Unit 4 - Application of Differentiation

Quiz - Mon. Jan. 16 Quiz - TUE. Jan. 31 Pretest - Fri. Feb. 10 Test - Thu. Feb. 16

min $\le$

- $f(x)$ has an absolute max at c if $f(c) \ge f(x)$ for all x in ID.

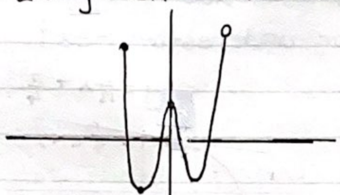
min

- $f(x)$ has a local max at c if $f(c)$ $f(x)$ when x is near c. (on some open interval containing c)

close

① open interval 로 가능?
② open & close end?

ex. $y = 3x^4 - 10x^2 + x + 5$, $-2 \le x < 2$



- No absolute max
- Absolute min near $f(-1)$
- local max near $f(0)$
- Local min near $f(1)$

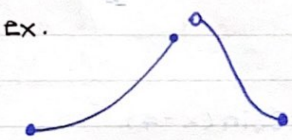
closed end?

Max & Min Values:

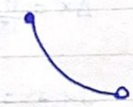
Extreme Value Theorem:

- If $f(x)$ is continous on a closed interval $[a, b]$, then $f(x)$ has an absolute max. and an absolute min. at same intervals in $[a, b]$

ex.



NO max
but abs. min



NO min
but abs. max

Fermat's Theorem:

- If $f(x)$ has a local max/min at c and $f'(c)$ exist, than $f'(c) = 0$
- A critical number of a function $f(x)$ is a value c in the ID. of $f(x)$ such that $F'(c) = 0$ or $f'(c)$ doesn't exist.
- If $f(c)$ is a local max/min, then c is a critical number of $f(x)$.

Closed Interval Method: (To find abs. max & min)

1. Find values of $f(x)$ at all critical numbers in (a, b).
2. Find $f(a)$ and $f(b)$.
3. Largest is absolute max, Smallest is absolute min.

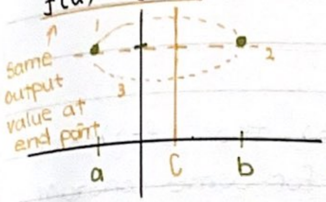
ex. Find absolute max & min values of $g(x) = x^3 - 2x^2 + 4$, $-\frac{1}{2} \le x \le 3$

1. $g'(x) = 3x^2 - 4x = 0 \rightarrow x(3x - 4) = 0 \rightarrow x = 0, \frac{4}{3}$
   $f(0) = 4$ $f(\frac{4}{3}) = (\frac{4}{3})^3 - 2(\frac{4}{3})^2 + 4 = \frac{76}{27}$
2. $f(-\frac{1}{2}) = (-\frac{1}{2})^3 - 2(-\frac{1}{2})^2 + 4 = \frac{27}{8}$ $f(3) = 3^3 - 2(3)^2 + 4 = 13$
3. abs. max $g(3) = 13$ abs. min $g(\frac{4}{3}) = \frac{76}{27}$

229 # 1, 33, 39, 43, 47, 51, 55, 63, 68

Rolle's Theorem:

• If $f(x)$ is continuous on $[a,b]$, and differentiable on (a,b), and $f(a) = f(b)$ then a number c exists in (a,b) such that $f'(c)=0$.



ex. Use Rolle's Theo. to show $x^3+x+2=0$ has 1 root.
exactly

- $f(x) = x^3+x+2$ $\quad f(0)=2$ & $f(-2)=-8$.

↳ There is 1 root by (intermediam value theo.)

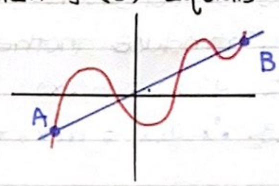
- Suppose there are 2 roots, a & b. Then $f(a)=f(b)=0$

- Rolle: "C" exists btw a & b where $f'(c)=0$

$f'(x) = 3x^2+1=0$ impossible!

Mean Value Theorem:

• Differentiable function contains, differentiable function contains $(a, f(a))$ and $(b, f(b))$, then there exists "c" on (a,b) for which $f'(c)$ equals the slope of the secant line, $\frac{f(b)-f(a)}{b-a}$.



☆: $f'(c) = \frac{f(b)-f(a)}{b-a}$

ex. $g(2)=5$, and $g'(x) \geq 2$ everywhere. Find the greatest possible value of $g(-1)$.

- Consider interval $[-1,2]$. Mean value theo. Says "c" exists, $-1<c<2$, for which $g'(c) = \frac{f(b)-f(a)}{b-a}$.

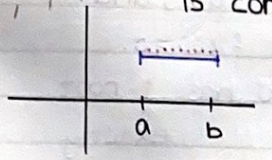
* $g'(c) = \frac{g(2)-g(-1)}{2-(-1)} \rightarrow g'(c) = \frac{5-g(-1)}{3}$

* $\frac{5-g(-1)}{3} \geq 2 \rightarrow 3\left(\frac{5-g(-1)}{3}\right) \geq 3(2) \rightarrow 5-g(-1) \geq 6 \rightarrow -g(-1) \geq 1$

↳ $-g(-1) \leq -1$

"−"

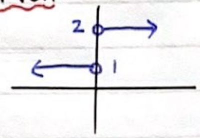
- Theorem: If $f'(x) = 0$ throughout an interval (a,b), then $f(x)$ is constant on (a,b)



- Corollary: If $f'(x) = g'(x)$ throughout on interval (a,b), than $(f-g)(x)$ is constant on (a,b); in other words, $f(x) = g(x) + c$ for some constant c.

