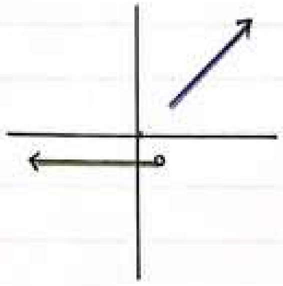
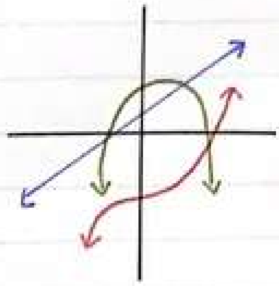
Unit 1 - Function

Quiz - Wed Sep. 21 Pretest - Mon. Sep. 26 Test - Mon. Oct. 3



$$f(x) = \begin{cases} x & \text{where } x \geq 1 \\ -1 & \text{where } x < 1 \end{cases}$$



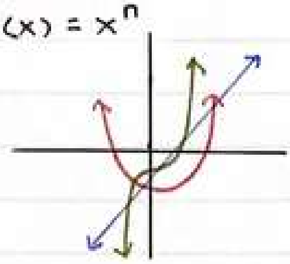
Linear:
ex. $y = 3x + 1$

Quadratic:
ex. $y = -(x-1)^2 + 2$

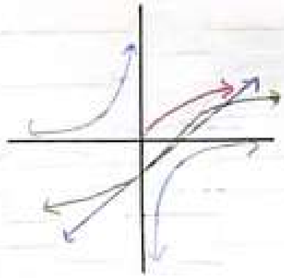
Cubic:
ex. $f(x) = x^3 - 3$

Power function: $f(x) = x^n$

ex. $f(x) = x^0 = 1$
$g(x) = x^1 = x$
$h(x) = x^2$
$k(x) = x^3$

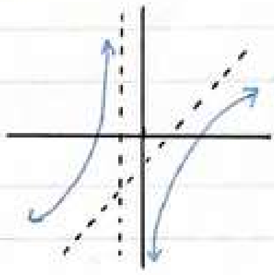


$g(x) = x^1 = x$
$h(x) = x^{\frac{1}{2}}$
$k(x) = x^{\frac{1}{3}}$
$m(x) = x^{-1}$



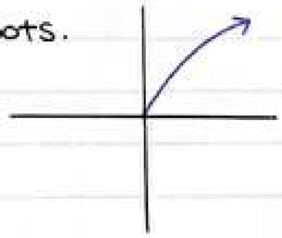
Rational Function: $f(x) = \frac{P(x)}{Q(x)}$

ex. $y = \frac{x^2 - 4}{x + 1}$

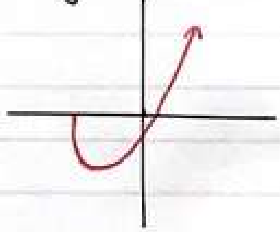


Algebraric function:
- Functions that can be constructed from polynomials using $+, -, \times, \div$ and extracting roots.

ex. $y = \frac{x}{2} + 3\sqrt{x}$

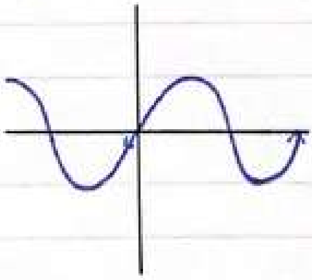


ex. $y = x\sqrt{x^3 + 64}$

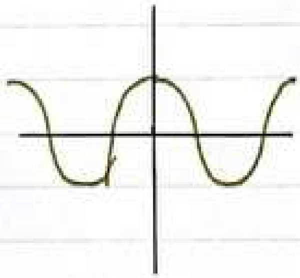


Trig:

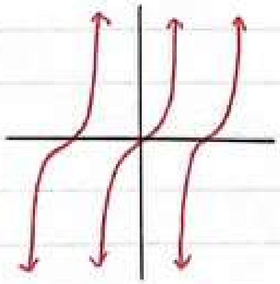
$y = \sin x$



$y = \cos x$

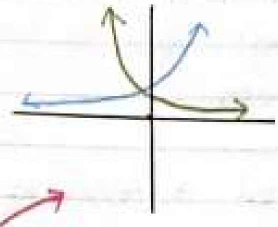


$y = \tan x$



1 radian $= \frac{180°}{\pi}$

Exponential Functions: $y = a^x$

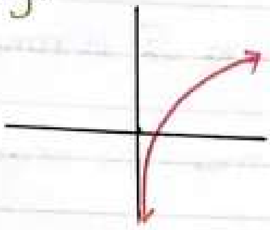


ex. $y = 2^x$ (increasing)

$y = (\frac{1}{4})^x$ (decreasing)

Logarithmic functions:

ex. $\log_3(x)$



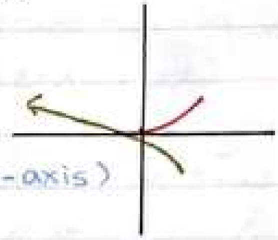
reflection of exponential functions.

Transcendental functions:

- Functions that are not algebraic: ex. Trig, log, exponential...

Absolute Value:

ex. $y = |\sqrt{3-x} - 2|$

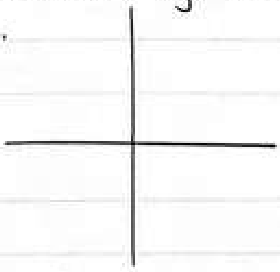


guide: $\sqrt{3-x} - 2 \rightarrow \sqrt{-(x-3)} - 2$ ($2\downarrow$, $3\rightarrow$, refl. y-axis)

$|\ | \rightarrow$ reflect below x-axis

Combining Function by graph:

ex.



ex. $f(x)$: ID $x \in \mathbb{R} \mid x > 3$ $\quad$ $g(x)$: ID $x \in \mathbb{R} \mid -7 < x < 20$

$\hookrightarrow (f+g)(x)$: ID $x \in \mathbb{R} \mid 3 < x < 20$

ex. $m(x)$: ID $x \in \mathbb{R} \mid x \geq 7$ $\quad$ $n(x)$: ID $x \in \mathbb{Z} \mid -50 < x < 27$

a) $(mn)(x) \rightarrow x \in \mathbb{Z} \mid 7 \leq x < 27$ $\quad$ b) $(\frac{m}{n})(x) \rightarrow x \in \mathbb{Z}$ $7 \leq x < 27$ or $0 < x < 27$

ex. $f(x) = \sqrt{x-1}$ $\quad$ $g(x) = \frac{2}{x}$

$f(g(x)) = \sqrt{(\frac{2}{x}) - 1} = \sqrt{\frac{2}{x} - 1}$ $\quad$ $g(f(x)) = \frac{2}{\sqrt{x-1}}$ $\quad$ $g(g(x)) = \frac{2}{(\frac{2}{x})} = x$

ID $f(g(x))$ is all values of x which (I) are are in ID of $g(x)$, & (II) for which the output $g(x)$ is in the ID of $f(x)$.

ex. $f(x)$ has ID: $x \in \mathbb{R} \mid -9 < x < 21$. $g(x) = \sqrt{x}$. Find ID of $f(g(x))$

ID of $g(x)$: $x \in \mathbb{R} \mid x \geq 0$

$\therefore$ ID of $f(g(x))$ is $x \in \mathbb{R} \mid 0 \leq x < 21$

$f(x) = \sqrt[3]{x-2}$ $\quad$ $g(x) = x - 2$ $\quad$ $f(x) = h(g(x))$ $\quad$ $f(x) = h(g(x))$

46 # 3,5, 16, 18, 20, 23, 30, 32, 37, 39, 49, 53 $\quad$ 1.1, 1.2

General problem solving:

1. Understand the problem.
2. Think of a plan.

ex. Proof $\sqrt{2}$ is irrational $\rightarrow$ Assume $\sqrt{2}$ is rational $\rightarrow \sqrt{2} = \frac{a}{b} \rightarrow (b\sqrt{2})^2 = a^2$

$\hookrightarrow 2b^2 \neq a^2$

Induction: show if true for x, then it's true for x+1. & show it's true for x=1.

3. Carry out the plan.
4. Look back.

ex. solve $|x-3| + |3x-5| = 10$.

Case 1 $x-3 \geq 0$

true only $x \geq 3$

case 1a $3x-5 \geq 0$

$\hookrightarrow x \geq \frac{5}{3}$

$\hookrightarrow x-3+3x-5=10$

$\Rightarrow 4x = 18 \quad x = \frac{9}{2}$

case 1b $3x-5<0$

$\hookrightarrow x < \frac{5}{3}$

$\hookrightarrow x-3-3x+5=10$

$\rightarrow -2x = 8 \quad \cancel{x=-4}$

Case 2 $x-3<0$

true only $x<3$

case 2a $3x-5 \geq 0$

$\hookrightarrow (x \geq \frac{5}{3})$

$\hookrightarrow -(x-3)+3x-5=10$

$2x = 12 \quad \cancel{x=6}$

case 2b $3x-5<0$

$\hookrightarrow (x < \frac{5}{3})$

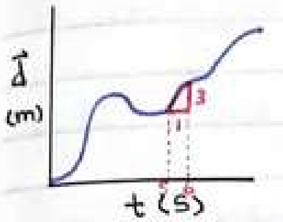
$-(x-3)-(3x-5)=10$

$-4x = 2 \quad x = -\frac{1}{2}$

63 # 1-7, 9, 11, 13

Unit 2 - Limits

QUIZ - FRI. Oct. 14 Pretest - THU. Oct. 20 Test - THU. Oct. 27



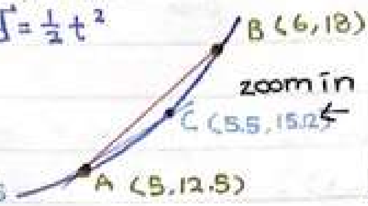
☆ $\vec{v} = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$

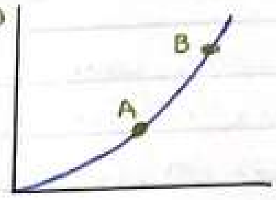
ex. A car stats at rest, then acclerates at $1 m/s^2$ f 10 s.

