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NB1230 Linear Algebra, Resit
Monday June 17 2023, 9:00-11:00

TU Delft

Student number :

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This exam consists of 9 main Grasple questions plus 6 main questions on paper, worth $35+46=$ 81 *points in total: Grade* $= 1 + \frac{\text{Point total}}{9}$. *You can use the build in Grasple calculator. Please write your answers in the answer boxes and provide explanations when requested. Raise your hand when you have a question!*

1. Consider the following matrix and vector 10

$$A = \begin{bmatrix} h^2-4 & 3 & 4 \\ 0 & h-1 & k \\ 4-h^2 & -3 & h-2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} k \\ h-1 \\ -k \end{bmatrix}$$

For each combination of values for h and k determine the number of solutions of the equation $A\mathbf{x} = \mathbf{b}$.

Answer:

No solutions	One solution	Infinitely many solutions
$h=-2$ & $k=-4$ $h=2$ & $k=\frac{4}{3}$	$h \neq -2, 1, 2$	$h=-2$ & $k \neq -4$ $h=1$ $h=2$ & $k \neq 4/3$

Examiner responsible: Iris Smit
Examination reviewer: Fokko van de Bult

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2. Consider the set S of vectors $\mathbf{x} \in \mathbb{R}^2$ for which $x_1^2 = x_2^2$. 5

Is S a linear subspace of $\mathbb{R}^2$?

Explain your answer: Investigate whether or not S satisfies the definition of a linear subspace.

Answer: Yes No

Explanation:

If a subspace is linear and $\vec{u}$ and $\vec{v}$ are in that subspace, $\vec{u}+\vec{v}$ should be in the subspace too.

For this set, for example

$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $1^2 = 1^2$ $\vec{v} = \begin{bmatrix} 2 \\ (-2) \end{bmatrix}$ $\therefore 2^2 = (-2)^2$

however $\vec{u}+\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3)^2 \neq (-1)^2$

$\therefore$ S is not a linear subspace.

3. Suppose that A is a 3×5 matrix whose first row consists only of zeroes. Based on this information, give the minimal and maximal possible value of the rank of A. 3

Explain your answers.

Answer: Min 0 Max 2

Explanation:

If one row is all zeros, then A can have at most two row pivots, and therefore, 2 column pivots, so maximum rank of A is 2.

It is also possible that the other rows might be all zeros too, in which case, the rank is zero (min).

4. Consider the following matrix and its echelon form. 9

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ -2 & -4 & -2 & -4 & -6 \\ 1 & 2 & 3 & 5 & 16 \\ -2 & -4 & 6 & 8 & 3 \end{bmatrix} \sim E = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 0 & 4 & 6 & 8 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following sets form a basis for Col(A)?

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 6 \end{bmatrix} \right\} \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ -6 \\ 16 \\ 3 \end{bmatrix} \right\} \quad \mathcal{B}_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \\ 0 \end{bmatrix} \right\}$$

Answer:

Yes/No: No , Yes/No: Yes , Yes/No: No

Explanation:

- There are 3 pivot columns in the echelon form of A, so the dim(Col(A)) should be equal to 3. Therefore β_1 is not a basis for Col(A).

Rank(A) = 3

- β_3 is is not a proper basis because these are the columns from the echelon form and they are not in the columnspace of A. for example,

$$\cancel{A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \cancel{\begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -2 \\ 1 & 2 & \\ -2 & -4 & \end{bmatrix}} \quad \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 5 & 7 & 1 \\ -2 & -4 & -2 & -4 & -6 & 0 \\ 1 & 2 & 3 & 5 & 16 & 0 \\ -2 & -4 & 6 & 8 & 3 & 0 \end{array}\right]$$

Not a linear comb, so not even a part of the set of vectors.

$\sim$ Row reduction leads to a Contradictory row :

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & -43/7 \end{array}\right]$$

- β_2 is a proper basis because dim of the set is 3 and the first and third vectors are in Col(A) - columns 1 and 5 of matrix A : which are the pivot columns in echelon form. They are a linearly independent set (not multiples of each other). ~~It forms a proper basis for a subspace spanning $\mathbb{R}^3$~~

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5. Consider the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and the linear subspace $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) Show that A is an orthogonal matrix. 3

Explanation:

Orthogonal matrix has orthonormal columns.

