ENGR-3000:

Renewable Energy, Technology, and Resource Economics

Supplemental Lecture: Betz Limit

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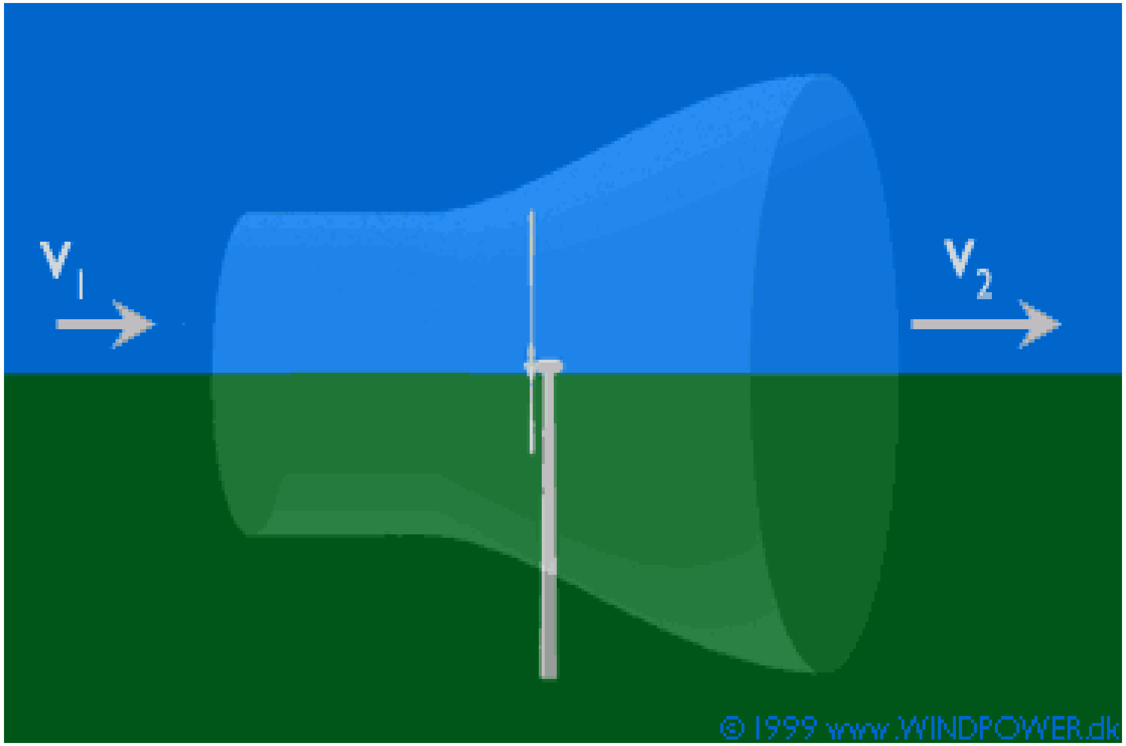
Turbine Energy Extraction

- As the turbine transforms the kinetic energy of the wind into rotational energy of the generator shaft, the wind velocity decreases:



Constant mass flow rate: $$\frac{\Delta m}{\Delta t} = \rho A_1 V_1 = \rho A_2 V_2$$

Calculating Turbine Power

$$\text{Power} = \frac{1}{2}\rho C_p A V^3$$

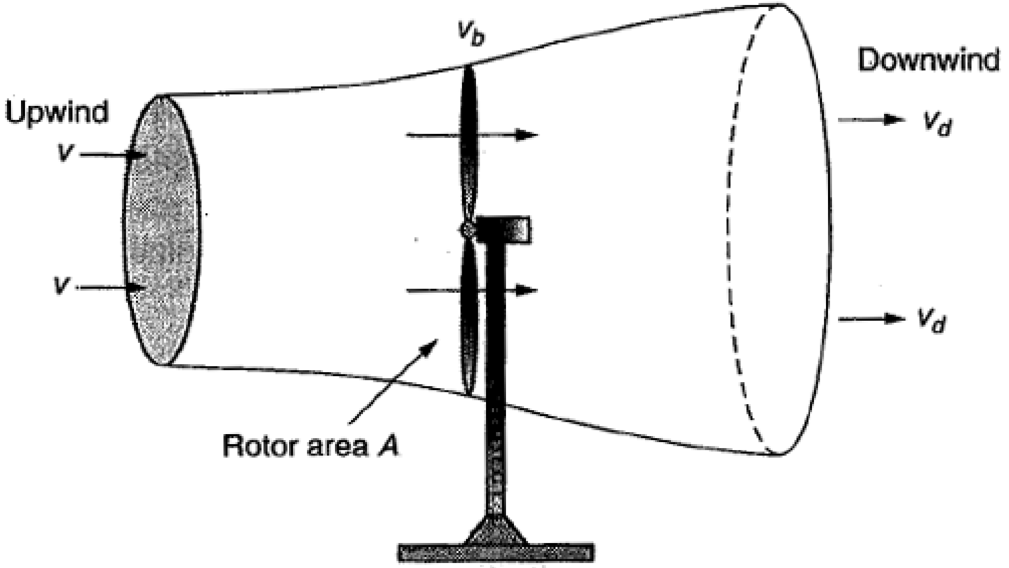
- Where:
  - ρ = air density (kg/m^3)
  - C_p = Power coefficient ($0 < C_p < 1$)
  - Swept rotor blade area = $\pi d^2/4$ (m^2) for HAWT
  - V = upstream (undisturbed) wind velocity
- For a conventional horizontal axis wind turbine HAWT, $A = (\pi/4)D^2$, so wind power is proportional to the blade diameter squared
- Cost is roughly proportional to blade diameter
- This explains why larger wind turbines are more cost effective

