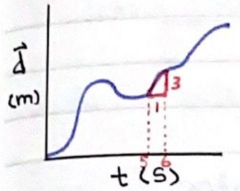
Unit 2 - Limits

QUIZ - FRI. Oct. 14 Pretest - THU. Oct. 20 Test - THU. Oct. 27



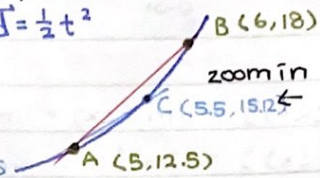
☆ $\vec{v} = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$

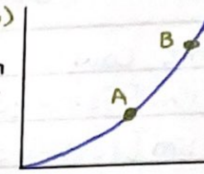
ex. A car stats at rest, then acclerates at $1 m/s^2$ f 10 s.

$\vec{d} = \frac{1}{2}at^2 \rightarrow \vec{d} = \frac{1}{2}(1)t^2 \rightarrow \vec{d} = \frac{1}{2}t^2$

$\vec{v}_1 = \frac{18 - 12.5}{1} = 5.5 m/s$

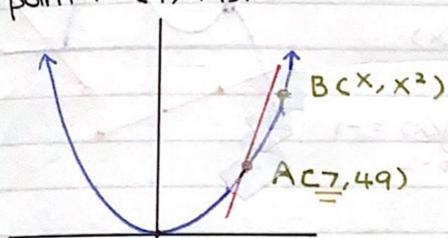
$\vec{v}_2 = \frac{15.12 - 12.25}{5.5 - 5} = 5.25 m/s$





$\lim_{B \to A} v = 5$ (As time gets shorter, it apporches 5.)

ex. Find the slope of the line tangent to the curve $y = x^2$, at point A (7,49).



$m_{AB} = \frac{x^2 - 49}{x - 7}$

$\lim_{x \to 7} \frac{x^2 - 49}{x - 7}$

x	y
8	15
7.5	14.5
7.1	14.1
7.01	14.01

x	y
6	13
6.5	13.5
6.9	13.9
6.99	13.99

x apporch to certain # → y closer to other # → grow
close to 7 some path closer to 14

ex. $f(x) = \frac{2x^2 - 1}{x^2}$ Can we plug ∞ ? $f(\infty) = \frac{2(\infty)^2 - 1}{(\infty)^2} \rightarrow \frac{2(\infty) - 1}{\infty} = \frac{\infty}{\infty}$

x	100	1000	1 000 000
$\frac{2x^2-1}{x}$	1.999	1.999999	1.999999999999

$\rightarrow \lim_{x \to \infty} \frac{2x^2 - 1}{x^2} = 2$

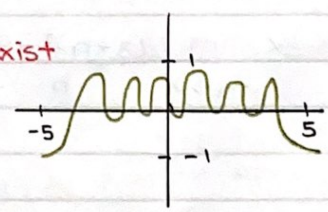
☆: $\lim_{x \to a} f(x) = L$ means that we can make f(x) as close to L as as we like, by macking x close enough to a (but ≠ a)

• For any tolerance $\varepsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ wherever $|x - a| < \delta$.

ex. Estimate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} \doteq 6$

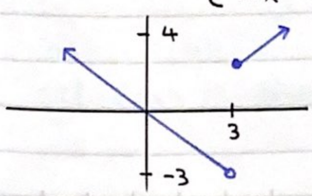
x	2.9	2.99	2.999	3.1	3.01	3.001
$\frac{x^2 - 9}{x - 3}$	5.9	5.99	5.999	6.1	6.01	6.001

ex. Estimate $\lim_{x \to 0} \cos\left(\frac{2\pi}{x}\right) \stackrel{?}{=} 1$ (no) → does not exist



x	1	0.5	0.1	0.01	0.001	0.0001	0.00003
$\cos\left(\frac{2\pi}{x}\right)$	1	1	1	1	1	1	-0.5

ex. $f(x) = \begin{cases} x + 1 & \text{if } x \geq 3 \\ -x & \text{if } x < 3 \end{cases}$ $\lim_{x \to} f(x) = ?$ → not exist



but, $\lim_{x \to 3^-} f(x) = -3$ & $\lim_{x \to 3^+} f(x) = 4$

☆: $\lim_{x \to a} f(x)$ exist only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

left (x < a) right (x > a)

ex. $\lim_{x \to 0} \frac{1}{x}$ = not exist → $\lim_{x \to 0} \frac{1}{x} = \infty$ (getting bigger)

