1.7

$x_t = w_{t-1} + 2w_t + w_{t+1}$, $w_t \sim N(0, \sigma_w^2)$

滯後函數變數h=s-t下算出自共變異數、自相關係數並畫出h的ACF

根據題目

$x_t = w_{t-1} + 2w_t + w_{t+1}$

得

$\mu_t = E(x_t) = E(w_{t-1} + 2w_t + w_{t+1}) = 0$

$V(x_t) = V\ (w_{t-1} + 2w_t + w_{t+1}) = \sigma_w^2 + 4\sigma_w^2 + \sigma_w^2 = 6\sigma_w^2$

s=t+h

$\mathrm{cov}(x_t, x_{t+h}) = E((x_t - \mu_t)\cdot(x_{t+h} - \mu_{t+h}))$

當h=0時$\Rightarrow \gamma(0) = cov(x_t, x_{t+0}) = V_x(x_t) = V_x(w_{t-1} + 2w_t + w_{t+1}) = 6\sigma_w^2$

當h=1時$\Rightarrow cov(x_t, x_{t+1}) = E((x_t - \mu_t)\cdot(x_{t+1} - \mu_{t+1})) = E(x_t \cdot x_{t+1}) =$

$E((w_{t-1} + 2w_t + w_{t+1})\cdot(w_t + 2w_{t+1} + w_{t+2})) = 2V(w_t) + 2V(w_{t+1}) = 4\sigma_w^2$

當h=2時$\Rightarrow cov(x_t, x_{t+2}) = E((x_t - \mu_t)\cdot(x_{t+2} - \mu_{t+2})) = E(x_t \cdot x_{t+2}) =$

$E((w_{t-1} + 2w_t + w_{t+1})\cdot(w_{t+1} + 2w_{t+2} + w_{t+3})) = V(w_{t+1}) = \sigma_w^2$

當h=3時$\Rightarrow cov(x_t, x_{t+3}) = E((x_t - \mu_t)\cdot(x_{t+3} - \mu_{t+3})) = E(x_t \cdot x_{t+3}) =$

$E((w_{t-1} + 2w_t + w_{t+1})\cdot(w_{t+2} + 2w_{t+3} + w_{t+4})) = 0$

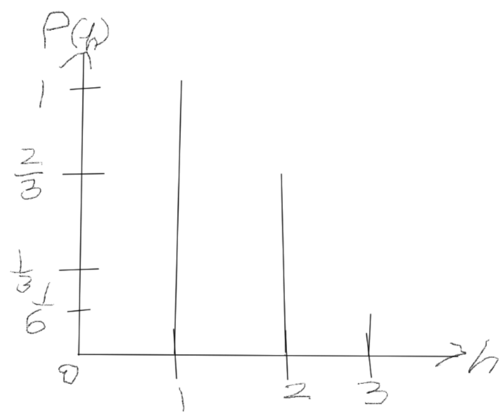
…

$x_t$為白噪音縣性組合$\Rightarrow x_t$平穩$\Rightarrow \gamma(h)$對稱於h=0

$x_t$ 的自共變異數$\Rightarrow \gamma(h)$={$6\sigma_w^2$,h=0, $4\sigma_w^2$,h=$\pm$1, $\sigma_w^2$,h=$\pm$2, 0,o.w.

$x_t$ 的自相關係數ACF:

$\rho_x(h) = \frac{\gamma(h)}{\gamma(0)}$={1,h=0, $\frac{2}{3}$,h=$\pm$1, $\frac{1}{6}$,h=$\pm$2, 0,o.w.



1.8

$x_t = \delta + x_{t-1} + w_t$ , $w_t \sim N\left(0, \sigma_w^2\right)$, $x_0 = 0$

(a)展現模型可以寫成$x_t = \delta t + \sum_{k=1}^{t} w_k$

$x_t = \delta + x_{t-1} + w_t$

$\Rightarrow x_t = \delta + \delta + x_{t-2} + w_{t-1} + w_t \Rightarrow x_t = \delta + \delta + \delta + x_{t-3} + w_{t-2} + w_{t-1} + w_t$....依此類推

$x_t = \delta t + x_0 + \sum_{k=1}^{t} w_k = \delta t + \sum_{k=1}^{t} w_k$

(b)找出$x_t$的mean function 和自共變異數函數

$E\left(x_t\right) = E\left(\delta t + \sum_{k=1}^{t} w_k\right) = \delta t + E\left(\sum_{k=1}^{t} w_k\right) = \delta t$

