RMSC5102 Simulation Methods for Risk Management Science and Finance

Tutorial Notes

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Contents

I) Probability and statistics

Discrete random variables

Random variables: numeric quantities that take different values with specified probabilities

Discrete random variable: a R.V. that takes value from a discrete set of numbers

Probability mass function: a pmf assigns a probability to each possible value x of the discrete random variable X, denoted by $f(x) = P(X = x)$

$\sum_{i=1}^{n} f(x_i) = 1$ (total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by $F(x) = P(X \leq x)$

Expected value: $\mu = E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$ (the idea is "probability weighted average")

Variance: $\sigma^2 = Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 P(X = x_i)$, alternatively $Var(X) = E(X^2) - [E(X)]^2$

Translation/rescale: $E(aX + b) = aE(X) + b$, $Var(aX + b) = a^2 Var(X)$

Linearity of expectation: $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$

Binomial distribution

Factorial: $n! = n \times (n-1) \times \ldots \times 1$, note that $0! = 1$

Permutation (order is important): $P_k^n = \frac{n!}{(n-k)!}$

Combination (order is not important): $C_k^n = \frac{n!}{k!(n-k)!}$, also denoted as $\binom{n}{k}$

Binomial distribution: probability distribution on the number of successes X in n independent experiments, each experiment has a probability of success p, then $X \sim B(n, p)$

Pmf: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$

Mean: $E(X) = np$

Variance: $Var(X) = np(1-p)$

Poisson distribution

Poisson distribution: probability distribution on the number of occurrence X (usually of a rare event) over a period of time or space with rate λ, then $X \sim Po(\lambda)$. Useful in modelling jump.

Pmf: $P(X = x) = \frac{e^{-\lambda}\mu^x}{x!}$ for $x = 0, 1, 2, \ldots$

Mean: $E(X) = \lambda$

Variance: $Var(X) = \lambda$

Continuous random variables

Continuous random variable: a R.V. that takes value over an interval of numbers

Probability density function: a pdf specifies the probability of the random variable falling within a particular range of values, denoted by $f(x)$

$P(a \le X \le b) = \int_a^b f(x)dx$, which is the area under the curve from a to b

$P(X = a) = \int_a^a f(x)dx = 0$ for all a

$\int_{-\infty}^{\infty} f(x)dx = 1$ (total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$

$P(a \le X \le b) = \int_a^b f(x)dx = F(b) - F(a)$ (by the fundamental theorem of calculus)

Expected value: $\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$

Variance: $\sigma^2 = Var(X) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

Uniform distribution

Uniform distribution: if X follows uniform distribution on the interval $[a, b]$, then it has the same probability density at any point in the interval and we denote it by $X \sim U(a, b)$. Basic R.V. in probability integral transform.

Pdf: $f(x) = \frac{1}{b-a}$ for $a \le x \le b$, otherwise 0

Cdf: $F(x) = \int_a^x \frac{1}{b-a} dt = \left[\frac{t}{b-a}\right]_a^x = \frac{x-a}{b-a}$ for $a \le x \le b$

Mean: $E(X) = \frac{a+b}{2}$

Variance: $Var(X) = \frac{(b-a)^2}{12}$

Normal distribution

Normal distribution: if X follows normal distribution with mean μ and variance σ^2, then $X \sim N(\mu, \sigma^2)$. Often used to represent continuous random variable with unknown distributions

Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ for $-\infty < x < \infty$

Standard normal distribution: $Z \sim N(0,1)$

Cdf of standard normal: denoted as $\Phi(z) = P(Z \leq z)$

$P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a) = \Phi(b) - \Phi(a)$

$\Phi(-z) = 1 - \Phi(z)$ by symmetric property

Percentile of standard normal: $\Phi(1.645) = 0.95, \Phi(1.96) = 0.975$

Standardization: if $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

Some remarks

Variance of sum: $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

Tower rule of expectation: $E(X) = E[E(X|Y)]$

Law of total variance (EVE): $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

