MAKERERE UNIVERSITY

CIT, 2010/2011 SEMESTER I

MTH 3105: DISCRETE MATHEMATICS, Test II Solutions

There are 4 (four) questions in this test, attempt all questions. Time allowed: 1 (one) hour.

1.

(a) Prove that $\sqrt{10}$ is irrational. *5 marks*

Assume $\sqrt{10}$ is rational, then we can write $\sqrt{10} = \frac{m}{n}$ such that m and n are both integers and they do not have common factors other than 1. Then we get $10n^2 = m^2$.
Since left hand side is even, m must be even, substitute m = 2k, we get $5n^2 = 2k^2$.
Since right hand side is even, n must be even. Hence m and n are both even, which contradicts the assumption that m and n do not have common factor other 1. Hence $\sqrt{10}$ is irrational

(b) A number *n* is a sum of two squares if $n = a^2 + b^2$ for some integers *a* and *b*. If *x* and *y* are both sum of two squares, prove that *xy* is also a sum of two squares. *5 marks*

Let $x = a^2 + b^2$ and $y = c^2 + d^2$. Then

$$\begin{aligned} xy &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= (a^2c^2 + b^2d^2 + 2abcd) + (a^2d^2 + b^2c^2 - 2abcd) \\ &= (ac + bd)^2 + (ad - bc)^2 \end{aligned}$$

2. Solve the followings, or give "unsolvable" if it has no solution.

(a) $19x \equiv 20 \pmod{77}$ *4 marks*

The congruence $ax \equiv b \pmod{n}$ has a solution for x if and only if b is divisible by gcd(a,n).
gcd(19, 77) = 1, which divides any b, and so there is just one solution in {0,1,2,…,76}: ***x=74***

(b) $49x \equiv 98 \pmod{21}$ *3 marks*

gcd(49, 21) = 7 (giving solutions ***2***,***5***,***8***,***11***,***14***,***17***, *and* ***20***), *we rewrite as $7x \equiv 14 \pmod{3}$*
gcd(7, 3) = 1, which divides any b. There is one solution in {0,1,2}: ***x = 2***

(c) $105x \equiv 143 \pmod{100}$ *3 marks*

gcd(105, 100) = 5 but $5 \nmid 143$. Therefore ***unsolvable***.

3.

(a) How many cards do you need to draw from a deck of 52 cards in order to ensure there is three-of-a-kind? *5 marks*

(Pigeonhole principle) The worst case is when you keep drawing until you have two of each kind, and on the next draw you will get three-of-a-kind. Therefore the number of cards to draw is 2 x 13 + 1 = ***27*** *(if a kind is interpreted as the suit, we have 2 x 4 + 1 =* ***9****)*

(b) Count the number of ways to place four pawns on a 10 x 10 chessboard so that no two pawns share a row or a column. *5 marks*

By generalized product rule there are $(10!/(10-4)!)^2$ ways to place four different pieces on a 10 x 10 chessboard. Then by division rule there are 4! permutations of the 4 pawns that map to the same configuration. Therefore there are $(10!/6!)^2/4!$ = ***1,058,400*** *ways.*

4. Given the functions below with the respective domain and range, complete this table that tells whether they are injective, surjective, or bijective:

10 marks

	Function	Domain	Codomain	Injective?	Surjective
(a)	$f(x) = e^x$	$\mathbb{R}$	$\mathbb{R}^+$	***Yes***	***Yes***
(b)	$f(x) = \cos(x)$	$\mathbb{R}$	$\mathbb{R}$	***No***	***No***
(c)	Reverse string	Bit strings of length n	Bit strings of length n	***Yes***	***Yes***
(d)	$f(x) = x^4$	$\mathbb{R}$	$\{0, \mathbb{R}^+\}$	***No***	***Yes***
(e)	$f(x) = $ is-prime(x)	$\mathbb{Z}^+$	{T, F}	***No***	***Yes***