$\vec{d} = \frac{1}{2}at^2 \rightarrow \vec{d} = \frac{1}{2}(1)t^2 \rightarrow \vec{d} = \frac{1}{2}t^2$

$\vec{v}_1 = \frac{18-12.5}{1} = 5.5 m/s$

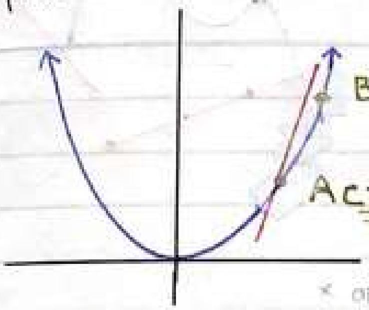
$\vec{v}_2 = \frac{15.12-12.25}{5.5-5} = 5.25 m/s$





↳ $\lim_{B \to A} v = 5$ (As time gets shorter, it apporches 5.)

ex. Find the slope of the line tangent to the curve $y = x^2$, at point A (7, 49).



$m_{AB} = \frac{x^2-49}{x-7}$

$\lim_{x \to 7} \frac{x^2-49}{x-7}$

x	y
8	15
7.5	14.5
7.1	14.1
7.01	14.01

x	y
6	13
6.5	13.5
6.9	13.9
6.99	13.99

x approch to certain # closer to other # → grow
close to 7 the closer to 14

ex. $f(x) = \frac{2x^2-1}{x^2}$ Can we plug ∞? $f(\infty) = \frac{2(\infty)^2-1}{(\infty)^2} \rightarrow \frac{2(\infty)-1}{\infty} = \frac{\infty}{\infty}$

x	100	1000	1 000 000
$\frac{2x^2-1}{x}$	1.999	1.999999	1.999999999999

$\rightarrow \lim_{x \to \infty} \frac{2x^2-1}{x^2} = 2$

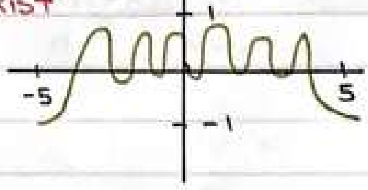
☆: $\lim_{x \to a} f(x) = L$ means that we can make $f(x)$ as close to L as as we like, by macking x close enough to a (but $\neq a$)

• For any tolerance $\varepsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ wherever $|x - a| < \delta$.

ex. Estimate $\lim_{x \to 3} \frac{x^2-9}{x-3} \doteq 6$

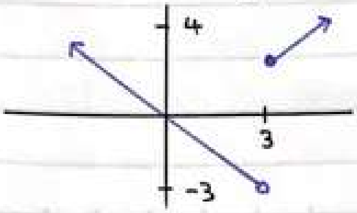
x	2.9	2.99	2.999	3.1	3.01	3.001
$\frac{x^2-9}{x-3}$	5.9	5.99	5.999	6.1	6.01	6.001

ex. Estimate $\lim_{x \to 0} \cos\left(\frac{2\pi}{x}\right) \stackrel{?}{=} 1$ (no) → does not exist



x	1	0.5	0.1	0.01	0.001	0.0001	0.00003
$\cos\left(\frac{2\pi}{x}\right)$	1	1	1	1	1	1	-0.5

ex. $f(x) = \begin{cases} x+1 & \text{if } x \geq 3 \\ -x & \text{if } x < 3 \end{cases}$ $\lim_{x \to} f(x) = ?$ → not exist



but, $\lim_{x \to 3^-} f(x) = -3$ & $\lim_{x \to 3^+} f(x) = 4$

☆: $\lim_{x \to a} f(x)$ exist only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

left $(x < a)$ right $(x > a)$

ex. $\lim_{x \to 0} \frac{1}{x}$ = not exist → $\lim_{x \to 0} \frac{1}{x} = \infty$ (getting bigger)

$\frac{1}{0.5} = 2$ $\frac{1}{0.1} = 10$ $\frac{1}{0.01} = 100$

Limit Law:

(1) $\lim [f(x) + g(x)] = \lim f(x) + \lim g(x)$

(2) $\lim [f(x) - g(x)] = \lim f(x) - \lim g(x)$

(3) $\lim [c f(x)] = c \lim f(x)$

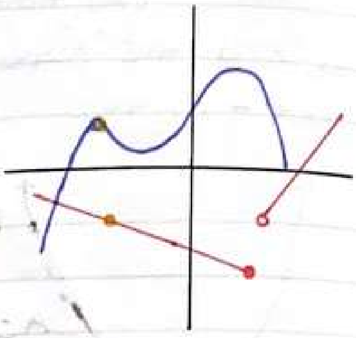
(4) $\lim [f(x) g(x)] = [\lim g(x)][\lim f(x)]$

(5) $\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$ $\lim g(x) \neq 0$

(6) $\lim [f(x)]^n = [\lim f(x)]^n$

(7) $\lim_{x \to a} c = c$

(8) $\lim_{x \to a} x = a$

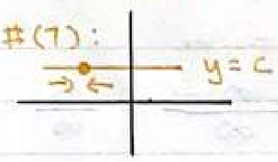


ex. $\lim_{x \to 2} [f(x) + 5g(x)]$ → $\lim_{x \to -2} f(x) + \lim_{x \to -2} 5g(x)$

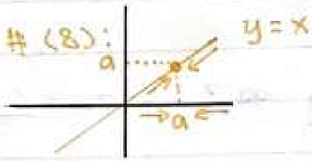
$= \lim_{x \to 2} f(x) + 5 \lim_{x \to 2} g(x)$ → $\lim_{x \to -2} f(x) = 1$ $\lim_{x \to -2} g(x) = -1$

↳ $1 + 5(-1) = 1 - 5 = -4$

#(7):



#(8):



ex. $\lim_{x \to a} 2x^2 - 3x + 4$ → $\lim_{x \to 2} 2x^2 - \lim_{x \to 2} 3x + \lim_{x \to 2} 4$ → $2\left(\lim_{x \to 2} x\right)^2 - 3\lim_{x \to 2} x + \lim_{x \to 2} 4$

use: (7) & (8) → $2(2)^2 - 3(2) + 4 = 6$

Direct Substitution:

If $f(x)$ is a polynomial or a rational function & a is in the domain of f then $\lim_{x \to a} f(x) = f(a)$

ex. $\lim_{x \to -1} \frac{4x^2 + 5x - 7}{3x^2 - 2x + 1}$ → $\frac{4(-1)^2 + 5(-1) - 7}{3(-1)^2 - 2(-1) + 1} = \frac{-4}{3}$

ex. $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$ → $\lim_{x \to 3} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}} = -6$ *Facto → cancel → sub

ex. $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$ → $\frac{h^2 + 6h + \cancel{9} - \cancel{9}}{\cancel{h}} = 6 + h = 6$

$\lim_{x \to 4} \frac{3 - x}{x^2 - 2x - 8}$ → $\frac{3 - x}{(x-4)(x+2)}$ → Doesn't exist

$\lim_{x \to 2} \frac{x^2 - x + 6}{x - 2}$ → $\frac{\text{can't factor}}{x - 2}$ → Doesn't exist

89 # 1, 2, 5, 11, 17, 24, 27, 35, 39, 41, 45, 47

Squeeze theorem:

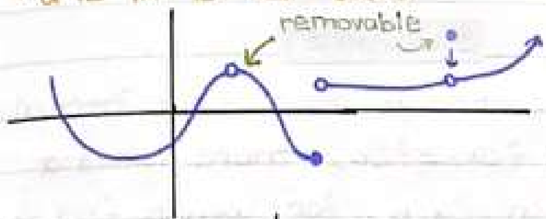
If $f(x) < g(x) < h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$

If $f(x) \leq g(x)$ when $\underline{x}$ is near $\underline{a}$, $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

Continuoity:

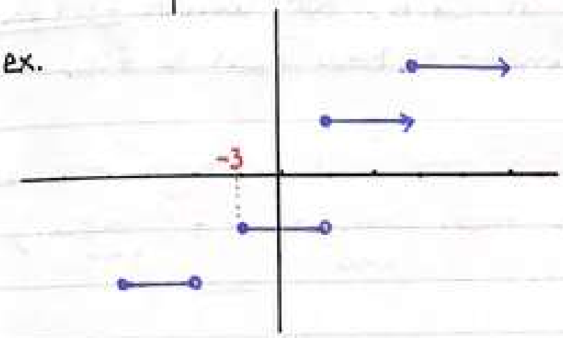
$f(x)$ is continuous at $\underline{a}$ if $\lim_{x \to \underline{a}} f(x) = f(\underline{a})$

• a is in ID of $f(x)$ • $\lim_{x \to a} f(x)$ exist • $\lim_{x \to a} f(x) = f(a)$



(ex. of 3 discontinuoity)

ex.



where $x = -3$, $f(x)$ is continuous from the right

$\lim_{x \to a^{-1}} f(x) = f(a)$

ex. Show $f(x) = \sqrt[3]{\frac{3-x^2}{x+5}}$ is continous on $(-3, 7)$.

where $-3 < x < 7$, $\lim_{x \to a} f(x) = \lim_{x \to a} \sqrt[3]{\frac{3-x^2}{x+5}} \to \sqrt[3]{\lim_{x \to a} \frac{3-x^2}{x+5}} \to \sqrt[3]{\frac{3-a^2}{a+5}}$

$= f(a)$

• If 2 functions are continued at a value, so is any multiple of those functions, their sum & difference & product, and their quotient. (as long as the divisor function $\neq 0$)

• If $f(x)$ & $g(x)$ are continuos at a, so is $(f+g)(x)$, $-3f(x)$, etc.

