

# STAT1012 Statistics for Life Sciences

## Quick Revision Notes

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(Reference: lecture and tutorial notes)

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## I) Descriptive Statistics

Data type: Qualitative (special: Categorical), Quantitative (Discrete, Continuous)

Population: the whole set of entities of interest

Sample: a subset of the population

### Central tendency

Sample mean:  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$

Sequential update property:  $\bar{X}_n = \frac{1}{n} [(n-1)\bar{X}_{n-1} + X_n]$

Mode: the value which has the greatest number of occurrence (may not be unique)

Median: the “middle” value, or the average of the two values closest to “middle” after sorting

Percentile: the p-th percentile ( $V_{\frac{p}{100}}$ ) is a value such that p% of the data are less than or equal to  $V_{\frac{p}{100}}$ . In particular, upper quantile =  $V_{0.75}$ , median =  $V_{0.5}$ , lower quantile =  $V_{0.25}$

Denote the sorted data by  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  where  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ . This is equivalent to saying that  $X_{(1)}$  is the smallest,  $X_{(2)}$  is the second smallest etc.

Median:  $V_{0.5} = X_{(\frac{n+1}{2})}$  if n is odd or  $\frac{1}{2} \left[ X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)} \right]$  if n is even

Percentile:  $V_{\frac{p}{100}} = X_{(k)}$  where  $k = \text{roundUp} \left( \frac{np}{100} \right)$  if  $\frac{np}{100}$  is not an integer

Otherwise,  $V_{\frac{p}{100}} = \frac{1}{2} \left[ X_{(\frac{np}{100})} + X_{(\frac{np}{100}+1)} \right]$

### Dispersion

Symmetric: the left hand side of the distribution mirrors the right hand side

Unimodal: the mode is unique

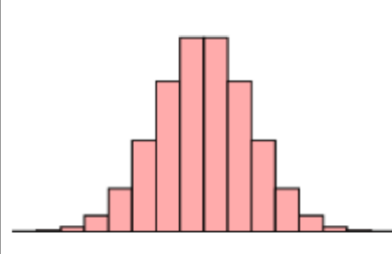
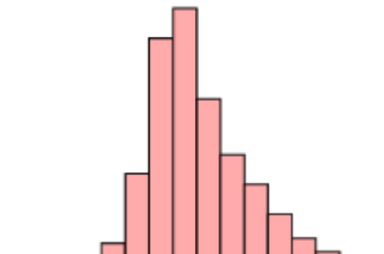
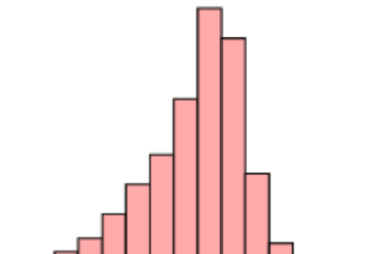
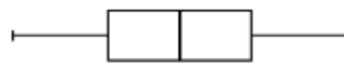

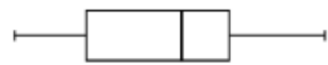
Skewness: measure of asymmetry

Left-skewed (negatively skewed): mean < median, have a few extreme small values

Right-skewed (positively skewed): mean > median, have a few extreme large values

Symmetric → mean = median (converse not true)

Symmetric + unimodal → mean = median = mode (converse not true)

Symmetric	Skewed right (positive)	Skewed left (negative)
		
		
$Q_1$ and $Q_3$ should be approximately equally spaced from the median ( $Q_2$ ).	$Q_3$ is farther from the median( $Q_2$ ) than $Q_1$	$Q_1$ is farther from the median( $Q_2$ ) than $Q_3$

Range: maximum – minimum ( $X_{(n)} - X_{(1)}$ )

Interquartile range:  $V_{0.75} - V_{0.25}$

Sample variance:  $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$  or  $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i^2 - n\bar{X}^2)$

Sample standard deviation:  $SD = \sqrt{S^2}$

### [Graphical methods](#)

Bar graph: use for categorical data, show the number of observations in each category

Histogram: use for quantitative data, showing the number of observations in each range

Stem-and-leaf plot: ordered the data into a tree-like structure

Boxplot: show 5 numbers (min,  $Q_1$ , median,  $Q_3$ , max), help locate outliers (As a rule of thumb, some people define outliers as values  $> Q_3 + 1.5 \cdot IQR$  or  $< Q_1 - 1.5 \cdot IQR$ )

