ES 666

Computer Vision

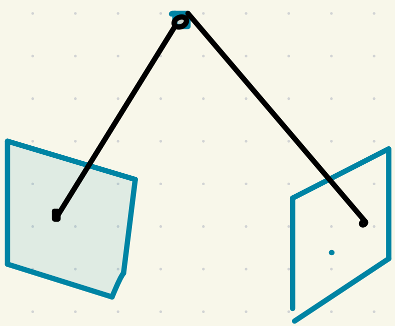
History of Computer Vision

Before 1970s → Blob/ world
Block

1980s → BKP Horn • Photometric Stereo

• Depth from Images
Shape from X

• Optical Flow   X → shading/ stereo



1990s → Multi-view stereo
→ Structure from Motion

1980's and 1990's

→ Projective Geometry

→ Zisserman, Hartley

1980's to 2000's

→ High level vision

- Recognition
- classification
- Detection

Engineered features

2010's to Present

Low, Mid, High] $\longrightarrow$ Learned Features

Traditional ML

Feature Engineering

SVM | DT | RF | MKL | LR

Image → Feature Extraction → ML → Prediction

Detection + Description

SIFT | SURF | ORB | LBP | KAZE

Deep Learning

Image → Deep Network → Prediction

AI

1. Basics of Math
    - linear algebra
    - Optimization
2. Classical Computer Vision
3. Learning Based Vision
4. Applications

Image Processing

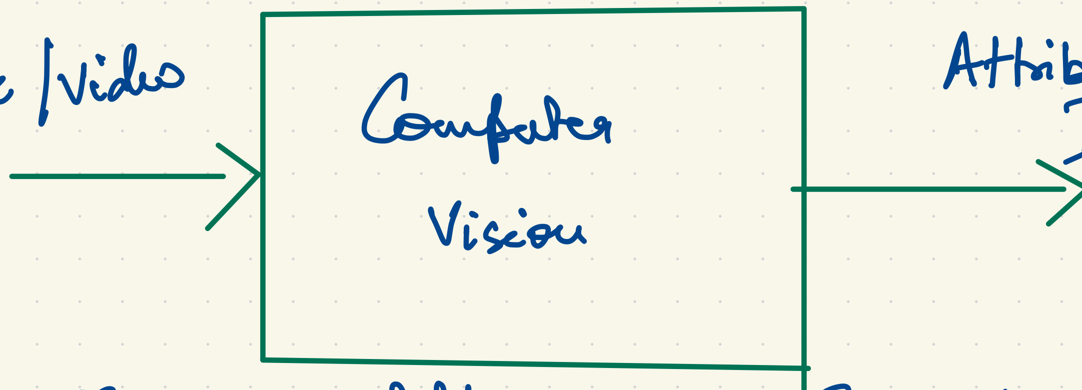
eg. Denoising
Image Enhancement
Cropping | Resizing
Colorization

Video Compression
Image Compression
Binary Image Processing

Computer Vision

Mimic
Human eye/brain

Image/Video

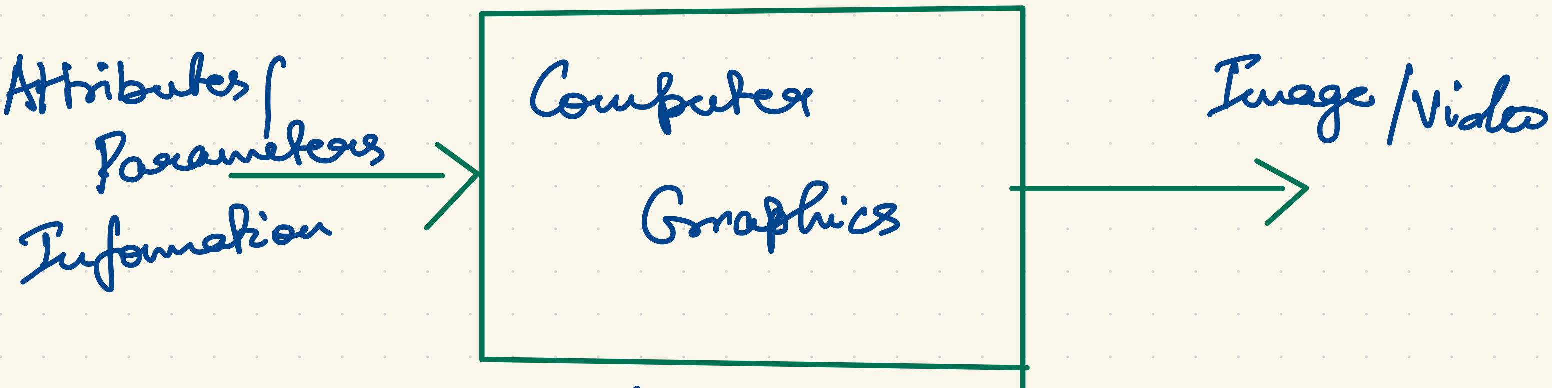


Attribute /
Information

eg.

