RMSC5102 Simulation Techniques in Risk Management and Finance

Tutorial Notes

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# I) Probability and statistics

## Discrete random variables

Random variables: numeric quantities that take different values with specified probabilities

Discrete random variable: a r.v. that takes value from a discrete set of numbers

Probability mass function: a pmf assigns a probability to each possible value of the discrete random variable , denoted by

(total probability rule)

Cumulative distribution function: a cdf gives the probability that X is less than or equal to the value x, denoted by

Expected value: (the idea is “probability weighted average”)

Variance: (the idea is “probability weighted distance from mean”)

Alternatively,

Translation/rescale: ,

Linearity of expectation:

Law of the unconscious statistician:

Moment generating function:

## Binomial distribution

Factorial: , note that

Permutation (order is important):

Combination (order is not important): , also denoted as

Binomial distribution: probability distribution on the number of successes in independent experiments, each experiment has a probability of success ; denoted by

If , then (sum of i.i.d. Bernoulli r.v.s is a binomial)

Pmf: for

Mgf:

Mean:

Variance:

## Poisson distribution

Poisson distribution: probability distribution on the number of occurrence (usually of a rare event) over a period of time or space with rate ; denoted by . Useful in modelling jump

Pmf: for

Mgf:

Mean:

Variance:

## Geometric distribution

Geometric distribution: probability distribution on the number of Bernoulli trials needed to get 1 success, each trial has a probability of success ; denoted by

Pmf: for

Mgf: for

Mean:

Variance:

## Continuous random variables

Continuous random variable: a r.v. that takes value over an interval of numbers

Probability density function: a pdf specifies the probability of the random variable falling within a particular range of values, denoted by

, which is the area under the curve from a to b

for all

(total probability rule)

Cumulative distribution function: a cdf gives the probability that is less than or equal to the value , denoted by

(by the fundamental theorem of calculus)

Expected value: (the idea is “probability weighted average”)

Variance: (the idea is “probability weighted distance from mean”)

Alternatively,

Translation/rescale: ,

Linearity of expectation:

Law of the unconscious statistician:

Moment generating function:

## Uniform distribution

Uniform distribution: if follows uniform distribution on the interval , then it has the same probability density at any point in the interval and we denote it by . Basic R.V. in inverse transform

Pdf: for , otherwise 0

Cdf: for

Mgf: for

Mean:

Variance:

## Normal distribution

Normal distribution: if follows normal distribution with mean and variance , then . Often used to represent continuous random variable with unknown distributions

Pdf: for

Mgf:

Standard normal distribution:

Cdf of standard normal: denoted as

by symmetric property

Percentile of standard normal:

Standardization: if , then

## Exponential distribution

Exponential distribution: if follows exponential distribution with rate , we denote it by . Continuous analogue of the geometric distribution

Pdf: for , otherwise 0

Cdf: for

Mgf: for

Mean:

Variance:

Memoryless property: for

Exponential distribution is the only continuous distribution that has this property

Useful representation: if , then

## Some remarks

Covariance:

Variance of sum:

Tower rule of expectation:

Law of total variance (EVE):

Sum of poisson: if independently, then

Sum of normal: if independently, then

Square of standard normal: if , the

Sum of chi square: if , then

# II) Financial derivative

## Forward

Payoff:

Pricing:

With known cash income:

With known dividend yield:

Minimum variance hedge ratio:

## Option

Upper bounds:

Lower bounds:

Put-call parity: (idea is call – put = forward)

Put call inequality:

European-American relationship: (for non-dividend-paying)

## Binomial tree

Risk neutral probability:

Pricing:

Backward induction: start from payoff as terminal prices (American: take max between payoff and )

## Black–Scholes–Merton model

Black-Scholes equation:

Black-Scholes formula: ,

where

Implied volatility: the value of volatility when back-solving an option pricing model (such as BS) with current market price

# III) Stochastic calculus

## Brownian motion

Wiener process: is called a Wiener process if the following holds

Stationary increment:

Independent increment:

Starts at zero:

Properties: (quadratic variation), nowhere differentiable

Itô’s process: is an Itô’s process if it is solution to the following stochastic differential equation

Where is known as the drift function and is known as the volatility function. **You may think and (useful in simulation)**

## Stochastic integral

Definition:

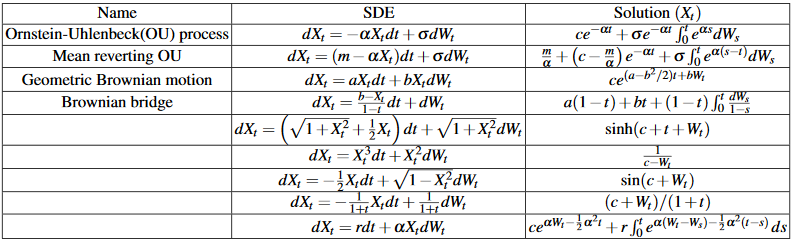
**Itô’s lemma:**

**Geometric Brownian motion:**

**Consequently,** where

Finding stochastic integral: “guess” the function such that it will contain the integrand in its SDE. Use Itô’s lemma to find SDE of the guess and then integrate both sides

Solving SDE: “guess” a solution and use Itô’s lemma to verify that the solution satisfies the SDE (the following table is borrowed from Prof. Yau Chun Yip’s notes on Stochastic Calculus)



Integrating factor: add to both sides of a SDE (target: cancel some terms)

Martingale property:

In particular,

Itô isometry:

Similarly,

Product rule:

# IV) Simulation methods

## Theoretical support

Sample mean:

Sample variance:

Law of large numbers (WLLN): Let be i.i.d. random variables with mean and variance , then   as

Central limit theorem: Let be i.i.d. random variables with mean and finite variance , then as

Modulo operation: find the remainder of a division; denoted by mod. Commonly used in generating pseudorandom number

Probability integral transform: if is a continuous random variable with cdf , then . In other words, if we can find the cdf inverse , then and have the same distributions

## Standard Monte Carlo

Idea: take average of independent replications/scenarios of the reality/future

Algorithm:

1. Generate random variable
2. Calculate , where is the target function
3. Repeat 1 and 2 for n times
4. (remember to do discounting if necessary)

Margin of error: terminate a simulation when  , where is the sample variance and is the maximum tolerable error

## Inverse transform

Idea: if we know (i.e., the cdf), we can generate out of uniform random numbers

Algorithm (discrete):

1. Generate
2. if

Algorithm (continuous):

1. Generate
2. assuming the inverse exists

## Rejection sampling

Idea: if we can simulate easily, we can use the proportional distribution (likelihood ratio) as a basis to simulate with pdf

Algorithm:

1. Find
2. Generate from a density g, e.g.,
3. Generate
4. If , set . Otherwise return to 2

Number of iterations needed:

# V) Variance reduction

## Antithetic variables

Idea: if we are able to generate negatively correlated underlying random variables, the estimator can have lower variance as compared with independent samples. This requires the target function to be monotone

Algorithm:

1. Generate
2. Set (note: want same distribution but negative correlation)
3. Repeat 1 and 2 for

Useful corollary: if is monotone, then where

## Stratified sampling

Idea: if we have information about grouping in the population, then we may use conditional mean (mean of subgroup) as the sample from the population

Algorithm:

1. Generate where for
2. Set
3. (average over subsamples and bins, remember to adjust for conditional probability)

## Control variate

Idea: if we combine the estimate of our target unknown quantity with estimates of some known quantities, we can exploit the known information

Algorithm:

1. Find for with a known distribution (or estimate via pilot simulation)
2. Generate for
3. Compute

Pilot simulation: we can run a simulation with a small sample size (e.g., ) and compute and based on this pilot sample. Then we can use their values when we compute for our target samples

Properties of effective control: evaluable from simulation data, known mean and high correlation with the simulation variable. Possible candidates are underlying random variable (e.g. uniform when we use inverse transform) and martingale transform (will not be tested)

## Importance sampling

Idea: if certain values of the simulation variable have more impact on the parameter of interest (e.g. probability of a rare event), we can try to “emphasize” those values by sampling them more frequently and reduce variance. This can be done by changing the probability measure using the likelihood ratio (technically it is called Radon–Nikodym derivative) as weight

Algorithm:

1. Find the likelihood ratio where is the original target pdf
2. Generate for

Maximum principle: choose such that both and take maximum values at the same

## Exponential tilting

Tilted density: where is the mgf of . Useful for rare event simulation such as choosing in rejection sampling or importance sampling

Choice of for importance sampling: choose such that the upper bound of is minimized. In particular, we first find (subscript because it may depend on ) such that for all in the support. Then we minimize with respect to