Center for Spatially Integrated Social Science

Kriging

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Outline

- Principles
- Kriging Models
- Spatial Interpolation

Principles

Spatial Prediction

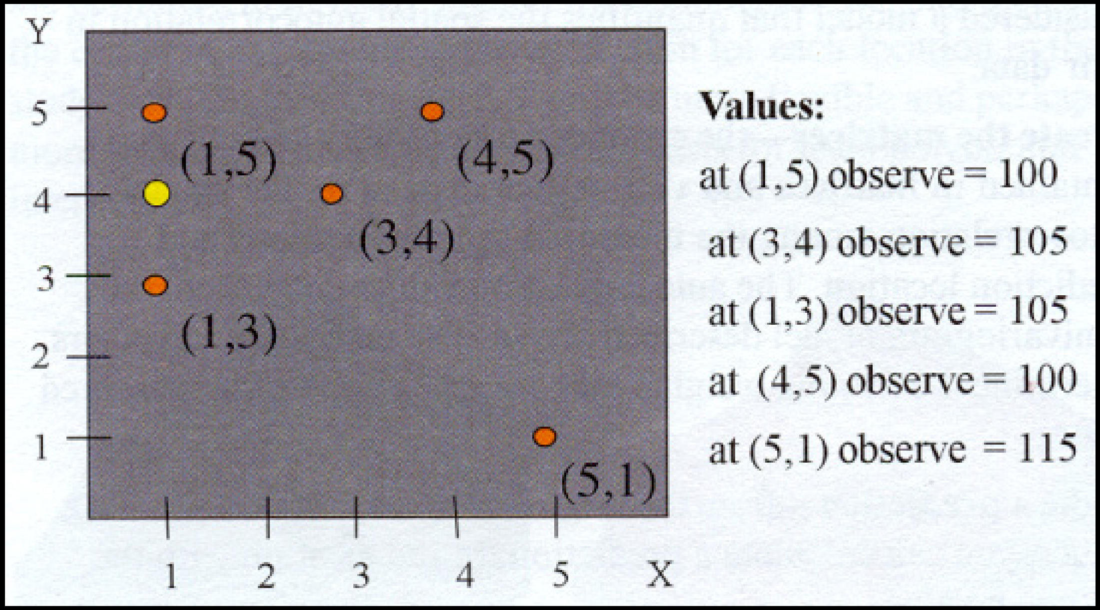
- Model of Spatial Variability
  - large scale trend + small scale autocorrelation
  - $Z(s) = \mu(s) + \varepsilon(s)$
- Predictor
  - model for large scale trend for unknown locations
  - use spatial structure in residuals to improve on prediction

Kriging

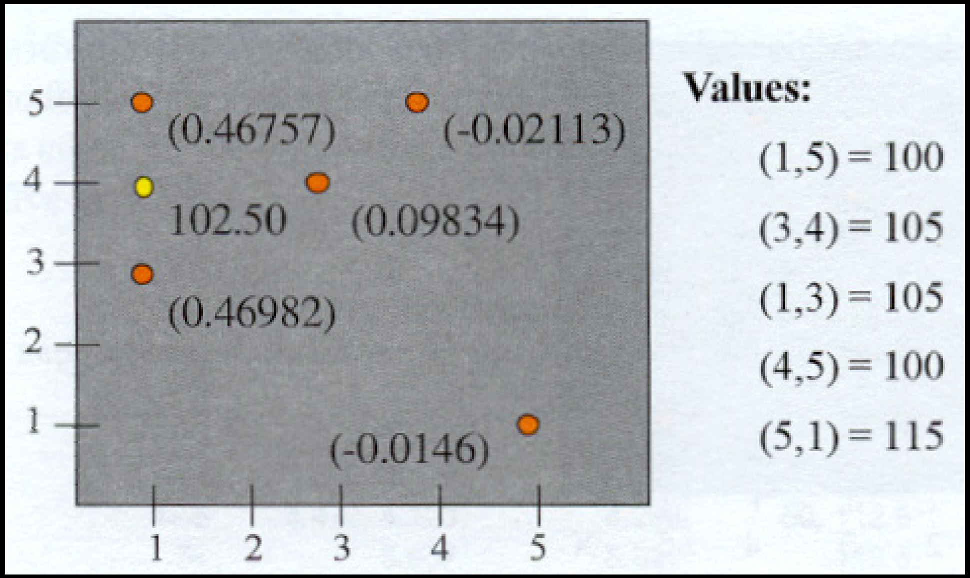
- Principle
  - obtain best linear unbiased predictor, BLUP
  - take into account covariance structure as a function of distance
- Best Predictor
  - unbiased: $E[y^p - y] = 0$ or no systematic error
  - minimum variance among the linear unbiased
  - some nonlinear predictors could be better