## II) Probability

### Notations

Sample space: the set of all possible outcomes, often denoted as  $\Omega$

Outcome: a possible type of occurrence

Event: any set of outcomes of interest, can be denoted as  $E \subset \Omega$

Probability (of an event): denoted by  $P(E)$ , always lies between 0 and 1 (both inclusive)

$$P(E) = \frac{\# \text{ of outcomes in } E}{\# \text{ of outcomes in } \Omega}$$

Union: either A or B occurs, or they both occurs, denoted by  $A \cup B$  (logically equivalent to OR)

Intersection: both A and B occur, denoted by  $A \cap B$  (logically equivalent to AND)

Complement: A does not occur, denoted by  $A^C$  (logically equivalent to NOT)

Commutativity:  $A \cup B = B \cup A$ ,  $A \cap B = B \cap A$

Associativity:  $(A \cup B) \cup C = A \cup (B \cup C)$ ,  $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive laws:  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ ,  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

DeMorgan's laws:  $(A \cup B)^C = A^C \cap B^C$ ,  $(A \cap B)^C = A^C \cup B^C$

### Probability theory

Mutually exclusive: A and B are mutually exclusive if  $P(A \cap B) = 0$  (cannot co-occur)

Independence:  $P(A \cap B) = P(A)P(B)$  iff A and B are independent. Their complements (A and  $B^C$ ;  $A^C$  and B;  $A^C$  and  $B^C$ ) will be pairwise independent as well

Addition law:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Multiplication law: if  $A_1, \dots, A_k$  are mutually independent, then  $P(A_1 \cap \dots \cap A_k) = P(A_1) \times \dots \times P(A_k)$

### Conditional probability

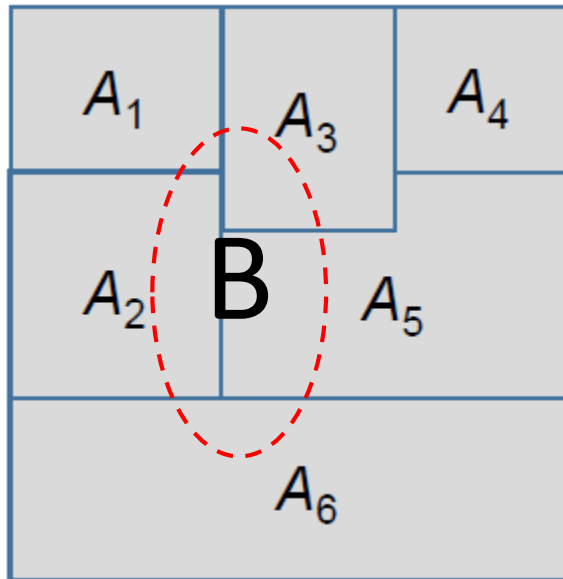
Conditional probability:  $P(B|A) = \frac{P(A \cap B)}{P(A)}$ , if  $P(B|A) = P(B)$ , then A and B are independent

Relative risk:  $RR(B|A) = \frac{P(B|A)}{P(B|A^c)}$

Total probability rule:  $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$

Exhaustive: if  $A_1, \dots, A_k$  are exhaustive, then  $A_1 \cup \dots \cup A_k = \Omega$  (at least one of them must occur)

Generalized total probability rule: let  $A_1, \dots, A_k$  be mutually exclusive and exhaustive events. For any event B, we have  $P(B) = \sum_{i=1}^k P(B|A_i)P(A_i)$



Bayes' theorem: conditional probability + generalized total probability rule. let  $A_1, \dots, A_k$  be mutually exclusive and exhaustive events. For any event B,

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^k P(B|A_j)P(A_j)}$$

### III) Discrete Probability Distributions

Random variables: numeric quantities that take different values with specified probabilities

Discrete random variable: a R.V. that takes value from a discrete set of numbers

Continuous random variable: a R.V. that takes value over an interval of numbers

#### Discrete random variables

Probability mass function: a pmf assigns a probability to each possible value  $x$  of the discrete random variable  $X$ , denoted by  $f(x) = P(X = x)$

$\sum_{i=1}^n f(x_i) = 1$  (total probability rule)

Cumulative distribution function: a cdf gives the probability that  $X$  is less than or equal to the value  $x$ , denoted by  $F(x) = P(X \leq x)$

