

## Abbot Chapter 1 Section 2

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### Exercise 1.2.1-a

*This follows the exercise\_1\_2\_1 section in Lean, so might be overly detailed. I'm also pretentiously going to use lemmas that correspond to the Lean theorems.*

**Lemma.** *First, we show that for  $n \in \mathbb{N}$ , we know that there is an  $x \in \mathbb{N}$  so that  $3x = n$ ,  $3x + 1 = n$ , or  $3x + 2 = n$ .*

*Proof.* We proceed by induction, showing that  $3 \cdot 0 = 0$ , then if the statement holds for  $n$ , we proceed by cases and show that for the same  $x$  we reached for  $n$ , we either have  $3x + 1 = n + 1$ ,  $3x + 2 = n + 1$ , or  $3(x + 1) = n + 1$ .  $\square$

**Lemma.** *If  $n \in \mathbb{N}$ , then  $3 \nmid n$  if and only if for some  $x \in \mathbb{N}$ ,  $3x + 1 = n$  or  $3x + 2 = n$ .*

*Proof.* We can prove the forward direction by the above statement: since  $3x = n$  would contradict the assumption that  $3 \nmid n$ .

The reverse direction is simplest by contradiction: if we have the  $x$  with remainder 1 or 2, we cannot find some  $y$  so that  $3y = n$ , since we'd form the equation  $3(y - x) = r$  for  $r$  being 1 or 2, which is absurd since 3 cannot divide a non-zero number less than itself.  $\square$

For the final lemma: we show that

**Lemma.** *For  $a, b \in \mathbb{N}$  if  $3 \mid ab$ ,  $3 \mid a$  or  $3 \mid b$ .*

*Proof.* We show the contrapositive: assuming  $3 \nmid a$  and  $3 \nmid b$ , we have  $3x_a + r_a = a$  and  $3x_b + r_b = b$  for  $x_a, x_b \in \mathbb{N}$  and  $r_a, r_b \in \{1, 2\}$  from the forward direction of the above. We compute  $ab$  with the above in all four cases:

1. if  $r_a, r_b = 1$ , then  $ab = 9x_ax_b + 3x_a + 3x_b + 1 = 3(3x_ax_b + x_a + x_b) + 1$ ;
2. if  $r_a = 1, r_b = 2$ , then  $ab = 9x_ax_b + 6x_a + 3x_b + 1 = 3(3x_ax_b + 2x_a + x_b) + 2$ ;
3. if  $r_a = 2, r_b = 1$ , then  $ab = 9x_ax_b + 3x_a + 6x_b + 1 = 3(3x_ax_b + x_a + 2x_b) + 2$ ;
4. if  $r_a, r_b = 2$ , then  $ab = 9x_ax_b + 6x_a + 6x_b + 4 = 3(3x_ax_b + 2x_a + 2x_b + 1) + 1$ .

In all the cases, we can express  $ab = 3y + r$  for  $y \in \mathbb{N}$  and  $r \in \{1, 2\}$  and apply the backwards direction of the lemma above to conclude  $3 \nmid ab$ , showing the contrapositive.  $\square$

**Lemma.**  $\sqrt{3}$  is irrational.

*Proof.* For contradiction, let  $a, b \in \mathbb{N}$  and  $\frac{a^2}{b^2} = 3$ . Without loss of generality, we can assume that  $a$  and  $b$  don't share factors. Rewriting this as  $a^2 = 3b^2$ , we see that  $3 \mid a^2$ , so the above gives us that  $3 \mid a$ . Hence, we write  $a = 3d$ , so  $a^2 = 9d^2 = 3b^2$ , which means that  $b^2 = 3d^2$ , so  $3 \mid b$ . This contradicts the assumption that  $a$  and  $b$  don't share factors.  $\square$

The proof would more-or-less work for  $\sqrt{6}$ .