$\frac{1}{0.5} = 2$ $\frac{1}{0.1} = 10$ $\frac{1}{0.01} = 100$

Limit Law:

(1) $\lim [f(x)+g(x)] = \lim f(x) + \lim g(x)$

(2) $\lim [f(x)-g(x)] = \lim f(x) - \lim g(x)$

(3) $\lim [cf(x)] = c \lim f(x)$

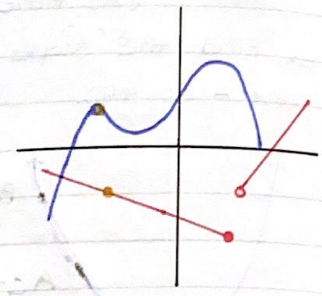
(4) $\lim [f(x)g(x)] = [\lim g(x)][\lim f(x)]$

(5) $\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$ $\lim g(x) \neq 0$

(6) $\lim [f(x)]^n = [\lim f(x)]^n$

(7) $\lim_{x \to a} c = c$

(8) $\lim_{x \to a} x = a$

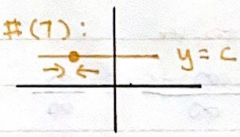


ex. $\lim_{x \to -2} [f(x) + 5g(x)]$ → $\lim_{x \to -2} f(x) + \lim_{x \to -2} 5g(x)$

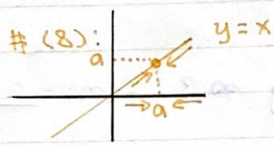
$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x)$ → $\lim_{x \to -2} f(x) = 1$ $\lim_{x \to -2} g(x) = -1$

↳ $1 + 5(-1) = 1 - 5 = -4$

#(7):



#(8):



ex. $\lim_{x \to a} 2x^2 - 3x + 4$ → $\lim_{x \to 2} 2x^2 - \lim_{x \to 2} 3x + \lim_{x \to 2} 4$ → $2\left(\lim_{x \to 2} x\right)^2 - 3\lim_{x \to 2} x + \lim_{x \to 2} 4$

use: (7) & (8) → $2(2)^2 - 3(2) + 4 = 6$

Direct Substitution:

If $f(x)$ is a polynomial or a rational function & a is in the domain of f then $\lim_{x \to a} f(x) = f(a)$

ex. $\lim_{x \to -1} \frac{4x^2+5x-7}{3x^2-2x+1}$ → $\frac{4(-1)^2+5(-1)-7}{3(-1)^2-2(-1)+1} = \frac{-4}{3}$

ex. $\lim_{x \to -3} \frac{x^2-9}{x+3}$ → $\lim_{x \to 3} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}} = -6$ *Facto → cancel → sub

ex. $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$ → $\frac{h^{\cancel{2}} + 6\cancel{h} + \cancel{9} - \cancel{9}}{\cancel{h}} = 6 + h = 6$

$\lim_{x \to 4} \frac{3-x}{x^2-2x-8}$ → $\frac{3-x}{(x-4)(x+2)}$ → Doesn't exist

$\lim_{x \to 2} \frac{x^2-x+6}{x-2}$ → $\frac{\text{can't factor}}{x-2}$ → Doesn't exist

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Squeeze theorem:

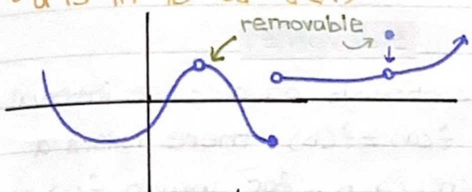
If $f(x) < g(x) < h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$

If $f(x) \leq g(x)$ when $\underline{x}$ is near $\underline{a}$, $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

Continuoity:

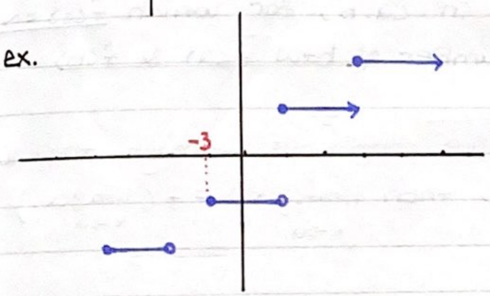
$f(x)$ is continuous at $\underline{a}$ if $\lim_{x \to \underline{a}} f(x) = f(\underline{a})$

