CSC 2105: Discrete Mathematics

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Relations and Number Theory – Rosen 4, 9

Overview

Relations

Introduction

Given sets S and T, a (binary) relation from S to T is just a subset R of $S \times T$, i.e., a set of ordered pairs (s,t).

Example 1

A mail-order record company has a list L of customers. Each customer indicates interest in certain categories of recordings: classical, easy-listening, Latin, religious, popular, rock, etc. Let C be the set of possible categories. The set of all ordered pairs (name, selected-category) is a relation R from L to C. This relation might contain such pairs as (K. A. Ros, classical), (C. R. B. Wright, classical) and (C. R. B. Wright, gangsta rap).

Relations and Functions

Example 2

A translator from decimal representation to binary representation can be viewed as the relation consisting of all ordered pairs whose first entries are allowable decimal representations and whose second entries are the corresponding binary representations. Actually, this relation is a function

- We say that s is R-related to t or that s is related to t by R in case $(s,t) \in R$
- As we might see, functions can be identified with their graphs and regarded as sets of ordered pairs. In fact,
  if $f : S \to T$, we identify f with the set:
  $R_f = \{(x, y) \in S \times T : y = f(x)\}$,
  which is a relation from S to T.
- Not all relations are functions.
- Function from S to T is a special kind of relation R from S to T, one such that for each $x \in S$ there exists exactly one $y \in T$ with $(x, y) \in R$.

Properties of Relations

- A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.
  $\forall a((a, a) \in R)$
- A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
  $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$
- A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.
  $\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$
- A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.
  $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$
- Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S, denoted by $S \circ R$, is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Thus $R^{n+1} = R^n \circ R$

Examples and Definitions

Examples

$x \leqslant x$ for all $x \in \mathbb{R}$,	Reflexive
$x \leqslant y$ and $y \leqslant x$ imply that $x = y$,	Antisymmetric
$x \leqslant y$ and $y \leqslant z$ imply that $x \leqslant z$,	Transitive
$x < x$ *never* holds,	Antireflexive
$x < y$ and $y < z$ imply that $x < z$.	(T)

n-ary Relations

- Let $A_1, A_2, \cdots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \cdots, A_n$ are called the *domains* of the relation, and n is called its *degree*
- Relations that link objects of the same type have quite a different flavor – we will say that a subset R of $S \times S$ is a relation on S.
- Every set S has the very basic "equality relation", $E = \{(x, x) : x \in S\}$. Two elements in S satisfy this relation if and only if they are identical. Thus $(x, y) \in E$ if and only if $x = y$

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More Definitions

1. The integers a and b are **relatively prime** if their greatest common divisor (gcd) is 1
2. If a and m are relatively prime integers and $m > 1$, then an **inverse of** a **modulo** m exists. Furthermore, this inverse is unique modulo m. (That is, there is a unique positive integer $\bar{a}$ less than m that is an inverse of a modulo m and every other inverse of a modulo m is congruent to $\bar{a}$ modulo m)
3. We say that m **is congruent to** n **modulo** p written as $m \equiv n \pmod{p}$ provided $m - n$ is a multiple of p (**congruent relation** on $\mathbb{Z}$).

$$m \equiv m \pmod{p} \text{ for all } m \in \mathbb{Z}, \qquad \text{(R)}$$
$$m \equiv n \pmod{p} \text{ implies } n \equiv m \pmod{p}, \qquad \text{(S)}$$
$$m \equiv n \pmod{p} \text{ and } n \equiv r \pmod{p} \text{ imply } m \equiv r \pmod{p}, \qquad \text{(T)}$$

Number Theorems (1)

1. Let a, b, and c be integers, where $a \neq 0$. Then
   (i) if $a|b$ and $a|c$, then $a|(b+c)$;
   (ii) if $a|b$, then $a|bc$ for all integers c;
   (iii) if $a|b$ and $b|c$, then $a|c$.

   Consequently, Corollary 1: $a|mb + nc$ whenever m and n are integers

2. The Division Algorithm: Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leqslant r < d$, such that $a = dq + r$

3. Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \textbf{ mod } m = b \textbf{ mod } m$

4. Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$

5. Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

   Corollary 2: Let m be a positive integer and let a and b be integers. Then $(a + b) \textbf{ mod } m = ((a \textbf{ mod } m) + (b \textbf{ mod } m)) \textbf{ mod } m$ and $ab \textbf{ mod } m = ((a \textbf{ mod } m)(b \textbf{ mod } m)) \textbf{ mod } m$

Number Theorems (2)

6. If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$
   Remark: The integer n is composite if and only if there exists an integer a such that $a|n$ and $1 < a < n$
7. There are infinitely many primes
8. The Prime Number Theorem: The ratio of the number of primes not exceeding x and $x/\ln x$ approaches 1 as x grows without bound.
   (Here $\ln x$ is the natural logarithm of x)

L[1]: Let $a = bq + r$, where a, b, q, and r are integers. Then $\gcd(a, b) = \gcd(b, r)$

9. Bézout's Theorem: If a and b are positive integers, then
   there exist integers s and t such that $\gcd(a, b) = sa + tb$

L[2]: If a, b, and c are positive integers such that $\gcd(a, b) = 1$ and $a|bc$, then $a|c$

L[3]: If p is a prime and $p|a_1 a_2 \cdots a_n$, where each a_i is an integer, then $p|a_i$ for some i

10. Let m be a positive integer and let a, b, and c be integers.
    If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$

[1]Lemma 1

Congruence Solutions

$ax \equiv b \pmod{n}$

The congruence $ax \equiv b \pmod{n}$ has a solution for x if and only if b is divisible by $gcd(a, n)$.