- Rational, root, and trig functions are continuous at every value. in their ID.

ex. On which interval(s) is each function continuous?

a) $f(x) = x^7 - 3x^5 + 5$ $(-\infty, \infty)$

b) $g(x) = \frac{x-7}{x-1}$ $x - 1 \neq 0 \to x \neq 1$ $(-\infty, 1)$ and $(1, \infty)$

c) $h(x)\ \frac{x-2}{x} + \sqrt[4]{x-1}$ $j(x) = \frac{x-2}{x} \to x \neq 0$ $K(x) = \sqrt[4]{x-1} \to x - 1 \geq 0 \to x \geq 1$

$h(x) = (j+k)(x) \to [1, \infty)$

• If $f(x)$ is continuous at b, and $\lim_{x \to a} g(x) = 6$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to 2} g(x))$

In a continuous function of a continuous function is continuous.

↳ If $g(x)$ is continuous at a, and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continous at a.

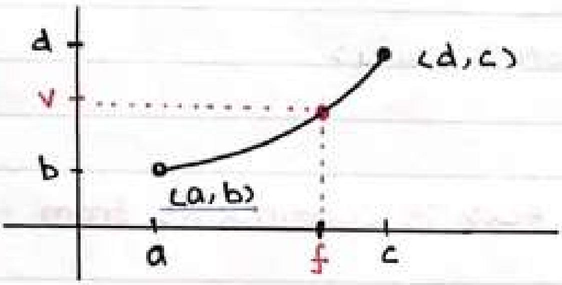
ex. where is $f(x) = \frac{3}{7+\sqrt{x^3-5}}$ continuous?

$a(x) = x^3 - 5$ $\quad$ $f(x) = d(c(b(a(x))))$

$b(x) = \sqrt[3]{x}$ $\quad$ • $f(x)$ is continuous if all its component are continuous.

$c(x) = 7 + x$ $\quad$ $x^3 - 5 \geq 0 \rightarrow x \geq \sqrt[3]{5}$

$d(x) = \frac{3}{x}$ $\quad$ • $[\sqrt[3]{5}, \infty)$



• If $f(x)$ is continuous on a close interval $[a,b]$, and $f(a) \neq f(b)$, there exists a number c in (a,b) for which $f(c) = N$, for any number N b+w $f(a)$ & $f(b)$

* Direct. Sub. for rational, root, trig → all are continuous through their D.

• If $f(x)$ is continuous at b, and $\lim_{x \to a} g(x) = b$ then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

• If $g(x)$ is continuous at a &

119 # 1, 7, 10, 11, 18, 19, 27, 28 $\quad$ Pretest THUR

$$\lim_{\Delta x \to 0} \frac{P(x+\Delta x) - P(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(0.03(x+\Delta x)^3 + (x+\Delta x) + 25) - (0.03x^3 + x + 25)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{0.03(x^3 + 3x^2\Delta x + 3x\Delta x^3) + x + \Delta x + 25 - 0.03x^3 - x - 25}{\Delta x}$$

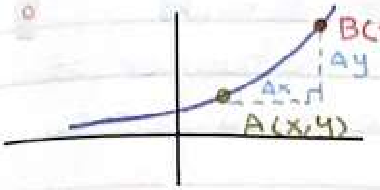
$$\lim_{\Delta x \to 0} 0.03x^3 + 0.09x^2\Delta x + 0.09x\Delta x^2 + 0.03\Delta x^3 + x + \Delta x + 25 - 0.03x^3 - x - 25$$

$$\lim_{\Delta x \to 0} \frac{0.09x^2\Delta x + 0.09x\Delta x^2 + 0.03\Delta x^3 + \Delta x}{\Delta x}$$

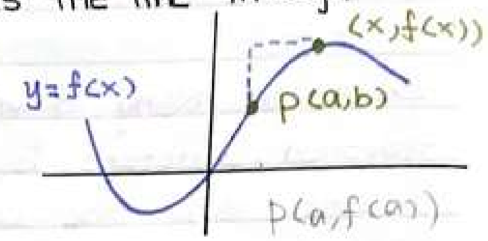
↳ $\frac{\Delta x(0.09x^2 + 0.09x\Delta x + 0.03\Delta x^2 + 1)}{\Delta x} \rightarrow \lim_{\Delta x \to 0} 0.09x^2 + 0.09x\Delta x + 0.03\Delta x^2 + 1$

$= 0.09x^2 + 1$

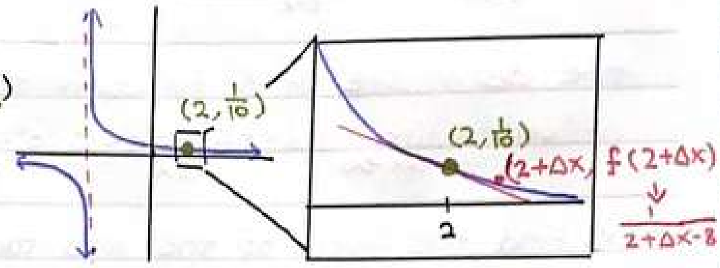
Tangent line - the tangent line to $\underline{f(x)}$ at $P(a,b)$ is the line through P with slope $m = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$



ex. Find tangent line to $y = \frac{1}{x+8}$ at $\underline{(2, \frac{1}{10})}$



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(2+\Delta x)-f(x)}{\Delta y}$$

$$\hookrightarrow \lim_{\Delta x \to 0} \frac{\frac{1}{2+\Delta x+8} - \frac{1}{2+8}}{\Delta x} \to \lim_{\Delta x \to 0} \frac{\frac{1}{\Delta x+10} - \frac{1}{10}}{\Delta x}$$

$$\hookrightarrow \lim_{\Delta x \to 0} \frac{\frac{10-(\Delta x+10)}{10(\Delta x+10)}}{\Delta x} \to \lim_{\Delta x \to} \frac{10-\Delta x-10}{10\Delta x(\Delta x+10)} \to \lim_{\Delta x \to 0} \frac{-\Delta x}{10\Delta x(\Delta x+10)} \to \lim_{\Delta x \to 0} \frac{-1}{100+10\Delta x}$$

$$\hookrightarrow \frac{-1}{100+10(0)} = \frac{-1}{100} \qquad \frac{1}{10} = -\frac{1}{100}(2)+b \to b = \frac{3}{25}$$

$$y = mx+b \qquad y = -\frac{1}{100}x + \frac{3}{25}$$

ex. A farmer started with 25 alfalfa plants in his field. After 10 days he had 65 plants. After 23 days he had 413 plants, and after 30 days, 865 plants. He used the cubic function $\underline{P(x) = 0.03x^3 + x + 25}$ to model how many plants he had after x days. Find the instaneous daily rate of in create in plants at:

a) day 3 b) day 7 c) day 28

$$\lim_{\Delta x \to 0} \frac{P(x+\Delta x) - P(x)}{\Delta x} \to \frac{(0.03(x+\Delta x)^3 + (x+\Delta x)+25) - (0.03x^3+x+25)}{\Delta x}$$

$$\hookrightarrow \frac{0.03(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3) + x + \Delta x + 25 - 0.03x^3 - x - 25}{\Delta x}$$

$$\hookrightarrow \frac{\cancel{0.03x^3} + 0.09x^2\Delta x + 0.03\Delta x^3 \cancel{+x} + \Delta x \cancel{+25} \cancel{-0.03x^3} - x - 25}{\Delta x}$$

$$\hookrightarrow \frac{\cancel{\Delta x}(0.09x^2 + 0.09x\Delta x + 0.03\Delta x^2 + 1)}{\cancel{\Delta x}} \to \lim_{\Delta x \to 0} 0.09x^2 + 0.09x\cancel{\Delta x}^{0} + 0.03\cancel{\Delta x^2}^{0} + 1$$

$$= 0.09x^2 + 1$$

a) $0.09(3)^2 + 1 = 1.81$ b) $0.09(7)^2 + 1 = 5.41$ c) $0.09(28)^2 + 1 = 71.56$

DEC. 12

Unit 3 - Derivatives

Quiz - Tue Nov. 15 Quiz - Wed Nov. 30 Pretest - Tue, Dec. 6 Test - MON ↑

ex. A ball rolling down a hill accelerate at $0.4 m/s^2$. It's distance travelled, d meters is given by $d = 0.2t^2$, where t is the number of seconds since the ball's release. Find its instaneous speed at t=5s.

$$d' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative of a function f(x) at a number n, f'(n), is given by

$$f'(n) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{x \to n} \frac{f(x) - f(n)}{x - n}$$

ex. Find the slope of the line tangent to the curve $y = \sqrt{x}$ at each point.

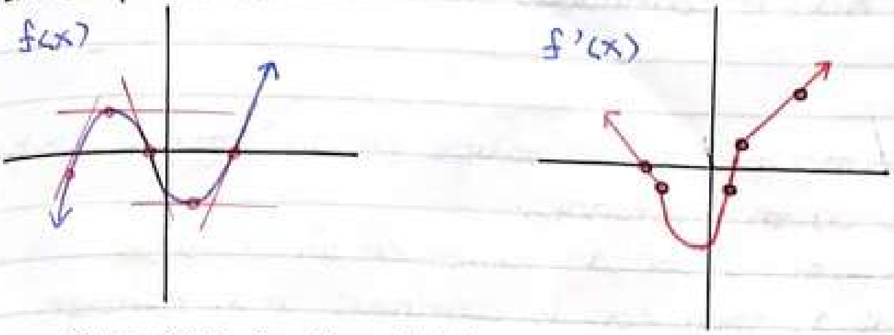
a) (1, 1) $\frac{1}{2}$ b) (9, 3) $\frac{1}{6}$ c) (54, 43617.31) $\rightarrow \frac{50}{1462}$

$$y' = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \rightarrow \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \left(\frac{\sqrt{x} + \Delta x + \sqrt{x}}{\sqrt{x} + \Delta x + \sqrt{x}}\right)$$

$$\hookrightarrow \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \rightarrow \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

132 # 3, 4, 5, 14, 15, 27, 28, 33

ex. Graph $f'(x)$



* take the tangent slope.

ex. Given $g(x)$ in the table, make an appox. table for $g'(x)$.

x	-80	-70	-60	-50	-40	-30	-20
y	4	1	0	-3	-1	2	7

x	-70	-60	-50	-40	-30
y	-0.2	-0.2	-0.05	0.25	0.4

• For $x = -70$: $g'(-70) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \to \frac{g(-70+\Delta x) - g(-70)}{\Delta x}$

1. ↳ let $\Delta x = 10$. $g'(-70) \doteq \frac{g(-70+10) - g(-70)}{10} \doteq \frac{0-1}{10} \doteq -0.1$

2. ↳ let $\Delta x = -10$. $g'(-70) \doteq \frac{g(-70-10) - g(-70)}{-10} \doteq \frac{4-1}{-10} \doteq -0.3$

*Aveg. $(-0.1 + -0.3) \div 2 = -0.2$

• For $x = -60$: 1. $g'(-60) \doteq \frac{g(-60+10) - g(-60)}{10} \doteq -0.3$

use 10 b/c it's diff. btw the x in T.O.V.

