MTH 3105: Discrete Mathematics Take Home Assignment I

1.

(a) Define the sequence $c_0, c_1, \ldots$ by the equations $c_0 = 0$ and $c_n = c_{\lfloor n/2 \rfloor} + 3$ for all $n > 0$. Prove that $c_n \le 2n$ for all $n \ge 3$. (5 Marks)

First of all, $c_3 = c_4 = c_5 = 6$, so the hypothesis is true for n = 3, 4 and 5. Now assume the hypothesis is true for n = 3, 4, …, k, where $k \ge 5$. Consider n = k + 1:

$$\begin{aligned} c_{k+1} &= c_{\lfloor (k+1)/2 \rfloor} + 3 \\ &\le 2\lfloor (k+1)/2 \rfloor + 3 && \text{by assumption} \\ &\le (k+1) + 3 \\ &\le 2(k+1) && \text{since } k \ge 5 \end{aligned}$$

Hence by induction the hypothesis is true for all $n \ge 3$.

(b) Prove that any positive integer N is divisible by 11 if and only if the difference between the sum of odd digits and the sum of even digits is divisible by 11. (5 Marks)

First, note that $10 \equiv -1\ (mod\ 11)$, and $100 \equiv 1\ (mod\ 11)$.
We observe that $10^{2k+1} \equiv -1\ (mod\ 11)$ and $10^{2k} \equiv 1\ (mod\ 11)$.
Let $N = \sum_{i=0}^{k} 10^i d_i$ so that k is the largest integer satisfying $10^k \le N$.
So $N \equiv \sum_{i=0}^{k} 10^i d_i\ (mod\ 11)$.
We set $d_{k+1} = 0$ if k is even.
It follows that we can write the previous modulo equation as

$$N \equiv \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (d_{2i} - d_{2i+1})\ (mod\ 11)$$

The right-hand side shows the difference of sum of odd digits and even digits and the proof completes.

2.

(a) Assume that $\forall x \exists y P(x, y)$ is false and the domain of them is nonempty. Which of the following must be false? (2 Marks)

(i)	$\forall x \forall y P(x, y)$	*False*
(ii)	$\exists x \forall y P(x, y)$	*False*
(iii)	$\exists x \exists y P(x, y)$	*True*

(b) Let $P(x)$ denote the statement "x is an accountant" and let $Q(x)$ denote the statement "x owns a Porsche". Write each statement below in first order logic.

(i)	All accountants own Porsches.	$\forall x(P(x) \longrightarrow Q(x))$	(2 Marks)
(ii)	Some accountant owns a Porsche.	$\exists x(P(x) \wedge Q(x))$	(2 Marks)
(iii)	All owners of Porsches are accountants.	$\forall x(Q(x) \longrightarrow P(x))$	(2 Marks)
(iv)	Someone who owns a Porsche is an accountant.	$\exists x(P(x) \wedge Q(x))$	(2 Marks)

3.

(a) Prove that $\sqrt{6}$ is irrational. (4 Marks)

Suppose $\sqrt{6}$ is rational, so we can write $\sqrt{6} = \frac{m}{n}$ so that m and n do not have common factor other than 1. So, we get $6 = \frac{m^2}{n^2}$

$6n^2 = m^2$

Since left hand side is even, m must be even, put n = 2k. Then

6n2 = $(2k)^2$

$3n^2 = 2k^2$

Now right-hand side is even, n must also be even. This contradict the assumption that n and m do not have common factor other than 1. Hence $\sqrt{6}$ is irrational.

(b) A detective has interviewed four witnesses to a crime. From their stories, the detective has concluded that:

(i) If the butler is telling the truth, then so is the cook.

(ii) The cook and the gardener cannot both be telling the truth.

(iii) The gardener and the handyman are not both lying.

(iv) If the handyman is telling the truth then the cook is lying.

Who must be lying? There may be more than one liar. Show your steps. (6 Marks)

Let B, C, G and H be the proposition that the butler, cook, gardener and handyman is telling the truth respectively. Then from the problem description we get

i. $B \longrightarrow C$

ii. $\neg (C \wedge G)$

iii. $G \vee H$

iv. $H \longrightarrow \neg C$

*Now suppose C is true, then by (iv) we get H is false, then by (iii) we get G is true. By (ii) we get C is false, which is a contradiction. As a result, C must be false. Furthermore, B must be false by (i). Here we cannot proceed any further, so we know **the cook and butler must be lying**.*

4.

(a) Simplify the following statement: (2 Marks)

$\neg (\neg q \wedge \neg (\neg q \vee s)) \vee (q \wedge (r \rightarrow r))$

$\equiv \neg (\neg q \wedge \neg (\neg q \vee s)) \vee q$ *identity laws*

$\equiv (q \vee (\neg q \vee s)) \vee q$

$\equiv (q \vee \neg q) \vee s \vee q$

$\equiv$ true

(b) Determine whether or not the following arguments are valid:

(i) $\dfrac{\neg p \rightarrow q, \neg q}{p}$ (4 Marks)

(ii) $\dfrac{\neg p \rightarrow \neg q}{p \rightarrow q}$ (4 Marks)

p	q	$\neg p$
F	F	T
F	T	T
T	F	F
T	T	F

$\neg p \longrightarrow q$	$\neg q$	p
F	T	F
T	F	F
T	T	T
T	F	T

Valid

$\neg q$	$\neg p \longrightarrow \neg q$	$p \longrightarrow q$
T	T	T
F	F	T
T	T	F
F	T	T

Invalid