STAT1012 Statistics for Life Sciences

Quick Revision Notes

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(Reference: lecture and tutorial notes)

Contents

I) Descriptive Statistics

Data type: Qualitative (Special: Categorical), Quantitative (Discrete, Continuous)

Population: the whole set of entities of interest

Sample: a subset of the population

Central tendency

Sample mean: $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Sequential update property: $\bar{X}_n = \frac{1}{n}[(n-1)\bar{X}_{n-1} + X_n]$

Mode: the value which has the greatest number of occurrence (may not be unique)

Median: the "middle" value, or the average of the two values closest to "middle" after sorting

Percentile: the p-th percentile ($V_{\frac{p}{100}}$) is a value such that p% of the data are less than or equal to $V_{\frac{p}{100}}$. In particular, upper quantile = $V_{0.75}$, median = $V_{0.5}$, lower quantile = $V_{0.25}$.

Denote the sorted data by $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ where $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. This is equivalent to saying that $X_{(1)}$ is the smallest, $X_{(2)}$ is the second smallest etc.

Median: $V_{0.5} = X_{\left(\frac{n+1}{2}\right)}$ if n is odd or $\frac{1}{2}\left[X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}\right]$ if n is even

Percentile: $V_{\frac{p}{100}} = X_{(k)}$ where $k = roundUp\left(\frac{np}{100}\right)$ if $\frac{np}{100}$ is not an integer.

Otherwise, $V_{\frac{p}{100}} = \frac{1}{2}\left[X_{\left(\frac{np}{100}\right)} + X_{\left(\frac{np}{100}+1\right)}\right]$

Dispersion

Symmetric: the left hand side of the distribution mirrors the right hand side

Unimodal: the mode is unique

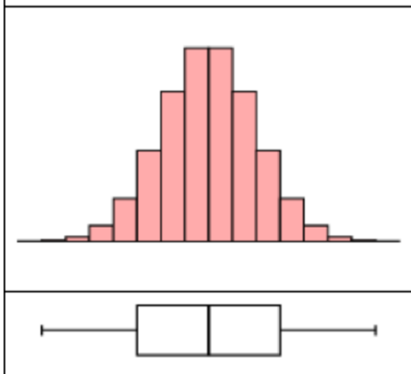
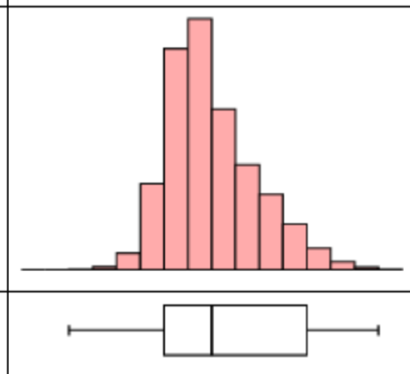
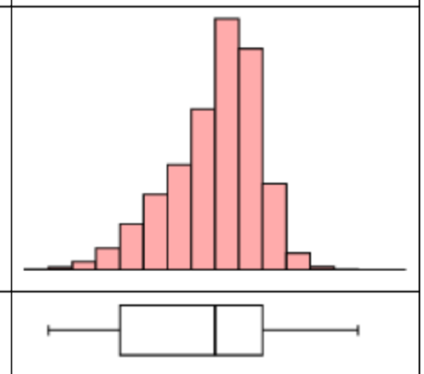
Skewness: measure of asymmetry

Left-skewed (negatively skewed): mean < median, have a few extreme small values

Right-skewed (positively skewed): mean > median, have a few extreme large values

Symmetric → mean = median (converse not true)

Symmetric + unimodal → mean = median = mode (converse not true)

Symmetric	Skewed right (positive)	Skewed left (negative)
		
Q_1 and Q_3 should be approximately equally spaced from the median (Q_2).	Q_3 is farther from the median(Q_2) than Q_1	Q_1 is farther from the median(Q_2) than Q_3

Range: maximum – minimum ($X_{(n)} - X_{(1)}$)

Interquartile range: $V_{0.75} - V_{0.25}$

Sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ or $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i^2 - n\bar{X}^2)$

Sample standard deviation: $SD = \sqrt{S^2}$

Graphical methods

Bar graph: use for categorical data, show the number of observations in each category

Histogram: use for quantitative data, showing the number of observations in each range