*The proof below is kept brief, emphasizing the differences between the case for  $\sqrt{3}$  and  $\sqrt{6}$ . Lean has all the details.*

**Lemma.**  $\sqrt{6}$  is irrational.

*Proof.* We follow the above structure of the proof for  $\sqrt{3}$ : we have that for  $n \in \mathbb{N}$ , there is an  $x \in \mathbb{N}$  and  $0 \leq r \leq 5$  with  $r \in \mathbb{N}$ . Furthermore,  $r = 0$  if and only if  $6 \mid n$ . Finally, instead of the prior lemma, we show  $6 \mid a^2$  implies  $6 \mid a$ . If we attempt the contrapositive, we have 5 cases for  $a = 6x + r$ , where we rewrite  $a^2 = 6(6x^2 + 2rx) + r^2 = 6y + r^2$ :

1. if  $r = 1$ ,  $r^2 = 1$ , so  $a^2 = 6y + 1$ ;
2. if  $r = 2$ ,  $r^2 = 4$ , so  $a^2 = 6y + 4$ ;
3. if  $r = 3$ ,  $r^2 = 9$ , so  $a^2 = 6(y + 1) + 3$ ;
4. if  $r = 4$ ,  $r^2 = 16$ , so  $a^2 = 6(y + 2) + 4$ ;
5. if  $r = 5$ ,  $r^2 = 25$ , so  $a^2 = 6(y + 4) + 1$ .

This shows that if  $a \nmid 6$ ,  $a^2 \nmid 6$ . We can proceed with this, showing that if  $a^2 = 6b^2$  and  $a$  and  $b$  don't share factors, we get  $6 \mid a$  and  $6 \mid b$ .  $\square$

**Exercise 1.2.1-b**

The proof of theorem 1.1.1 breaks down at the first supposition that  $p^2 = 4q^2$  as  $p = 2, q = 1$  already suffices. Furthermore,  $4 \mid 6^2$  but  $4 \nmid 6$ , so in the next step, we cannot conclude anything from the fact that  $4 \mid p^2$ .

**Exercise 1.2.2**

*Proof.* If we suppose that  $2^r = 3$  for some rational  $r$ , we can show that  $2^p = 3^q$  for some  $p, q \in \mathbb{N}$  with  $p > 1$ . However, we can show that all such powers of 2 are even whereas all powers of 3 are odd. Therefore, such a  $p, q$  cannot exist.  $\square$

**Exercise 1.2.3-a**

We use  $A_n = \{2^{(n+k)} : k \in \mathbb{N}\}$ . The  $\cap_{n=1}^{\infty} A_n = \emptyset$ .

**Exercise 1.2.3-b**

*I couldn't find the well-ordering principle in the text, but I think that's clearly the way to prove this.*

Since all  $A_n$  are finite, and each  $A_n$  is a subset of the prior ones, there must be one of a minimal cardinality  $c$ , and  $c \geq 1$  since even the smallest set must be nonempty. All sets will contain this smallest set, so it will be the intersection, which is, therefore, finite and nonempty.

**Exercise 1.2.3-c**

Take  $A = \{1, 2, 3\}$ ,  $B = \{2, 4\}$ , and  $C = \{3, 6\}$  to observe that  $A \cap (B \cup C) = \{2, 3\} \neq (A \cap B) \cup C = \{2, 3, 6\}$ .

**Exercise 1.2.3-d**

Yes: if  $x \in A \cap (B \cap C)$ ,  $x$  is in all of  $A, B$ , and  $C$ , so  $x \in (A \cap B) \cap C$  and similarly in the reverse direction.

**Exercise 1.2.3-e**

Yes: if  $x \in A \cap (B \cup C)$ , we know  $x \in A$  and  $x \in B$  or  $x \in C$ , so either  $x \in A \cap B$  since  $x \in B$ , or  $x \in A \cap C$  otherwise. In reverse, if  $x \in A \cap B$ , in  $x \in A$  and  $B$ , so  $x \in A \cap (B \cup C)$  and similarly if  $x \in C$ .

**Exercise 1.2.4**

Let  $A_i$  be the subset of  $\mathbb{N}$  formed by numbers with  $i$  prime factors (forgetting exponents – so  $12 \in A_2$ ). Then, by unique factorization, each  $n \in \mathbb{N}$  is in exactly one  $A_i$ . Hence, if  $i \neq j$ ,  $A_i \cap A_j = \emptyset$ , and  $\cup_{i=1}^{\infty} A_i = \mathbb{N}$ .