2. $g'(-60) \doteq \frac{g(-60-10) - g(-60)}{-10} \doteq -0.1$

*Avg. $(-0.1 + -0.3) \div 2 = 0.2$

ex. where is the function $h(x) = \sqrt{x}$ differentiable?

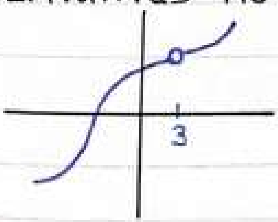
ID: $x \geq 0$ $h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \to \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right)$

↳ $\frac{(x+\Delta x) - x}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})} \to \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \to \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \to \frac{1}{2\sqrt{x}}$

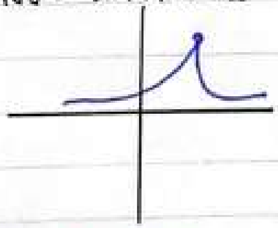
ID: $x > 0$, $x \in \mathbb{R}$ $h(x)$ is differentiable where $x > 0$

Differentiable = where the derivaties exist

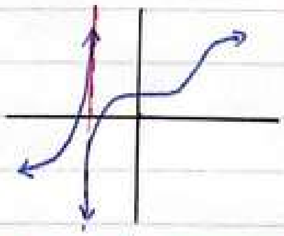
Derivatives not differentiable:



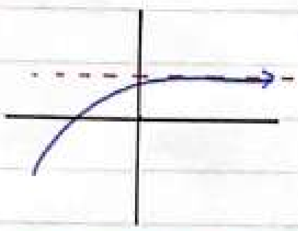
Discountinuity



A kink



vertical tangent line



horizontal tangent line is differentiable

If $f(x)$ is diferentiable at a value a, the $f(x)$ is countinous at a.

ex. $h(x) = \sqrt{x}$ $\lim_{x \to 0} \sqrt{x} = 0$, $h(x)$ is continuous at 0 but NOT differentiable.

Differentiability:

- A function $f(x)$ is differentiable for all values on an interval (a, b), then $f(x)$ is differentiable on that intervals.
- A function $f(x)$ is differentiable at a number a if $f'(a)$ exists
- If $f(x)$ is differentiable at a, then $f(x)$ is continous at a. However, the converse is not always ture ; a function can be continous at a but not differentiable at a.

142 # 1, 4, 7, 10, 19, 25, 29, 35, 46.

Product & Quotient Rules:

1. $\frac{d}{dx}(C) = 0$ 2. $\frac{d}{dx}(x) = 1$ 3. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

4. $\frac{d}{dx} x^n = nx^{n-1}$ 5. $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$

6. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$2(1x^0) = 2$

ex. Find $f'(x)$. $f(x) = x^5 + x^4 + 2x + 1$ (4) $\rightarrow 5x^4 + 4x^3 + 2 + 0$

ex. Find $f'(x)$. $f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$ (4)

$\hookrightarrow 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1x^0) + 0 = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

ex. $f'(x)$ of $f(x) = (\underset{f}{3x - 2x^2})(\underset{g}{5 + 4x})$ (5) $f(x)g'(x) + g(x)f'(x)$

$\hookrightarrow f'(x) = (3 - 4x)(5 + 4x) + (3x - 2x^2)(4)$

ex. $f'(x)$ of $y = \frac{x^2 + x - 2}{x^3 + 6}\begin{matrix} f \\ g \end{matrix}$ (6) $\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$\hookrightarrow \frac{(x^3+6)(2x+1) - (x^2 + x - 2)(3x^2)}{(x^3+6)^2}$

ex. $f'(x)$ of $\frac{1}{x^2} \rightarrow x^{-2} \rightarrow -2x^{-3} = \frac{-2}{x^3}$

ex. $f'(x)$ of $y = \sqrt{x} = x^{\frac{1}{2}} \rightarrow \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

ex. Find the equation of the tangent line to the curve at (1,1) $y = \frac{2x}{x+1}\begin{matrix} f \\ g \end{matrix}$

$y' = \frac{(x+1)(2) - (2x)(1)}{(x+1)^2} = \frac{2x + 2 - 2x}{(x+1)^2} = \frac{2}{(x+1)^2} \rightarrow \frac{2}{(1+1)^2} = \frac{1}{2}$

154 # 2, 3, 5, 7, 11, 13, 17, 19
21, 25, 35, 41, 53, 67, 72

Apply in Science: ' is speed = $\vec{v}$ in. v = m.r.o.f.c.

ex. The location of a particle is given by the equation $s(t) = \frac{2}{3}t^3 - 7t^2 + 20t$, where s metres is its location and t is time in seconds.

a) Find the particle's velocity at time t.

$v(t) = s'(t) = \frac{2}{3}(3t^2) - 7(2t) + 20(1) \rightarrow 2t^2 - 14t + 20 = v(t)$

b) How fast is the particle moving at time 3 sec? 10 sec?

3s: $v(\underline{3}) = 2(3)^2 - 14(3) + 20 = -4 m/s$

10s: $v(\underline{10}) = 2(10)^2 - 14(10) + 20 = 80 m/s$

c) when is the particle at <u>rest</u>?

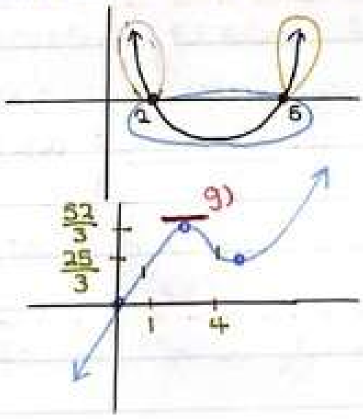
$0 = 2t^2 - 14t + 20 \rightarrow 2(t^2 - 7t + 10) \rightarrow 2(t-5)(t-2) = 5 \text{ sec}, 2 \text{ sec}$

d) when is it moving forward? backward?

$2(t-5)(t-2) > 0$

Forward: $t < 2$ & $t > 5$

Backward: $2 < t < 5$



e) Sketch a graph of the particle's motion.

$s(t) = \frac{2}{3}t^3 - 7t^2 + 20t \rightarrow t(\frac{2}{3}t^2 - 7t + 20)$

f) How far does it travel in the first 6 seconds?

where $0 \le t \le 2$: $\frac{52}{3}$

$s(2) = \frac{2}{3}(2^3) - 7(2)^2 + 20(2) = \frac{52}{3}$

where $2 < t \le 5$: $\frac{52}{3} - \frac{25}{3} = 9$

$s(5) = \frac{2}{3}(5)^3 - 7(5)^2 + 20(5) = \frac{25}{3}$

where $5 < t \le 6$: 12

$s(6) = \frac{2}{3}(6)^3 - 7(6)^2 + 20(6) = 12$

total: $\frac{52}{3} + 21 = \frac{115}{3}$ m

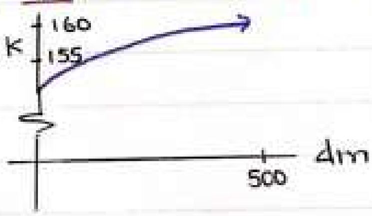
g) what is the greatest $\vec{d}$ btw $t = 1$ & $t = 4$

ex. In a chemical reaction, a liquid in a full 50-cm deep beaker is heated unevenly. Suppose the kelvin temperature of the liquid at depth d mm is given by $K(d) = 150 + \sqrt[3]{d}$, where $0 \le d \le 500$.

a) Find the kelvin temp at 6mm & 123mm.

$K(\underline{6}) = 150 + \sqrt[3]{6} = 151.817K$

$K(\underline{123}) = 150 + \sqrt[3]{123} = 154.973K$

b) 

Q: Sketch a graph showing the temp. at depth d mm.

(use T.O.V.)

c) Find the rate of temp. change per mm of depth, as a stir stick lowered into the beaker.

$K'(d) = 0 + \frac{1}{3}d^{-\frac{2}{3}} \rightarrow \frac{1}{3\sqrt[3]{d^2}}$

d) what's the rate of temperature increase per mm when the tip of the stick is 47 mm deep?

e) At what day th will the temperature be changing by 0.05 K/mm

$0.05 = \frac{1}{3\sqrt[3]{d^2}} \rightarrow 0.05(3\sqrt[3]{d^2}) = 1 \rightarrow \sqrt[3]{d^2} = \frac{1}{0.15} \rightarrow d^2 = (\frac{1}{0.15})^3$

$\hookrightarrow d = \pm\sqrt{(\frac{1}{0.15})^3} = \pm 17.213 \rightarrow 17^{th}$ day

ex. A beaver population increases until its food supply begins to run out, then it decreases. Suppose the population can be approximatly model by $P(t) = -3t^2 + 150t + 902$, where t is the number of days since Jan. 1, 2022.

a) what's the rate of population change per day at time t days?

$P'(t) = -6t + 150$

b) when does the population hit its maximum? what is the max. population?

$0 = -6t + 150 \rightarrow t = \frac{-150}{-6} = 25$

$P(25) = -3(25)^2 + 150(25) + 902 = 2777$

c) what's the rate of population increase / decrease on Feb. 9^{th}?

$P'(31+9) = -6(40) + 150 = -90$

d) what is the population increasing by 19 beavers a day?