Use of Covariance

- Covariance a Function of Distance
  - predict for new location s on the basis of distance between pairs
    - covariance between new and observed
      - uses distance between s and all s_i
    - covariance between observed
      - uses distance between and all s_i, s_j
  - $y^p(s) = \Sigma_i \lambda_i(s)y(s_i)$
    - linear predictor in y
    - weights λ must be obtained



Observed Values at s_i



Predicted Values for s_0

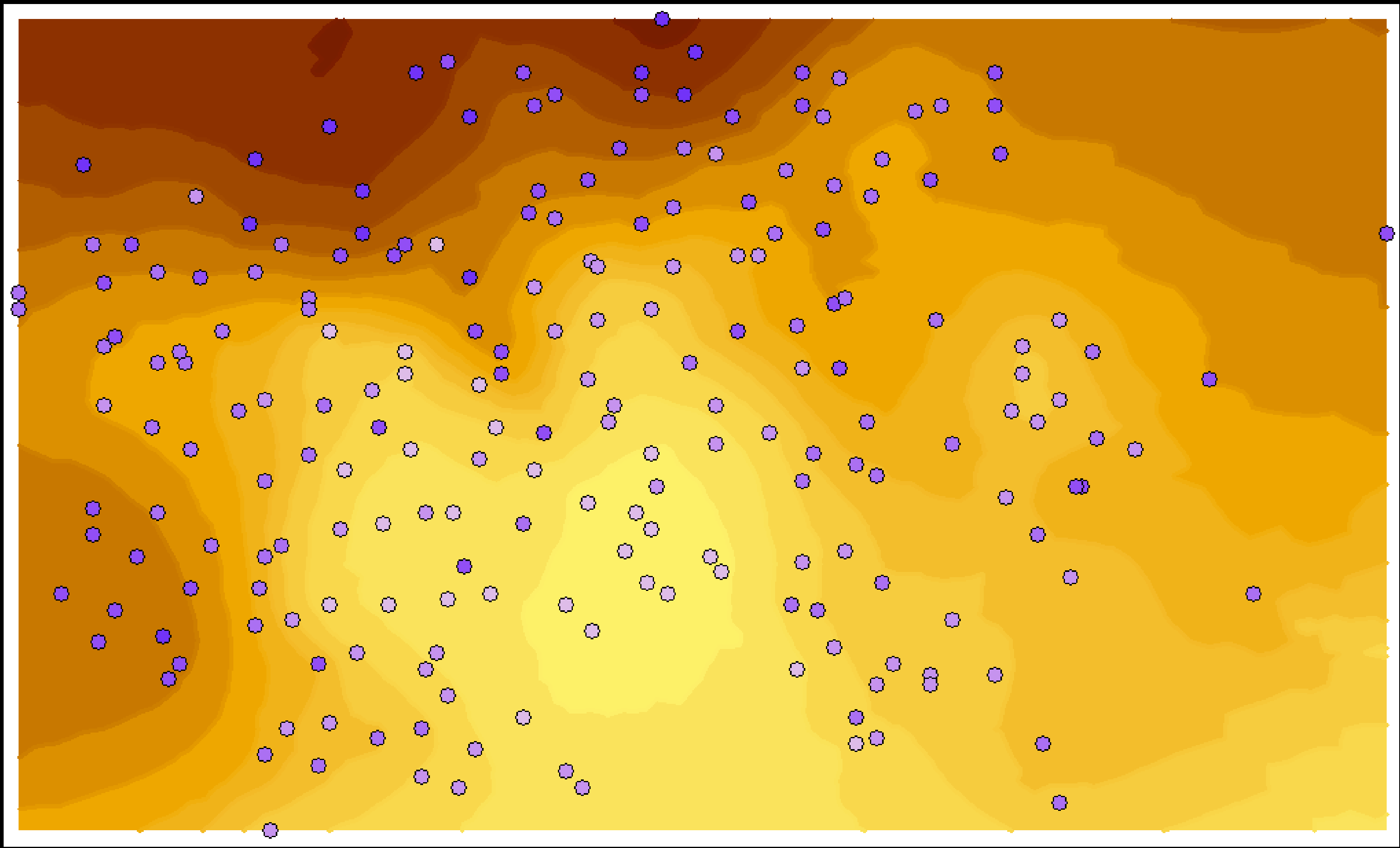
Kriging as a Linear Interpolator (Source: ESRI 2001)

