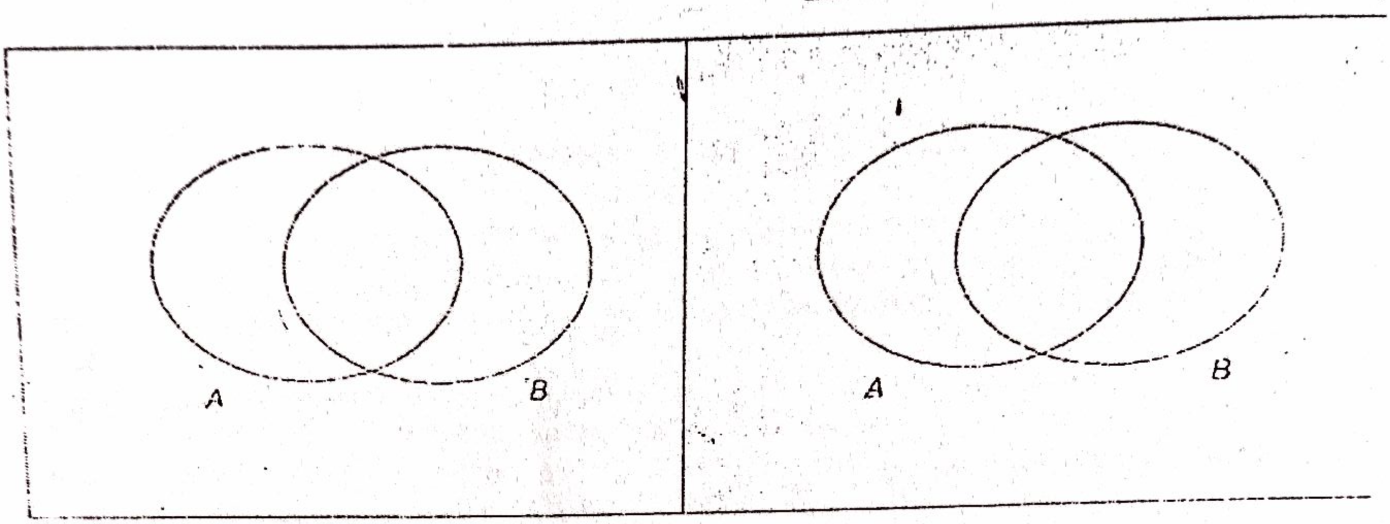
③ $= 26 - (1 \cdot 23)$.

$-1 \cdot 23 + 8(26 - 1 \cdot 23)$



6 (a) One of the shaded regions in question 5 represents the set $A - B$. Identify which one it is, and hence write a definition of $A - B$ using only symbols from the list $\cap$, $\cup$ and $'$.

(b) Again using one of your answers to question 5, write a definition of $A \Delta B$ using only symbols from the list $\cap$, $\cup$ and $'$. (There are two possibilities here – see if you can find them both!)

Set Theory Exercise 4

1(a) If $A = \{1, 2, 3, 4\}$, write down $P(A)$ by listing its elements. What is the value of $|P(A)|$?

(b) If $|A| = 5$, what is the value of $|P(A)|$?
(c) If $|A| = 10$, what is the value of $|P(A)|$?

2 Prove the following identities, stating carefully which of the set laws you are using at each stage of the proof.

(a) $B \cup (\emptyset \cap A) = B$
(b) $(A' \cap U)' = A$
(c) $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$
(d) $(A \cap B) \cup (A \cap B') = A$
(e) $(A \cap B) \cup (A \cup B')' = B$
(f) $A \cap (A \cup B) = A$

Set Theory Exercise 5

1 $X = \{a, c\}$ and $Y = \{a, b, e, f\}$.
Write down the elements of:

Profs Accepts Bribes

(a) $X \times Y$
(b) $Y \times X$
(c) X^2 ($= X \times X$)
(d) What could you say about two sets A and B if $A \times B = B \times A$?