$19 = -6t + 150 \rightarrow t = \frac{19-150}{-6} = 22$

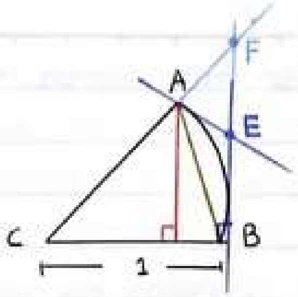
e) $0.05 = \frac{1}{3\sqrt[3]{d^2}} = \pm 17.213$

d) 17.213

47

Trig:

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$



$\widehat{AB} = \theta$

$\sin\theta = \frac{|AD|}{1} \rightarrow \sin\theta = |AD|$

$\frac{\sin\theta}{\theta} < 1$

$|AD| < |AB| < \widehat{AB} \rightarrow \sin\theta < |AB| < \theta$ # $\sin\theta < \theta$

$\widehat{AB} < |EB| + |AE| \rightarrow \theta < |EB| + |AE|$, $\theta < |EB| + |FE|$, $\theta < |BF|$. # $\theta < \tan\theta$

$\hookrightarrow \theta < \frac{\sin\theta}{\cos\theta}$)

$\cos\theta < \frac{\sin\theta}{\theta}$ ☆: $\cos\theta < \frac{\sin\theta}{\theta} < 1$

$\lim_{\theta \to 0} \cos\theta = 1$ $\lim_{\theta \to 0} 1 = 1$ $\rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (squeeze theorem)

- $y = \frac{\sin\theta}{\theta}$ is even function

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

$y = \sin x \rightarrow y' = \cos x$ $y = \cos x \rightarrow y' = -\sin x$ $y = \tan x \rightarrow y' = \sec^2 x$

$y = \csc x \rightarrow y' = -\csc x \cot x$ $y = \sec x \rightarrow y' = \sec x \tan x$ $y = \cot x \rightarrow y' = -\csc^2 x$

ex. Find equation of line tangent to $y = \cos^2 x$ at $(\frac{\pi}{8}, 0.854)$

$y = (\cos x)(\cos x) \rightarrow y' = (\cos x)(-\sin x) + (-\sin x)(\cos x)$

$\hookrightarrow y' = -2\sin x\cos x = -\sin 2x$

$m = -\sin 2x$ $y = mx + b \rightarrow 0.854 = \frac{-\sqrt{2}}{2}(\frac{\pi}{8}) + b \rightarrow b = 1.132$

$\downarrow -\sin(2(\frac{\pi}{8})) = \frac{-\sqrt{2}}{2}$

$\rightarrow y = \frac{-\sqrt{2}}{2}x + 1.132$

Chain Rule:

$y = f(g(x)) \rightarrow y' = [f'(g(x))][g'(x)]$ 181 # 6, 11, 13, 19, 23, 33, 39, 40, 57, 65, 72

ex. Differentiate each function:

a) $y = \sqrt{\cos x}$ $f(x) = \sqrt{x}$ $g(x) = \cos x$ $y = f(g(x))$ $g'(x) = -\sin x$

$y = (\cos x)^{\frac{1}{2}}$ $y' = \frac{1}{2}\cos x^{-\frac{1}{2}}(-\sin x) \rightarrow \dfrac{-\sin x}{2\sqrt{\cos x}}$

b) $y = (x^3 - 2x^2 + 5)^{78}$ $f(x) = x^{78}$ $g(x) = x^3 - 2x^2 + 5 \rightarrow g'(x) = 3x^2 - 2(2x) + 0$

$y' = 78(x^3 - 2x^2 + 5)^{77}(3x^2 - 4x) \rightarrow 78(3x^2 - 4x)(x^3 - 2x^2 + 5)^{77}$

c) $y = \sqrt[3]{\dfrac{x^2 - 5}{2\sqrt{x}}}$ $f(x) = \sqrt[3]{x}$ $g(x) = \dfrac{x^2 - 5}{2\sqrt{x}}$ $2x - 0$ $\rightarrow g'(x) = \dfrac{(2x)(2\sqrt{x}) - (x^{-\frac{1}{2}})(x^2 - 5)}{(2x^{\frac{1}{2}})^2}$

$2x^{\frac{1}{2}} \rightarrow 2\sqrt{x}$ $\cancel{2}(\frac{1}{\cancel{2}}x^{-\frac{1}{2}})$

$\hookrightarrow \dfrac{4x\sqrt{x} - \frac{x^2}{\sqrt{x}} + \frac{5}{\sqrt{x}}}{4x}$ $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$y' = \frac{1}{3}\left(\frac{x^2 - 5}{2\sqrt{x}}\right)^{-\frac{2}{3}}\left(4x\sqrt{x} - \frac{x^2}{\sqrt{x}} + \frac{5}{\sqrt{x}}\right)\left(\frac{1}{4x}\right) = \left(\frac{2\sqrt{x}}{x^2 - 5}\right)^{\frac{2}{3}}\left(\frac{3x^2 + 5}{12x\sqrt{x}}\right)$

$\dfrac{4x^2 - x^2 + 5}{\sqrt{x}} \div 4x$

ex. $y = \cos((\sin x)^5)$ $y = f(g(h(x)))$

$f(x) = \cos x$ $g(x) = x^5$ $h(x) = \sin x$

$y' = -\sin((\sin x)^5) \cdot 5(\sin x)^4 \cdot \cos x$

Implicit : 188 #1, 4, 9, 13, 17, 26, 29, 36, 39

$y = f(x)$ $\quad x = y^2 \rightarrow y_1 = \sqrt{x} \rightarrow \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\quad\quad\quad\quad\quad\quad\quad \searrow y_2 = -\sqrt{x} \rightarrow -\frac{1}{2}x^{-\frac{1}{2}} = \frac{-1}{2\sqrt{x}}$

or $\quad x = y^2 \rightarrow 1 = 2y\frac{dy}{dx} \rightarrow \frac{1}{2y} = \frac{dy}{dx} \rightarrow \frac{dy}{dx} = \boxed{\frac{1}{2\sqrt{x}}}$

$\searrow \frac{dy}{dx} = \frac{1}{2(-\sqrt{x})} = \boxed{\frac{-1}{2\sqrt{x}}}$

ex. differentiate $x^2 + y^2 = 16$

$2x + 2y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \rightarrow \frac{dy}{dx} = \boxed{-\frac{x}{y}}$

ex. Find the tangent to the hyperbola. $\frac{(x-3)^2}{16} - \frac{(y+5)^2}{9} = -1$ at point $(3+4\sqrt{3}, 1)$.

$\frac{1}{16}(x-3)^2 - \frac{1}{9}(y+5)^2 = -1 \rightarrow \frac{2}{16}(x-3)(1) - \frac{2}{9}(y+5)\left(\frac{dy}{dx}\right) = 0$

$\hookrightarrow \frac{1}{8}(x-3) - \frac{2}{9}(y+5)\frac{dy}{dx} = 0 \rightarrow 72\left(\frac{1}{8}(x-3)\right) - \frac{2}{9}(y+5)\frac{dy}{dx}) = 72(0)$

$\hookrightarrow 9(x-3) - 16(y+5)\frac{dy}{dx} = 0 \rightarrow -16(y+5)\frac{dy}{dx} = -9(x-3)$

$\hookrightarrow \frac{dy}{dx} = \frac{9(x-3)}{16(y+5)}$ $\quad m = \frac{9(3+4\sqrt{3}-3)}{16(1+5)} = \frac{3\sqrt{3}}{8}$

ex. Find y' if $\sin(2x+y^2) = y^2\cos x$.

$\cos(2x+y^2)(2+2y\frac{dy}{dx}) = y^2(-\sin x) + \cos x(2y\frac{dy}{dx})$

$2\cos(2x+y^2) + 2y\cos(2x+y^2)\frac{dy}{dx} = -y^2\sin x + 2y\cos x\frac{dy}{dx}$

$2\cos(2x+y^2) + 2y\cos(2x+y^2)\frac{dy}{dx} - 2y\cos x\frac{dy}{dx} = -y\sin x$

$\frac{dy}{dx} = \frac{-y^2\sin x - 2\cos(2x+y^2)}{2y(\cos(2x+y^2) - \cos x}$

Second Derivative:

$y = 7x^6 - 3x^3 + 2x^2$

$y' = 42x^5 - 6x^2 + 4x$

$y'' = 210x^4 - 18x + 4$

$y''' = 840x^3 - 18$

$y^{(4)} = 2520x^2$

$y^{(5)} = 5040x$

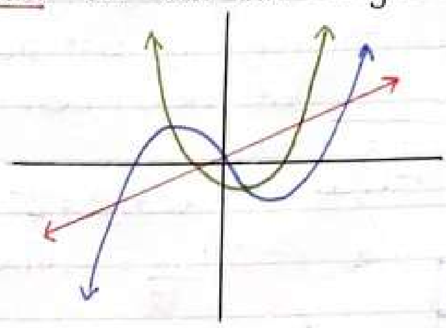
$y^{(6)} = 5040$

$y^{(7)} = 0$

ex. $f(x) = x^3 - 3x^2 - 9x$. Graph $f(x)$, $f'(x)$ and $f''(x)$ on the same grid.

$f(x) = x(x^2 - 3x - 9)$

x	-4	-2	0	2	4	6
y	-76	-2	0	-22	-20	54



$f'(x) = 3x^2 - 6x - 9 \Rightarrow 3(x^2 - 2x + 1) + 3(-1) - 9$

$= 3(x-1)^2 - 12$

$f''(x) = 6x - 6$

ex. A particle's position is given by $f(t) = t^3 - 7t^2 + 14t - 5$, where t is the time in second.

a) Find the $\vec{v}$ at time t.

$f'(t) = 3t^2 - 14t + 14$

b) How fast is the particle moving at time $t = 3$ sec?

$f'(3) = 27 - 42 + 14 = -1 m/s$

c) Find $\vec{a}$ at time t.

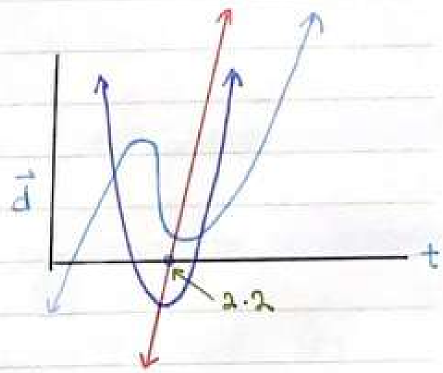
$f''(t) = 6t - 14$

d) what is $\vec{a}$ after 2 seconds?