Kriging Weights

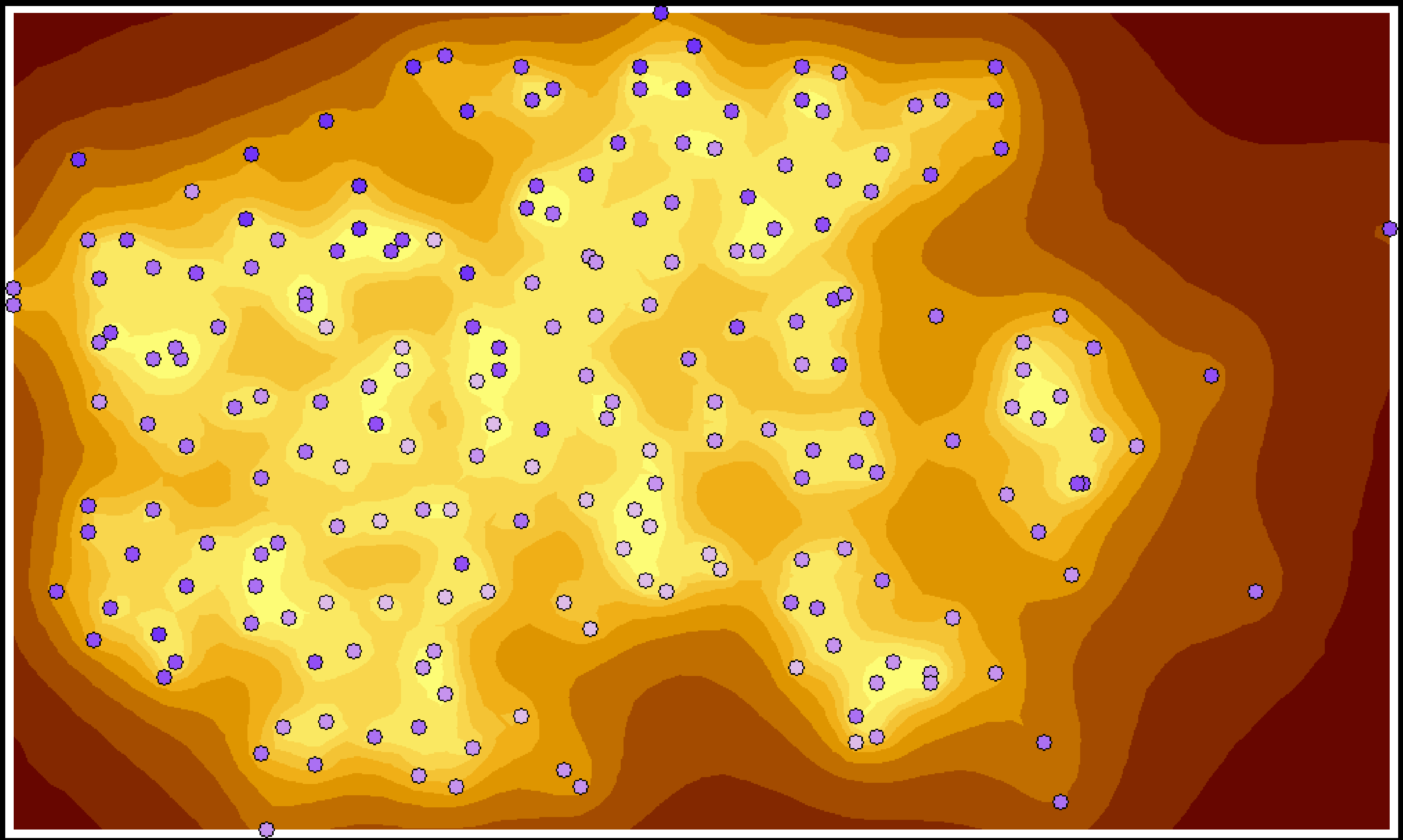
- Optimal Weights
  - as the solution of an optimization process
  - unbiased and min mean squared error
- Simple Kriging (ignore mean)
  - $\lambda(s) = C^{-1}c(s)$
    - C is covariance matrix for all i, j
    - in practice, use a moving window (dimensionality)
    - c(s) is covariance between s and s_i as a function of distance between s and s_i from variogram model

