

RMSC5102 Simulation Methods for Risk Management Science and Finance

Quick Revision Notes

Spring, 2020

LEUNG Man Fung, Heman

(Reference: lecture notes)

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I) Probability and statistics

Discrete random variables

Random variables: numeric quantities that take different values with specified probabilities

Discrete random variable: a R.V. that takes value from a discrete set of numbers

Probability mass function: a pmf assigns a probability to each possible value x of the discrete random variable X , denoted by $f(x) = P(X = x)$

$$\sum_{i=1}^n f(x_i) = 1 \text{ (total probability rule)}$$

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x , denoted by $F(x) = P(X \leq x)$

Expected value: $\mu = E(X) = \sum_{i=1}^n x_i P(X = x_i)$ (the idea is “probability weighted average”)

Variance: $\sigma^2 = Var(X) = \sum_{i=1}^n (x_i - \mu)^2 P(X = x_i)$, alternatively $Var(X) = E(X^2) - [E(X)]^2$

Translation/rescale: $E(aX + b) = aE(X) + b$, $Var(aX + b) = a^2 Var(X)$

Linearity of expectation: $E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i)$

Binomial distribution

Factorial: $n! = n \times (n - 1) \times \dots \times 1$, note that $0! = 1$

Permutation (order is important): $P_k^n = \frac{n!}{(n-k)!}$

Combination (order is not important): $C_k^n = \frac{n!}{k!(n-k)!}$, also denoted as $\binom{n}{k}$

Binomial distribution: probability distribution on the number of successes X in n independent experiments, each experiment has a probability of success p , then $X \sim B(n, p)$

Pmf: $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$ for $x = 0, 1, 2, \dots, n$

Mean: $E(X) = np$

Variance: $Var(X) = np(1 - p)$

Poisson distribution

Poisson distribution: probability distribution on the number of occurrence X (usually of a rare event) over a period of time or space with rate μ , then $X \sim Po(\mu)$

Pmf: $P(X = x) = \frac{e^{-\mu} \mu^x}{x!}$ for $x = 0, 1, 2, \dots$

Mean: $E(X) = \mu$

Variance: $Var(X) = \mu$

Continuous random variables

Continuous random variable: a R.V. that takes value over an interval of numbers

Probability density function: a pdf specifies the probability of the random variable falling within a particular range of values, denoted by $f(x)$

$P(a \leq X \leq b) = \int_a^b f(x)dx$, which is the area under the curve from a to b

$P(X = a) = \int_a^a f(x)dx = 0$ for all a

$\int_{-\infty}^{\infty} f(x)dx = 1$ (total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x , denoted by $F(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt$

$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$ (by the fundamental theorem of calculus)

Expected value: $\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$

Variance: $\sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

Uniform distribution

Uniform distribution: if X follows uniform distribution on the interval $[a, b]$, then it has the same probability density at any point in the interval and we denote it by $X \sim U(a, b)$

Pdf: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$, otherwise 0

Cdf: $F(x) = \int_a^x \frac{1}{b-a} dt = \left[\frac{t}{b-a} \right]_a^x = \frac{x-a}{b-a}$ for $a \leq x \leq b$

Mean: $E(X) = \frac{a+b}{2}$

Variance: $Var(X) = \frac{(b-a)^2}{12}$

Normal distribution

Normal distribution: if X follows normal distribution with mean μ and variance σ^2 , then $X \sim N(\mu, \sigma^2)$, often used to represent continuous random variable with unknown distributions

Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ for $-\infty < x < \infty$

Standard normal distribution: $Z \sim N(0,1)$

Cdf of standard normal: denoted as $\Phi(z) = P(Z \leq z)$

$$P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a) = \Phi(b) - \Phi(a)$$

$\Phi(-z) = 1 - \Phi(z)$ by symmetric property

Percentile of standard normal: $\Phi(1.645) = 0.95$, $\Phi(1.96) = 0.975$

Standardization: if $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Some remarks

Variance of sum: $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Tower rule of expectation: $E(X) = E[E(X|Y)]$

Law of total variance (EVE): $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

Sum of poisson: if $X \sim Po(\lambda_1)$, $Y \sim Po(\lambda_2)$ independently, then $X + Y \sim Po(\lambda_1 + \lambda_2)$