**Exercise 1.2.5-a**

*Proof.* Let  $x \in (A \cap B)^c$ . Then,  $x \notin A \cap B$ , so it is not the case that  $x \in A$  and  $x \in B$ ; therefore, by the propositional DeMorgan's rule,  $x \notin A$  or  $x \notin B$ , so  $x \in A^c \cup B^c$ .  $\square$

**Exercise 1.2.5-b**

*Proof.* Put  $x \in A^c \cup B^c$ . Then,  $x \notin A$  or  $x \notin B$ , which, by the propositional DeMorgan's rule, gives  $x \notin (A \cap B)^c$ .  $\square$

**Exercise 1.2.5-c**

*Proof.* If  $x \in (A \cup B)^c$ , then  $x$  is not in  $A$  or  $B$ , so by the propositional DeMorgan's rule,  $x \in A^c \cap B^c$ . Similarly, if  $x \in A^c \cap B^c$ ,  $x \notin A$  and  $x \notin B$ , so by DeMorgan's rule,  $x \notin A \cup B$ , thus,  $x \in (A \cup B)^c$ .  $\square$

**Exercise 1.2.6-a**

*Proof.* Suppose  $a, b \geq 0$ . Then  $|a + b| = a + b = |a| + |b|$  so  $|a + b| \leq |a| + |b|$ . If  $a, b \leq 0$ , then we can rewrite  $a = -x$  and  $b = -y$  where  $x, y \geq 0$ , so  $|-x - y| = |-1||x + y| = x + y = |-x| + |-y|$ , so  $|a + b| \leq |a| + |b|$ .  $\square$

**Exercise 1.2.6-b**

*Proof.* If  $a, b \geq 0$ ,  $|a| = a$ ,  $|b| = b$ , so  $(a+b)^2 = (|a|+|b|)^2$ .

If  $a, b \leq 0$ , we can factor out the -1, so  $(-a-b)^2 = (a+b)^2$  and since  $|a| = -a$  and  $|b| = -b$ , we get  $(a+b)^2 = (|a|+|b|)^2$ .

Finally suppose that  $a$  and  $b$  have different signs. Then, without loss of generality, suppose  $a \geq 0$  and  $b \leq 0$ , so that  $b = -c$  for  $c \geq 0$ . Then  $(a+b)^2 = a^2 - 2bc + c^2$  while  $(|a|+|b|)^2 = a^2 + 2ac + c^2$ , and since  $a, c \geq 0$ , we know that  $(a+b)^2 \leq (|a|+|b|)^2$ .

$0 \leq a \leq b$  if and only if  $0 \leq a^2 \leq b^2$ , so because  $|a+b| \geq 0$  and  $|a|+|b| \geq 0$ ,  $|a+b| \leq |a|+|b|$  if and only if  $|a+b|^2 \leq (|a|+|b|)^2$ . Then, since  $(a+b)^2 \leq (|a|+|b|)^2$  and  $|a+b|^2 = (a+b)^2$  (because if  $a+b < 0$ , the -1 multiplies out), we know that  $|a+b| \leq |a|+|b|$ . □

**Exercise 1.2.6-c**

*Proof.*  $a-b = a-c+c-d+d-b$ , applying  $|\cdot|$  to both sides,  $|a-b| \leq |a-c|+|c-d+d-b|$  by the triangle inequality, and the triangle inequality gives  $|c-b| \leq |c-d|+|d-b|$ , so  $|a-b| \leq |a-c+c-d+d-b|$ . □

**Exercise 1.2.6-d**

*Proof.*  $||a|-|b|| = ||a-b+b|-|b|| \leq ||a-b|+|b|-|b|| = ||a-b|| = |a-b|$ , where the inequality is from the triangle inequality. □

**Exercise 1.2.7-a**

$f(A) = [0, 4]$  and  $f(B) = [1, 16]$ .  $f(A \cap B) = [1, 4] = f(A) \cap f(B)$ .  $f(A \cup B) = [0, 16] = [0, 4] \cup [1, 16]$ .

