

## Abbot Chapter 1 Section 2

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2024-10-22

### Exercise 1.2.1-a

*This follows the exercise\_1\_2\_1 section in Lean, so might be overly detailed. I'm also pretentiously going to use lemmas that correspond to the Lean theorems.*

**Lemma.** *First, we show that for  $n \in \mathbb{N}$ , we know that there is an  $x \in \mathbb{N}$  so that  $3x = n$ ,  $3x + 1 = n$ , or  $3x + 2 = n$ .*

*Proof.* We proceed by induction, showing that  $3 \cdot 0 = 0$ , then if the statement holds for  $n$ , we proceed by cases and show that for the same  $x$  we reached for  $n$ , we either have  $3x + 1 = n + 1$ ,  $3x + 2 = n + 1$ , or  $3(x + 1) = n + 1$ .  $\square$

**Lemma.** *If  $n \in \mathbb{N}$ , then  $3 \nmid n$  if and only if for some  $x \in \mathbb{N}$ ,  $3x + 1 = n$  or  $3x + 2 = n$ .*

*Proof.* We can prove the forward direction by the above statement: since  $3x = n$  would contradict the assumption that  $3 \nmid n$ .

The reverse direction is simplest by contradiction: if we have the  $x$  with remainder 1 or 2, we cannot find some  $y$  so that  $3y = n$ , since we'd form the equation  $3(y - x) = r$  for  $r$  being 1 or 2, which is absurd since 3 cannot divide a non-zero number less than itself.  $\square$

For the final lemma: we show that

**Lemma.** *For  $a, b \in \mathbb{N}$  if  $3 \mid ab$ ,  $3 \mid a$  or  $3 \mid b$ .*

*Proof.* We show the contrapositive: assuming  $3 \nmid a$  and  $3 \nmid b$ , we have  $3x_a + r_a = a$  and  $3x_b + r_b = b$  for  $x_a, x_b \in \mathbb{N}$  and  $r_a, r_b \in \{1, 2\}$  from the forward direction of the above. We compute  $ab$  with the above in all four cases:

1. if  $r_a, r_b = 1$ , then  $ab = 9x_ax_b + 3x_a + 3x_b + 1 = 3(3x_ax_b + x_a + x_b) + 1$ ;
2. if  $r_a = 1, r_b = 2$ , then  $ab = 9x_ax_b + 6x_a + 3x_b + 1 = 3(3x_ax_b + 2x_a + x_b) + 2$ ;
3. if  $r_a = 2, r_b = 1$ , then  $ab = 9x_ax_b + 3x_a + 6x_b + 1 = 3(3x_ax_b + x_a + 2x_b) + 2$ ;
4. if  $r_a, r_b = 2$ , then  $ab = 9x_ax_b + 6x_a + 6x_b + 4 = 3(3x_ax_b + 2x_a + 2x_b + 1) + 1$ .

In all the cases, we can express  $ab = 3y + r$  for  $y \in \mathbb{N}$  and  $r \in \{1, 2\}$  and apply the backwards direction of the lemma above to conclude  $3 \nmid ab$ , showing the contrapositive.  $\square$

**Lemma.**  $\sqrt{3}$  is irrational.

*Proof.* For contradiction, let  $a, b \in \mathbb{N}$  and  $\frac{a^2}{b^2} = 3$ . Without loss of generality, we can assume that  $a$  and  $b$  don't share factors. Rewriting this as  $a^2 = 3b^2$ , we see that  $3 \mid a^2$ , so the above gives us that  $3 \mid a$ . Hence, we write  $a = 3d$ , so  $a^2 = 9d^2 = 3b^2$ , which means that  $b^2 = 3d^2$ , so  $3 \mid b$ . This contradicts the assumption that  $a$  and  $b$  don't share factors.  $\square$

The proof would more-or-less work for  $\sqrt{6}$ .