2 A chess board's 8 rows are labelled 1 to 8, and its 8 columns a to h. Each square of the board is described by the ordered pair (column letter, row number).
(a) A knight is positioned at (d, 3). Write down its possible positions after a single move of the knight.
(b) If $R = \{1, 2, \ldots, 8\}$, $C = \{a, b, \ldots, h\}$, and P = {coordinates of all squares in the chess board}, use set notation to express P in terms of R and C.
(c) A rook is positioned at (g, 2). If $T = \{2\}$ and $G = \{g\}$, express its possible positions after one move of the rook in terms of R, C, T and G.

3 In a certain programming language, all variable names have to be 3 characters long. The first character must be a letter from 'a' to 'z'; the others can be letters or digits from 0 to 9.
If $L = \{a, b, c, \ldots, z\}$, $D = \{0, 1, 2, \ldots, 9\}$, and V = {permissible variable names}, use a Cartesian product to complete: $V = \{pqr \mid (p, q, r) \in \ldots\}$

Discrete Mathematics/Set theory/Answers

Answers to Set Theory Exercise 1

1 (a) Yes; alphanumeric characters are A..Z, a...z and 0..9
(b) No, 'tall' is not well-defined
(c) Yes: the set is {12.5}
(d) Yes: the empty set
(e) No; 'good' is not well-defined

2 (a) T
(b) F
(c) T
(d) F: J is a *subset* of U (which we meet in the next section)
(e) F: {even numbers} means the set of *all* the even numbers, not just those between 2 and 10

3 (a) {4, 33, √9}
(b) {4, -5, 33, √9}
(c) {4, 2/3, -2.5, -5, 33, √9}
(d) {√2, π}

4 (a) T
(b) T
(c) F
(d) T

fe3
90 F

7p

SECTION A (40 Marks)

Question A

Describe the following sets in both formal and informal ways. (1 Mark @)

Formal Set Notation Description	*Informal English Description*
(i)	The set of all positive odd integers $\forall o \in Z^+:$
(ii)	The set of all even integers $\forall n \in Z$
(iii)	The set of all negative even integers (using the convention that 0 is not a natural number)
(iv)	The set of all positive multiples of 6
(v)	The set containing the numbers 2, 20, and 200
(vi) $\{n \mid n \in \mathbb{Z} \text{ and } n > 42\}$	$\forall n \in Z \cdot$
(vii) $\{n \mid n \in \mathbb{Z} \text{ and } n < 42 \text{ and } n > 0\}$ $= \{n \mid n \in \mathbb{N} \text{ and } n < 42\}$	$\forall o \in Z^+: odd(o)$
(viii) {hello}	
(ix) {bba, bab}	
(x) $\varphi = \{\}$	

$\forall n \in Z$ even(n)

Question B

Using the product rule notation, prove that if $|A| = n$ and $|B| = m$, then $|AxB| = mn$ (6 Marks)

Question C

Represent the following statements using predicate logic (2 Marks @)

(i) There is someone in this class who doesn't have a good attitude $\exists x \neg P(x)$

(ii) Some drivers do not obey speed limits $\exists x \neg P(x)$

(iii) Every bird can fly. $\forall x\, P(x)$

(iv) No one can keep a secret.

$\forall x \neg P(x)$

Question D

Determine whether the function f, in each of the following cases is surjective or not. Give a reason for your answer.

i) f from {a, b, c, d} to {1,2,3} defined by f(a) = 3, f(b) = 2, f(c) = 1 and f(d) = 3 (4 Marks)

ii) $f(x) = x^2$ from the set of integers to the set of integers (2 Marks)

Question E

Briefly expound on the different types of graphs (please use diagrams to do so) (10 Marks)

$\forall n \in Z^+:$

{0, 1, 2, 3, ...}

(1, 2, 3, ...

SCHOOL OF COMPUTING AND INFORMATICS TECHNOLOGY

2012/2013 SEMESTER I

MTH 3105: DISCRETE MATHEMATICS

Mid-Semester Test

There are 5 (five) questions in this test, answer all questions.

Time allowed: 1 (one) hour.

$(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

$(p \wedge q) \vee [\sim p \wedge (q \vee \sim q)]$

$(p \wedge q) \vee [\sim p \wedge T]$

$(p \wedge q) \vee \sim p$

1. Let p, q, r be the following propositions:

   p = "it is raining,"

   q = "the sun is shining,"

   r = "there are clouds in the sky."