Kriging Predictor

- Predicted Value
  - similarity with least squares solution
  - $y^p(s) = c^T(s)C^{-1}y$
    - with c, y as vectors, C matrix
- Kriging Variance
  - uncertainty of interpolated value
  - $\sigma^2{}_p = \sigma^2 - c^T(s)C^{-1}c(s)$
    - σ^2 is variance of process C(h=0)
- Practical Considerations
  - account for uncertainty in estimation of C
  - remove trend (estimate)



Predicted Value Map



Standard Errors of Spatial Interpolation

Kriging Models

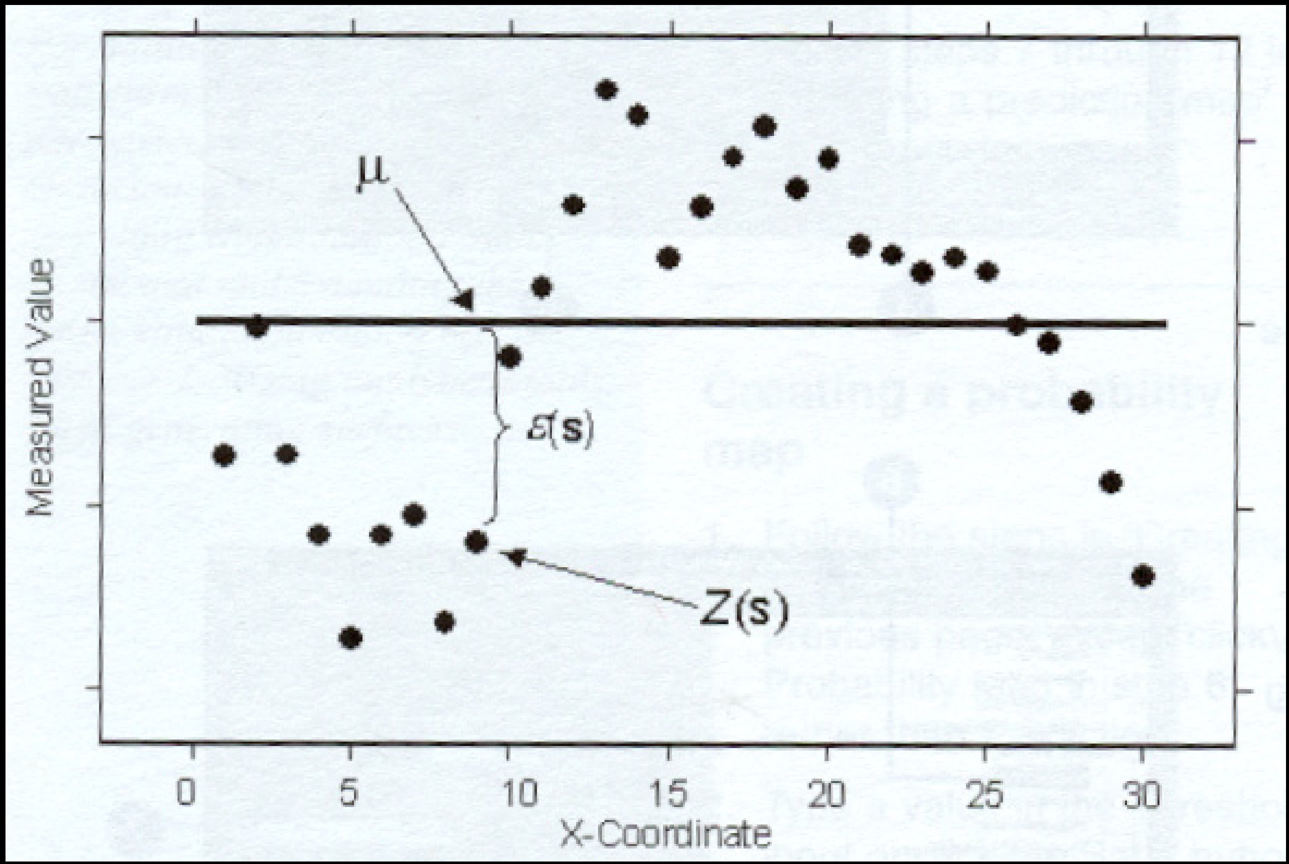
Kriging Models

- Classification
  - different formulations for Z(s) and μ(s)