$f''(2) = 6(2) - 14 = -2 m/s^2$

e) Graph the position, $\vec{v}$, & $\vec{a}$ curves for $0 \leq t \leq 5$

t	0	1	2	3	4	5
f(t)	-5	3	3	1	3	15
f'(t)	14	3	-2	-1	6	19
f''(t)	-14	-8	-2	4	10	16



f) When is the position speeding up? slowing down?

$t > 2.2$ sec. $\quad t < 2.2$ sec.

ex. $f(x) = \sin x$. Find $f^{(100)}x$.

$f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x \quad f^{(4)}(x) = \sin x \quad f^{(5)}(x) = \cos x \ldots$

$\hookrightarrow f^{(100)}(x) = \sin x$

195 #1 5, 8, 10, 13, 17, 20, 23, 29, 32, 43, 53, 54, 55

3.5 - 3.7 (trig d') (chain rule) (implicit diff.)

Relate Rates:

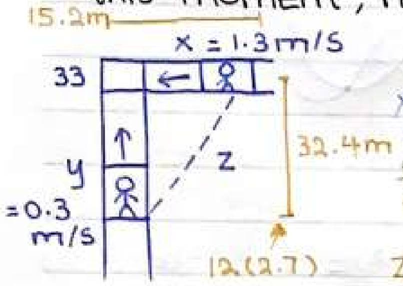
ex. A circle's radius is increasing at 4cm/s. How fast is the circle's area increasing when the diameter is 7.8 cm?

$\frac{7.8}{2} = 3.9 = r$

We seek $\frac{dA}{dt}$, know $\frac{dr}{dt} = 4$ cm/s.

$A = \pi r^2 \rightarrow \frac{dA}{dt} = \pi(2r)\left(\frac{dr}{dt}\right) = \pi(2)(3.9)(4) = 31.2\pi$ sq.cm/sec

ex. At the Brull building, each floor is 2.7m high. Jack is on the 33rd floor, walking toward the elevator at 1.3 m/s. Jill is riding the elevator up from the lobby toward the 33rd floor, at 0.3m/s. When Jill hits the 21st floor, Jack is 15.2m from the elevator. At this moment, how much is the distance btw Jack & Jill changing?



$x^2 + y^2 = z^2 \rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$\frac{dx}{dt} = 1.3$ m/s $\quad \frac{dy}{dt} = 0.3$ m/s $\quad \frac{dz}{dt} = ?$

$z = \sqrt{32.4^2 + 15.2^2} = 35.78\cdots$

$\frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{dz}{dt} \rightarrow \frac{15.2(-1.3) + 32.4(-0.3)}{35.78\cdots} = -0.823$ m/s $= \frac{dz}{dt}$

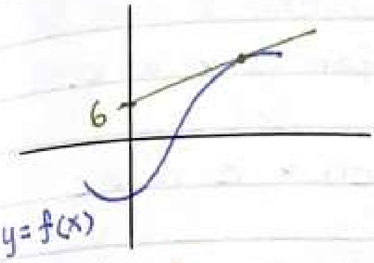
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{2t}$

$\frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{dz}{dt}$

$\frac{15.2(-1.3) + 32.4(-0.3)}{35.78826623} = \frac{dz}{dt}$

202 #1,3,6,7,13,

Linear Aproximation:



At point $(a, f(a))$, tangent line is $y = mx + b$

$y = f'(a) + b \rightarrow b = f(a) - f'(x)a$

$\hookrightarrow y = f'(x) + f(a) - f'(x)a \rightarrow f(a) + f'(x)(x-a)$

☆: the approximation $f(x) \approx f(a) + f'(x)(x-a)$ is the linear or tangent line appoximation of f(x) at a.

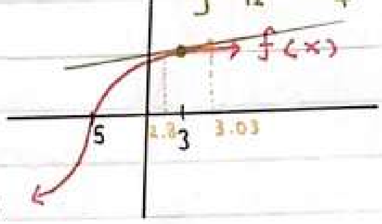
★: The function $L(x) = f(a) + f'(x)(x-a)$ is the linearization of f(x) at a.

ex: $f(x) = \sqrt[3]{x+5}$. Find the linearization of f(x) at a=3, and use it to estimate $\sqrt[3]{8.03}$ and $\sqrt[3]{7.96}$. Are these overestimates or underestimates?

$L(x) = f(a) + f'(a)(x-a)$

$f(a) = f(3) = \sqrt[3]{3+5} = 2$ $\quad f'(x) = \frac{1}{3}(x+5)^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{(x+5)^2}}$

$L(x) = 2 + \frac{1}{3\sqrt[3]{(3+5)^2}}(x-3) \rightarrow f(x) \approx \frac{1}{12}x + \frac{7}{4}$

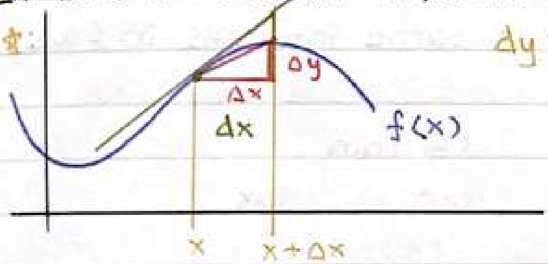


$8.03 = 3.03 + 5$ $\quad \sqrt[3]{8.03} = \sqrt[3]{3.03+5} \approx \frac{1}{12}(3.03) + \frac{7}{4} = 2.0025$

$7.96 = 2.96 + 5$ $\quad \sqrt[3]{7.96} = \sqrt[3]{2.96+5} \approx \frac{1}{12}(2.96) + \frac{7}{4} = 1.99\overline{6}$

ex: If $f(x) = 3\cos(x-4)$, find Δy and dy where x changes from $\frac{\pi}{16}$ to $\frac{\pi}{15}$.

☆:



$dy = f'(x)dx$

$\Delta x = \frac{\pi}{15} - \frac{\pi}{16} = \frac{\pi}{240} = dx$

$f'(x) = 3(-\sin(x-4))(1) = -3\sin(x-4)$

$\Delta y = f(\frac{\pi}{15}) - f(\frac{\pi}{16}) = 3\cos(\frac{\pi}{15}) - 3\cos(\frac{\pi}{16} - 4) = -0.023937424$

$dy = f'(x)dx \rightarrow -3\sin(\frac{\pi}{16} - 4)(\frac{\pi}{240}) = -0.024140831$ close!

- We can use linear approximation to estimate the value of a function at a value a using $f(x) \approx f(a) + f'(a)(x-a)$ [x close to a]
- The corresponding linear function $L(x) = f(a) + f'(a)(x-a)$ is the linearization of f(x) at a.
- If we increment x by Δx in a function y = f(x), we define the differential dx as that increment (Δx). The dependent variable dy is another differential, representing the change in linearization of f(x) over the increment Δx. Thus $dy = f'(x)dx$

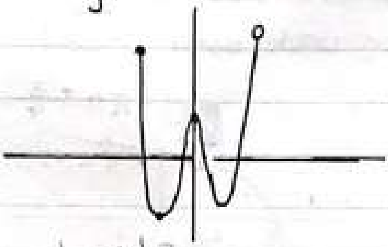
Unit 4 - Application of Differentiation

Quiz - Mon. Jan. 16 Quiz - Tue. Jan. 31 Pretest - Fri. Feb. 10 Test - Thu. Feb. 16

- $f(x)$ has an absolute max (min) at c if $f(c) \geq (\leq) f(x)$ for all x in ID.
- $f(x)$ has a local max (min) at c if $f(c)$ $f(x)$ when x is near c. (on some open interval containing c)

① close
① open interval 로 가능?
② open & close end?

ex. $y = 3x^4 - 10x^2 + x + 5$, $-2 \leq x < 2$



- No absolute max
- Absolute min near $f(-1)$
- local max near $f(0)$
- Local min near $f(1)$

closed end?

Max & Min Values:

Extreme Value Theorem:
- If $f(x)$ is continous on a closed interval $[a, b]$, then $f(x)$ has an absolute max. and an absolute min. at some intervals in $[a, b]$

ex.

NO max
but abs. min

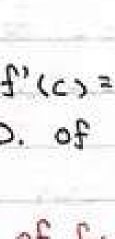
NO min
but abs. max

Fermat's Theorem:
- If $f(x)$ has a local max/min at c and $f'(c)$ exist, than $f'(c) = 0$
- A critical number of a function $f(x)$ is a value c in the ID. of $f(x)$ such that $F'(c) = 0$ or $f'(c)$ doesn't exist.
- If $f(c)$ is a local max/min, then c is a critical number of $f(x)$.

Closed Interval Method: (To find abs. max & min)
1. Find values of $f(x)$ at all critical numbers in (a, b).
2. Find $f(a)$ and $f(b)$.
3. Largest is absolute max, Smallest is absolute min.

ex. Find absolute max & min values of $g(x) = x^3 - 2x^2 + 4$, $\frac{-1}{2} \leq x \leq 3$

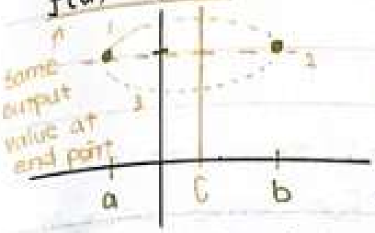
1. $g'(x) = 3x^2 - 4x = 0 \rightarrow x(3x-4) = 0 \rightarrow x = 0, \frac{4}{3}$
   $f(0) = 4$ $\quad f(\frac{4}{3}) = (\frac{4}{3})^3 - 2(\frac{4}{3})^2 + 4 = \frac{76}{27}$
2. $f(-\frac{1}{2}) = (-\frac{1}{2})^3 - 2(-\frac{1}{2})^2 + 4 = \frac{27}{8}$ $\quad f(3) = 3^3 - 2(3)^2 + 4 = 13$
3. abs. max $g(3) = 13$ $\quad$ abs. min $g(\frac{4}{3}) = \frac{76}{27}$



229 # 2, 33, 39, 43, 47, 51, 55, 63, 68

Rolle's Theorem:

- If $f(x)$ is continuous on $[a,b]$, and differentiable on (a,b), and $f(a)=f(b)$ then a number c exists in (a,b) such that $f'(c)=0$.



ex. Use Rolle's Theo. to show $x^3+x+2=0$ has 1 root.
exactly

- $f(x) = x^3+x+2$ $\quad f(0)=2$ & $f(-2)=-8$.