   Translate the following into logical notation, using p, q, r and logical connectives.

   a) It is raining and the sun is shining. $p \wedge q$ (2 Marks each)

   b) If it is raining, then there are clouds in the sky. $p \rightarrow q$ $p \Rightarrow r$

   c) If it is not raining, then the sun is not shining and there are clouds in the sky.

   d) The sun is shining if and only if it is not raining $q \leftrightarrow \sim p$

   e) If there are no clouds in the sky, then the sun is shining. $\sim r \rightarrow q$

2. Construct truth tables for (2 Marks each)

   a) $\neg(p \vee q) \rightarrow r$

   b) $\neg((p \vee q) \rightarrow r)$

   c) $[(p \vee q) \wedge r] \rightarrow (p \wedge \neg q)$

   d) $(p \rightarrow q) \rightarrow [(p \vee \neg q) \rightarrow (p \vee q)]$

   e) $[(p \leftrightarrow q) \vee (p \rightarrow r)] \rightarrow (\neg q \wedge p)$

3. a) Convert the following to First-Order Logic (2 Marks each)

   i. All numbers are bigger than themselves divided by two

   ii. Every even number is the sum of two prime numbers

   iii. There is an antivirus program killing every computer virus

   b) Simplify $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$ (4 Marks)

4. For the following relations on $S = \{0, 1, 2, 3\}$, specify which of the properties (R), (AR), (S), (AS), and (T) the relations satisfy. (2 Marks each)

   a) $(m, n) \in R_1$ if $m + n = 3$

   b) $(m, n) \in R_2$ if $m - n$ is even

   c) $(m, n) \in R_3$ if $m \leq n$

   d) $(m, n) \in R_4$ if $m + n \leq 4$

   e) $(m, n) \in R_5$ if $\max(m, n) = 3$

5. State whether (a) to (e) in (4) above are equivalence relations. (2 Marks each)

$10 = 7 + 3$

$\forall x\, (x > x/2)$

4. (a) Use diagrams to state whether each of the following functions is injective or surjective.

   i. $f(x) = |x - 2|$ (5 marks).

   ii. $g(x) = x^2 - 2$. (5 marks).

   (b) Let $f : A \to B$ be a function where A and B are finite sets and $|A| = |B|$. Prove that f is injective if and only if it is surjective. (5 marks).

   (c) Find the inverse of f(x)= $-\frac{1}{3}$ x + 1 (5 marks)

5. (a) You are required to specify a library system where registered readers can borrow books from the collection. Some of the issues we may need our system to keep track of include; how many books do we have in our collection?, how many registered readers do we have in our system?, and which reader has borrowed which book?. You are also required to impose some restrictions on our system like; no more than maxloan books can be borrowed per reader ; no reader can borrow a book unless he/she is registered. The operations on the system include; issuing a copy to a reader, returning a copy by a reader, adding a copy to the collection, removing a copy from the collection. Using Z-specification Language, come up with all the required schemas for the above scenario (20 marks).

6. (a) Given that set A={1,2,3,4,5} and set B ={4,5,6,7}. Compute the following.

   i. $|\mathbb{P}(A \cup B)|$ (4 marks)

   ii. $A \triangle B$ (3 marks)

   iii. A x B (2 marks)

   iv. $|(A - B) \cup (B - A)|$ (3 marks)

   v. (B x A)∪A (3 marks)

   (b) Using Sets A and B in a) above, prove (or disprove) that $|A \triangle B)| = |A| + |B| - |A \cap B|$ (5 marks).