$\left\|\begin{bmatrix}0\\1\\0\end{bmatrix}\right\| = 1 \qquad \left\|\begin{bmatrix}0\\0\\1\end{bmatrix}\right\| = 1 \qquad \left\|\begin{bmatrix}1\\0\\0\end{bmatrix}\right\| = 1$

$b_1 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad b_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad b_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ • It is a square matrix (3×3)

then, $b_1 \cdot b2 = 0$
$b_2 \cdot b_3 = 0$ ∴ they are all orthogonal
$b_1 \cdot b_3 = 0$ to each other.

For this matrix, $A^T = A^{-1}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = A^{-1}$ $\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$

(b) Show that $A\mathbf{w} \in W$ for all $\mathbf{w} \in W$. 4

Explanation:

Since $W = \text{span}\{v_1, v_2\}$ of two linearly independent vectors, vector $\vec{w} = c_1 v_1 + c_2 v_2$

$Av_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \boxed{v_2}$

$Av_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = (v_1 + v_2) \times -1$
$= \boxed{-v_1 - v_2}$

Since Av_1 and Av_2 both result in vectors in W,

$A(c_1 v_1 + c_2 v_2) = c_2 A v_1 + c_2 A v_2$

which is still some constants times Av_1 and Av_2 ∴ will be in W.

~~Span of~~

(c) Compute the eigenvalues of A. 4

Answer: 1

Explanation:
Since A is an orthogonal matrix, A only has one eigen value : 1.

Since columns of are A are linearly independent they span all of $\mathbb{R}^3$. Therefore for any vector in $\mathbb{R}^3$, the eigenvalue is 1 with arithmetic mutiplicity of 3 and geometric multiplicity of 3.

$$\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \quad -\lambda \begin{bmatrix} \lambda^2 \end{bmatrix} + 1 \begin{bmatrix} 1 \end{bmatrix} = \quad -\lambda^3 + 1 = 0$$

$$\lambda^3 = 1$$

$$\boxed{\lambda = 1}$$

(d) The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix A is a rotation. Determine a vector that spans the axis of rotation. 3

Answer:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Explanation:
A vector that spans the axis of rotation has eigenvalue 1, because it does not change.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathrm{Nul}\left(\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\right) = \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{aligned} -x_1 + x_3 &= 0 \\ x_1 &= x_3 \\ x_2 &= x_3 \end{aligned} \quad x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is}$$

in E_1 of.

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(e) Bonus: Determine the angle of rotation +3

Explanation:

6. Consider an *injective* linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^5$ and a *linearly independent* set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^4$. Explain why $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ in $\mathbb{R}^5$ is linearly independent. 5

Explanation:

Injective linear transformation is one-to-one transformation. which means there is only one value in domain that maps onto one value in the range. for lin. independent set $\{v_1, v_2, v_3\}$. $\{T(v_1), T(v_2), T(v_3)\}$ will be unique too, making $\{T(v_1), T(v_2), T(v_3)\}$ a linearly independent set.

v_1, v_2, v_3 are not multiples of each other or a linear combination.

>> $\{v_1, v_2, v_3\}$ will have pivot in every column, so $\{T(v_1), T(v_2), T(v_3)\}$ will have pivot in every colum too

Extra space 1

6. Let

$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ $A\vec{x} = \vec{0}$ $\vec{x} = \vec{0}$

$Nul(A) = \{0\}$ since the vectors are linearly independent.

$T(A) = \begin{bmatrix} T(v_1) & T(v_2) & T(v_3) \end{bmatrix}$ only has the ~~trivitia~~ trivial solution as a solution to $T(A)\vec{x} = 0$. therefore, these colums are linearly indepent too.

→ ~~Substitution $\vec{x} = \vec{0}$~~

Extra space 2