Sum of normal: if $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ independently, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Square of standard normal: if $X \sim N(\mu, \sigma^2)$, the $Z^2 = \left[\frac{X-\mu}{\sigma}\right]^2 \sim \chi_1^2$

Sum of chi square: if $X \sim \chi_n^2$, $Y \sim \chi_m^2$, then $X + Y \sim \chi_{n+m}^2$

II) Financial derivative

Forward

Payoff: $S_t - K$ (*long*), $K - S_t$ (*short*)

Pricing: $f = S - Ke^{-r(T-t)}$, $F = Se^{r(T-t)}$

With known cash income: $f = S - I - Ke^{-r(T-t)}$, $F = (S - I)e^{r(T-t)}$, $I = PV(\text{income})$

With known dividend yield: $f = Se^{-q(T-t)} - Ke^{-r(T-t)}$, $F = Se^{(r-q)(T-t)}$

Minimum variance hedge ratio: $h^* = \rho \times \frac{\sigma_S}{\sigma_F} \Rightarrow N_F^* = h^* \times N_A$ (since $h = \frac{N_F}{N_S}$)

Option

Upper bounds: $C_E \leq C_A \leq S$, $P_E \leq Ke^{-r(T-t)}$, $P_A \leq K$

Lower bounds: $\max(S - Ke^{-r(T-t)}, 0) \leq C_E \leq C_A$, $\max(Ke^{-r(T-t)} - S, 0) \leq P_E \leq P_A$

Put-call parity: $C_E - P_E = S - I - Ke^{-r(T-t)}$ (call - put = forward)

Put call inequality: $S - K < C_A - P_A < S - Ke^{-r(T-t)}$, $P_A > P_E$, $C_A = C_E$ (for non-dividend-paying)

Binomial tree

Risk neutral probability: $q = \frac{e^{r\delta t} - d}{u - d}$, $u = e^{\sigma\sqrt{\delta t}}$, $d = u^{-1} = e^{-\sigma\sqrt{\delta t}}$

Pricing: $f = e^{-r\delta t}[qf_u + (1 - q)f_d]$

Backward induction: starts from payoff as terminal prices (American: take max between payoff and f)

III) Stochastic calculus

Brownian motion

Wiener process: $W_t - W_s \sim N(0, t - s)$, $W_{t_4} - W_{t_3} \perp W_{t_2} - W_{t_1}$, $P(W_{t_0} = 0) = 1$

Properties: $\text{Cov}(W_s, W_t) = \min(s, t)$, $[dW_t]^2 = dt$ (quadratic variation), nowhere differentiable

Itô's process: $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$, $X_0 = a$

Stochastic integral

Definition: $\int_0^T f(s, W_s) dW_s = \lim_{\delta(\pi) \rightarrow 0} \sum_{j=0}^{N(\pi)-1} f(t_j, W_{t_j})(W_{t_{j+1}} - W_{t_j})$

Martingale property: $E \left[\int_0^T f(t, W_t) dW_t \middle| \mathcal{F}_s \right] = \int_0^s f(t, W_t) dW_t$. In particular, $E \left(\int_0^T f(t) dW_t \right) = 0$

Itô isometry: $E \left[\left(\int_0^T f(t, W_t) dW_t \right)^2 \right] = \int_0^T E[f(t, W_t)^2] dt$ (applicable on product of stochastic integrals)

Itô's lemma: $df(t, X_t) = \left[\frac{\partial f}{\partial t} + \mu(t, X_t) \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2 f}{\partial x^2} \right] dt + \sigma(t, X_t) \frac{\partial f}{\partial x} dW_t$

Product rule: $d(X_t Y_t) = X_t dY_t + Y_t dX_t + d[\sigma(t, X_t)W_t, \bar{\sigma}(t, Y_t)\bar{W}_t]$

Geometric Brownian motion: $dS_t = rS_t dt + \sigma S_t dW_t \Rightarrow S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$

Leibniz integral rule: $\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x)) \cdot b'(x) - f(x, a(x)) \cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt$

Integrating factor: add e^{rt} to both sides of a SDE (target: cancel some terms)