Good Luck

A structuring mechanism, fragmented? op that allows spec in data & operators that across that

SECTION B:(60 Marks)

2. (a) Prove whether the following sequents are valid or not without using truth tables

   i. $\dfrac{(P \vee Q) \wedge R}{(P \wedge R) \vee (Q \wedge R)}$ (5 marks)

   ii. $\dfrac{(P \wedge Q) \rightarrow R}{[(P \rightarrow Q) \rightarrow P] \rightarrow [(P \rightarrow Q) \rightarrow R]}$ (5 marks)

   $Fa \vee Ga \rightarrow Ha$

   $\forall_x \sim (Gx \wedge Hx)$

   $\sim (Ga \wedge Ha)$

   (b) Construct a formal derivation of the conclusion for each of the following arguments

   i. $\forall x[(Fx \vee Gx) \rightarrow Hx];\ \forall x[Hx \rightarrow (Jx \wedge Kx)]\ /\ Fa \rightarrow Ka$ (5 marks)

   ii. $\forall x(Fx \rightarrow Gx);\ \sim \exists x(Gx \wedge Hx)/\ \forall(Fx \rightarrow \sim Hx)$ (5 marks)

3. (a) Using the suggested abbreviations (the capitalized words), translate each of the following into the language of predicate logic.

   i. Not everyone is PERFECT. (2 marks).

   ii. All HORSES and COWS are FARM animals. (2 marks).

   iii. At least one thing is neither MATERIAL nor SPIRITUAL. (2 marks).

   iv. CATS and DOGS that have RABIES are not SUITABLE pets. (2 marks).

   v. At least one thing is neither MATERIAL nor SPIRITUAL. (2 marks).

   (b) Translate each of the following sentences into first-order logic. The statements describe situations involving Toddlers, Nannies, and Blocks. Billy is a constant referring to a particular toddler. You may only use the following predicates:

   IsSuper(x, y) – x is supervising y

   Toddler(x) – x is a toddler

   Sleeping(x) – x is sleeping

   Nanny(x) – x is a nanny

   Block(x) – x is a block

   FightsWith(x, y, z)– x fights with y over z

   i. Nobody who is supervising can be sleeping or fighting. (2 marks).

   ii. Some nanny is supervising (at least) two toddlers. (2 marks).

   iii. Billy fights with another toddler over a block. (2 marks).

   iv. Every toddler who is sleeping is being supervised by some nanny or it is the case that all nannies are sleeping. (2 marks).

   v. The only sleeping nanny is the one supervising Billy. (2 marks).

SECTION B *(60 Marks)*

Question 1

a) There are 200 students in a school. Out of these, 100 students play baseball, 50 students play hockey, and 60 students play basketball. 30 students play both baseball and hockey, 35 students play both hockey and basketball, and 45 students play both basketball and baseball. Compute the following: H = 50 B = 100 U = 200

(i) The maximum number of students who play at least one game (2 Marks)
(ii) The maximum number of students who play all the 3 games (3 Marks)
(iii) The minimum number of students playing at least in one game (3 Marks)
(iv) The minimum number of students playing all the 3 games (2 Marks)

b) Let U = {x∈Z | 1 <= x <= 50

A = {x∈U | x is divisible by 2}, B = {x∈U | x is divisible by 5}
A∩B = {x∈U | x is divisible by 10}
A∪B = {x∈U | x is divisible by 2 or is divisible by 5 (or both)}
A – B = {x∈U | x is divisible by 2 but is not divisible by 5}

Compute: (i) |A|, (ii) |B|, (iii) |A∩B|, (iv) |A∪B|, (v) |A – B|. (2 Marks @)

Question 2

How many passwords satisfy the following requirements? (12 Marks)

- between 6 & 8 characters long
- starts with a letter
- case sensitive
- other characters: digits or letters

~~Determine the subsets of size k of an n-element set~~

b) In how many ways can we select three students from a group of five students to stand in line for a picture? (4 Marks)

c) In how many ways can we arrange all five of these students in a line for a picture? (4 Marks)

Question 3

a) Let p and q be the propositions

p : I bought a lottery ticket this week.
q : I won the million dollar jackpot.

Express each of these propositions as an English sentence. (8 Marks)

i) $\neg p$	ii) $p \vee q$	iii) $p \rightarrow q$
iv) $p \wedge q$	v) $p \leftrightarrow q$	vi) $\neg p \rightarrow \neg q$
vii) $\neg p \wedge \neg q$	viii) $\neg p \vee (p \wedge q)$	

b) Determine whether each of these conditional statements is true or false. (2 Marks)

i) If 1 + 1 = 2, then 2 + 2 = 5. true
ii) If 1 + 1 = 3, then 2 + 2 = 4. false
iii) If 1 + 1 = 3, then 2 + 2 = 5. true
iv) If monkeys can fly, then 1 + 1 = 3.

