Abbot Chapter 1 Section 2

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Exercise 1.2.1-a

This follows the exercise_1_2_1 section in Lean, so might be overly detailed. I'm also pretentiously going to use lemmas that correspond to the Lean theorems.

Lemma. *First, we show that for $n \in \mathbb{N}$, we know that there is an $x \in \mathbb{N}$ so that $3x = n$, $3x + 1 = n$, or $3x + 2 = n$.*

Proof. We proceed by induction, showing that $3 \cdot 0 = 0$, then if the statement holds for n, we proceed by cases and show that for the same x we reached for n, we either have $3x + 1 = n + 1$, $3x + 2 = n + 1$, or $3(x + 1) = n + 1$. □

Lemma. *If $n \in \mathbb{N}$, then $3 \nmid n$ if and only if for some $x \in \mathbb{N}$, $3x + 1 = n$ or $3x + 2 = n$.*

Proof. We can prove the forward direction by the above statement: since $3x = n$ would contradict the assumption that $3 \nmid n$.

The reverse direction is simplest by contradiction: if we have the x with remainder 1 or 2, we cannot find some y so that $3y = n$, since we'd form the equation $3(y - x) = r$ for r being 1 or 2, which is absurd since 3 cannot divide a non-zero number less than itself. □

For the final lemma: we show that

Lemma. *For $a, b \in \mathbb{N}$ if $3 \mid ab$, $3 \mid a$ or $3 \mid b$.*

Proof. We show the contrapositive: assuming $3 \nmid a$ and $3 \nmid b$, we have $3x_a + r_a = a$ and $3x_b + r_b = b$ for $x_a, x_b \in \mathbb{N}$ and $r_a, r_b \in \{1, 2\}$ from the forward direction of the above. We compute ab with the above in all four cases:

1. if $r_a, r_b = 1$, then $ab = 9x_ax_b + 3x_a + 3x_b + 1 = 3(3x_ax_b + x_a + x_b) + 1$;
2. if $r_a = 1, r_b = 2$, then $ab = 9x_ax_b + 6x_a + 3x_b + 1 = 3(3x_ax_b + 2x_a + x_b) + 2$;
3. if $r_a = 2, r_b = 1$, then $ab = 9x_ax_b + 3x_a + 6x_b + 1 = 3(3x_ax_b + x_a + 2x_b) + 2$;
4. if $r_a, r_b = 2$, then $ab = 9x_ax_b + 6x_a + 6x_b + 4 = 3(3x_ax_b + 2x_a + 2x_b + 1) + 1$.

In all the cases, we can express $ab = 3y + r$ for $y \in \mathbb{N}$ and $r \in \{1, 2\}$ and apply the backwards direction of the lemma above to conclude $3 \nmid ab$, showing the contrapositive. □

Lemma. *$\sqrt{3}$ is irrational.*

Proof. For contradiction, let $a, b \in \mathbb{N}$ and $\frac{a^2}{b^2} = 3$. Without loss of generality, we can assume that a and b don't share factors. Rewriting this as $a^2 = 3b^2$, we see that $3 \mid a^2$, so the above gives us that $3 \mid a$. Hence, we write $a = 3d$, so $a^2 = 9d^2 = 3b^2$, which means that $b^2 = 3d^2$, so $3 \mid b$. This contradicts the assumption that a and b don't share factors. □

The proof would more-or-less work for $\sqrt{6}$.

The proof below is kept brief, emphasizing the differences between the case for $\sqrt{3}$ and $\sqrt{6}$. Lean has all the details.

