RMSC5102 Simulation Methods for Risk Management Science and Finance

Tutorial Notes

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# I) Probability and statistics

## Discrete random variables

Random variables: numeric quantities that take different values with specified probabilities

Discrete random variable: a R.V. that takes value from a discrete set of numbers

Probability mass function: a pmf assigns a probability to each possible value x of the discrete random variable X, denoted by

(total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by

Expected value: (the idea is “probability weighted average”)

Variance: (the idea is “probability weighted distance from mean”)

Alternatively

Translation/rescale: ,

Linearity of expectation:

## Binomial distribution

Factorial: , note that

Permutation (order is important):

Combination (order is not important): , also denoted as

Binomial distribution: probability distribution on the number of successes in independent experiments, each experiment has a probability of success , then

Pmf: for

Mean:

Variance:

## Poisson distribution

Poisson distribution: probability distribution on the number of occurrence (usually of a rare event) over a period of time or space with rate , then . Useful in modelling jump

Pmf: for

Mean:

Variance:

## Continuous random variables

Continuous random variable: a R.V. that takes value over an interval of numbers

Probability density function: a pdf specifies the probability of the random variable falling within a particular range of values, denoted by

, which is the area under the curve from a to b

for all

(total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by

(by the fundamental theorem of calculus)

Expected value: (the idea is “probability weighted average”)

Variance: (the idea is “probability weighted distance from mean”)

## Uniform distribution

Uniform distribution: if follows uniform distribution on the interval , then it has the same probability density at any point in the interval and we denote it by . Basic R.V. in probability integral transform

Pdf: for , otherwise 0

Cdf: for

Mean:

Variance:

## Normal distribution

Normal distribution: if follows normal distribution with mean and variance , then . Often used to represent continuous random variable with unknown distributions

Pdf: for

Standard normal distribution:

Cdf of standard normal: denoted as

by symmetric property

Percentile of standard normal:

Standardization: if , then

## Some remarks

Variance of sum:

Tower rule of expectation:

Law of total variance (EVE):

Sum of poisson: if independently, then

Sum of normal: if independently, then

Square of standard normal: if , the

Sum of chi square: if , then

# II) Financial derivative

## Forward

Payoff:

Pricing:

With known cash income:

With known dividend yield:

Minimum variance hedge ratio:

## Option

Upper bounds:

Lower bounds:

Put-call parity: (idea is call – put = forward)

Put call inequality:

European-American relationship: (for non-dividend-paying)

## Binomial tree

Risk neutral probability:

Pricing:

Backward induction: start from payoff as terminal prices (American: take max between payoff and f)

## Black–Scholes–Merton model

Black-Scholes equation:

Black-Scholes formula: ,

where

Implied volatility: the value of volatility when back-solving an option pricing model (such as BS) with current market price

# III) Stochastic calculus

## Brownian motion

Wiener process: is called a Wiener process if the following holds

Stationary increment:

Independent increment:

Starts at zero:

Properties: (quadratic variation), nowhere differentiable

Itô’s process: is an Itô’s process if it is solution to the following stochastic differential equation

Where is known as the drift function and is known as the volatility function. **You may think and (useful in simulation)**

## Stochastic integral

Definition:

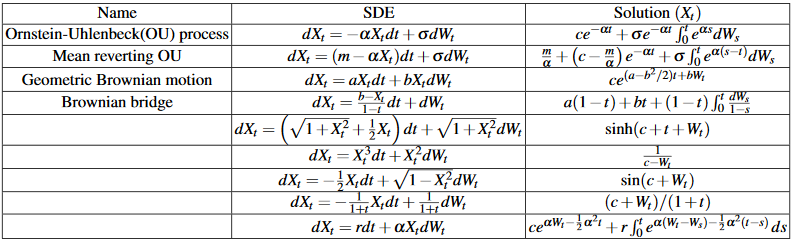
**Itô’s lemma:**

**Geometric Brownian motion:**

**Consequently,** where

Finding stochastic integral: “guess” the function such that it will contain the integrand in its SDE. Use Itô’s lemma to find SDE of the guess and then integrate both sides

Solving SDE: “guess” a solution and use Itô’s lemma to verify that the solution satisfies the SDE (the following table is borrowed from Prof. Yau Chun Yip’s notes on Stochastic Calculus)



Integrating factor: add to both sides of a SDE (target: cancel some terms)

Martingale property:

In particular,

Itô isometry:

Similarly,

Product rule:

# IV) Simulation methods

## Theoretical support

Sample mean:

Sample variance:

Law of large numbers (WLLN): Let be i.i.d. random variables with mean and variance , then for any given , as

Central limit theorem (CLT, Lindeberg–Lévy): Let be i.i.d. random variables with mean and finite variance , then as

## Standard Monte Carlo

Idea: take average of independent replications/scenarios of the reality/future

Algorithm:

1. Generate random variable
2. Calculate , where is the target function
3. Repeat 1 and 2 for n times
4. (remember to do discounting if necessary)

## Inverse transform

Idea: if we know (i.e. the cdf), we can generate out of uniform random numbers

Algorithm (discrete):

1. Generate
2. if

Algorithm (continuous):

1. Generate
2. assuming the inverse exists

## Rejection sampling

Idea: if we can simulate easily, we can use the proportional distribution as a basis to simulate with pdf

Algorithm:

1. Find
2. Generate from a density g:
3. Generate
4. If , set , otherwise return to 2

Number of iterations needed:

# V) Variance reduction

## Antithetic variables

Idea: if we are able to generate negatively correlated underlying random variables, the estimator can have lower variance as compared with independent samples. This requires the target function to be monotone

Algorithm:

1. Generate
2. Set (note: want X, Y same distribution but negative correlation)
3. Repeat 1 and 2 for n times

Useful corollary: if is monotone, then where

## Stratified sampling

Idea: if we have information about grouping in the population, then we may use conditional mean (mean of subgroup) as the sample from the population

Algorithm:

1. Generate where for
2. Set
3. (average over subsamples and bins, remember to adjust for conditional probability)

## Control variate

Idea: if we combine the estimate of our target unknown quantity with estimates of some known quantities, we can exploit the known information

Algorithm:

1. Find for with a known distribution (or estimate via pilot simulation)
2. Generate for
3. Compute