Expected value:  $\mu = E(X) = \sum_{i=1}^n x_i P(X = x_i)$  (the idea is “probability weighted average”)

Variance:  $\sigma^2 = Var(X) = \sum_{i=1}^n (x_i - \mu)^2 P(X = x_i)$ , alternatively  $Var(X) = E(X^2) - [E(X)]^2$

Translation/rescale:  $E(aX + b) = aE(X) + b$ ,  $Var(aX + b) = a^2 Var(X)$

Linearity of expectation:  $E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i)$

#### Binomial distribution

Factorial:  $n! = n \times (n - 1) \times \dots \times 1$ , note that  $0! = 1$

Permutation (order is important):  $P_k^n = \frac{n!}{(n-k)!}$

Combination (order is not important):  $C_k^n = \frac{n!}{k!(n-k)!}$ , also denoted as  $\binom{n}{k}$

Binomial distribution: probability distribution on the number of successes  $X$  in  $n$  independent experiments, each experiment has a probability of success  $p$ , then  $X \sim B(n, p)$

Pmf:  $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$  for  $x = 0, 1, 2, \dots, n$

Mean:  $E(X) = np$

Variance:  $Var(X) = np(1 - p)$

Skewness: right-skewed if  $p < 0.5$ , symmetric if  $p = 0.5$ , left-skewed if  $p > 0.5$

Poisson distribution

Poisson distribution: probability distribution on the number of occurrence  $X$  (usually of a rare event) over a period of time or space with rate  $\mu$ , then  $X \sim Po(\mu)$

$$\text{Pmf: } P(X = x) = \frac{e^{-\mu} \mu^x}{x!} \text{ for } x = 0, 1, 2, \dots$$

$$\text{Mean: } E(X) = \mu$$

$$\text{Variance: } Var(X) = \mu$$

Skewness: right-skewed

Poisson limit theorem (poisson approximation to binomial): if  $X \sim B(n, p)$  where  $n \geq 20$ ,  $p < 0.1$  and  $np < 5$ , then  $X \approx Y \sim Po(\mu)$  where  $\mu = np$

Hypergeometric distribution (not required)

Hypergeometric distribution: probability distribution on the number of success  $X$  in  $n$  trials without replacement, from a finite population of size  $N_1 + N_2 = N \geq n$  that contains  $N_1$  trials classified as success, then  $X \sim Hypergeometric(N_1, N_2, n)$

$$\text{Pmf: } P(X = x) = \frac{\binom{N_1}{x} \binom{N_2}{n-x}}{\binom{N}{n}} \text{ for } x = \max(0, n - N_2), \dots, \min(n, N_1)$$

$$\text{Mean: } E(X) = n \left( \frac{N_1}{N} \right)$$

$$\text{Variance: } Var(X) = n \left( \frac{N_1}{N} \right) \left( \frac{N_2}{N} \right) \left( \frac{N-n}{N-1} \right)$$

Geometric distribution (not required)

Geometric distribution: probability distribution on the number of trials  $X$  when the first success occurs, each trial has a probability of success  $p$ , then  $X \sim Geo(p)$

$$\text{Pmf: } P(X = x) = (1 - p)^{x-1} p \text{ for } x = 1, 2, \dots$$

$$\text{Mean: } E(X) = \frac{1}{p}$$

$$\text{Variance: } Var(X) = \frac{1-p}{p^2}$$

Memoryless:  $P(X > k + j | X > k) = P(X > j)$ . Geometric distribution is the only discrete distribution with this property



Negative binomial distribution (not required)

Negative binomial distribution: probability distribution on the number of times  $X$  when the  $r$  success occurs, each trial has a probability of success  $p$ , then  $X \sim NB(r, p)$

Pmf:  $P(X = x) = \binom{x-1}{r-1} (1-p)^{x-r} p^r$  for  $x = r, r+1, \dots$

Mean:  $E(X) = \frac{r}{p}$

Variance:  $Var(X) = \frac{r(1-p)}{p^2}$

## IV) Continuous Probability Distributions

### Continuous random variables

Probability density function: a pdf specifies the probability of the random variable falling within a particular range of values, denoted by  $f(x)$

$$P(a \leq X \leq b) = \int_a^b f(x)dx, \text{ which is the area under the curve from } a \text{ to } b$$

$$P(X = a) = \int_a^a f(x)dx = 0 \text{ for all } a$$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \text{ (total probability rule)}$$

