Thus, the complexity of the problem depends on a number of problems a, size of the problem (n / b) and the division/ combination cost f(n).

4.2 Recurrence and Different Methods to Solve Recurrence

Definition : Recurrence equation recursively defines a sequence of function with a different argument. The behavior of the recursive algorithm is better represented using recurrence equations.

- Running time of the recursive algorithm is often described by the recurrence equation. Recurrence equation is like a recursive function. By finding the running time of smaller sub-problems recursively, we can find the running time of a larger problem.
- Recurrence is normally of the form :

$$T(n) = T(n-1) + f(n), \quad \text{for } n > 1 \qquad \ldots(4.2.1)$$

$$T(n) = 0, \quad \text{for } n = 0$$

- The function f(n) may represent constant or any polynomial in n.
- Equation (4.2.1) is called the recurrence equation. T(n) is interpreted as the time required to solve the problem of size n.
- On recursively solving T(n) for n = n − 1, recurrence will hit the base case. T(n) = 0, for n = 0. Values will be back propagated and the final value of T(n) is computed.
- Recurrence equation for some of the problems is mentioned here. Later in the book, we will explore the recurrence equation in mode depth.
- Recurrence of linear search/ factorial:
  $T(n) = T(n-1) + 1$
- Recurrence of selection/ bubble sort :
  $T(n) = T(n-1) + n$
- Recurrence of binary search : $T(n) = T(n/2) + 1$
- Recurrence of merge sort : $T(n) = 2T(n/2) + n$

Use

- Recurrence relation can effectively represent the running time of recursive algorithms.
- The time complexity of certain recurrence can be easily solved using master methods.
- There are several ways to solve the recurrence equation. Following are the widely known methods for solving recurrence.
  - Unfolding method
    (a) Forward substitution
    (b) Backward substitution
  - Homogenous equations
  - Inhomogeneous equations
  - Master method
  - Change of variable
  - Let us discuss various methods to solve the recurrence equation.

4.2.1 Unfolding Methods

In this section, we will discuss two unfolding methods which are widely used to solve a large class of recurrence. It is also known **iteration methods**. There are two ways to solve such equations.

Ways to solve recurrence using unfolding method

1. Forward substitution
2. Backward substitution

Fig. 4.2.1 : Ways to solve unfolding methods

4.2.1 (A) Forward Substitution

Forward substitution method finds the solution of the smallest problem using base condition. A solution to the bigger problem is obtained using the previously computed solution of the smaller problem. This process is repeated until the solution for the original problem of size *n* is achieved.

Ex. 4.2.1 : Solve the recurrence of linear search using a forward substitution method.

Soln. :

Recurrence equation of linear search is,

$$T(n) = T(n-1) + 1, \text{ for } n > 0 \text{ and}$$

$$T(n) = 0, \text{ for } n = 0 \text{ (Base condition)}$$

Inductive case of the recurrence is

$$T(n) = T(n-1) + 1 \quad .....(1)$$

Forward substitution finds the solution of T(n) by solving the progressively bigger problems, starting from the smallest possible case.

Here, the base case is

$$T(n) = 0, \text{ for } n = 0 \quad ...(2)$$

Put n = 1 in Equation (1) to solve the problem of size 1.

$$\therefore \ T(1) = T(1-1) + 1 = T(0) + 1 = 0 + 1$$

(From Equation (2))

$$\therefore \ T(1) = 1$$

$$\text{For } n = 2,$$

$$T(2) = T(2-1) + 1 = T(1) + 1 = 1 + 1$$

$$\therefore \ T(2) = 2$$

After k substitutions,

$$T(k) = k,$$

$$\text{And for } k = n,$$

$$T(n) = O(n)$$

4.2.1 (B) Backward Substitution

This method substitutes the value of n by n – 1 recursively, to solve the smaller and smaller sub problem. It works exactly in reverse order of forward substitution.

Ex. 4.2.2 : Solve the recurrence of linear search using a backward substitution method.

Soln. :

Recurrence equation of linear search is,

$$T(n) = T(n-1) + 1, \text{ for } n > 0 \text{ and}$$

$$T(n) = 0, \text{ for } n = 0 \text{ (Base condition)}$$

Inductive case of the recurrence is

$$T(n) = T(n-1) + 1 \quad .....(1)$$

The problem can be solved if T(n – 1) is known.