Stem-and-leaf plot: ordered the data into a tree-like structure

Boxplot: show 5 numbers (min, Q1, median, Q3, max), help locate outliers (As a rule of thumb, some people define outliers as values > Q3 + 1.5*IQR or < Q1 – 1.5*IQR)

II) Probability

Notations

Sample space: the set of all possible outcomes, often denoted as Ω

Outcome: a possible type of occurrence

Event: any set of outcomes of interest, can be denoted as $E \subset \Omega$

Probability (of an event): denoted by $P(E)$, always lies between 0 and 1 (both inclusive)

$$P(E) = \frac{\#\ of\ outcomes\ in\ E}{\#\ of\ outcomes\ in\ \Omega}$$

Union: either A or B occurs, or they both occurs, denoted by $A \cup B$ (logically equivalent to OR)

Intersection: both A and B occur, denoted by $A \cap B$ (logically equivalent to AND)

Complement: A does not occur, denoted by A^C (logically equivalent to NOT)

DeMorgan's laws: $(A \cup B)^C = A^C \cap B^C, (A \cap B)^C = A^C \cup B^C$

Probability theory

Mutually exclusive: A and B are mutually exclusive if $P(A \cap B) = 0$ (cannot co-occur)

Independence: $P(A \cap B) = P(A)P(B)$ iff A and B are independent. Their complements (A and B^C; A^C and B; A^C and B^C) will be pairwise independent as well

Addition law: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Multiplication law: if $A_1, \dots, A_k$ are mutually independent, then $P(A_! \cap \dots \cap A_k) = P(A_1) \times \dots \times P(A_k)$

Conditional probability

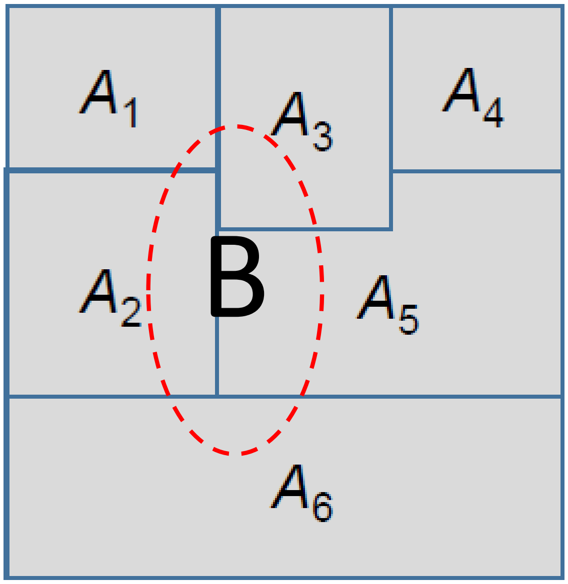
Conditional probability: $P(B|A) = \frac{P(A \cap B)}{P(A)}$, if $P(B|A) = P(B)$, then A and B are independent

Relative risk: $RR(B|A) = \frac{P(B|A)}{P(B|A^C)}$

Total probability rule: $P(B) = P(B|A)P(A) + P(B|A^C)P(A^C)$

Exhaustive: if $A_1, \dots, A_k$ are exhaustive, then $A_1 \cup \dots \cup A_k = \Omega$ (at least one of them must occur)

Generalized total probability rule: let $A_1, \dots, A_k$ be mutually exclusive and exhaustive events. For any event B, we have $P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$



Bayes' theorem: conditional probability + generalized total probability rule. let $A_1, \dots, A_k$ be mutually exclusive and exhaustive events. For any event B,

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{k} P\big(B\big|A_j\big)P(A_j)}$$

III) Discrete Probability Distributions

Random variables: numeric quantities that take different values with specified probabilities

Discrete random variable: a R.V. that takes value from a discrete set of numbers

Continuous random variable: a R.V. that takes value over an interval of numbers

Discrete random variables

Probability mass function: a pmf assigns a probability to each possible value x of the discrete random variable X, denoted by $f(x) = P(X = x)$

$\sum_{i=1}^{n} f(x_i) = 1$ (total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by $F(x) = P(X \leq x)$

Expected value: $\mu = E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$ (the idea is "probability weighted average")

Variance: $\sigma^2 = Var(X) = \sum_{i=1}^{n}(x_i - \mu)^2 P(X = x_i)$, alternatively $Var(X) = E(X^2) - [E(X)]^2$

Translation/rescale: $E(aX + b) = aE(X) + b$, $Var(aX + b) = a^2 Var(X)$

Linearity of expectation: $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$