- simple kriging, ordinary kriging, universal kriging
  - mean μ(s) known, constant or variable
- disjunctive kriging, indicator kriging, probability kriging
  - transformations of Z(s)
    - to model threshold effects
- block kriging
    - areal aggregate

Formal Kriging Models

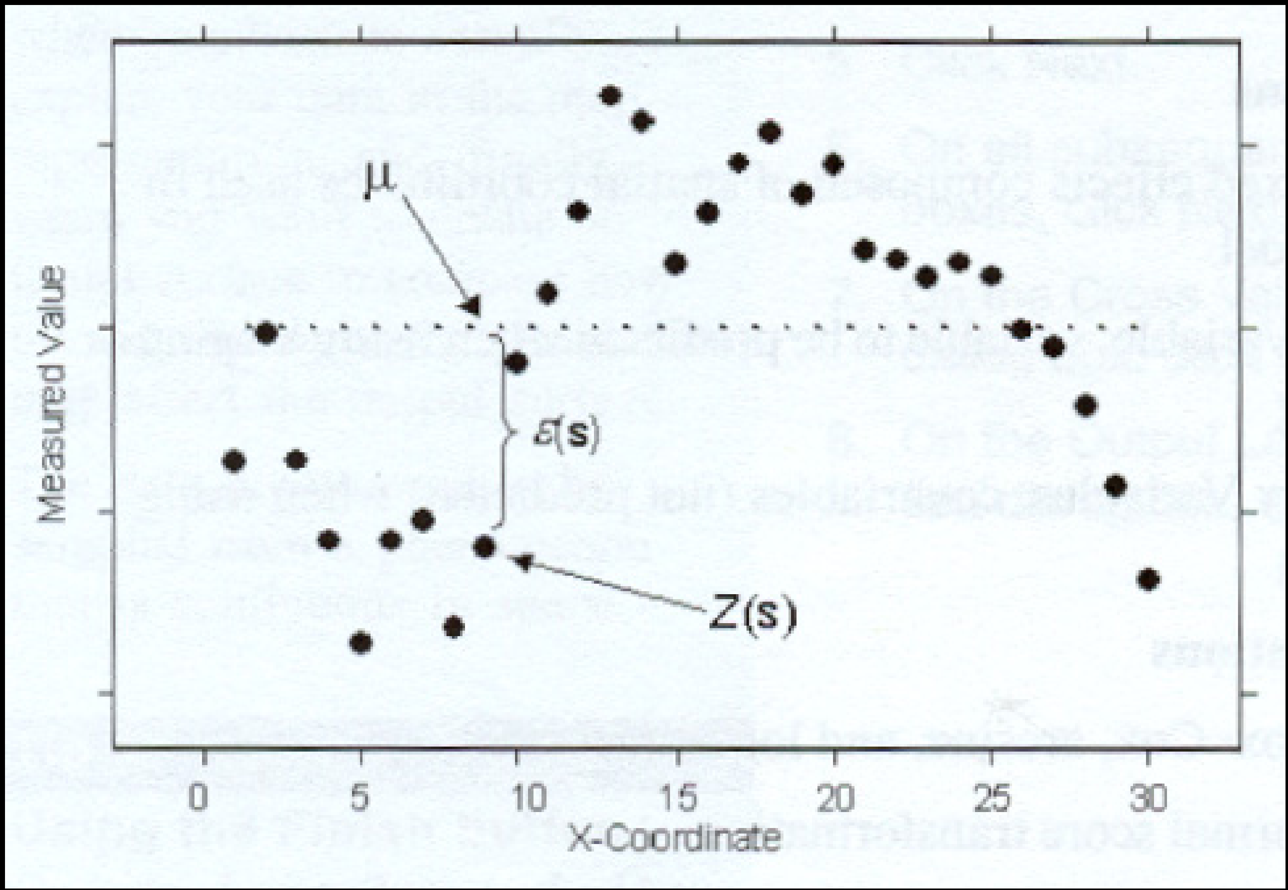
- Simple Kriging
  - $Z(s) = \mu + e(s)$
  - μ known and fixed (no estimation)
- Ordinary Kriging
  - $Z(s) = \mu + e(s)$
  - μ fixed but not known (requires estimation)
- Universal Kriging
  - $Z(s) = \mu(s) + e(s)$
  - μ varies: trend surface, regression model
  - requires estimation, variogram on residuals