- a is in ID of $f(x)$
- $\lim_{x \to a} f(x)$ exist
- $\lim_{x \to a} f(x) = f(a)$



(ex. of 3 discontinuoity)

ex.



where $x = -3$, $f(x)$ is continuous from the right

$\lim_{x \to a^{-1}} f(x) = f(a)$

ex. Show $f(x) = \sqrt[3]{\dfrac{3-x^2}{x+5}}$ is continous on $(-3, 7)$.

where $-3 < x < 7$, $\lim_{x \to a} f(x) = \lim_{x \to a} \sqrt[3]{\dfrac{3-x^2}{x+5}} \to \sqrt[3]{\lim_{x \to a} \dfrac{3-x^2}{x+5}} \to \sqrt[3]{\dfrac{3-a^2}{a+5}}$

$= f(a)$

- If 2 functions are continued at a value, so is any multiple of those functions, their sum & difference & product, and their quotient. (as long as the divisor function $\neq 0$)
- If $f(x)$ & $g(x)$ are continuos at a, so is $(f+g)(x)$, $-3f(x)$, etc.

- Rational, root, and trig functions are continuous at every value in their ID.

ex. On which interval(s) is each function continuous?

a) $f(x) = x^7 - 3x^5 + 5$ $\quad (-\infty, \infty)$

b) $g(x) = \dfrac{x-7}{x-1}$ $\quad x - 1 \neq 0 \to x \neq 1$ $\quad (-\infty, 1)$ and $(1, \infty)$

c) $h(x)$ $\dfrac{x-2}{x} + 4\sqrt{x-1}$ $\quad j(x) = \dfrac{x-2}{x} \to x \neq 0$ $\quad K(x) = 4\sqrt{x-1} \to x - 1 \geq 0 \downarrow x \geq 1$

$h(x) = (j+k)(x) \to [1, \infty)$

- If $f(x)$ is continuous at b, and $\lim_{x \to a} g(x) = 6$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to 2} g(x))$

In a continuous function of a continuous function is continuous.
↳ If $g(x)$ is continuous at a, and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

ex. where is $f(x) = \frac{3}{7+\sqrt{x^3-5}}$ continuous?

$a(x) = x^3 - 5$

$b(x) = \sqrt[3]{x}$

$c(x) = 7 + x$

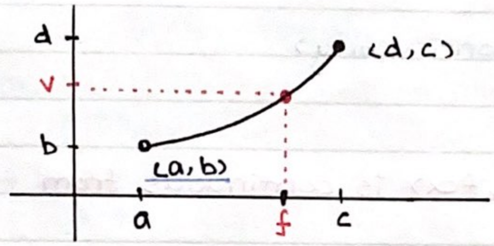
$d(x) = \frac{3}{x}$

$f(x) = d(c(b(a(x))))$

- $f(x)$ is continuous if all its component are continuous.

  $x^3 - 5 \geq 0 \rightarrow x \geq \sqrt[3]{5}$

- $[\sqrt[3]{5}, \infty)$



- If $f(x)$ is continuous on a close interval $[a,b]$, and $f(a) \neq f(b)$, there exists a number c in (a,b) for which $f(c) = N$, for any number N b+w $f(a)$ & $f(b)$

* Direct. Sub. for rational, root, trig → all are continuous through their ID.

- If $f(x)$ is continuous at b, and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$

- If $g(x)$ is continuous at a &

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$$\lim_{\Delta x\to 0} \frac{P(x+\Delta x) - P(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(0.03(x+\Delta x)^3 + (x+\Delta x) + 25) - (0.03x^3 + x + 2\ldots}{\Delta x}$$

$$\lim_{\Delta x\to 0} \frac{0.03(x^3 + 3x^2\Delta x + 3x\Delta x^3) + x + \Delta x + 25 - 0.03x^3 - x - 25}{\Delta x}$$

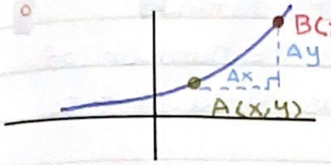
$$\lim_{\Delta x\to 0} 0.03x^3 + 0.09x^2\Delta x + 0.09x\Delta x^2 + 0.03\Delta x^3 + x + \Delta x + 25 - 0.03x^3 - x - 25$$

$$\lim_{\Delta x\to 0} \frac{0.09x^2\Delta x + 0.09x\Delta x^2 + 0.03\Delta x^3 + \Delta x}{\Delta x}$$