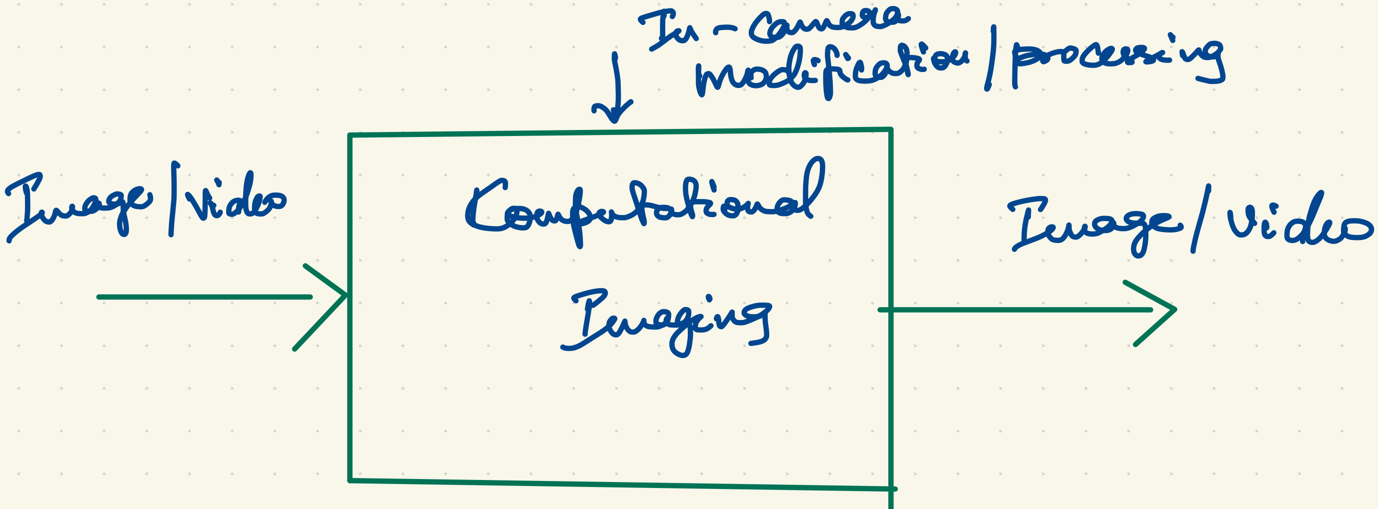
- Segmentation
- Detection (Object)
- Classification
- Counting of crowd
- Pose Estimation
- Tracking
- Depth Estimation (Normal)
- Summarization
- Captioning

Computer Graphics



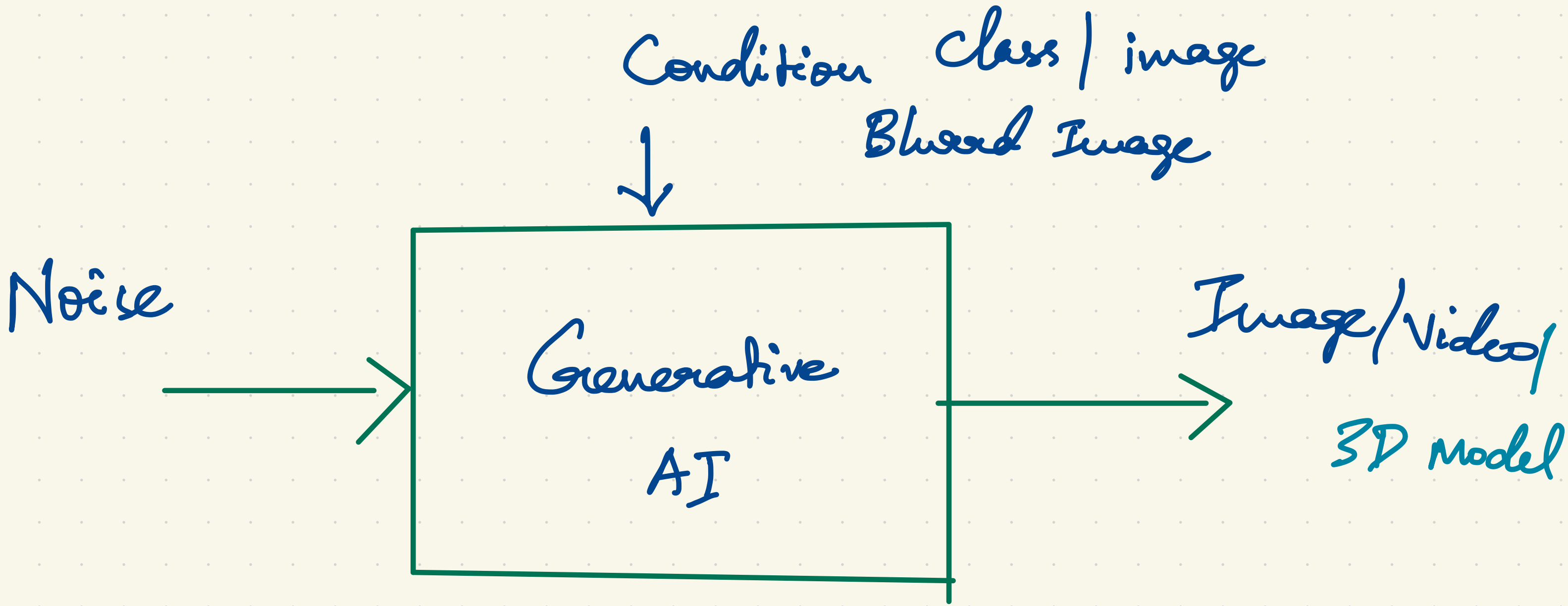
eg. Rendering
Animation
Visual Effects
AR/VR