Simple Kriging



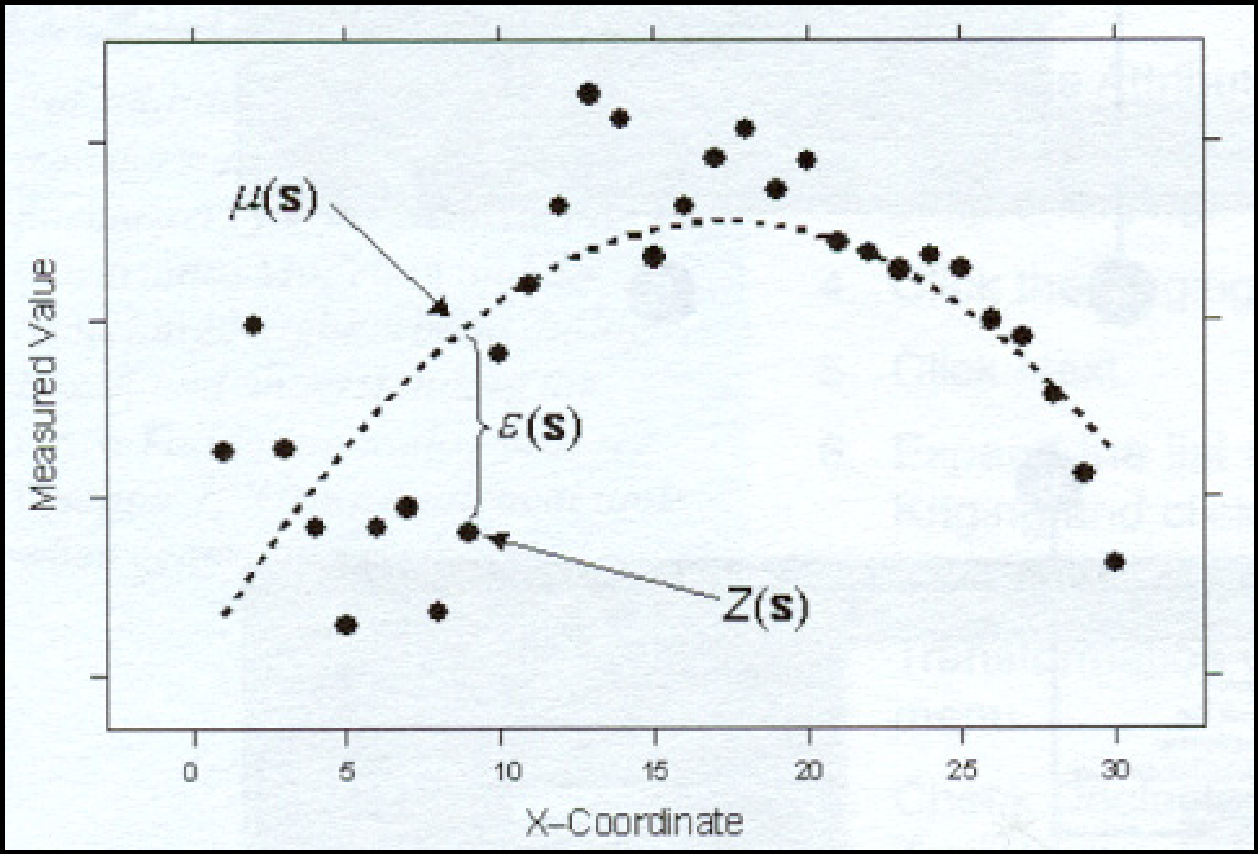
Source: ESRI (2001)