*The proof below is kept brief, emphasizing the differences between the case for  $\sqrt{3}$  and  $\sqrt{6}$ . Lean has all the details.*

**Lemma.**  $\sqrt{6}$  is irrational.

*Proof.* We follow the above structure of the proof for  $\sqrt{3}$ : we have that for  $n \in \mathbb{N}$ , there is an  $x \in \mathbb{N}$  and  $0 \leq r \leq 5$  with  $r \in \mathbb{N}$ . Furthermore,  $r = 0$  if and only if  $6 \mid n$ . Finally, instead of the prior lemma, we show  $6 \mid a^2$  implies  $6 \mid a$ . If we attempt the contrapositive, we have 5 cases for  $a = 6x + r$ , where we rewrite  $a^2 = 6(6x^2 + 2rx) + r^2 = 6y + r^2$ :

1. if  $r = 1$ ,  $r^2 = 1$ , so  $a^2 = 6y + 1$ ;
2. if  $r = 2$ ,  $r^2 = 4$ , so  $a^2 = 6y + 4$ ;
3. if  $r = 3$ ,  $r^2 = 9$ , so  $a^2 = 6(y + 1) + 3$ ;
4. if  $r = 4$ ,  $r^2 = 16$ , so  $a^2 = 6(y + 2) + 4$ ;
5. if  $r = 5$ ,  $r^2 = 25$ , so  $a^2 = 6(y + 4) + 1$ .

This shows that if  $a \nmid 6$ ,  $a^2 \nmid 6$ . We can proceed with this, showing that if  $a^2 = 6b^2$  and  $a$  and  $b$  don't share factors, we get  $6 \mid a$  and  $6 \mid b$ .  $\square$

**Exercise 1.2.1-b**

The proof of theorem 1.1.1 breaks down at the first supposition that  $p^2 = 4q^2$  as  $p = 2, q = 1$  already suffices. Furthermore,  $4 \mid 6^2$  but  $4 \nmid 6$ , so in the next step, we cannot conclude anything from the fact that  $4 \mid p^2$ .

**Exercise 1.2.2**

*Proof.* If we suppose that  $2^r = 3$  for some rational  $r$ , we can show that  $2^p = 3^q$  for some  $p, q \in \mathbb{N}$  with  $p > 1$ . However, we can show that all such powers of 2 are even whereas all powers of 3 are odd. Therefore, such a  $p, q$  cannot exist.  $\square$

**Exercise 1.2.3-a**

We use  $A_n = \{2^{(n+k)} : k \in \mathbb{N}\}$ . The  $\bigcap_{n=1}^{\infty} A_n = \emptyset$ .

**Exercise 1.2.3-b**

*I couldn't find the well-ordering principle in the text, but I think that's clearly the way to prove this.*

Since all  $A_n$  are finite, and each  $A_n$  is a subset of the prior ones, there must be one of a minimal cardinality  $c$ , and  $c \geq 1$  since even the smallest set must be nonempty. All sets will contain this smallest set, so it will be the intersection, which is, therefore, finite and nonempty.

**Exercise 1.2.3-c**

Take  $A = \{1, 2, 3\}$ ,  $B = \{2, 4\}$ , and  $C = \{3, 6\}$  to observe that  $A \cap (B \cup C) = \{2, 3\} \neq (A \cap B) \cup C = \{2, 3, 6\}$ .

**Exercise 1.2.3-d**

Yes: if  $x \in A \cap (B \cap C)$ ,  $x$  is in all of  $A, B$ , and  $C$ , so  $x \in (A \cap B) \cap C$  and similarly in the reverse direction.

**Exercise 1.2.3-e**

Yes: if  $x \in A \cap (B \cup C)$ , we know  $x \in A$  and  $x \in B$  or  $x \in C$ , so either  $x \in A \cap B$  since  $x \in B$ , or  $x \in A \cap C$  otherwise. In reverse, if  $x \in A \cap B$ , in  $x \in A$  and  $B$ , so  $x \in A \cap (B \cup C)$  and similarly if  $x \in C$ .