★ ID must be an interval.

ex. $b(x) \begin{cases} 1 \text{ if } x < 0 \\ 2 \text{ if } x > 0 \end{cases}$



ID: $x \in \mathbb{R} \mid x \neq 0$
$b'(x) = 0$ for all x in ID, but $b(x)$ is NOT constant function.

ex. If $g'(x) = 3$ over the interval (a,b), show that $g(x) = 3x + d$, where d is a constant.

1. introduce another function where derivative is 3. Say $h(x) = 3x + r$
2. $g'(x) = h'(x)$, so $g(x) - h(x) = C$ is constant on (a,b) by corollary.
3. $\therefore g(x) = h(x) + c = 3x + r + c \rightarrow$ Let $d = r + c$
$= 3x + d$

ex. $f(x) = 3x^2 - \sin 3x$. Show there exists a number r, where $\frac{3\pi}{4} < r < \pi$, for which $f'(r) = \frac{21\pi^2 + 8\sqrt{2}}{4\pi}$.

1. $f\left(\frac{3\pi}{4}\right) = \frac{27\pi^2}{16} - \frac{1}{\sqrt{2}} = \frac{27\pi^2 - 8\sqrt{2}}{16}$ $\quad f(\pi) = (3\pi)^2 - 0 = 3\pi^2$

2. M.V.T.: There is a c on $\left(\frac{3\pi}{4}, \pi\right)$ sutch that $f'(c) = \frac{f(\pi) - f\left(\frac{3\pi}{4}\right)}{\pi - \frac{3\pi}{4}}$

$\hookrightarrow \frac{3\pi^2 - \frac{27\pi^2 - 8\sqrt{2}}{16}}{\frac{\pi}{4}} = \frac{\left(\frac{48\pi^2}{16} - \frac{27\pi^2 - 8\sqrt{2}}{16}\right)\left(\frac{4}{3}\right)}{\frac{\pi}{4}} = \frac{21\pi^2 + 8\sqrt{2}}{4\pi}$

Increasing / decreasing test :

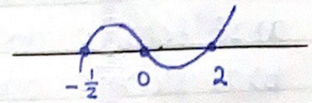
- If $f'(x) > 0$ on (a,b), then $f(x)$ is increasing on (a,b).
- If $f'(x) < 0$ on (a,b), then $f(x)$ is decreasing (a,b).

ex. $f(x) = \frac{3}{2}x^4 - 3x^3 - 3x^2 + 7$, where is $f(x)$ increasing? decreasing?

$f'(x) = 6x^3 - 9x^2 - 6x \rightarrow 3x(x-2)(2x+1)$

$f'(x) < 0$ where $x < -\frac{1}{2}$ or $0 < x < 2$

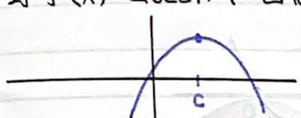
$f'(x) > 0$ where $-\frac{1}{2} < x < 0$ or $x > 2$

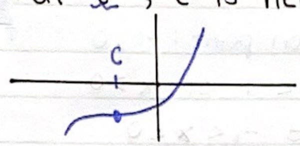


$\therefore$ $f(x)$ is increasing on $(-\frac{1}{2}, 0)$ and $(2, \infty)$ decreasing on $(-\infty, -\frac{1}{2})$ & $(0,2)$

First Derivative Test :

- If c is a critical number of a continous function $f(x)$,
  - a) If $f'(x)$ changes from + to − at c, c is a local max.
  - b) If $f'(x)$ changes from − to + at c, c is a local min.
  - c) If $f'(x)$ doesn't change sign at c, c is neither a local max or min.



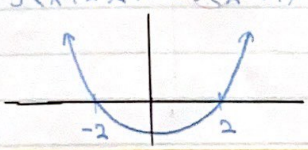


ex. Find local max/min of $f(x) = x^5 - 5x^3 - 20x + 15$.

$f'(x) = 5x^4 - 15x^2 - 20$

$= 5(x^2-4)(x^2+1)$

$= 5(x+2)(x-2)(x^2+1)$



- local max: -2

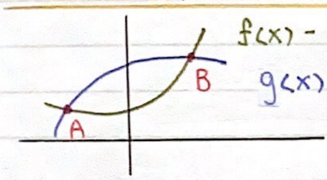
$f(-2) = (-2)^5 - 5(-2)^3 - 20(-2) + 15$

$= 63$ $(-2, 63)$

- local min: 2

$f(2) = (2)^5 - 5(2)^3 - 20(2) + 15$

$= -37$ $(2, -37)$



$f(x)$ - concave upward

$g(x)$ - concave downward

- If the graph of $f(x)$ lies above all its tangents on (a,b), it is concave upwards on (a,b).
- If its graph lies below all its tangents on (a,b), it concave downwards on (a,b).

Concavity :

- If $f''(x) > 0$ throughout (a,b), then $f(x)$ is concave upwards on (a,b)
- If $f''(x) < 0$ throughout (a,b), then $f(x)$ is concave downwards on (a,b).