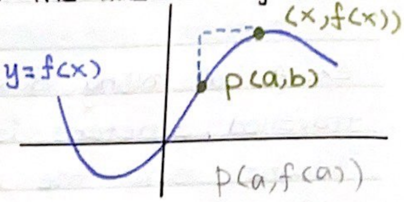
↳ $\frac{\Delta x(0.09x^2 + 0.09x\Delta x + 0.03\Delta x^2 + 1)}{\Delta x} \rightarrow \lim_{\Delta x\to 0} 0.09x^2 + 0.09x\Delta x + 0.03\Delta x^2 + 1$

$= 0.09x^2 + 1$

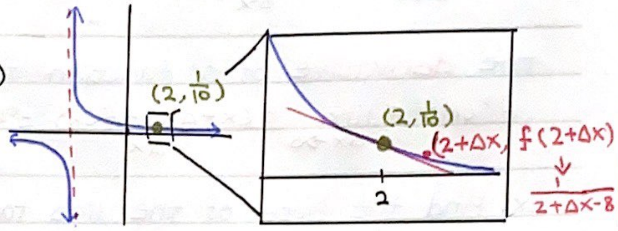
Tangent line - the tangent line to $\underline{f(x)}$ at $P(a,b)$ is the line through P with slope $m = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



ex. Find tangent line to $y = \dfrac{1}{x+8}$ at $(2, \frac{1}{10})$



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(2+\Delta x)-f(x)}{\Delta y}$$

$$\hookrightarrow \lim_{\Delta x \to 0} \frac{\frac{1}{2+\Delta x+8} - \frac{1}{2+8}}{\Delta x} \to \lim_{\Delta x \to 0} \frac{\frac{1}{\Delta x+10} - \frac{1}{10}}{\Delta x}$$

$$\hookrightarrow \lim_{\Delta x \to 0} \frac{\frac{10-(\Delta x+10)}{10(\Delta x+10)}}{\Delta x} \to \lim_{\Delta x \to} \frac{10-\Delta x-10}{10\Delta x(\Delta x+10)} \to \lim_{\Delta x \to 0} \frac{-\Delta x}{10\Delta x(\Delta x+10)} \to \lim_{\Delta x \to 0} \frac{-1}{100+10\Delta x}$$

$$\hookrightarrow \frac{-1}{100+10(0)} = \frac{-1}{100} \qquad \frac{1}{10} = -\frac{1}{100}(2) + b \to b = \frac{3}{25}$$

$y = mx+b \qquad y = -\frac{1}{100}x + \frac{3}{25}$

ex. A farmer started with 25 alfalfa plants in his field. After 10 days he had 65 plants. After 23 days he had 413 plants, and after 30 days, 865 plants. He used the cubic function $\underline{P(x) = 0.03x^3 + x + 25}$ to model how many plants he had after x days. Find the instaneous daily rate of in create in plants at:

a) day 3 b) day 7 c) day 28

$$\lim_{\Delta x \to 0} \frac{P(x+\Delta x) - P(x)}{\Delta x} \to \frac{(0.03(x+\Delta x)^3 + (x+\Delta x)+25) - (0.03x^3+x+25)}{\Delta x}$$

$$\hookrightarrow \frac{0.03(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3) + x + \Delta x + 25 - 0.03x^3 - x - 25}{\Delta x}$$

$$\hookrightarrow \frac{\cancel{0.03x^3} + 0.09x^2\Delta x + 0.03\Delta x^3 \cancel{+x} + \Delta x \cancel{+25} \cancel{-0.03x^3} \cancel{-x} \cancel{-25}}{\Delta x}$$

$$\hookrightarrow \frac{\cancel{\Delta x}(0.09x^2 + 0.09x\Delta x + 0.03\Delta x^2 + 1)}{\cancel{\Delta x}} \to \lim_{\Delta x \to 0} 0.09x^2 + 0.09x\cancel{\Delta x} + 0.03\cancel{\Delta x^2} + 1$$

$= 0.09x^2 + 1$

a) $0.09(3)^2 + 1 = 1.81$ b) $0.09(7)^2 + 1 = 5.41$ c) $0.09(28)^2 + 1 = 71.56$