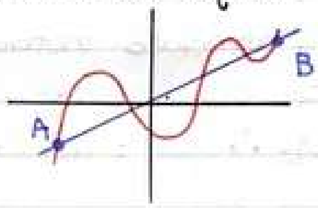
↳ There is 1 root by (intermediam value theo.)

- Suppose there are 2 roots, a & b. Then $f(a)=f(b)=0$
- Rolle: "c" exists btw a & b where $f'(c)=0$

$f'(x) = 3x^2+1=0$ impossible!

Mean Value Theorem:

- Differentiable function contains, differentiable function contains $(a, f(a))$ and $(b, f(b))$, then there exists "c" on (a,b) for which $f'(c)$ equals the slope of the secant line, $\frac{f(b)-f(a)}{b-a}$.



ϕ: $f'(c) = \frac{f(b)-f(a)}{b-a}$

ex. $g(2)=5$, and $g'(x) \geq 2$ everywhere. Find the greatest possible value of $g(-1)$.

- Consider interval $[-1,2]$. Mean value theo. says "c" exists, $-1<c<2$, for which $g'(c) = \frac{f(b)-f(a)}{b-a}$.

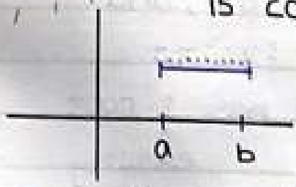
* $g'(c) = \frac{g(2)-g(-1)}{2-(-1)} \rightarrow g'(c) = \frac{5-g(1)}{3}$

* $\frac{5-g(-1)}{3} \geq 2 \rightarrow 3\left(\frac{5-g(-1)}{3}\right) \geq 3(2) \rightarrow 5-g(-1) \geq 6 \rightarrow -g(-1) \geq 1$

↳ $-g(-1) \leq -1$

"−"

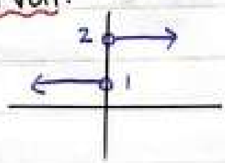
• Theorem : If $f'(x)=0$ throughout an interval (a,b), then $f(x)$ is constant on (a,b)



• Corollary : If $f'(x)=g'(x)$ throughout on interval (a,b), than $(f-g)(x)$ is constant on (a,b); in other words, $f(x)=g(x)+c$ for some constant c.

☆ ID must be an interval.

ex. $b(x)\begin{cases} 1 \text{ if } x<0 \\ 2 \text{ if } x>0 \end{cases}$



ID: $x \in \mathbb{R} \mid x \neq 0$
$b'(x)=0$ for all x in ID, but $b(x)$ is NOT constant function.

ex. If $g'(x)=3$ over the interval (a,b), show that $g(x)=3x+d$, where d is a constant.

1. introduce another function where derivative is 3. say $h(x)=3x+r$
2. $g'(x)=h'(x)$, so $g(x)-h(x)=C$ is constant on (a,b) by corollary.
3. $\therefore g(x)=h(x)+c = 3x+r+c \rightarrow$ Let $d=r+c$
$= 3x+d$

ex. $f(x)=3x^2-\sin 3x$. Show there exists a number r, where $\frac{3\pi}{4}<r<\pi$, for which $f'(r)=\frac{21\pi^2+8\sqrt{2}}{4\pi}$.

1. $f\left(\frac{3\pi}{4}\right)=\frac{27\pi^2}{16}-\frac{1}{\sqrt{2}}=\frac{27\pi^2-8\sqrt{2}}{16}$ $\quad f(\pi)=(3\pi)^2-0=3\pi^2$

2. M.V.T. : There is a c on $\left(\frac{3\pi}{4},\pi\right)$ surch that $f'(c)=\frac{f(\pi)-f\left(\frac{3\pi}{4}\right)}{\pi-\frac{3\pi}{4}}$

$\hookrightarrow \frac{3\pi^2-\frac{27\pi^2-8\sqrt{2}}{16}}{\frac{\pi}{4}} = \frac{\left(\frac{48\pi^2}{16}-\frac{27\pi^2-8\sqrt{2}}{16}\right)\left(\frac{4}{3}\right)}{\frac{\pi}{4}} = \frac{21\pi^2+8\sqrt{2}}{4\pi}$

Increasing / decreasing test:

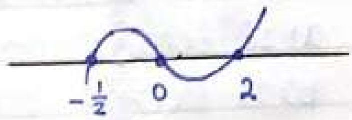
- If $f'(x) > 0$ on (a,b), then $f(x)$ is increasing on (a,b).
- If $f'(x) < 0$ on (a,b), then $f(x)$ is decreasing (a,b).

ex. $f(x) = \frac{3}{2}x^4 - 3x^3 - 3x^2 + 7$, where is $f(x)$ increasing? decreasing?

$f'(x) = 6x^3 - 9x^2 - 6x \rightarrow 3x(x-2)(2x+1)$

$f'(x) < 0$ where $x < -\frac{1}{2}$ or $0 < x < 2$

$f'(x) > 0$ where $-\frac{1}{2} < x < 0$ or $x > 2$



$\therefore$ $f(x)$ is increasing on $(-\frac{1}{2}, 0)$ and $(2, \infty)$ decreasing on $(-\infty, -\frac{1}{2})$ & $(0,2)$

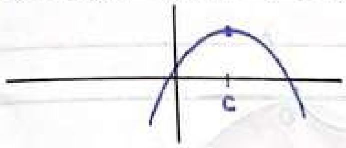
First Derivative Test:

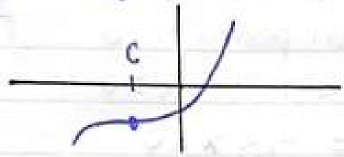
- If c is a critical number of a continous function $f(x)$,

a) If $f'(x)$ changes from + to − at c, c is a local max.

b) If $f'(x)$ changes from − to + at c, c is a local min.

c) If $f'(x)$ doesn't change sign at c, c is neither a local max or min.



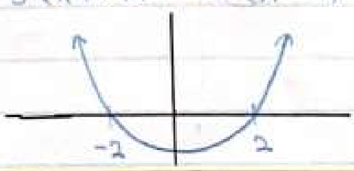


ex. Find local max/min of $f(x) = x^5 - 5x^3 - 20x + 15$.

$f'(x) = 5x^4 - 15x^2 - 20$

$= 5(x^2 - 4)(x^2 + 1)$

$= 5(x+2)(x-2)(x^2+1)$



- local max: −2

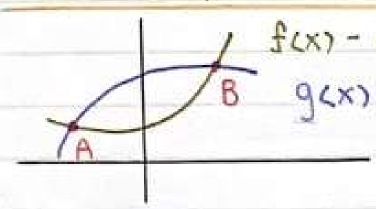
$f(-2) = (-2)^5 - 5(-2)^3 - 20(-2) + 15$

$= 63$ $(-2, 63)$

- local min: 2

$f(2) = (2)^5 - 5(2)^3 - 20(2) + 15$

$= -37$ $(2, -37)$



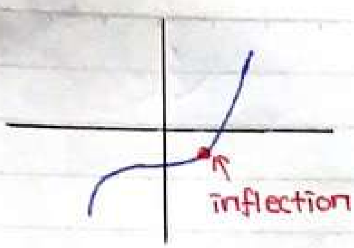
$f(x)$ - concave upward

$g(x)$ - concave downward

- If the graph of $f(x)$ lies above all its tangents on (a,b), it is concave upwards on (a,b).
- If its graph lies below all its tangents on (a,b), it concave downwards on (a,b).

Concavity:

- If $f''(x) > 0$ throughout (a,b), then $f(x)$ is concave upwards on (a,b)
- If $f''(x) < 0$ throughout (a,b), then $f(x)$ is concave downwards on (a,b).



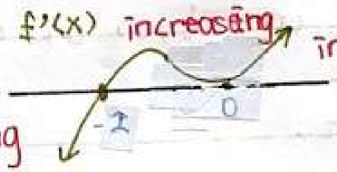
∴ When the second derivative changes sign, it is an inflection point.

Second derivative test:

a) If $f'(c)=0$ and $f''(c)>0$, c is a local min.

b) If $f'(c)=0$ and $f''(c)<0$, c is a local max.

ex. Graph $\underline{f(x)=x^4+2x^3}$.



① $f(x)=x^3(x+2)$ → zeros : 0, -2

$f'(x)=4x^3+6x^2$

↳ CV : -1.5, 0

• First derivative test : local min at -1.5 & no max/min at 0

$f(-1.5)=\left(-\frac{3}{2}\right)^4+2\left(-\frac{3}{2}\right)^3=\frac{-27}{16}$ → $\left(-\frac{3}{2}, -\frac{27}{16}\right)$

② $f''(x)=12x^2+12x$ → inflection point : -1, 0

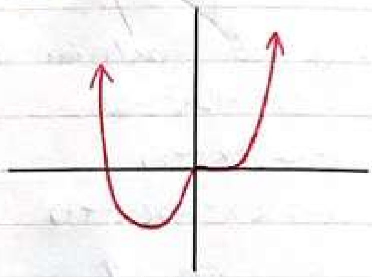
concave upward where $x<-1$ or $x>0$

concave downward where $-1<x<0$

$f(-1)=(-1)^4+2(-1)^3=-1$ → $(-1,-1)$ & $(0,0)$

• Note : At $-\frac{3}{2}$, $f'(x)=0$ and $f''(0)>0$.

∴ By 2nd derivative test, $-\frac{3}{2}$ is a local min.