Lemma. *$\sqrt{6}$ is irrational.*

Proof. We follow the above structure of the proof for $\sqrt{3}$: we have that for $n \in \mathbb{N}$, there is an $x \in \mathbb{N}$ and $0 \leq r \leq 5$ with $r \in \mathbb{N}$. Futhermore, $r = 0$ if and only if $6 \mid n$. Finally, instead of the prior lemma, we show $6 \mid a^2$ implies $6 \mid a$. If we attempt the contrapositive, we have 5 cases for $a = 6x + r$, where we rewrite $a^2 = 6(6x^2 + 2rx) + r^2 = 6y + r^2$:

1. if $r = 1$, $r^2 = 1$, so $a^2 = 6y + 1$;
2. if $r = 2$, $r^2 = 4$, so $a^2 = 6y + 4$;
3. if $r = 3$, $r^2 = 9$, so $a^2 = 6(y + 1) + 3$;
4. if $r = 4$, $r^2 = 16$, so $a^2 = 6(y + 2) + 4$;
5. if $r = 5$, $r^2 = 25$, so $a^2 = 6(y + 4) + 1$.

This shows that if $a \nmid 6$, $a^2 \nmid 6$. We can proceed with this, showing that if $a^2 = 6b^2$ and a and b don't share factors, we get $6 \mid a$ and $6 \mid b$. □

Exercise 1.2.1-b

The proof of theorem 1.1.1 breaks down at the first supposition that $p^2 = 4q^2$ as $p = 2, q = 1$ already suffices. Furthermore, $4 \mid 6^2$ but $4 \nmid 6$, so in the next step, we cannot conclude anything from the fact that $4 \mid p^2$.

Exercise 1.2.2

Proof. If we suppose that $2^r = 3$ for some rational r, we can show that $2^p = 3^q$ for some $p, q \in \mathbb{N}$ with $p > 1$. However, we can show that all such powers of 2 are even whereas all powers of 3 are odd. Therefore, such a p, q cannot exist. □

Exercise 1.2.3-a

We use $A_n = \{2^{(n+k)} : k \in \mathbb{N}\}$. The $\cap_{n=1}^{\infty} A_n = \emptyset$.

Exercise 1.2.3-b

I couldn't find the well-ordering principle in the text, but I think that's clearly the way to prove this.

Since all A_n are finite, and each A_n is a subset of the prior ones, there must be one of a minimal cardinality c, and $c \geq 1$ since even the smallest set must be nonempty. All sets will contain this smallest set, so it will be the intersection, which is, therefore, finite and nonempty.

Exercise 1.2.3-c

Take $A = \{1, 2, 3\}, B = \{2, 4\}$, and $C = \{3, 6\}$ to observe that $A \cap (B \cup C) = \{2, 3\} \neq (A \cap B) \cup C = \{2, 3, 6\}$.

Exercise 1.2.3-d

Yes: if $x \in A \cap (B \cap C)$, x is in all of A, B, and C, so $x \in (A \cap B) \cap C$ and similarly in the reverse direction.

Exercise 1.2.3-e

Yes: if $x \in A \cap (B \cup C)$, we know $x \in A$ and $x \in B$ or $x \in C$, so either $x \in A \cap B$ since $x \in B$, or $x \in A \cap C$ otherwise. In reverse, if $x \in A \cap B$, in $x \in A$ and B, so $x \in A \cap (B \cup C)$ and similarly if $x \in C$.

Exercise 1.2.4

Let A_i be the subset of $\mathbb{N}$ formed by numbers with i prime factors (forgetting exponents – so $12 \in A_2$). Then, by unique factorization, each $n \in \mathbb{N}$ is in exactly one A_i. Hence, if $i \neq j$, $A_i \cap A_j = \emptyset$, and $\cup_{i=1}^{\infty} A_i = \mathbb{N}$.

Exercise 1.2.5-a

Proof. Let $x \in (A \cap B)^c$. Then, $x \notin A \cap B$, so it is not the case that $x \in A$ and $x \in B$; therefore, by the propositional DeMorgan's rule, $x \notin A$ or $x \notin B$, so $x \in A^c \cup B^c$. □

Exercise 1.2.5-b

Proof. Put $x \in A^c \cup B^c$. Then, $x \notin A$ or $x \notin B$, which, by the propositional DeMorgan's rule, gives $x \notin (A \cap B)^c$. □