Cumulative distribution function: a cdf gives the probability that  $X$  is less than or equal to the value  $x$ , denoted by  $F(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt$

$$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a) \text{ (by the fundamental theorem of calculus)}$$

$$\text{Expected value: } \mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$\text{Variance: } \sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

(Note: Calculus is NOT required in our course)

### Uniform distribution

Uniform distribution: if  $X$  follows uniform distribution on the interval  $[a, b]$ , then it has the same probability density at any point in the interval and we denote it by  $X \sim U(a, b)$

$$\text{Pdf: } f(x) = \frac{1}{b-a} \text{ for } a \leq x \leq b, \text{ otherwise } 0$$

$$\text{Cdf: } F(x) = \int_a^x \frac{1}{b-a} dt = \left[ \frac{t}{b-a} \right]_a^x = \frac{x-a}{b-a} \text{ for } a \leq x \leq b$$

$$\text{Mean: } E(X) = \frac{a+b}{2}$$

$$\text{Variance: } Var(X) = \frac{(b-a)^2}{12}$$

### Normal distribution

Normal distribution: if  $X$  follows normal distribution with mean  $\mu$  and variance  $\sigma^2$ , then  $X \sim N(\mu, \sigma^2)$ , often used to represent continuous random variable with unknown distributions

Pdf:  $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$  for  $-\infty < x < \infty$

Shape: bell-shape, symmetric about the mean, unimodal

Standard normal distribution:  $Z \sim N(0,1)$

Cdf of standard normal: denoted as  $\Phi(z) = P(Z \leq z)$

$P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a) = \Phi(b) - \Phi(a)$

$\Phi(-z) = 1 - \Phi(z)$  by symmetric property

Percentile of standard normal:  $\Phi(1.645) = 0.95$ ,  $\Phi(1.96) = 0.975$

Standardization: if  $X \sim N(\mu, \sigma^2)$ , then  $\frac{X-\mu}{\sigma} \sim N(0,1)$

$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

De Moivre–Laplace theorem (normal approximation to binomial): if  $X \sim B(n, p)$ ,  $P(a < X < b) \approx P(a + 0.5 \leq Y \leq b - 0.5)$  where  $Y \sim N(np, np(1 - p))$ . The 0.5s are continuity correction

Normal approximation to poisson: if  $X \sim Po(\lambda)$ ,  $P(X \leq a) \approx P(Y \leq a + 0.5)$  where  $Y \sim N(\lambda, \lambda)$

### Some remarks (not required)

Statistical parameter: a numerical characteristic of a statistical population or a statistical model. We are given these numbers (e.g.  $p, \lambda, \mu$ ) in previous chapters but in reality we do not know these numbers. These lead to the next part of our course: Statistical Inference

Why approximation: one major reason is that calculating binomial probability involves combination and large factorials are hard/costly to compute in previous centuries

Variance of sum:  $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Tower rule of expectation:  $E(X) = E[E(X|Y)]$

Law of total variance (EVE):  $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

Sum of poisson: if  $X \sim Po(\lambda_1)$ ,  $Y \sim Po(\lambda_2)$  independently, then  $X + Y \sim Po(\lambda_1 + \lambda_2)$

Sum of normal: if  $X \sim N(\mu_1, \sigma_1^2)$ ,  $Y \sim N(\mu_2, \sigma_2^2)$  independently, then  $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Square of standard normal: if  $X \sim N(\mu, \sigma^2)$ , the  $Z^2 = \left[\frac{X-\mu}{\sigma}\right]^2 \sim \chi_1^2$

Sum of chi square: if  $X \sim \chi_n^2$ ,  $Y \sim \chi_m^2$ , then  $X + Y \sim \chi_{n+m}^2$

## V) Point Estimation

Statistical inference: process of drawing conclusions from data that are subject to random variations

Estimation: estimate the values of specific population parameters based on the observed data

Hypothesis testing: test on whether the value of a population parameter is equal to some specific value based on the observed data

### Sampling

Sample: the data obtained after the experiments are performed, usually denoted by  $x_1, \dots, x_n$

Random sample: the data before the experiments are performed, usually denoted by  $X_1, \dots, X_n$

Non-probability sample: some elements of the population have no chance of being selected

Probability sample: all elements in the population has known nonzero chance to be selected

Simple random sample: all elements in the population has the same probability to be selected

Systematic sample: elements are selected at regular intervals through certain order