Steps

1. Check if b is divisible by $d = \gcd(a, n)$
2. Compute s and t in $sa + tn = d$ (Theorem 9 & Euclidean Algorithm)
3. $x = \frac{sb}{d}$
4. Find d solution(s) in the range $\{0, 1, 2, ..., n - 1\}$

Example 3

Solve for x in $19x \equiv 20 \pmod{77}$

Solution

1. $\gcd(19, 77) = 1$, which divides any b so there is one solution in $\{0, 1, 2, ..., 76\}$
2. $19s + 77t = 1$ gives s = -4 and t = 1.
3. $x = \frac{-4 \cdot 20}{1} = -80$
4. Giving x = 74 in the range
5. More solutions (if any) are obtained by adding n to x

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Congruence Solutions

Using Modular Inverse

Example 4

Solve for x in $3x \equiv 4 \pmod 7$

Solution

- Because $\gcd(3, 7) = 1$, an inverse of 3 modulo 7 exists
- $7 = 2 \cdot 3 + 1$ [Theorem 2]
- $\Rightarrow -2 \cdot 3 + 1 \cdot 7 = 1$
- By Theorem 9, $s = -2$, is an inverse of 3 modulo 7.
- We have $-2 \cdot 3x \equiv -2 \cdot 4 \pmod 7$.
- Because $-6 \equiv 1 \pmod 7$ and $-8 \equiv 6 \pmod 7$, it follows that if x is a solution, then $x \equiv 6 \pmod 7$ [Congruence Definition]
- By Theorem 5, $3x \equiv 3 \cdot 6 = 18 \equiv 4 \pmod 7$.
- Thus $x = \{6, 13, 20, \ldots\}$ and $\{-1, -8, -15, \ldots\}$ or simply $x = 6$.

Congruence Solutions

Finding s and t – Bézout's Theorem

Example 5

Find an inverse of 101 modulo 4620

Phase I: Euclidean algorithm

$4620 = 45 \cdot 101 + 75$
$101 = 1 \cdot 75 + 26$
$75 = 2 \cdot 26 + 23$
$26 = 1 \cdot 23 + 3$
$23 = 7 \cdot 3 + 2$
$3 = 1 \cdot 2 + 1$

- Because the last nonzero remainder is 1, we know that $gcd(101, 4620) = 1$

Phase II: Bézout coefficients

$1 = 3 - 1 \cdot 2$
$= 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$
$= -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$
$= 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = -9 \cdot 75 + 26 \cdot 26$
$= -9 \cdot 75 + 26 \cdot (101 - 1 \cdot 75) = 26 \cdot 101 - 35 \cdot 75$
$= 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$
$= 1601 \cdot 101 - 35 \cdot 4620 \equiv 101s + 4620t$

- Thus 35 and 1601 are Bézout coefficients of 4620 and 101, and 1601 is an inverse of 101 modulo 4620

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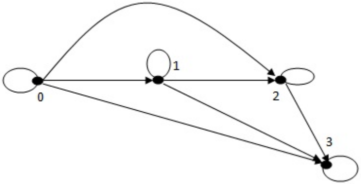
Converse Relation

- Consider $R \subseteq S \times T$, the converse relation $R^{\leftarrow}$ is the relation from T to S defined by:
  $(t, s) \in R^{\leftarrow}$ if and only if $(s, t) \in R$
- Since every function $f : S \to T$ is a relation, its converse $f^{\leftarrow}$ always exists:
  As a relation $f^{\leftarrow} = \{(y, x) \in T \times S : y = f(x)\}$

Example 6

- Consider the relation R_1 on the set $\{0,1,2,3\}$ defined by $\leqslant$; thus $(m,n) \in R_1$ if and only if $m \leqslant n$. A picture R_1 is given besides...
- Observe that we have drawn an arrow from m to n whenever $(m,n) \in R_1$, though we left off the arrowheads on the "loops" $0 \to 0$, $1 \to 1$, etc.



R_1

Challenge 1 for the bored

1. Find an inverse of 5 modulo 67, i.e.,
   Find s such that $5s + 67t = 1$.
2. Solve for x in Example 3 using the modular inverse.
3. Solve for x in Example 4 using the gcd method.
4. Let R_2 be the relation on $\{1,2,3,4,5\}$ defined by mR_2n if and only if $m - n$ is even. Draw a picture for this relation.
5. Obtain a picture of the converse relation $R_1^{\leftarrow}$ by reversing all the arrows in that on previous slide.
6. Similarly, obtain a picture of the converse relation $R_2^{\leftarrow}$. Compare this with the picture of R_2. Give reasons to support your observation.