Exercise 1.2.5-c

Proof. If $x \in (A \cup B)^c$, then x is not in A or B, so by the propositional DeMorgan's rule, $x \in A^c \cap B^c$. Similarly, if $x \in A^c \cap B^c$, $x \notin A$ and $x \notin B$, so by DeMorgan's rule, $x \notin A \cap B$, thus, $x \in (A \cap B)^c$. □

Exercise 1.2.6-a

Proof. Suppose $a, b \geq 0$. Then $|a + b| = a + b = |a| + |b|$ so $|a + b| \leq |a| + |b|$. If $a, b \leq 0$, then we can rewrite $a = -x$ and $b = -y$ where $x, y \geq 0$, so $| - x - y| = | - 1||x + y| = x + y = | - x| + | - y|$, so $|a + b| \leq |a| + |b|$. □

Exercise 1.2.6-b

Proof. If $a, b \geq 0$, $|a| = a$, $|b| = b$, so $(a+b)^2 = (|a| + |b|)^2$.

If $a, b \leq 0$, we can factor out the -1, so $(-a-b)^2 = (a+b)^2$ and since $|a| = -a$ and $|b| = -b$, we get $(a+b)^2 = (|a|+|b|)^2$.

Finally suppose that a and b have different signs. Then, without loss of generality, suppose $a \geq 0$ and $b \leq 0$, so that $b = -c$ for $c \geq 0$. Then $(a+b)^2 = a^2 - 2bc + c^2$ while $(|a| + |b|)^2 = a^2 + 2ac + c^2$, and since $a, c \geq 0$, we know that $(a+b)^2 \leq (|a| + |b|)^2$.

$0 \leq a \leq b$ if and only if $0 \leq a^2 \leq b^2$, so because $|a+b| \geq 0$ and $|a| + |b| \geq 0$, $|a+b| \leq |a| + |b|$ if and only if $|a+b|^2 \leq (|a| + |b|)^2$. Then, since $(a+b)^2 \leq (|a| + |b|)^2$ and $|a+b|^2 = (a+b)^2$ (because if $a + b < 0$, the -1 multiplies out), we know that $|a+b| \leq |a| + |b|$. □

Exercise 1.2.6-c

Proof. $a - b = a - c + c - d + d - b$, applying $|\cdot|$ to both sides, $|a-b| \leq |a-c| + |c-d+d-b|$ by the triangle inequality, and the triangle inequality gives $|c-b| \leq |c-d| + |d-b|$, so $|a-b| \leq |a-c+c-d+d-b|$. □

Exercise 1.2.6-d

Proof. $||a| - |b|| = ||a - b + b| - |b|| \leq ||a-b| + |b| - |b|| = ||a-b|| = |a-b|$, where the inequality is fron the triangle inequality. □

Exercise 1.2.7-a

$f(A) = [0, 4]$ and $f(B) = [1, 16]$. $f(A \cap B) = [1, 4] = f(A) \cap f(B)$. $f(A \cup B) = [0, 16] = [0, 4] \cup [1, 16]$.

Exercise 1.2.7-b

$A = [-1, 0]$ and $B = [0, 1]$. Then while $f(A \cap B) = \{0\}$ but $f(A) \cap f(B) = [0, 1]$.

Exercise 1.2.7-c

Proof. If $x \in g(A \cap B)$, then for some $y \in A$ and $y \in B$, $y = g(x)$, so $x \in g(A)$ and $x \in g(B)$, so $x \in g(A) \cap g(B)$. □

Exercise 1.2.7-d

Lemma. *Let $g : \mathbb{R} \to \mathbb{R}$ and $A, B \subset \mathbb{R}$. Then, $g(A \cup B) = g(A) \cup g(B)$.*

Proof. If $x \in g(A \cup B)$, then for some $y \in A$ or $y \in B$, $y = g(x)$, so $x \in g(A)$ or $x \in g(B)$, so $x \in g(A) \cup g(B)$.

In reverse, without loss of generality, suppose $x \in g(A)$, so there is a $y \in A$ so that $g(y) = x$, and so $y \in A \cup B$, putting $x \in g(A \cup B)$. □