To find T(n – 1), substitute n = n – 1 in Equation (1),

$$T(n-1) = T(n-2) + 1$$

Replace this T(n – 1) in Equation (1)

$$\therefore \ T(n) = [\,T(n-2) + 1\,] + 1$$

$$= T(n-2) + 2 \quad ...(2)$$

The problem can be solved if T(n – 2) is known.

To find T(n – 2), substitute n = n – 1 in Equation (1),

$$T(n-2) = T(n-3) + 1 \quad ...(3)$$

Replace this T(n – 2) in Equation (2)

$$\therefore \ T(n) = [\,T(n-3) + 1\,] + 2$$

$$= T(n-3) + 3$$

After k substitutions,

$$T(n) = T(n-k) + k,$$

To match the base case, assume that k = n,

$$\therefore \ T(n) = T(n-n) + n = T(0) + n = 0 + n$$

(From base case)

$$\therefore \ T(n) = O(n)$$

Ex. 4.2.3 : Solve the following recurrence using the backward substitution method : $T(n) = T(n-1) + n^4$

Soln. :

$$\text{Given that, } T(n) = T(n-1) + n^4 \quad ...(1)$$

Substitute n by n – 1 in Equation (1),

$$\therefore \ T(n-1) = T(n-2) + (n-1)^4$$

Put this value of T(n – 1) in Equation (1)

$$\therefore \ T(n) = [\,T(n-2) + (n-1)^4\,] + n^4 \quad ...(2)$$

Substituting n by n – 2 in Equation (1),

$$T(n-2) = T(n-3) + (n-2)^4$$

To match the base case, consider $k = n - 1$

$$\therefore \quad T(n) = 2^{n-1} \cdot T(n - n + 1) = 2^{n-1} \cdot T(1)$$

$$\therefore \quad T(n) = O(2^{n-1}) \qquad (\because T(1) = 1)$$

4.2.2 Homogeneous Equations

- Homogeneous recurrence equation has the form,

$$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \dots + a_k t_{n-k} = 0$$

1. This is a linear homogeneous equation with constant coefficients.

2. It is linear because it involves t_i with power one only.
3. It is homogeneous because $\sum t_{n-i} = 0$.

- Homogeneous recurrences equation is converted into the characteristic equation by substituting $t_n = x^n$.
- Characteristic equation of above homogenous recurrence would be,

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_k x^{n-k} = 0$$

This equation is satisfied if $x = 0$, or else if

$$a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} + \dots + a_k = 0$$

Note : If characteristic equation has two different roots r_1 and r_2, then general solution of recurrence is given as, $t_n = c_1 (r_1)^n + c_2 (r_2)^n$

write

$$f_n = \frac{1}{\sqrt{5}}[(\Phi)^n - (\Phi^{-1})^n]$$

Ex. 4.2.6 : Solve following homogenous recurrence :

$$t_n = \begin{cases} 0 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 3t_{n-1} + 4t_{n-2} & \text{otherwise} \end{cases}$$

Soln. : Rewrite the given recurrence in homogenous from,

$$t_n - 3t_{n-1} - 4t_{n-2} = 0$$

$$t_n = 3t_{n-1} + 4t_{n-2}$$

Compare it with the general characteristic equation,

$$a_0 x^k + a_1 \cdot x^{k-1} + \ldots + a_k = 0, \text{ we get } k = 2$$

Characteristic equation of the given homogenous equation would be,

$$x^2 - 3x - 4 = 0$$

$$\therefore (x + 1)(x - 4) = 0$$

Two roots are $r_1 = -1$ and $r_2 = 4$, both roots are different. So general solution of homogenous equation would be,

$$t_n = c_1 (r_1)^n + c_2 (r_2)^n$$

$$= c_1 (-1)^n + c_2 (4)^n)$$

From base condition, $t_n = 0$ for $n = 0$.

For $n = 0$, From Equation (1),

$$0 = c_1 (-1)^0 + c_2 (4)^0$$

$$0 = c_1 + c_2$$

From base condition, $t_n = 5$ for $n = 1$.