F T F
F T F
T F F
T → F

c) Translate the following English statements into logical expressions (5 Marks)

i) You can access the Internet from campus only if you are a computer science major or you are not a freshman

ii) You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

d) Determine whether these system specifications are consistent; (5 Marks)

"The diagnostic message is stored in the buffer or it is retransmitted."

"The diagnostic message is not stored in the buffer."

"If the diagnostic message is stored in the buffer, then it is retransmitted."

Question 4

a) Using an appropriate technique, prove the theorem $\sqrt{2}$ is irrational. (8 Marks)

b) Show that if n is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. (12 Marks)

Question 5

a) Briefly illustrate and discuss the Seven Bridges of Königsberg problem is relation to graph theory and the Eularian solution (7 Marks)

b) Visualize each of the following graphs, and state whether or not it is simple. (3 Marks)

(i) G1 = (V1, E1), where V1 = {a, b, c, d, e} and E1 = {ab, bc, ac, ad, de}.

(ii) G2 = (V2, E2), where V2 = {P, Q, R, S, T} and E2 = {PQ, PR, PS, PT, TR, PR}.

(iii) G3 = (V3, E3), where V3 = {v1, v2, v3, v4, v5} and E3 = {v1v1, v1v2, v2v3, v3v4, v5v4, v4v5}.

c) Determine whether or not each of the following pairs of graphs are isomorphic or not: (10 Marks)

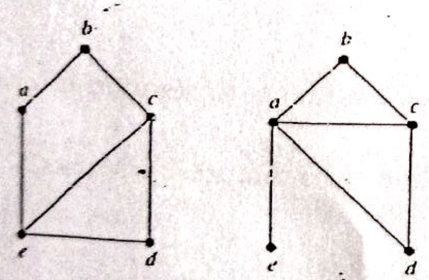


Figure 1

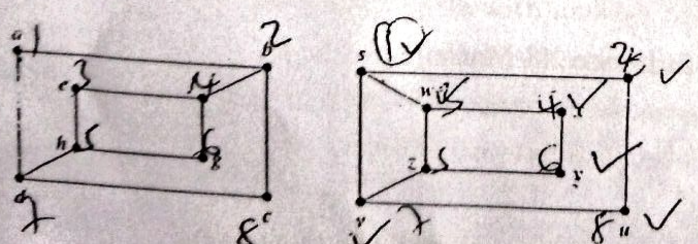


Figure 2

=END=

Question 1

a. Express these system specifications using the propositions: [2 Each]

p "The user enters a valid password",

q "Access is granted" and

r "The user has paid the subscription fee"

and logical connectives (including negations). $p \wedge \neg p$

i. "The user has paid the subscription fee, but does not enter a valid password."

ii. "Access is granted whenever the user has paid the subscription fee and enters a valid password." $r \wedge p \to q$

iii. "Access is denied if the user has not paid the subscription fee." $\neg r \to \neg q$

iv. "If the user has not entered a valid password but has paid the subscription fee, then access is granted." $\neg p \wedge r \to q$

b. Express the following statements using predicate logic [4 Each]

i. There is a smallest positive integer $\exists x\, P(x)$

ii. There is no smallest positive real number $\neg \exists x\, P(x)$

c. Differentiate between a fallacy and a tautology. Give an example in each case [4]

$p \wedge \neg p$ $\quad p \vee \neg p$

Question 2

a. Find the truth set of each of these predicates where the domain is the set of integers. [2 Each]

i. $P(x): x^2 < 3$

ii. $Q(x): x^2 > x$

iii. $R(x): 2x + 1 = 0$

b. Use algebraic proof techniques to determine whether the following expression holds [6]

$$\overline{((A \cup B) \cap (B \cup C))} = (\bar{A} \cup \bar{B}) \cup \bar{C}$$

c. Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 3$.