Ordinary Kriging



Source: ESRI (2001)

Universal Kriging



Source: ESRI (2001)

Spatial Interpolation by Kriging
An Example

Spatial Interpolation

- Consider 3 Baltimore Locations
  - #67 x=908.5 y=565.0 r=-4.44 p=53.5
  - #69 x=907.5 y=563.0 r=-3.82 p=53.0
  - #65 x=910.0 y=562.0 r=-8.14 p=48.0
- Predict for x=909 y=564
  - trend surface prediction (mean)
  - p = -166.01 - 0.148 (909) + 0.634 (564) = 57.024

Step 1

- Compute distances between sample points and between sample and prediction point
  - $D(s_i, s_j) = \begin{matrix} 0 & 2.236 & 3.354 \\ & 0 & 2.693 \\ & & 0 \end{matrix}$
  - $D(s_0, s_i) = 1.118 \quad 1.803 \quad 2.236$

Step 2

- Compute covariogram values
  - $C(h) = C(0) - \gamma(h)$
  - using exponential variogram
    $\gamma(h) = C(0)[\ 1 - e^{-3h/a}\ ]$ st. $C(h) = C(0).e^{-3h/a}$
    - with $C(0) = 440$ and $a=28.2$
    - note: $C(0)$ cancels out in $c'C^{-1}$
  - $C^*(s_i, s_j) = \begin{matrix} 1 & 0.788 & 0.700 \\ & 1 & 0.751 \\ & & 1 \end{matrix}$
  - $c^*(s_0,s_i) = 0.888 \quad 0.825 \quad 0.788$

Step 3

- Compute kriging weights
  - $\lambda = c.C^{-1}$
  - $\lambda = [\ 0.888 \quad 0.825 \quad 0.788\ ]$

$$\times \begin{matrix} 2.743 & -1.793 & -0.491 \\ & 3.466 & -1.401 \\ & & 2.381 \end{matrix}$$

  - $\lambda = 0.569\ 0.163\ 0.284$
    - (some rounding errors, sum is ~ 1.01)

Step 4

- Kriging Predictor
  - error predictor
  - e = 0.57x(-4.44) + 0.16x(-3.82) + 0.28x(-8.14) = -5.46
- Spatial Predicted Value
  - p = trend surface prediction + kriged residual
  - p = 57.02 - 5.46 = 51.6
- Plot Predicted Values on Map
  - contour or surface map of predicted values

Step 5

- Prediction Error
  - Kriging Variance
    - $C(0) - c'C^{-1}c = (1 - 0.864)x440 = 59.9$
  - Standard Error
    - $\sqrt{59.9} = 7.7$
  - Uncertainty
    - assuming normality (1.96 approx. 95%)
    - 51.6 +/- 1.96*7.7

Step 6

- Interpolated Map
  - repeat Kriging exercise for a grid of regularly spaced points
  - visualize by means of Grid Map, Contours, 3D elevation maps, TIN, etc.
  - map uncertainty, confidence intervals

[redacted]