Stratified sample: all elements are classified into different strata and each stratum is sampled as an independent sub-population

Cluster sample: all elements are divided into different clusters and a simple random sample of clusters is selected

Coverage error: exists if some groups are excluded from the frame and have no chance of being selected

Non-response error: people who do not respond may be different from those who do respond

Measurement error: due to weaknesses in question design, respondent error, and interviewer's impact on the respondent

Sampling error: Chance (luck of the draw) variation from sample to sample

### Point estimator

Point estimator: a rule for calculating a single value to "best guess" an unknown population parameter of interest based on the observed data

(Note: estimator  $\hat{\theta}(X)$  is random, estimate  $\hat{\theta}(x)$  is fixed, estimand  $\theta$  is the unknown parameter)

Unbiasedness:  $E(\hat{\theta}) = \theta$

Minimum variance:  $Var(\hat{\theta}) \leq Var(\tilde{\theta}) \quad \forall \tilde{\theta} \in \Theta$

Independent and identically distributed (i.i.d.): an assumption where the random variables  $X_1, \dots, X_n$  are sampled such that they are independent and follows the same distribution

Central limit theorem (CLT, Lindeberg–Lévy): Let  $X_1, \dots, X_n$  be i.i.d. random variables with mean  $\mu$  and finite variance  $\sigma^2$ , then as  $n$  tends to infinity ( $>30$  in practice),  $\bar{X} \xrightarrow{d} N\left(\mu, \frac{\sigma^2}{n}\right)$

### Mean

Estimand:  $\theta = \mu = E(X)$

Sample mean (estimator):  $\hat{\theta} = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$

Expectation:  $E(\bar{X}) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{n\mu}{n} = \mu$  (unbiased)

Variance:  $Var(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$  (by i.i.d.)

Distribution:  $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ . If  $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ , then this follows from the fact that sum of independent normal is normal (remarks in section IV). If  $X_1, \dots, X_n$  follows some other distribution, then this follows from the CLT when  $n$  is large (usually  $>30$ )

### Variance

Estimand:  $\theta = \sigma^2 = Var(X)$

Sample variance (estimator):  $\hat{\theta} = S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$

Expectation:  $E(S^2) = \sigma^2$  (unbiased)

Variance:  $Var(S^2) = \frac{2\sigma^4}{n-1}$  (not required)

Distribution:  $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2 \Rightarrow S^2 \sim \frac{\sigma^2}{n-1} \chi_{n-1}^2$  (right-skewed)

## VI) Interval Estimation

### Confidence interval

Confidence interval: an interval associated with a confidence level  $1 - \alpha$  that may contain the true value of an unknown population parameter

Meaning of confidence level: in the long run,  $100(1 - \alpha)\%$  of all the confidence intervals that can be constructed will contain the unknown true parameter (NOT the probability that an interval will contain the parameter)

Elements of confidence interval:  $\{\hat{\theta}, c_\alpha, se(\hat{\theta})\}$ , where  $\hat{\theta}$  is the point estimate,  $c_\alpha$  is the critical value from some distribution under the confidence level  $1 - \alpha$ ,  $se(\hat{\theta})$  is the standard error of the point estimate

### Mean

Confidence interval ( $\sigma$  is known):  $\bar{x} \pm z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$

Confidence interval ( $\sigma$  is unknown,  $n > 30$ ):  $\bar{x} \pm z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$

Confidence interval ( $\sigma$  is unknown,  $n \leq 30$ ):  $\bar{x} \pm t_{n-1, \frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$

Margin of error:  $E = c_\alpha \times se(\hat{\theta})$  (width is  $2E$  which helps determine sample size)

Critical values: standard normal and t-distribution are symmetric around 0  $\Rightarrow c_{1-\frac{\alpha}{2}} = c_{\frac{\alpha}{2}}$

### Variance

Confidence interval ( $\mu$  is known):  $\left( \frac{\sum_{i=1}^n (X_i - \mu)^2}{\chi^2_{n-1, 1-\frac{\alpha}{2}}}, \frac{\sum_{i=1}^n (X_i - \mu)^2}{\chi^2_{n, \frac{\alpha}{2}}} \right)$

Confidence interval ( $\mu$  is unknown):  $\left( \frac{(n-1)s^2}{\chi^2_{n-1, 1-\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{n-1, \frac{\alpha}{2}}} \right)$

Critical values: chi-squared distribution is not symmetric, so cannot simplify