Computational Photography / Imaging



eg. Long Exposure Photography
HDR imaging
Slow-Mo video
Coded Aperture Imaging

Generative AI



eg. GAN

Diffusion

Image

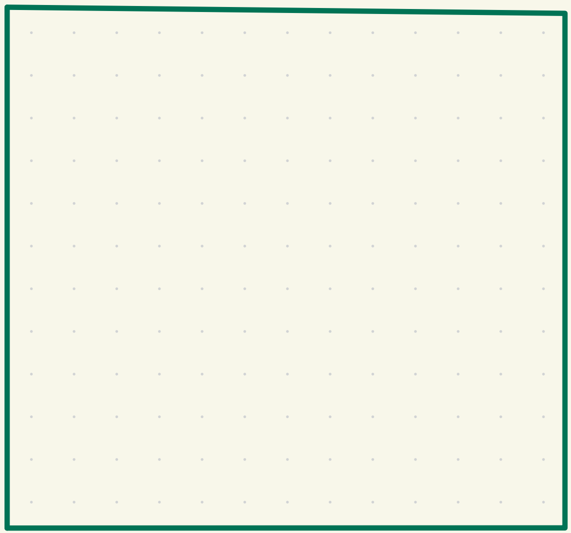
(8-bit) $\rightarrow [0 \text{ to } \underbrace{2^8 - 1}_{255}]$

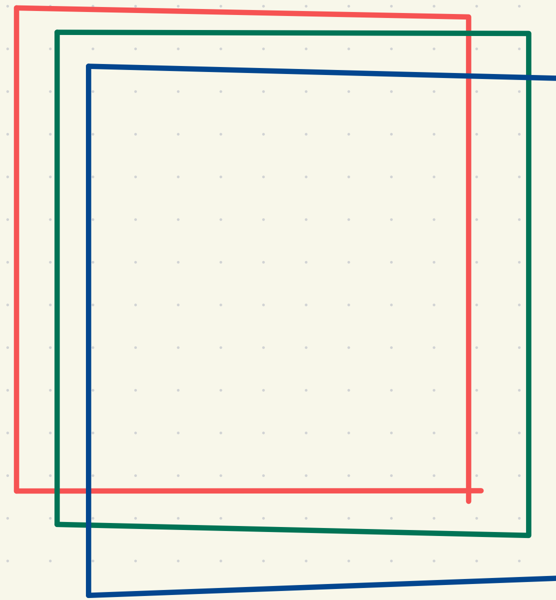
Gray Scale I

Color I_c

$j \rightarrow$

$i \downarrow$





$M \times N$ $\quad I(i,j) \in [0, 255]$

$i \in \{1, 2, \ldots, M\}$

$j \in \{1, 2, \ldots, N\}$

locations (pixels)

$M \times N \times 3$ $\quad I_c(i,j,k) \in [0, 255]$

locations (pixels)

$k \in \{1, 2, 3\}$

Gray Scale Image

$I(i,j)$ $M \times N$

$g(i,j)$ $K \times K$

filter

weights

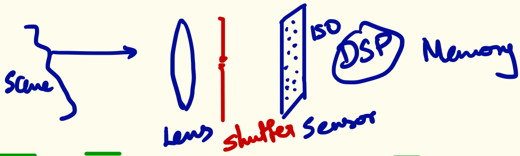
- fixed
- learned

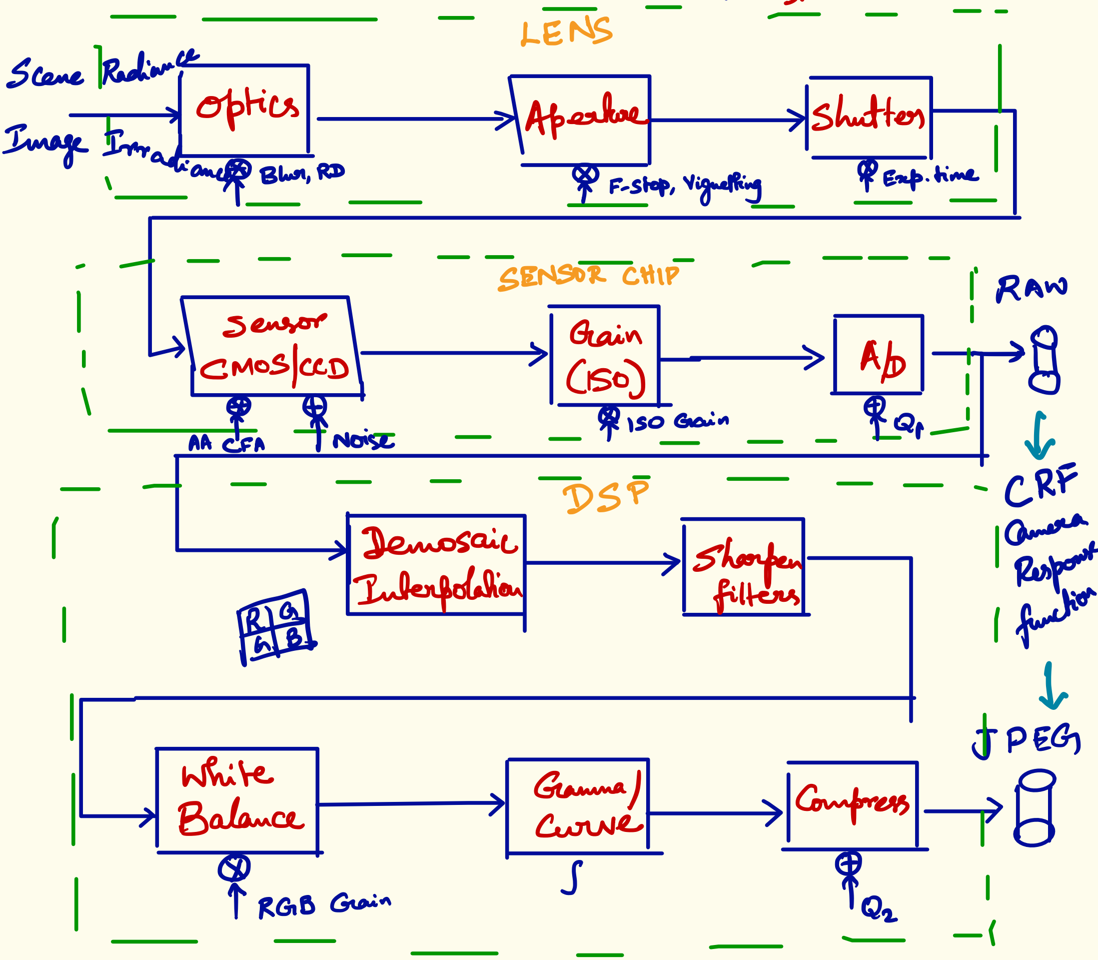
Why Computer Vision is hard?