$V\left(x_t\right) = V\left(\delta t + \sum_{k=1}^{t} w_k\right) = V\left(\sum_{k=1}^{t} w_k\right) = \sum_{k=1}^{t} V\left(w_k\right) = t\sigma_w^2$

$\gamma(0) = cov\left(x_t, x_{t+0}\right) = V\left(x_t\right) = t\sigma_w^2$

$\gamma(1) = cov\left(x_t, x_{t+1}\right) = E\left((x_t - \mu_t) \cdot (x_{t+1} - \mu_{t+1})\right)$

$= E\left((\delta t + \sum_{k=1}^{t} w_k - \mu_t) \cdot (\delta(t + 1) + \sum_{k=1}^{t+1} w_k - \mu_{t+1})\right)$

$= E\left(\delta t + \sum_{k=1}^{t} w_k - \mu_t\right) \cdot E\left(\delta(t + 1) + \sum_{k=1}^{t+1} w_k - \mu_{t+1}\right)$

$= E\left(\sum_{k=1}^{t} w_k \cdot \sum_{k=1}^{t+1} w_k\right) = \sum_{k=1}^{t} V\left(w_k\right) = t\sigma_w^2$

…

$\gamma(-1) = cov\left(x_t, x_{t-1}\right) = E\left((x_t - \mu_t) \cdot (x_{t-1} - \mu_{t-1})\right) = (t - 1)\sigma_w^2$

$\gamma(-2) = cov\left(x_t, x_{t-2}\right) = E\left((x_t - \mu_t) \cdot (x_{t-2} - \mu_{t-2})\right) = (t - 2)\sigma_w^2$

…

$\gamma(-t) = cov\left(x_t, x_0\right) = 0$

$\Rightarrow \gamma(h) = \{ t\sigma_w^2, h \in Z_0^+$ ， $(t - h)\sigma_w^2, h \in Z^-, h \geq -t$ ， $0, o.w.$

(c)說明$x_t$不平穩

$E\left(x_t\right) = \delta t$期望值是與t有關的，所以不平穩。

(d)展現$\rho_x(t-1,t)$=$\sqrt{\frac{t-1}{t}}$當$t\to\infty$時會有甚麼意義。

$\rho_x(t-1,t)$=

$$\frac{cov(x_t,x_{t-1})}{\sqrt{V(x_t)V(x_{t-1})}}=\frac{(t-1)\sigma_w^2}{\sqrt{t\sigma_w^2\cdot(t-1)\sigma_w^2}}=\frac{(t-1)\sigma_w^2\sqrt{t\sigma_w^2\cdot(t-1)\sigma_w^2}}{t\sigma_w^2\cdot(t-1)\sigma_w^2}=\frac{\sqrt{t\sigma_w^2\cdot(t-1)\sigma_w^2}}{t\sigma_w^2}=\frac{\sqrt{t\cdot(t-1)}}{t}=\sqrt{\frac{t\cdot(t-1)}{t^2}}$$

$$=\sqrt{\frac{t-1}{t}}$$

當$t\to\infty$時，$\rho_x(t-1,t)$=$\sqrt{\frac{t-1}{t}}$會趨近於1，表示當觀測值$x_t$的t越大，會$x_t$與前一個值$x_{t-1}$越接近完美的線性關係。

(e)找一轉換使資料平穩並證明

$$x_t=\delta+x_{t-1}+w_t=\delta t+\sum_{k=1}^{t}w_k$$

$$x_{t-1}=\delta+x_{t-2}+w_{t-1}=(\delta t-1)+\sum_{k=1}^{t-1}w_k$$

$$\nabla x_t=x_t-x_{t-1}=\delta t+\sum_{k=1}^{t}w_k-\delta t+\sum_{k=1}^{t-1}w_k=\delta+w_t$$

(1)

$$E\left(\nabla x_t\right)=E\left(\delta+w_t\right)=\delta$$

與t無關

(2)

$$cov\left(\nabla x_t,\nabla x_{t+1}\right)=E\left((\nabla x_t-E\left(\nabla x_t\right))\cdot(\nabla x_{t+1}-E\left(\nabla x_{t+1}\right))\right)=E\left((\delta+w_t-\delta)\cdot(\delta+w_{t+1}-\delta)\right)$$

$$=0$$

與t無關

(3)

$$V(\nabla x_t)=cov\left(\nabla x_t,\nabla x_t\right)=E\left((\delta+w_t-\delta)\cdot(\delta+w_t-\delta)\right)=\sigma_w^2<\infty$$

所以由(1)、(2)、(3)結果，$\nabla x_t$是平穩的。