**Exercise 1.2.7-b**

$A = [-1, 0]$  and  $B = [0, 1]$ . Then while  $f(A \cap B) = \{0\}$  but  $f(A) \cap f(B) = [0, 1]$ .

**Exercise 1.2.7-c**

*Proof.* If  $x \in g(A \cap B)$ , then for some  $y \in A$  and  $y \in B$ ,  $y = g(x)$ , so  $x \in g(A)$  and  $x \in g(B)$ , so  $x \in g(A) \cap g(B)$ . □

**Exercise 1.2.7-d**

**Lemma.** Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $A, B \subset \mathbb{R}$ . Then,  $g(A \cup B) = g(A) \cup g(B)$ .

*Proof.* If  $x \in g(A \cup B)$ , then for some  $y \in A$  or  $y \in B$ ,  $y = g(x)$ , so  $x \in g(A)$  or  $x \in g(B)$ , so  $x \in g(A) \cup g(B)$ .

In reverse, without loss of generality, suppose  $x \in g(A)$ , so there is a  $y \in A$  so that  $g(y) = x$ , and so  $y \in A \cup B$ , putting  $x \in g(A \cup B)$ . □

**Exercise 1.2.8-a**

Let  $f(x) = x+1$ . Then if  $f(x) = f(y)$  for  $x, y \in \mathbb{N}$ ,  $x+1 = y+1$ , so  $x = y$ ; meaning  $f$  is 1-1. Furthermore, there is no  $x \in \mathbb{N}$  so that  $x+1 = 0$ , so  $f$  is not onto.

**Exercise 1.2.8-b**

Let  $f(x) = \begin{cases} \frac{x}{2} & 2 \mid x \\ \frac{x-1}{2} & 2 \nmid x \end{cases}$ , then  $f(2y) = y$  for  $y \in \mathbb{N}$  so  $f$  is onto but  $f(0) = f(1) = 0$  so  $f$  is not 1-1.

**Exercise 1.2.8-b**

Put  $f(x) = \begin{cases} \frac{x}{2} & 2 \mid x \\ -\frac{x+1}{2} & 2 \nmid x \end{cases}$ .  $f$  is onto since if  $y \in \mathbb{Z}$ ,  $y \geq 0$ ,  $f(2y) = y$  and if  $y < 0$ ,  $f(-2y-1) = y$ . Let  $x, y \in \mathbb{N}$  such that  $f(x) = f(y)$ . Then  $2 \mid x$  if and only if  $2 \mid y$ . If  $2 \mid x$ ,  $\frac{x}{2} = \frac{y}{2}$  so that  $x = y$ . Similarly, if  $2 \nmid x$ ,  $-\frac{x+1}{2} = -\frac{y+1}{2}$ , so  $x+1 = y+1$  meaning  $x = y$ . Hence  $f$  is 1-1.

**Exercise 1.2.9-a**

$f^{-1}(A) = [-2, 2]$  and  $f^{-1}(B) = [-1, 1]$ .

$$f^{-1}(A \cap B) = [-1, 1] = [-2, 2] \cap [1, -1] = f^{-1}(A) \cap f^{-1}(B).$$

$$f^{-1}(A \cup B) = [-2, 2] = [-2, 2] \cup [1, -1] = f^{-1}(A) \cup f^{-1}(B).$$

**Exercise 1.2.9-b**

Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $A, B \subset \mathbb{R}$ .

**Lemma.**  $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ .

*Proof.*  $x \in g^{-1}(A \cap B)$  if and only if  $f(x) \in A \cap B$  meaning  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$  so  $x \in g^{-1}(A) \cap g^{-1}(B)$  by definition.  $\square$

**Lemma.**  $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ .

*Proof.*  $x \in g^{-1}(A \cup B)$  if and only if  $f(x) \in A \cup B$  meaning  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$  so  $x \in g^{-1}(A) \cup g^{-1}(B)$  by definition.  $\square$