- AI
- Variation in Data (Images)
  - Illumination change
  - Camera settings (exposure)
  - Viewpoint change
  - Appearance change (position)
  - Scale change
  - Occlusion
  - Background clutter
  - Motion
  - Sensing Noise
  - Resolution
  - Depth Perception (Defocus Blur)
- Huge variation in same class appearance (Diversity of Universe)
- Limitations of active sensors
  - Range, Power, Cost (kinect, LiDAR)
- Inverse Problem.

GIAN IITGN
Marco Callieri
MeshLab

Image Sensing Pipeline





⊛ → Convolution

⊕ → Addition

⊗ → Multiplication

Q_1, Q_2 → Quantization Noise

RD → Radial Distortion

$$DR = 20\log_{10}\left(\frac{V_{max}}{V_{min}}\right)$$

System of Linear Equations

$$\underset{M\times N}{A}\ \underset{N\times 1}{\vec{x}} = \underset{M\times 1}{\vec{b}}$$

$\times$

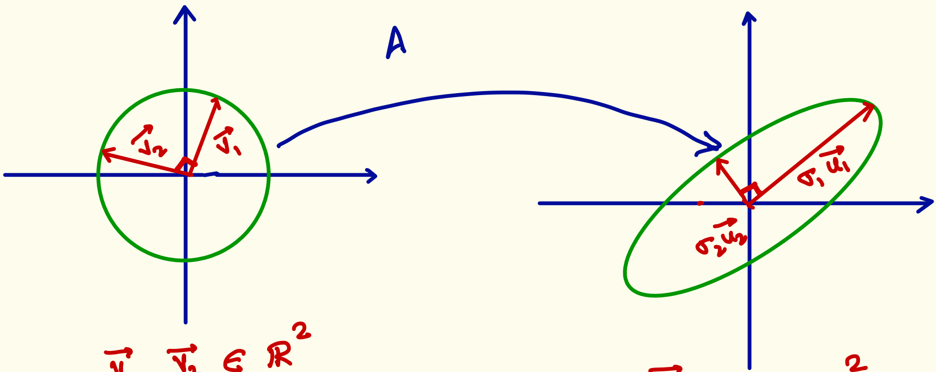
$$\underset{M\times N}{\begin{bmatrix} \rule{3cm}{0.4pt} \\ \rule{3cm}{0.4pt} \\ \rule{3cm}{0.4pt} \\ \rule{3cm}{0.4pt} \\ \vdots \\ \rule{3cm}{0.4pt} \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

Linear Algebra

Singular Value Decomposition (SVD)

Numerical Linear Algebra
– Trefethen and Bau

Consider $A_{2\times 2}$



$\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$

$\|\vec{v}_1\|_2 = \|\vec{v}_2\|_2 = 1$

$\vec{v}_1^{\,T}\vec{v}_2 = 0$

$\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$

$\|\vec{u}_1\|_2 = \|\vec{u}_2\|_2 = 1$

$\vec{u}_1^{\,T}\vec{u}_2 = 0$

$$A\vec{v}_1 = \sigma_1\vec{u}_1$$

$$A\vec{v}_2 = \sigma_2\vec{u}_2$$

$$A\underbrace{\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma}$$

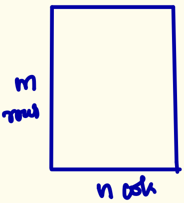
$$A = U\Sigma V^T$$

$U, V \rightarrow$ Orthogonal Matrices

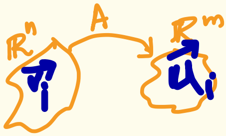
$V^TV = VV^T = I$

$U^TU = UU^T = I$

$A \in \mathbb{R}^{m\times n}$, $m \geqslant n$ $\quad$ $\operatorname{rank}(A) \leqslant \min(m,n)$



$$\underset{m\times n}{A}\;\underset{n\times 1}{\vec{v}_i} = \sigma_i\, \underset{m\times 1}{\vec{u}_i}\,, \quad i = 1,2,\dots,n$$