For $n = 1$, From Equation (1),

$$5 = c_1 (-1)^1 + c_2 (4)^1$$

$$5 = -c_1 + 4 c_2$$

From Equation (2) $c_1 = -c_2$, put it in Equation (3)

$$\therefore c_2 + 4 c_2 = 5$$

$$\therefore c_2 = 1 \text{ and hence } c_1 = -1$$

Thus, the general solution of recurrence equation would be,

$$t_n = (-1)(-1)^n + (1)(4)^n = 4^n - (-1)^n$$

Tip : If characteristic equation has three roots r_1, r_2 and r_3, such that $r_2 = r_3$, then general solution of recurrence is given as,

$$t_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n + c_3 \cdot n\, r_3^n$$

If characteristic equation has five roots r_1, r_2, r_3, r_4 and r_5, such that $r_1 = r_2$ and $r_3 = r_4 = r_5$, then general solution of recurrence is given as,

$$t_n = c_1 \cdot r_1^n + c_2 \cdot n\, r_1^n + c_3 \cdot r_3^n + c_4 \cdot n\, r_4^n + c_5 \cdot n^2 r_3^n$$

$$t_n = 2^{n+1} - n \cdot 2^{n} - 2$$

4.2.3 Inhomogeneous Equation

Inhomogeneous recurrence equation has the form,

$$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \ldots + a_k t_{n-k} = b^n \cdot p(n)$$

- This is linear inhomogeneous equation with constant coefficients. Where
  1. b is a constant
  2. p(n) is polynomial of degree d.
- It is linear because it involves t_i with power one only.
- It is non-homogeneous because $\sum t_{n-i} \neq 0$.
- To solve the inhomogeneous equation, we should first convert it into homogenous form and then proceed in the same way we did in the Section 4.2.2.

Ex. 4.2.8 : Solve the recurrence equation : $t_n - 2t_{n-1} = (n+5)3^n$ for $n \geq 1$

Soln. :

Given that, $t_n - 2t_{n-1} = (n+5)3^n$ ...(1)

Rewriting it, $t_n - 2t_{n-1} = n3^n + 5 \cdot 3^n$

This is not a homogeneous equation because $\sum a_i \cdot t_{n-i} \neq 0$. Let us convert it into homogenous form.

Replacing n by (n – 1) in Equation (1),

$t_{n-1} - 2t_{n-2} = (n+4)3^{n-1}$

$t_{n-1} - 2t_{n-2} = n \cdot 3^{n-1} + 4 \cdot 3^{n-1}$ ...(2)

Replacing n by n – 2 in Equation (1),

$t_{n-2} - 2t_{n-3} = (n+3) \cdot 3^{n-2}$

$t_{n-2} - 2t_{n-3} = n \cdot 3^{n-2} + 3 \cdot 3^{n-2}$ ...(3)

To find characteristic equation, we should eliminate polynomial terms

Multiply Equation (3) by 9 and add in Equation (1),

Equation (1) * 9 ⇒

$$9\,t_{n-2} - 18\,t_{n-3} = 9\,n \cdot 3^{n-2} + 9 \cdot 3 \cdot 3^{n-2}$$

$$9\,t_{n-2} - 18\,t_{n-3} = n \cdot 3^n + 3 \cdot 3^n$$

Adding Equation (1)

$$+t_n - 2\,t_{n-1} = n \cdot 3^n + 5 \cdot 3^n$$

$$t_n - 2\,t_{n-1} + 9\,t_{n-2} - 18\,t_{n-3} = 2\,n \cdot 3^n + 8 \cdot 3^n \quad ...(4)$$

Multiply Equation (2) by – 6 and add it to Equation (4)

Equation (2) * – 6 ⇒

$$-6\,t_{n-1} + 12\,t_{n-2} = -6 \cdot n \cdot 3^{n-1} - 24 \cdot 3^{n-1}$$

$$-6\,t_{n-1} + 12\,t_{n-2} = -2\,n \cdot 3^n - 8 \cdot 3^n$$

Adding Equation (4)

$$\Rightarrow +t_n - 2\,t_{n-1} + 9\,t_{n-2} - 18\,t_{n-3} = 2\,n \cdot 3^n + 8 \cdot 3^n$$

$$t_n - 8\,t_{n-1} + 21\,t_{n-2} - 18\,t_{n-3} = 0,$$

which is a homogeneous equation.

