = 0$, then the **Taylor series** converts to the **Maclaurin series**

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Example

- Use **Taylor** series to expand the function

$$f(x) = e^x \text{ with } x_0 = 0$$

- Then prove that $0 \leq (1+x)^m \leq e^{mx}$ for positive m and $x \geq -1$.

Solution

$$f(x) = e^x \text{ with } x_0 = 0$$

$$f(x) = e^x \quad \rightarrow \quad f(0) = e^0 = 1$$

$$f'(x) = e^x \quad \rightarrow \quad f'(0) = e^0 = 1$$

$$f''(x) = e^x \quad \rightarrow \quad f''(0) = e^0 = 1$$

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$1 + x \le e^x$$

$$0 \le 1 + x \le e^x$$

$$0 \le (1 + x)^m \le e^{mx}$$

Example

- Use **Taylor** series to expand the function

$$f(x) = \sqrt[3]{x+1} \text{ with } x_0 = 0$$

$$f(x) = \sqrt[3]{x+1} \qquad \rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{3}(1+x)^{\frac{-2}{3}} \qquad \rightarrow f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{-2}{9}(1+x)^{\frac{-5}{3}} \qquad \rightarrow f''(0) = \frac{-2}{9}$$

$$f'''(x) = \frac{10}{27}(1+x)^{\frac{-8}{3}} \qquad \rightarrow f''(0) = \frac{10}{27}$$

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$f(x) = \sqrt[3]{x+1} = 1 + \frac{1}{3}x + \frac{-1}{9}x^2 + \frac{5}{81}x^3 + \cdots$$

Exercises

Use Maclaurin series to approximate the following functions:

$$f(x) = \sin x$$
$$f(x) = \cos x$$
$$f(x) = \sin 2x + \cos 2x$$

Power Series

$$f(x) = a_0 + a_1(x - x_0)^1 + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n + \cdots$$

- $f(x)$ and x_0 are given
- where the constants a_0, a_1, a_2,… are called **coefficients of the power series**

- When $x_0 = 0$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Example

❑Find a **power series for** the function

$$f(x) = e^x$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \quad \rightarrow \quad f(0) = a_0$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \quad \rightarrow \quad f'(0) = a_1$$

$$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \quad \rightarrow \quad f''(0) = 2a_2$$

$$f'''(x) = 6a_3 + 24a_4 x + \cdots \quad \rightarrow \quad f'''(0) = 6a_3$$

On the other hand

$$f(x) = e^x \quad \rightarrow \quad f(0) = e^0 = 1$$

$$f'(x) = e^x \quad \rightarrow \quad f'(0) = e^0 = 1$$

$$f''(x) = e^x \quad \rightarrow \quad f''(0) = e^0 = 1$$

$$a_0 = 1, \qquad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}$$

Thus, the function can be approximated as:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots$$

Example

- Find a **power series for** the function

$$f(x) = \sqrt[3]{x+1}$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots & \rightarrow \quad f(0) &= a_0 \\
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots & \rightarrow \quad f'(0) &= a_1 \\
f''(x) &= 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots & \rightarrow \quad f''(0) &= 2a_2 \\
f'''(x) &= 6a_3 + 24a_4 x + \cdots & \rightarrow \quad f'''(0) &= 6a_3
\end{aligned}$$

On the other hand

$$f(x) = \sqrt[3]{x+1} \qquad \rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{3}(1+x)^{\frac{-2}{3}} \qquad \rightarrow f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{-2}{9}(1+x)^{\frac{-5}{3}} \qquad \rightarrow f''(0) = \frac{-2}{9}$$

$$f'''(x) = \frac{10}{27}(1+x)^{\frac{-8}{3}} \qquad \rightarrow f''(0) = \frac{10}{27}$$

$$a_0 = 1, \qquad a_1 = \frac{1}{3}, \quad a_2 = \frac{-1}{9}, \quad a_3 = \frac{5}{81}$$

Thus, the function can be approximated as:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = \sqrt[3]{x+1} = 1 + \frac{1}{3}x + \frac{-1}{9}x^2 + \frac{5}{81}x^3 + \cdots$$