$\vec{v}_i \in \mathbb{R}^n$

$\vec{u}_i \in \mathbb{R}^m$

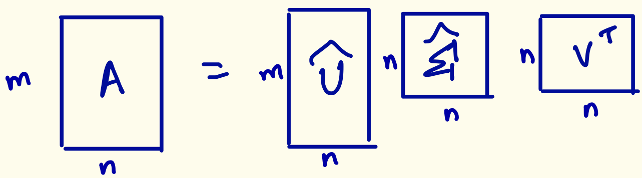
$$\left.\begin{matrix}\vec{v}_i^{\,T}\vec{v}_j \\ \vec{u}_i^{\,T}\vec{u}_j\end{matrix}\right\} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j\end{cases}$$

$$\underset{m\times n}{A}\underbrace{\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}}_{V}{}_{n\times n} = \underbrace{\begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n\end{bmatrix}}_{\hat{U}}{}_{m\times n}\underbrace{\begin{bmatrix}\sigma_1 & & 0 \\ & \sigma_2 & \\ & & \ddots \\ 0 & & & \sigma_n\end{bmatrix}}_{\hat{\Sigma}}{}_{n\times n}$$

$V^T V = V V^T = I$ $\qquad$ $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$

$$A \underbrace{V V^T} = \hat{U}\hat{\Sigma} V^T$$

$$\underset{m\times n}{A} = \underset{m\times n}{\hat{U}}\;\underset{n\times n}{\hat{\Sigma}}\;\underset{n\times n}{V^T}$$ Reduced SVD

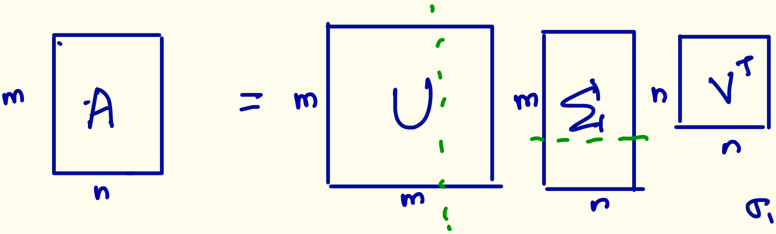


$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V^T_{n \times n}$$ Full SVD

adding $(m-n)$ orth. cols. $\quad$ $(m-n)$ zero rows.

$U^T U = U U^T = I$ $\quad$ cols of U span $\mathbb{R}^m$

$V^T V = V V^T = I$ $\quad$ rows of V^T span $\mathbb{R}^n$



Full SVD

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

SVD exists for all matrices.

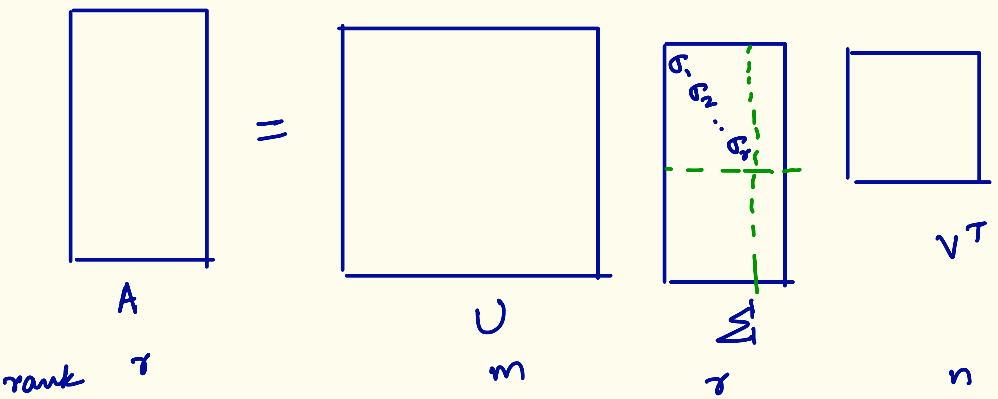
$\sigma_1, \sigma_2, \ldots, \sigma_n \rightarrow$ Singular values of A. real $\in \mathbb{R}$

$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m \rightarrow$ Left singular vectors of A. $\in \mathbb{R}^m$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \rightarrow$ Right singular vectors of A. $\in \mathbb{R}^n$

$\text{rank}(A) = \#$ of Non-zero singular values of A.

$A = X \Lambda X^{-1}$



$AA^T, A^TA \to$ Symmetric, positive semi-definite $\vec{x}^T C \vec{x} \geq 0$

$$AA^T_{m\times m} = (U\Sigma V^T)(U\Sigma V^T)^T = V\Sigma \underbrace{V^T V}_{I} \Sigma^T U^T$$

$$= U\Sigma^2 U^T$$ Eigenvalue decomposition

$$A^TA_{n\times n} = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T \underbrace{U^T U}_{I} \Sigma V^T$$

$$= V\Sigma^2 V^T$$ Eigenvalue decomposition

$\sigma_1, \sigma_2, \ldots, \sigma_n \to$ +ve Square roots of eigenvalues of A^TA or AA^T

$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m \to$ eigenvectors of AA^T cols of U

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \to$ eigenvectors of A^TA cols of V

$A \rightarrow \mathbb{R}^{n \times n}$

$$A \vec{x}_i = \lambda_i \vec{x}_i \qquad i = 1, 2, \ldots, n$$

$$A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$A \quad X = X \Lambda$$

$$A = X \Lambda X^{-1}$$ eigenvalue decomposition

$$A = U \Sigma V^T$$

- Trefethen & Bau
- Numerical Linear Algebra

Matrix Properties through SVD

$A \in \mathbb{R}^{m \times n}$, $m \geq n$ Proofs →

1. $\mathrm{rank}(A) = r$ (Number of non-zero singular values).