Characteristic equation of this homogeneous equation is written as,

$$x^3 - 8\,x^2 + 21\,x - 18 = 0$$

Put $x = 1 \Rightarrow 1 - 8 + 21 - 18 \neq 0$, so $(x - 1) = 0$ is not the solution

Put $x = 2 \Rightarrow 8 - 8\,(4) + 21\,(2) - 18$

$$8 - 32 + 42 - 18 = 0$$

∴ $(x - 2) = 0$ is one the solution of given characteristic equation.

Rewriting it as factor of $(x - 2)$

$$(x-2)\,x^2 - (x-2)\,6\,x + (x-2)\,9 = 0$$

$$(x-2)\,(x^2 - 6x + 9) = 0$$

$$(x-2)\,(x-3)^2 = 0$$

Roots of equation are, $r_1 = 2$ and $r_2 = r_3 = 3$

Generalized solution of recurrence Equation is,

$$t_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n + n \cdot c_3 \cdot r_3^n \quad ...(5)$$

By putting $n = 1$ in Equation (1), i.e. in $t_n - 2\,t_{n-1} = n\,3^n + 5 \cdot 3^n$

$$t_1 - 2t_0 = 3 + 15$$

$$\therefore\ t_1 = 18 + 2t_0 \quad ...(6)$$

By putting $n = 2$ in Equation (1)

$$t_2 - 2\,t_1 = 18 + 45 = 63$$

$$t_2 = 63 + 2\,t_1$$

From Equation (6),

$$t_2 = 63 + 2\,(18 + 2\,t_0)$$

$$t_2 = 4\,t_0 + 99 \quad ...(7)$$

Put $n = 0$ in Equation (5),

i.e. $t_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n + n \cdot c_3 \cdot r_3^n$

$$t_0 = c_1 \cdot 2^0 + c_2 \cdot 3^0 + 0$$

$$t_0 = c_1 + c_2 \quad ...(8)$$

Put $n = 1$ in Equation (5)

$$t_1 = 2 \cdot c_1 + 3 \cdot c_2 + 3\,c_3$$

But from Equation (6), $t_1 = 18 + 2t_0$

$$\therefore\ 2\,t_0 + 18 = 2 \cdot c_1 + 3\,c_2 + 3\,c_3 \quad ...(9)$$

Put $n = 2$ in Equation (5)

$$t_2 = 4c_1 + 9\,c_2 + 18\,c_3$$

But from Equation (7), $t_2 = 4\,t_0 + 99$

$$\therefore\ 63 + 2t_1 = 4\,c_1 + 9\,c_2 + 18\,c_3$$

From Equation (6),

$$63 + 2\,(t_0 + 18) = 4\,c_1 + 9\,c_2 + 18\,c_3$$

$$\therefore\ 4\,c_1 + 9\,c_2 + 18\,c_3 = 4\,t_0 + 99 \quad ...(10)$$

From Equation (8) and (9)

$$2\,t_0 + 18 = 2\,(t_0 - c_2) + 3\,c_2 + 3\,c_3$$

$$2\,t_0 + 18 = 2\,t_0 - 2\,c_2 + 3\,c_2 + 3\,c_3$$

$$c_2 = 18 - 3\,c_3 \quad ...(11)$$

From Equation (8) and (10),

$$4\,t_0 + 99 = 4\,(t_0 - c_2) + 9\,c_2 + 18\,c_3$$

$$4\,t_0 + 99 = 4\,t_0 - 4\,c_2 + 9\,c_2 + 18\,c_3$$

$$5\,c_2 = 99 - 18\,c_3 \quad ...(12)$$

Multiply Equation (11) by – 5 and add it to Equation (12),

$$5\,c_2 = 99 - 18\,c_3$$

$$-5\,c_2 = -90 + 15\,c_3$$

$$0 = 9 - 3\,c_3$$

$$\therefore\ c_3 = 3$$

From Equation (11), $c_2 = 18 - 3\,c_3 = 18 - 9 = 9$

From Equation (8), $c_1 = c_2$ to $- c_2$

$$c_1 = t_0 - 9$$

So, generalized solution of recurrence equation is given as,

$$t_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n + n \cdot c_3 \cdot r_3^n$$

$$= (t_0 - 9)\, 2^n + 3 \cdot 9^n + n \cdot 3 \cdot 3^n$$

$$= (t_0 - 9) \cdot 2^n + 3 \cdot 3 \cdot 3^{n+1} + n \cdot 3^{n+1}$$

$$= (t_0 - 9)\, 2^n + (n + 3)\, 3^{n+1}$$

Ex. 4.2.9 : Find the solution for the following recurrence: $t_n - 2t_{n-1} = 3^n$