i. Is f an onto function? [2]

ii. Justify your answer in (i) above. [2]

d. How can we produce the terms of a sequence if the first 10 terms are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123? Use your understanding of recurrence relations to give your solution [4]

Question 3

a. Using an appropriate technique, prove the theorem $\sqrt{10}$ is irrational. [6]

b. Using induction principle, prove that

i. $\forall n \geq 0, \quad 1 + r + \ldots + r^n = \dfrac{r^{n+1} - 1}{r - 1}$, where $r \neq 1$ [7]

ii. $\forall n \geq 0, \quad \sum_{i=1}^{n} i(i-1) = \left(\dfrac{n(n-1)(n+1)}{3}\right)$ [7]

Question 4

a. Show that $m^{13} - m$ is divisible by 13. [4]

b. Prove that for any integer n, $n^{6k} - 1$ is divisible by 7 if $\gcd(n, 7) = 1$ and k is a positive integer. [4]

c. Solve the following congruencies, or give "unsolvable" if it has no solution. [4 Each]

i. $10x \equiv 35 \pmod{42}$

ii. $49x \equiv 98 \pmod{21}$

iii. $14x \equiv 7 \pmod{28}$

Question 5

a. There are 7 glasses on a table, all standing upside down. You are allowed to turn over any 4 of them in one move. Is it possible to have all the glasses right-side-up? Give reasons to support your answer. [5]

b. Two Pairs is a hand with two cards of one value, two cards of another value, and one card of a third value. How many different hands contain Two Pairs from a deck of 52 cards? [5]

c. A particular random number generator is able to give a bunch of random integers from 0 to 65535. How many numbers do you need from this generator to guarantee that there exists two subsets of integers that sum to the same number? Show your steps. [5]

d. Count the number of ways to place four pawns on a 10x10 chessboard so that no two pawns share a row or a column. [5]

=END=

4620 101, 1+2+...+n

101
75

4620 101 · 45 + 75

101 75×1 + 26

75 26·2 + 23

26 23·1 + 3

23 3·7 + 2

7 2·3 + 1

2 2·1 + 0

SET theory exercise 2

2 If U = {letters of the alphabet}, A = {a, a, a, b, b, a, c}, B = {c, b, a, c} and C = {a, b, c}, what can be said about A, B and C?

3 U = {natural numbers}; A = {2, 4, 6, 8, 10}; B = {1, 3, 6, 7, 8}
State whether each of the following is true or false:
(a) A ⊂ U
(b) B ⊆ A
(c) Ø ⊂ U

4 U = {a, b, c, d, e, f, g, h}; P = {c, f}; Q = {a, c, d, e, f, h}; R = {c, d, h}
(a) Draw a Venn diagram, showing these sets with all the elements entered into the appropriate regions. If necessary, redraw the diagram to eliminate any empty regions.
(b) Which of sets P, Q and R are proper subsets of others? Write your answer(s) using the ⊂ symbol.
(c) P and R are disjoint sets. True or False?

5 Sketch Venn diagrams that show the universal set, U, the sets A and B, and a single element x in each of the following cases:
(a) $x \in A$; $A \subset B$
(b) $x \in A$; A and B are disjoint
(c) $x \in A$; $x \notin B$; $B \subset A$
(d) $x \in A$; $x \in B$; A is not a subset of B; B is not a subset of A