247 1,3,6,8,11,17,19-21,25,29,33,37,49,50

$\lim_{x \to \infty} f(x) = L$ $\qquad \lim_{x \to -\infty} f(x) = L$

$y = L$ is a horizontal asymptote of $f(x)$, if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

Theorem : If $r > 0$ is rational, then $\lim_{x \to \infty} \frac{1}{x^r} = 0$

If $r > 0$ is rational such that x^r is defined for all x.

ex. Find $\lim_{x \to \infty} \frac{3x^3 - 4x + 1}{x^3 + 2x} \frac{(\frac{1}{x^3})}{(\frac{1}{x^3})} = \lim_{x \to \infty} \frac{3 - \frac{4}{x^2} + \frac{1}{x^2}}{1 + \frac{2}{x^2}} = \frac{\lim_{x \to \infty} (3 - \frac{4}{x^2} + \frac{1}{x^3})}{\lim_{x \to +\infty} (1 + \frac{2}{x^2})}$

$= \frac{\lim_{x \to \infty} 3 - 4 \lim_{x \to \infty} \frac{1}{x^2} + \lim_{x \to \infty} \frac{1}{x^3}}{\lim_{x \to \infty} 1 + 2 \lim_{x \to \infty} \frac{1}{x^2}} = \frac{3 - 4(0) + 0}{1 + 2(0)} = 3$

ex. Find $\lim_{x \to -\infty} \frac{x^4 + 1}{2x^4 - 3x^2} \frac{(\frac{1}{x^4})}{(\frac{1}{x^4})} = \lim_{x \to -\infty} \frac{1 + \frac{1}{x^4}}{2 - \frac{3}{x^2}} = \frac{\lim_{x \to -\infty} 1 + \lim_{x \to -\infty} \frac{1}{x^4}}{\lim_{x \to -\infty} 2 - 3 \lim_{x \to -\infty} \frac{1}{x^2}}$

$= \frac{1 + 0}{2 + 3(0)} = \frac{1}{2}$

ex. $\lim_{x \to \infty} f(x) = \infty$, find a) $\lim_{x \to \infty} x^4$ $\qquad$ b) $\lim_{x \to -\infty} x^4$ $\qquad$ c) $\lim_{x \to \infty} x^2 - 3x$

a) $= \infty$ $\qquad$ b) $= \infty$ $\qquad$ c) $\lim_{x \to \infty} x(x-3) = \lim_{x \to \infty} x \cdot \lim_{x \to \infty} (x-3) = \infty \cdot \infty = \infty$

☆ : c) NOT $\lim_{x \to \infty} x^2 - \lim_{x \to \infty} 3x = \underbrace{\infty - \infty}_{\text{not } \infty} = \infty$ ← but undefine

Limits at Infinity :

- $\lim_{x \to \infty} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by choosing sufficently large x.
- $\lim_{x \to -\infty} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by choosing sufficently large negative.
- If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of $f(x)$.
- If x^r is defined for all x, then $\lim_{x \to -\infty} \frac{1}{x^r} = 0$ ($r > 0$ is rational, $\lim_{x \to \infty} \frac{1}{x^r} = 0$)

Steps to graph a function:

1. Find ID.
2. Find intercepts.
3. Check if $f(x)$ is even, odd, periodic.
4. Asymptotes : • horizontal if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

   • Vertical if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$

5. inc/dec test : • $f'(x) > 0 \rightarrow$ inc.

   • $f'(x) < 0 \rightarrow$ dec.

6. min/max : Find critical #s , use 1st derivative test.

   $- f'(x) + \rightarrow$ max , $f'(x) - \rightarrow$ min

   and/or use 2nd derivative test.

7.

8. Sketch the graph.

★: even: $f(-x) = f(x)$

odd : $f(-x) = -f(x)$

ex. Graph $f(x) = \dfrac{x^2}{2x^2-4}$.

1. ID : $\{x \in \mathbb{R} \mid x \neq \pm\sqrt{2}\}$ $\quad 2x^2-4 \neq 0 \rightarrow x^2 \neq 2 \rightarrow x \neq \pm\sqrt{2}$
2. y-int : 0 x-int : 0

   $f(0) = \dfrac{0^2}{2(0)^2-4} = 0$ (y-int) $\quad 0 = \dfrac{x^2}{2x^2-4}$ $[x \neq \pm\sqrt{2}] \rightarrow 0 = x^2 \rightarrow x = 0$ (x-int)

3. Even fn.
4. Horiz. asy. $y = \frac{1}{2}$, vert. asy. $x = \sqrt{2}, -\sqrt{2}$

   $\lim_{x\to\infty} \dfrac{x^2}{2x^2-4} = \dfrac{1}{2}$ $\quad \lim_{x\to-\infty} \dfrac{x^2}{2x^2-4}$

5. Increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$

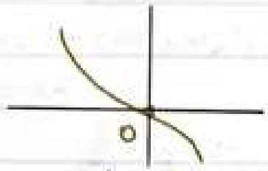
   $f'(x) = \dfrac{(2x^2-4)(2x) - x^2(4x)}{(2x^2-4)^2} \rightarrow \dfrac{-8x}{(2x^2-4)^2}$ $\quad f'(x) \overset{\text{pos.}}{>} 0$ where $x < 0$

   $f'(x) \overset{\text{neg.}}{<} 0$ where $x > 0$

6. max at 0

   critical #s : $\pm\sqrt{2}$, 0

   $\dfrac{-8x}{(2x^2-4)^2} = 0$ $[x \neq \pm\sqrt{2}] \rightarrow -8x = 0 \rightarrow x = 0$



$f''(x) = \dfrac{(2x^2-4)^2(-8) - (-8x)(2(2x^2-4)(4x))}{(2x^2-4)^4} = \dfrac{32x^2 - 16x^2 + 32}{(2x^2-4)^3} = \dfrac{16x^2+32}{(2x^2-4)^3}$

$f''(0) = -\frac{1}{2} < 0$ (max)

7. No inflection. Concave-upward $(-\infty,\sqrt{2})$ & $(\sqrt{2},\infty)$ Concave-downward $(-\sqrt{2},\sqrt{2})$

$\frac{16x^2+32}{(2x^2-4)^3}=0 \rightarrow 16x^2+32=0 \rightarrow x^2=-2$ (no real #)

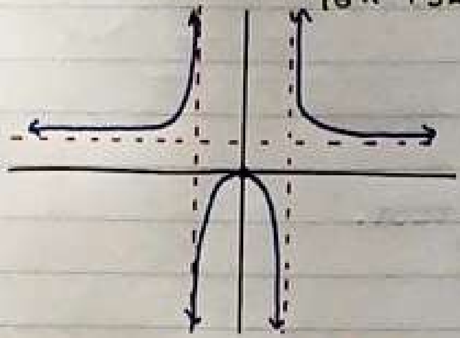
$\frac{16x^2+32}{(2x^2-4)^3}>0$ case 1. $(2x^2-4)^3>0 \rightarrow x<-\sqrt{2}$ or $x>\sqrt{2}$

$16x^2+32>0 \rightarrow x^2+2>0$ always ture.

case 2. $(2x^2-4)^3<0 \rightarrow -\sqrt{2}<x<\sqrt{2}$

$16x^2+32<0$ never ture

8.



$f(3)=\frac{9}{14}$

$f(1)=\frac{1}{-2}$

$f(2)=1$

ex. Graph $f(x)=\frac{x^2-4}{2x-5}$ (2nd deg. / 1st deg.) $\rightarrow \frac{\frac{1}{2}x^2-2}{x-\frac{5}{2}}$

$-\frac{5}{2}$	$\frac{1}{2}$	0	-2
	$-$	$-\frac{5}{4}$	$-\frac{25}{8}$
	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{9}{8}$

Q: $\frac{1}{2}x+\frac{5}{4}$

R: $\frac{9}{8}$

4. S.A. $y=\frac{1}{2}x+\frac{5}{4}$

V.A. $x=\frac{5}{2}$

$f(0)=\frac{-4}{-5}=\frac{4}{5}$ $0=\frac{x^2-4}{2x-5}=\pm 2$

3. not odd, even, periodic.

2. y-int $\frac{4}{5}$ x-int ± 2

1. $\{x\in\mathbb{R} \mid x\neq\frac{5}{2}\}$

$f'(x)=\frac{(2x-5)(2x)-(x^2-4)(2)}{(2x-5)^2}=\frac{2(x-4)(x-1)}{(2x-5)^2}$

$y=2(x-4)(x-1)$ $f'(x)>0 \rightarrow x<1, x>4$

$f'(x)<0 \rightarrow 1<x<4$

5. increasing on $(-\infty,1)$ & $(4,\infty)$. decreasing on $(1,4)$.

6. max at $x=1$, min at $x=4$

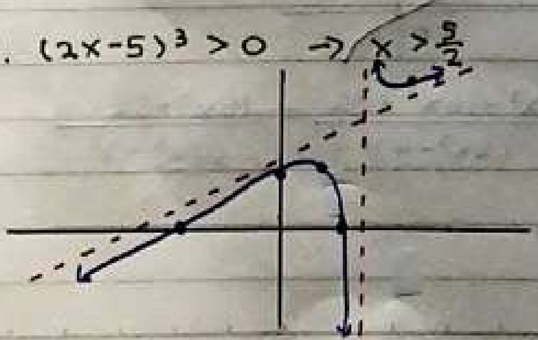
c.v. $=\frac{5}{2},1,4$ $\frac{2x^2-10x+8}{(2x-5)^2}=0 \rightarrow 2(x-4)(x-1)=0 \rightarrow x=\frac{5}{2},1,4$

7. no I.P. C.U. $(x>\frac{5}{2})$ C.D. $(x<\frac{5}{2})$

$f''(x)=\frac{(2x-5)^2(4x-10)-(2x^2-10x+8)(2)(2x-5)(2)}{(2x-5)^4}=\frac{18}{(2x-5)^3}\neq 0$

1. $(2x-5)^3>0 \rightarrow x>\frac{5}{2}$

2. $(2x-5)^3<0 \rightarrow x<\frac{5}{2}$



$f(1)=1$

$f(4)=4$