2. $\mathrm{Range}(A) = \mathrm{colspace}(A) = \langle \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r \rangle$ $\quad r < n$

$\mathrm{Nullspace}(A) = \langle \vec{v}_{r+1}, \ldots, \vec{v}_n \rangle$

$A\vec{x} = \vec{0}$ $\qquad A \vec{v}_i = \sigma_i \vec{u}_i$

rowspace(A) =

nullspace(A^T) =

$\dim(\mathrm{Range}(A)) = r$

$\dim(\mathrm{Null}(A)) = n - r$

3. Norms of A

$$\|A\|_2 = \sigma_1 \qquad \|A\|_2 = \sup_{\vec{x}} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} \quad \text{2-Norm (induced norm)}$$

$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \dots + \sigma_r^2} \quad \text{Frobenius Norm}$$

$$= \sqrt{\sum_i \sum_j a_{ij}^2}$$

4. Non-zero singular values of A are the +ve square roots of the non-zero eigenvalues of A^TA or AA^T.

5. If $A = A^T$ (symmetric), singular values of A are the absolute values of eigenvalues of A. $\sigma_i = |\lambda_i|$

A

6. For any $A \in \mathbb{R}^{m \times m}$ (square matrix)

$$|\det(A)| = \prod_{i=1}^{m} \sigma_i$$

If $\operatorname{rank}(A) = r < m$ as $\sigma_{r+1}, \dots, \sigma_m = 0$

$$|\det(A)| = 0$$

7. $A \in \mathbb{R}^{m \times n}$, $m \geq n$, $\text{rank}(A) = r$

$$B = \vec{u}\,\vec{v}^{T}$$
$$m \times n \quad n \times 1 \quad 1 \times n$$

$$A = \sum_{j=1}^{r} \sigma_j \vec{u_j} \vec{v_j}^{T}$$

sum of r $\left\{ \begin{array}{l} \text{rank-1} \\ \text{matrices} \end{array} \right.$ scaled outer products of $\vec{u_j}$ and $\vec{v_j}^{T}$

$$A = \begin{bmatrix} | & | & & \\ \vec{u_1} & \vec{u_2} & \cdots & \vec{u_m} \\ | & | & & \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ 0 & & & & 0 \end{bmatrix} \begin{bmatrix} - & \vec{v_1}^{T} & - \\ - & \vec{v_2}^{T} & - \\ & \vdots & \\ & \vec{v_n}^{T} & \end{bmatrix}$$

8. For any γ, with $0 \leq \gamma \leq r$ $\quad A \to$ rank

$$A_\gamma = \sum_{j=1}^{\gamma} \sigma_j \vec{u_j} \vec{v_j}^{T}$$

Summing only top γ singular values

Then $$\|A - A_\gamma\|_2 = \sigma_{\gamma+1}$$

A_γ is the closest matrix with rank γ to A

$$\|A - A_\gamma\|_F = \sqrt{\sigma_{\gamma+1}^2 + \cdots + \sigma_r^2}$$

Low Rank Approximation of A.

Given a matrix A with rank r, to find the closest matrix A_q to A with rank q, set the least $r-q$ singular values to zero.

Solve Over-determined System of Equations

$$A\vec{x} = \vec{b}$$

Given $A, \vec{b}$

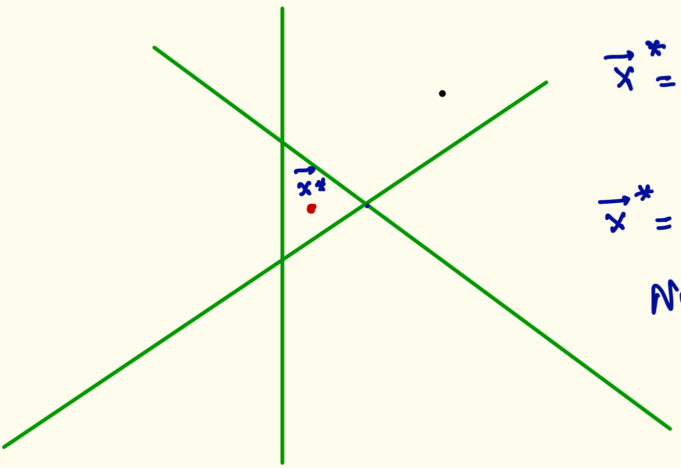
find $\vec{x}$

$A \in \mathbb{R}^{m\times n}$ $\quad m > n$ (More equations than unknowns)

$\vec{x} \in \mathbb{R}^n$ $\quad$ Equations $\rightarrow$ rows of A

$\vec{b} \in \mathbb{R}^m$ $\quad$ Unknowns $\rightarrow$ $\vec{x}$

Lines in $\mathbb{R}^2$



$$\vec{x}^* = \underset{\vec{x}}{\operatorname{argmin}} \|A\vec{x}-\vec{b}\|_2^2$$

$$\frac{d}{d\vec{x}} = 0$$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

Normal equation

Using SVD,

$$\underset{m\times n}{A} = \underset{m\times n}{\hat{U}}\ \underset{n\times n}{\hat{\Sigma}}\ \underset{n\times n}{V^T} \qquad \text{Reduced SVD}$$

$$A\vec{x} = \vec{b} \qquad A, \vec{b} \to \text{known}$$

$$\hat{U}\hat{\Sigma}V^T\vec{x} = \vec{b}$$

$$\underbrace{\hat{U}^T\hat{U}}\hat{\Sigma}V^T\vec{x} = \hat{U}^T\vec{b}$$

$$\hat{\Sigma}\underbrace{V^T\vec{x}}_{\vec{y}} = \vec{b'} \qquad , \vec{b'} = \hat{U}^T\vec{b}$$

$$V^T\vec{x} = \vec{y} \implies \vec{x} = V\vec{y}$$

1. $\vec{b'} = \hat{U}^T\vec{b}$

2. Solve $\hat{\Sigma}\vec{y} = \vec{b'}$

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ & & & \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ \vdots \end{bmatrix}$$