Binomial distribution

Factorial: $n! = n \times (n-1) \times \dots \times 1$, note that $0! = 1$

Permutation (order is important): $P_k^n = \frac{n!}{(n-k)!}$

Combination (order is not important): $C_k^n = \frac{n!}{k!(n-k)!}$, also denoted as $\binom{n}{k}$

Binomial distribution: probability distribution on the number of successes X in n independent experiments, each experiment has a probability of success p, then $X \sim B(n, p)$

Pmf: $P(X = x) = \binom{n}{k} p^x (1-p)^{1-x}$ for $x = 0, 1, 2, \dots, n$

Mean: $E(X) = np$

Variance: $Var(X) = np(1-p)$

Skewness: right-skewed if p<0.5, symmetric if p=0.5, left-skewed if p>0.5

Poisson distribution

Poisson distribution: probability distribution on the number of occurrence X (usually of a rare event) over a period of time or space with rate μ, then $X{\sim}Po(\mu)$

Pmf: $P(X = x) = \frac{e^{-\mu}\mu^k}{k!}$ for $k = 0, 1, 2, \ldots$

Mean: $E(X) = \mu$

Variance: $Var(X) = \mu$

Skewness: right-skewed

Poisson limit theorem (poisson approximation to binomial): if $X{\sim}B(n, p)$ where $n \geq 20,\ p < 0.1$ and $np < 5$, then $X \approx Y{\sim}Po(\mu)$ where $\mu = np$

IV) Continuous Probability Distributions

Continuous random variables

Probability density function: a pdf specifies the probability of the random variable falling within a particular range of values, denoted by $f(x)$

$P(a \leq X \leq b) = \int_a^b f(x)dx$, which is the area under the curve from a to b

$P(X = a) = \int_a^a f(x)dx = 0$ for all a

$\int_{-\infty}^{\infty} f(x)dx = 1$ (total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$

$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$ (by the fundamental theorem of calculus)

Expected value: $\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$

Variance: $\sigma^2 = Var(X) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

Uniform distribution

Uniform distribution: if X follows uniform distribution on the interval $[a, b]$, then it has the same probability density at any point in the interval and we denote it by $X \sim U(a, b)$

Pdf: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, otherwise 0

Cdf: $F(x) = \int_a^x \frac{1}{b-a} dt = \left[\frac{t}{b-a}\right]_a^x = \frac{x-a}{b-a}$ for $a \leq x \leq b$

Mean: $E(X) = \frac{a+b}{2}$

Variance: $Var(X) = \frac{(b-a)^2}{12}$

Normal distribution

Normal distribution: if X follows normal distribution with mean μ and variance σ^2, then $X \sim N(\mu, \sigma^2)$, often used to represent continuous random variable with unknown distributions

Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ for $-\infty < x < \infty$

Shape: bell-shape, symmetric about the mean, unimodal

Standard normal distribution: $Z \sim N(0,1)$

Cdf of standard normal: denoted as $\Phi(z) = P(Z \leq z)$

$P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a) = \Phi(b) - \Phi(a)$

$\Phi(-z) = 1 - \Phi(z)$ by symmetric property

Percentile of standard normal: $\Phi(1.645) = 0.95, \Phi(1.96) = 0.975$

Standardization: if $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

De Moivre–Laplace theorem (normal approximation to binomial): if $X \sim B(n,p)$, $P(a < X < b) \approx P(a + 0.5 \leq Y \leq b - 0.5)$ where $Y \sim N(np, np(1-p))$. The 0.5s are continuity correction

Normal approximation to poisson: if $X \sim Po(\lambda), P(X \leq a) \approx P(Y \leq a + 0.5)$ where $Y \sim N(\lambda, \lambda)$

Some remarks (not required)

Statistical parameter: a numerical characteristic of a statistical population or a statistical model. We are given these numbers (e.g. p, λ, μ) in previous chapters but in reality we do not know these numbers. These lead to the next part of our course: Statistical Inference

Why approximation: one major reason is that calculating binomial probability involves combination and large factorials are hard/costly to compute in previous centuries

Variance of sum: $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Tower rule of expectation: $E(X) = E[E(X|Y)]$

Law of total variance (EVE): $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

Sum of poisson: if $X \sim Po(\lambda_1), Y \sim Po(\lambda_2)$ independently, then $X + Y \sim Po(\lambda_1 + \lambda_2)$

Sum of normal: if $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$ independently, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$