Maximum Rotor Efficiency

- Two extreme cases, and neither makes sense-
  - Downwind velocity is the same as the upwind velocity – turbine extracted no power
  - Downwind velocity is zero – turbine extracted all of the power
- Albert Betz 1919 - There must be some ideal slowing of the wind so that the turbine extracts the maximum power
- There is a limit on the ability of a wind turbine to convert kinetic energy in the wind into mechanical power extracted by the turbine

Maximum Rotor Efficiency

- As wind passes thru a turbine- it slows down and the pressure is reduced so it expands:



Turbine Power:

$$P_b = \frac{1}{2}\dot{m}\left(v^2 - v_d^{\,2}\right)$$

  - $\dot{m}$ = mass flow rate of air within stream tube
  - v = upwind undisturbed windspeed
  - v_d = downwind windspeed
  - From the difference in kinetic energy between upwind and downwind air flows

Determining Mass Flow Rate

- Easiest to determine at the plane of the rotor because we know the cross sectional area A
- Then, the mass flow rate is

$$\dot{m} = \rho A v_b$$

- Assume the velocity through the rotor v_b is the average of upwind velocity *v* and downwind velocity v_d :

$$v_b = \frac{v + v_d}{2} \quad \Rightarrow \quad \dot{m} = \rho A \left(\frac{v + v_d}{2} \right)$$

Power Extracted by the Blades

- Then the turbine power equation becomes

$$P_b = \frac{1}{2}\rho A\left(\frac{v+v_d}{2}\right)\left(v^2 - v_d{}^2\right)$$

- Define λ (lambda) as the ratio of exit to inlet velocities:

$$\lambda = \frac{v_d}{v} \quad \Rightarrow \quad v_d = \lambda v$$

Thus: $$P_b = \frac{1}{2}\rho A\left(\frac{v+\lambda v}{2}\right)\left(v^2 - \lambda^2 v^2\right)$$

Power Extracted by the Blades

$$P_b = \frac{1}{2}\rho A\left(\frac{v+\lambda v}{2}\right)\left(v^2-\lambda^2 v^2\right)$$

$$\left(\frac{v+\lambda v}{2}\right)\left(v^2-\lambda^2 v^2\right) = \frac{v^3}{2}-\frac{\lambda^2 v^3}{2}+\frac{\lambda v^3}{2}-\frac{\lambda^3 v^3}{2}$$

$$= \frac{v^3}{2}\left[\left(1+\lambda\right)-\lambda^2\left(1+\lambda\right)\right]$$

$$= \frac{v^3}{2}\left[\left(1+\lambda\right)\left(1-\lambda^2\right)\right]$$

$$\Rightarrow \quad P_b = \underbrace{\frac{1}{2}\rho A v^3}_{P_W} \cdot \underbrace{\frac{1}{2}\left[\left(1+\lambda\right)\left(1-\lambda^2\right)\right]}_{C_P}$$

P_W = Power in the wind $\qquad$ C_P = Power Coefficient

Maximum Rotor Efficiency

- Find the speed windspeed ratio λ which maximizes the power coefficient, C_P
- From the previous slide

$$C_P = \frac{1}{2}\left[\left(1+\lambda\right)\left(1-\lambda^2\right)\right] = \frac{1}{2} - \frac{\lambda^2}{2} + \frac{\lambda}{2} - \frac{\lambda^3}{2}$$

- Set the derivative $\partial C_p/\partial \lambda$ to zero and solve for λ:

$$\frac{\partial C_P}{\partial \lambda} = -\lambda + \frac{1}{2} - \frac{3\lambda^2}{2} = 0$$

$$3\lambda^2 + 2\lambda - 1 = \left(3\lambda - 1\right)\left(\lambda + 1\right) = 0$$

$$\left(3\lambda - 1\right) = 0 \quad \Rightarrow \quad \lambda = \frac{1}{3}$$

⇦ maximizes rotor efficiency

Maximum Rotor Efficiency: Betz Limit

- Plug the optimal value of λ = 1/3 back into C_P to find the maximum rotor efficiency:

$$C_P = \frac{1}{2}\left[\left(1+\frac{1}{3}\right)\left(1-\frac{1}{3^2}\right)\right] = \frac{16}{27} = 59.3\%$$

- The maximum efficiency of 59.3% occurs when air is slowed to 1/3 of its upstream rate
- Called the "Betz efficiency" or "Betz' law"

Betz Limit

- The maximum fraction of available wind energy that can be extracted by a wind turbine rotor.

- The down stream velocity (v_2) cannot be zero, in fact, the smallest is can be is $v_1/3$.

- This corresponds to a Cp_{max} = 0.5926