Equivalence Relations

Let S be a set of marbles. We might regard s and t as equivalent if they have the same color, in which case we might write $s \sim t$

$\sim$ satisfies three properties:

$$
\begin{aligned}
&s \sim s \text{ for all marbles s,} && \text{(R)}\\
&\text{if } s \sim t \text{ then } t \sim s, && \text{(S)}\\
&\text{if } s \sim t \text{ and } t \sim u \text{, then } s \sim u. && \text{(T)}
\end{aligned}
$$

- For the same set S of marbles, we might regard marbles s and t as equivalent if they are of the same size, and write $s \approx t$ in this case.
- Other notations sometimes used for equivalence relations are $s \approxeq t$, $s \equiv t$ and $s \leftrightarrow t$.

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Equivalence Relations

Examples

Example 7

Consider a machine that accepts input strings in Σ^* for some alphabet Σ and that generates output strings. We can define an equivalence relation $\sim$ on Σ^* by letting $w_1 \sim w_2$ if the machine generates the same output string for either w_1 or w_2.

Example 8

For $m, n \in \mathbb{Z}$, define $m \sim n$ in case $m - n$ is odd. The relation $\sim$ is symmetric but is highly nonreflexive and nontransitive. In fact,

$$m \nsim m \text{ for all } m \in \mathbb{Z}$$

and

$$m \sim n \text{ and } n \sim p \text{ always imply } m \nsim p.$$

Partitions

Let f be a function from a set T. Then the set $\{f^{\leftarrow}(y) : y \in T\}$ for all inverse images $f^{\leftarrow}$ partitions S. Also, the union $\bigcup_{y \in T} f^{\leftarrow}(y)$ of all the sets $f^{\leftarrow}(y)$ is S.

- Consider again an equivalence relation $\sim$ on a set S. For each s in S we define:
  $[s] = \{t \in S : s \sim t\}$
  the equivalence class (or set) or $\sim$-class containing s.
- The set of all equivalence classes of S is then denoted by $[S]$, i.e., $[S] = \{[s] : s \in S\}$

In the marble example, the equivalence class $[s]$ of a given marble s is the set of all marbles that are the same color as s

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Exercise

Problem

Determine which relations are reflexive, transitive, symmetric, or antisymmetric on the following – there may be more than one characteristic. xRy if

1. $x = y$
2. $x < y$
3. $x^2 = y^2$
4. $x \leqslant y$

Problem

Partition $\{x | 1 \leqslant x \leqslant 9\}$ into equivalence classes under the equivalence relation $x \equiv y \pmod 6$

Solution

[1]={1,7}, [2]={2,8}, [3]={3,9}, [4]={4}, [5]={5}, [6]={6}

Solution

1. Symmetric, Reflexive, Transitive and Antisymmetric
2. Transitive
3. Symmetric, Reflexive, Transitive and Antisymmetric ($x^2 = y^2$ is just a special case of equality, so all properties that apply to $x = y$ also apply to this case)
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Example 9

If S is our set of marbles and $f(s)$ is the color of s, then $v(s)$ is the class $[s]$ of all marbles the same color as s. We could think of $v(s)$ as a bag of marbles, with v the function that puts each marble in its proper bad. With this interpretation, $[S]$ is a collection of bags, each with at least one marble in it. The number of bags is the number of colors used

Challenge 2 for the bored

Let R_1 and R_2 be relations from a set S to a set T

1. Show that $(R_1 \cup R_2)^{\leftarrow} = R_1^{\leftarrow} \cup R_2^{\leftarrow}$
2. Show that $(R_1 \cap R_2)^{\leftarrow} = R_1^{\leftarrow} \cap R_2^{\leftarrow}$
3. Show that if $R_1 \subseteq R_2$ then $R_1^{\leftarrow} \subseteq R_2^{\leftarrow}$

Example 9

If S is our set of marbles and $f(s)$ is the color of s, then $v(s)$ is the class $[s]$ of all marbles the same color as s. We could think of $v(s)$ as a bag of marbles, with v the function that puts each marble in its proper bad. With this interpretation, $[S]$ is a collection of bags, each with at least one marble in it. The number of bags is the number of colors used

Challenge 2 for the bored

Let R_1 and R_2 be relations from a set S to a set T

1. Show that $(R_1 \cup R_2)^{\leftarrow} = R_1^{\leftarrow} \cup R_2^{\leftarrow}$
2. Show that $(R_1 \cap R_2)^{\leftarrow} = R_1^{\leftarrow} \cap R_2^{\leftarrow}$
3. Show that if $R_1 \subseteq R_2$ then $R_1^{\leftarrow} \subseteq R_2^{\leftarrow}$