OR Solve following recurrence : $t(n) - 2\, t(n-1) = 3^n$

Soln. :

Given that

Generalized solution would be

$$t_n = (-2)\cdot(1)^n + (-1)\,n\cdot(1)^n + (2)\cdot(2)^n + n\,(1)\cdot 2$$

$$= -2 - n + 2^{n+1} + n\cdot 2^n$$

$$= n\cdot 2^n + 2^{n+1} - n - 2$$

4.2.4 Master's Theorem

- Divide and conquer strategy uses recursion. The time complexity of the recursive program is described using recurrence. In the Sections 4.2.1, we have studied various methods for solving the recurrence.
- The master method is used to quickly solve the recurrence of the form $T(n) = a\cdot T\left(\frac{n}{b}\right) + f(n)$.

  The master method finds the solution without substituting the values of $T(n / b)$. In the above equation,

  n = Size of the problem

  a = Number of subproblems created in the recursive solution

  n / b = Size of each sub problem

  f(n) = Work done outside recursive call. This includes the cost of the division of problem and merging of the solution.

Let $T(n) = a\,T(n/b) + f(n)$

Where $a \geq 1, b \geq 1, f(n) = \Theta(n^d \log^m n)$

The solution of the recurrence equation using the master method is obtained as,

Case – 1 : if $b^d < a$

- $T(n) = \Theta(n^{\log_b a})$

Case – 1 : if $b^d = a$

- If $m > -1$ then $T(n) = \Theta(n^d \log^{m+1} n)$
- If $m = -1$ then $T(n) = \Theta(n^d \log n)$
- If $m < -1$ then $T(n) = \Theta(n^d)$

Case – 1 : if $b^d > a$

- If $m \geq 0$ then $T(n) = \Theta(n^d \log^m n)$
- If $m < 0$ then $T(n) = \Theta(n^d)$

Examples of the Master method

Ex. 4.2.13 : Solve the given recurrence using master method : $T(n) = 2T(n/2) + n^2 \log n$

Soln. :

Compare this equation with $T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$

where $f(n) = n^d \log^m n$

Here, $a = 2, b = 2, d = 2, m = 1$

$b^d = 2^2 = 4$

Here $b^d > a$ (Case – III, $m \geq 0$)

$T(n) = \Theta(n^d \log^m n)$

$= \Theta(n^2 \log^n)$

Ex. 4.2.14 : Solve given recurrence using Master method : $T(n) = 2T(n/2) + n^2 \log^2 n$

Soln. :

Compare this equation with $T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$

where $f(n) = n^d \log^m n$

Here, $a = 2, b = 2, d = 2, m = 2$

$b^d = 2^2 = 4$

Here, $b^d > a$ (Case – III, $m \geq 0$)

$\therefore T(n) = \Theta(n^d \log^m n)$

$= \Theta(n^2 \log^2 n)$

Ex. 4.2.15 : Solve given recurrence using Master method : $T(n) = 4T(n/2) + n^3$

Soln. :

Compare this equation with $T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$,

where $f(n) = n^d \log^m n$

Here, $a = 4, b = 2, d = 3, m = 0$

$b^d = 2^3 = 8$

Here, $b^d > a$ (Case – III, $m \geq 0$)

$\therefore T(n) = \Theta(n^d \log^m n)$

$= \Theta(n^3)$

Ex. 4.2.16 : Solve given recurrence using Master method : $T(n) = 4T(n/2) + n^3/\log n$

Soln. :

Compare this equation with $T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$,

where $f(n) = n^d \log^m n$

Here, $a = 4, b = 2, d = 3, m = -1$

$b^d = 2^3 = 8$

Here, $b^d > a$ (Case – III, $m < 0$)

$\therefore T(n) = \Theta(n^d) = \Theta(n^3)$