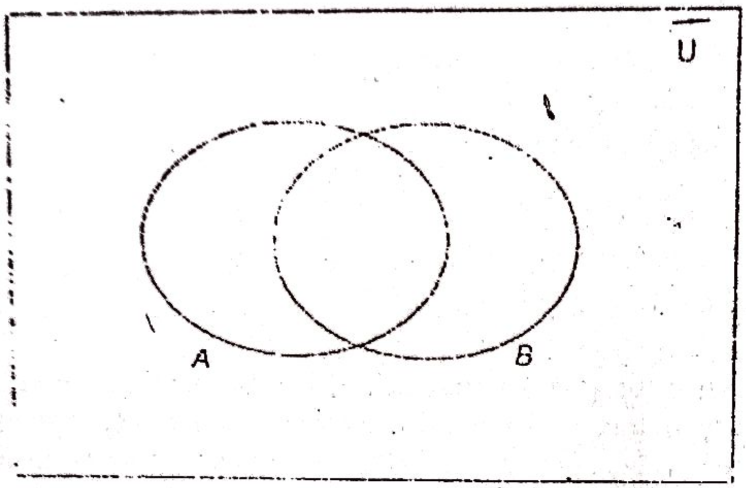
Set Theory Exercise 3

1 U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
A = {2, 4, 6, 8, 10}
B = {1, 3, 6, 7, 8}
C = {3, 7}
(a) Illustrate the sets U, A, B and C in a Venn diagram, marking all the elements in the appropriate places. (Note: if any region in your diagram does not contain any elements, re-draw the set loops to correct this.)
(b) *Using your Venn diagram*, list the elements in each of the following sets:
$A \cap B$, $A \cup C$, A', B', $B \cap A'$, $B \cap C'$, $A - B$, $A \Delta B$
(c) Complete the statement using a single symbol: $C - B$ = ... ϕ

2 True or false?
(a) |ø| = 1 F
(b) | { x, x } | = 2 F
(c) | U ∩ ø | = 0 T

3 What can you say about two sets P and Q if:
(a) $P \cap Q' = \emptyset$
(b) $P \cup Q = P$?

$A \cap B = \{6, 8\}$
$A \cup C = \{2, 3, 4, 6, 7, 8, 10\}$



Question 4

1 Make six copies of the Venn diagram shown alongside, and then shade the areas represented
for (a) $A' \cup B$

(b) $A \cap B'$

(c) $(A \cap B)'$

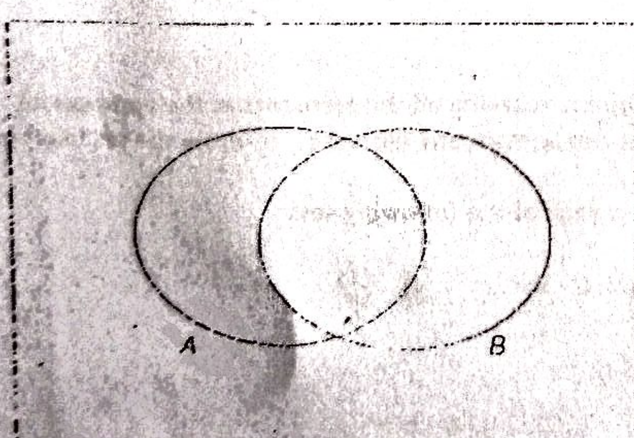
(d) $A' \cup B'$

(e) $(A \cup B)'$

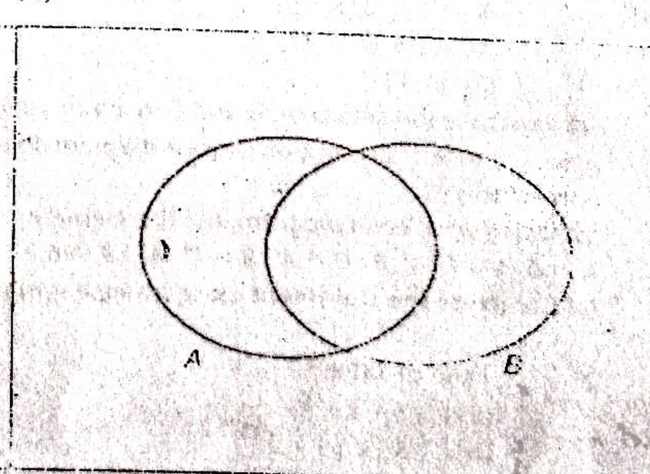
(f) $A' \cap B'$

2 Identify the sets represented by each of the shaded areas below, using the set notation symbols ∩, ∪ and '.

(a)



(b)



(c)

(d)

23
21

$3 = ~~1\cdot 2 + 1~~$

$1 = 3 - 1\cdot 2$ → $1 = 3 - 1(23 - 7\times 3)$

$(23 - 7\cdot 3) = 2$