Image Compression Using Truncated SVD

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I. Introduction

Singular Value Decomposition (SVD) is a fundamental mathematical technique used to break down a matrix into three simpler matrices.It expresses any matrix A as a product of three matrices in which two are orthogonal matrices and a diagonal matrix consisting singular values as diagonal entries.

II. Summary of Strang's video

A. Key points in Strang video

- Any matrix can be decomposed into three different matrices
  - Orthogonal matrix of left singular vectors
  - Diagonal matrix
  - Orthogonal matrix of right singular vectors
- Suppose **A** be a matrix of order mxn, then **A** can be writtten as

$$\mathbf{A} = \mathbf{U\Sigma V^T} \tag{0.1}$$

  where U and V are orthogonal matrices and Σ is a diagonal matrix.

- This decomposition of a matrix is called Singular Value Decomposition.
- Consider a square symmetric matrix B, then B can be written as

$$\mathbf{B} = \mathbf{Q}\lambda\mathbf{Q^T} \tag{0.2}$$

  where Q is a matrix containing eigen vectors of B and λ is a diagonal matrix containing eigen values as diagonal entries.

- This is called Eigenvalue Decomposition. This is the basis for the proof of SVD.

B. Proof of SVD

Consider a orthogonal matrix V in row space and a orthogonal matrix U consisting of unit vectors in column space

$$\mathbf{V} = \left(v_1, v_2, ..., v_n\right) \tag{0.3}$$

$$\mathbf{U} = \left(u_1, u_2, ..., u_m\right) \tag{0.4}$$

Consider a matrix A such that, u_i can be written interms of v_i

$$\sigma_i\mathbf{u_i} = \mathbf{Av_i} \tag{0.5}$$

$$\mathbf{U}\Sigma = \mathbf{AV} \tag{0.6}$$

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V^{-1}} \tag{0.7}$$

Since $\mathbf{U^TU} = \mathbf{I}$ and $\mathbf{V^TV} = \mathbf{I}$

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V^T} \tag{0.8}$$

C. Calculating U and V

Consider

$$\mathbf{A^T A = V\Sigma^{\mathit{T}} U^T U \Sigma V^T} \tag{0.9}$$

$$\mathbf{A^T A = V\Sigma^T \Sigma V^T} \tag{0.10}$$

From this we can say that V is a matrix of eigen vectors of matrix $A^T A$, and similarly U is a matrix of eigen vectors of matrix AA^T and the diagonal matrix is given by,

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{pmatrix} \tag{0.11}$$

For a 4x4 matrix, where

$$\sigma_i = \sqrt{\lambda_i} \tag{0.12}$$

where λ_i are the eigen values of matrix $A^T A$.

III. Explanation of the algorithm

The given c code compresses an image using the SVD technique. It uses the Power Iteration method to find the largest singular values and their corresponding singular vectors.

- The input image is read using the STB image library and converted into a matrix A, where each element represents a pixel intensity (0-255).
- A symmetric matrix $A^T A$ is calculated and used for finding right singular vectors of A.
- A random vector v is created as the starting point for the iteration.
- Some random vector v is assumed and it is repeatedly multiplied with $A^T A$ and normalized by the algorithm:

  $$v = \frac{A^T(Av)}{||A^T(Av)||} \tag{0.13}$$

  After several repetitions by approximation, v becomes the right singular vector corresponding to the largest singular value.
- Once v is found, the left singular vector is calculated as:

  $$u = \frac{Av}{||Av||} \tag{0.14}$$

- The singular value is given by:

  $$\sigma = ||Av|| \tag{0.15}$$

- Using the values of u, v, and σ, a rank-1 matrix is formed:

  $$A_1 = \sigma u v^T \tag{0.16}$$

- Subtracting this rank-1 approximation from the original matrix to remove its effect:

  $$A = A - A_1 \tag{0.17}$$

  This step allows finding the next largest singular value and vector.
- The above steps are repeated k times to form a rank-k approximation:

  $$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \tag{0.18}$$

- The reconstructed matrix A_k is converted back to pixel form and saved as a new compressed image.