3. Solve $\vec{x}^* = V\vec{y}$

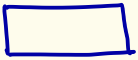
$\vec{x}^*$ is the optimal solution $\to$ least squares solution.

Least Squares Minimization

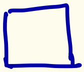
Inhomogeneous System $\vec{b} \neq \vec{0}$

$$A\vec{x} = \vec{b} \qquad A \in \mathbb{R}^{m \times n},\ \vec{x} \in \mathbb{R}^{n},\ \vec{b} \in \mathbb{R}^{m}$$

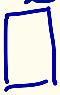
Cases

a) $m < n$ A  fat   Under determined

- More unknowns than equations
- No unique solution
- Vector space of solutions (infinite)

b) $m = n$ A  square   Properly Determined

- Unique solution if A is invertible or A has full rank.
  (Equal number of equations and unknowns)
- Otherwise a) if A is rank deficient.

c) $m > n$ A  tall   Over Determined

- More equations than unknowns
- Unique solution if $\vec{b}$ lies in colspace of A
- No solution otherwise.

Consider $m \geq n$ $\qquad\qquad$ $\vec{a}^T\vec{a} = \|\vec{a}\|_2^2$

$\text{rank}(A) = n$

Find $\vec{x}$ closest to the system $A\vec{x} = \vec{b}$

Least squares

$$\vec{x}^* = \underset{\vec{x}}{\text{argmin}} \; \|A\vec{x} - \vec{b}\|_2^2$$

1×1 $\qquad \underset{1\times m \; m\times n \; n\times 1}{\vec{b}^T A \vec{x}}$ $\qquad \underset{1\times n \; n\times m \; m\times 1}{\vec{x}^T A^T \vec{b}}$ $\qquad 1\times1$

$$\|A\vec{x} - \vec{b}\|_2^2 = (A\vec{x} - \vec{b})^T (A\vec{x} - \vec{b})$$

$$= (\vec{x}^T A^T - \vec{b}^T)(A\vec{x} - \vec{b})$$

$-\vec{b}^T A \vec{x}$ $\qquad -\vec{x}^T A^T \vec{b}$

$$= \vec{x}^T A^T A \vec{x} - 2\vec{x}^T A^T \vec{b} + \vec{b}^T \vec{b}$$

$$\frac{\partial(\;)}{\partial \vec{x}} = 2A^TA\vec{x}^* - 2A^T\vec{b} = 0$$

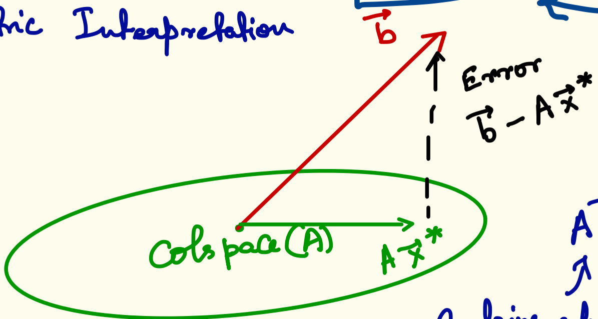
$$A^TA\vec{x}^* = A^T\vec{b}$$

$\vec{x}^* = A^{-1}\vec{b}$ if $A \to n\times n$, $\text{rank}(A) = n$

$$\vec{x}^* = \underbrace{(A^TA)^{-1}A^T}_{\text{Pseudo inverse}}\vec{b}$$

Normal Equation

Geometric Interpretation



$$A^T(\vec{b} - A\vec{x}^*) = \vec{0}$$

Contains cols of A as rows $\qquad \vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

Affine ~~Linear~~ Regression

$$y = w_1 x_1 + w_2 x_2 + w_3 x_3 + \beta$$

$$y = \vec{w}^T \vec{x} + \beta \qquad \underline{y} \in \mathbb{R}$$

$$\begin{bmatrix} x_1^1 & x_2^1 & x_3^1 & 1 \\ \vdots & & & \vdots \\ x_1^5 & x_2^5 & x_3^5 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \beta \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \\ y^5 \end{bmatrix}$$

$$A \qquad \vec{w} = \vec{Y}$$

$$\arg\min_{\vec{x}} \|A\vec{x} - \vec{b}\|_2 = \arg\min_{\vec{x}} \|U\Sigma V^T \vec{x} - \vec{b}\|_2$$