Sum of poisson: if $X \sim Po(\lambda_1), Y \sim Po(\lambda_2)$ independently, then $X + Y \sim Po(\lambda_1 + \lambda_2)$

Sum of normal: if $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$ independently, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Square of standard normal: if $X \sim N(\mu, \sigma^2)$, the $Z^2 = \left[\frac{X-\mu}{\sigma}\right]^2 \sim \chi_1^2$

Sum of chi square: if $X \sim \chi_n^2, Y \sim \chi_m^2$, then $X + Y \sim \chi_{n+m}^2$

II) Financial derivative

Forward

Payoff: $\mathrm{S_t} - K\ (long), K - S_t\ (short)$

Pricing: $\mathrm{f = S - Ke^{-r(T-t)}, F = Se^{r(T-t)}}$

With known cash income: $\mathrm{f = S - I - Ke^{-r(T-t)}, F = (S - I)e^{r(T-t)}}, I = PV(income)$

With known dividend yield: $\mathrm{f = Se^{-q(T-t)} - Ke^{-r(T-t)}, F = Se^{(r-q)(T-t)}}$

Minimum variance hedge ratio: $\mathrm{h^* = \rho \times \frac{\sigma_S}{\sigma_F}} \Rightarrow N_F^* = h^* \times N_A\ (since\ h = \frac{N_F}{N_S})$

Option

Upper bounds: $\mathrm{C_E} \leq C_A \leq S, P_E \leq Ke^{-r(T-t)}, P_A \leq K$

Lower bounds: $\max\left(\mathrm{S - Ke^{-r(T-t)}}, 0\right) \leq C_E \leq C_A, \max\left(\mathrm{Ke^{-r(T-t)}} - S, 0\right) \leq P_E \leq P_A$

Put-call parity: $\mathrm{C_E} - P_E = S - I - Ke^{-r(T-t)}$ (idea is call – put = forward)

Put call inequality: $\mathrm{S - K \leq C_A} - P_A \leq S - Ke^{-r(T-t)}$

European-American relationship: $\mathrm{P_A} > P_E, C_A = C_E$ (for non-dividend-paying)

Binomial tree

Risk neutral probability: $\mathrm{q} = \frac{\mathrm{e^{r\delta t}} - d}{u - d}, \mathrm{u = e^{\sigma\sqrt{\delta t}}}, d = u^{-1} = \mathrm{e^{-\sigma\sqrt{\delta t}}}$

Pricing: $\mathrm{f = e^{-r\delta t}}[qf_u + (1 - q)f_d]$

Backward induction: start from payoff as terminal prices (American: take max between payoff and f)

III) Stochastic calculus

Brownian motion

Wiener process: W_t is called a Wiener process if the following holds

Stationary increment: $W_t - W_s \sim N(0, t-s)$

Independent increment: $W_{t_4} - W_{t_3} \perp W_{t_2} - W_{t_1}$

Starts at zero: $P\left(W_{t_0} = 0\right) = 1$

Properties: $\mathrm{Cov}(W_s, W_t) = \min(s,t)$, $[dW_t]^2 = dt$ (quadratic variation), nowhere differentiable

Itô's process: X_t is an Itô's process if it is solution to the following stochastic differential equation

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = a \end{cases}$$

Where $\mu(t, X_t)$ is known as the drift function and $\sigma(t, X_t)$ is known as the volatility function. **You may think $dX_t \approx X_{t+\delta t} - X_t$ and $dt \approx \delta t$ (useful in simulation)**

Stochastic integral

Definition: $\int_0^T f(s, W_s)dW_s = \lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} f(t_j, W_{t_j})(W_{t_{j+1}} - W_{t_j})$

Itô's lemma: $\mathrm{d}f(\mathrm{t}, \mathrm{X_t}) = \left[\frac{\partial f}{\partial t} + \mu(t, X_t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}\right]\mathrm{dt} + \sigma(t, X_t)\frac{\partial f}{\partial x}dW_t$