$U^TU = I$

$$= \arg\min_{\vec{x}} \|\Sigma V^T\vec{x} - U^T\vec{b}\|_2 \qquad \vec{b}' = U^T\vec{b}$$

$$\underset{m\times 1}{\vec{b}'} = \underset{m\times m}{U^T}\ \underset{m\times 1}{\vec{b}}$$

$$\underset{n\times 1}{\vec{y}} = \underset{n\times n}{V^T}\ \underset{n\times 1}{\vec{x}}$$

$$\iff \arg\min_{\vec{y}} \|\Sigma\vec{y} - \vec{b}'\|_2$$

$\Sigma_{m\times n} \longrightarrow$ diagonal matrix

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \\ & & & \end{bmatrix}_{m\times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} \approx \begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_n' \\ \vdots \\ b_m' \end{bmatrix}_{m\times 1}$$

$$y_1 = \frac{b_1'}{d_1}$$

$$y_i = \frac{b_i'}{d_i} \qquad i = 1, 2, \ldots, n$$

$\Sigma\vec{y}$ can approach to $\vec{b}'$

$= (b_1', b_2', \ldots, b_n', 0, 0 \ldots 0)$

$$\vec{x}^* = V\vec{y}$$

Reduced SVD

$$\| \underset{n\times n}{\hat{\Sigma}_1} \underset{n\times 1}{\vec{y}} - \underset{n\times 1}{\vec{b}} \|$$

$$\underset{n\times 1}{\vec{y}} = \underset{n\times n}{V^T} \underset{n\times 1}{\vec{x}}$$

$$\underset{n\times 1}{\vec{b}'} = \underset{n\times m}{\hat{U}^T} \underset{m\times 1}{\vec{b}}$$

$$\vec{x}^* = V \vec{y}$$

$A\vec{x}^* \neq \vec{b}$
if $\vec{b}$ is not in colspace(A)

$A\vec{x} = \vec{b}$ rank(A) = n

1. Find SVD $A = U \Sigma_1 V^T$

2. Set $\vec{b}' = U^T \vec{b}$

3. solve $\Sigma_1 \vec{y} = \vec{b}'$ $y_i = \frac{b'_i}{d_i}, i = 1, 2, \ldots, n$

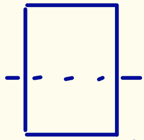
4. $\vec{x}^* = V\vec{y}$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$\text{rank}(A) = r < n$ (rank deficient systems $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$)

$m \geq n$

1. Find SVD of $A = U \Sigma_1 V^T$



$m\times n$

2. $\vec{b}' = U^T \vec{b}$

3. Find $\vec{y}$ s.t. $y_i = \dfrac{b_i'}{d_i}$ $\quad i = 1, 2, \dots, r$

$y_i = 0$ $\quad$ o/w

4. General solution (infinite)

$$\vec{x} = V\vec{y} + \lambda_{r+1}\vec{v}_{r+1} + \dots + \lambda_n \vec{v}_n .$$

5. $\vec{x}$ with minimum norm $\|\vec{x}\|_2$

$$\vec{x}^* = V\vec{y}$$

Solution of Homogeneous System $A\vec{x} = \vec{0}$

$$A\vec{x} = \vec{0}$$

$A \in \mathbb{R}^{m\times n}$, $\vec{x} \in \mathbb{R}^n$ $\qquad A = U\Sigma V^T$ rows of V^T

A $m = n$, a) $\vec{x}$ is in nullspace (A). $<\vec{v}_{r+1}, \ldots, \vec{v}_n>$ if exists $(\text{rank}(A) < n)$ b) $\vec{x} = \vec{0}$ if $\text{rank}(A) = n$

B Over-determined

$m > n$ $\qquad A_{m\times n}\vec{x}_{n\times 1} = \vec{0}_{m\times 1}$

a) if $\text{rank}(A) < n$, $\vec{x}$ is in nullspace (A)

$m > n$ b) $\text{rank}(A) = n$, $A\vec{x} = \vec{0}$ has only trivial solution $\vec{x} = \vec{0}$

For cases b)

Solve $\vec{x}^* = \underset{\vec{x}}{\arg\min} \|A\vec{x}\|_2$, subject to $\|\vec{x}\|_2 = 1$.

$$A = U\Sigma V^T \qquad \|U\|_2 = \|V\|_2 = 1$$

$$\min \|A\vec{x}\| = \min \|U\Sigma V^T\vec{x}\|_2 \text{ s.t. } \|\vec{x}\|_2 = 1$$

$$= \min \|\Sigma V^T\vec{x}\|_2 \text{ s.t. } \|\vec{x}\|_2 = 1$$

Let $V^T\vec{x} = \vec{y}$ $\qquad \|\vec{x}\|_2 = 1 \Rightarrow \|V\vec{y}\|_2 = 1$

$\vec{x} = V\vec{y}$ $\qquad \Rightarrow \|\vec{y}\|_2 = 1$

$$\min \|\Sigma \vec{y}\|_2 \quad s.t \quad \|V\vec{y}\|_2 = 1$$

$$= \min \|\underset{m\times n}{\Sigma}\ \underset{n\times 1}{\vec{y}}\|_2 \quad s.t. \quad \|\vec{y}\|_2 = 1$$

$$\vec{y}^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \begin{array}{l} d_1, d_2, \ldots, d_{n-1} \longrightarrow 0 \\ d_n \checkmark \end{array}$$

$$\vec{x}^* = V\vec{y}^* = \vec{v}_n \quad (\text{last column of } V)$$

$$\vec{x}^* = \underset{\vec{x}}{\operatorname{argmin}} \|A\vec{x}\|_2 \quad s.t. \quad \|\vec{x}\|_2 = 1$$

$$\vec{x}^* = \vec{v}_n \quad (\text{last column of } V)$$