Geometric Brownian motion: $\mathrm{dS_t} = rS_t dt + \sigma S_t dW_t \Rightarrow \mathrm{S_t} = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$

Solving SDE: "guess" a solution and use Itô's lemma to verify that the solution satisfies the SDE (the following table is borrowed from Prof. Yau Chun Yip's notes on Stochastic Calculus):

Name	SDE	Solution (X_t)
Ornstein-Uhlenbeck(OU) process	$dX_t = -\alpha X_t dt + \sigma dW_t$	$ce^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW_s$
Mean reverting OU	$dX_t = (m - \alpha X_t)dt + \sigma dW_t$	$\frac{m}{\alpha} + \left(c - \frac{m}{\alpha}\right)e^{-\alpha t} + \sigma\int_0^t e^{\alpha(s-t)}dW_s$
Geometric Brownian motion	$dX_t = aX_t dt + bX_t dW_t$	$ce^{(a-b^2/2)t + bW_t}$
Brownian bridge	$dX_t = \frac{b - X_t}{1-t}dt + dW_t$	$a(1-t) + bt + (1-t)\int_0^t \frac{dW_s}{1-s}$
	$dX_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}dW_t$	$\sinh(c + t + W_t)$
	$dX_t = X_t^3 dt + X_t^2 dW_t$	$\frac{1}{c - W_t}$
	$dX_t = -\frac{1}{2}X_t dt + \sqrt{1 - X_t^2}dW_t$	$\sin(c + W_t)$
	$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dW_t$	$(c + W_t)/(1+t)$
	$dX_t = rdt + \alpha X_t dW_t$	$ce^{\alpha W_t - \frac{1}{2}\alpha^2 t} + r\int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)}ds$

Martingale property: $\mathrm{E}\left[\int_0^T f(t,W_t)dW_t \middle| \mathcal{F}_s\right] = \int_0^s f(t,W_t)dW_t$

In particular, $\mathrm{E}\left(\int_0^T f(t,W_t)dW_t\right) = 0$

Itô isometry: $\mathrm{E}\left[\left(\int_0^T f(t,W_t)dW_t\right)^2\right] = \int_0^T E[f(t,W_t)^2]dt$

Similarly, $\mathrm{E}\left[\left(\int_0^T f(t,W_t)dW_t\right)\left(\int_0^T g(t,W_t)dW_t\right)\right] = \int_0^T E[f(t,W_t)g(t,W_t)]dt$

Product rule: $\mathrm{d}(\mathrm{X_t}Y_t) = X_t dY_t + Y_t dX_t + d[\sigma(t,X_t)W_t, \bar{\sigma}(t,Y_t)\bar{W}_t]$

Integrating factor: add e^{rt} to both sides of a SDE (target: cancel some terms)

IV) Simulation methods

Theoretical support

Sample mean: $\bar{X}_n = \sum_{i=1}^{n} X_i$

Sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$

Law of large numbers (WLLN): Let $X_1, \dots, X_n$ be i.i.d. random variables with mean θ and variance σ^2, then for any given $\epsilon > 0$, $P(|\bar{X}_n - \theta| > \epsilon) \to 0$ as $n \to \infty$

Central limit theorem (CLT, Lindeberg–Lévy): Let $X_1, \dots, X_n$ be i.i.d. random variables with mean θ and finite variance σ^2, then $\bar{X}_n \xrightarrow{d} N\left(\theta, \frac{\sigma^2}{n}\right)$ as $n \to \infty$

Standard Monte Carlo

Procedure:

1) Generate random variable $\mathrm{X_i}$
2) Calculate $\mathrm{h_i} = h(X_i)$, where h is the target function
3) Repeat 1 and 2 for n times
4) $\hat{\theta} = \frac{1}{n}\sum_{j=1}^{n} h_j$ (remember to do discounting if necessary)