Quantum Mechanics Homework Questions Angular Momentum Operator and Measurement Probabilities

Question 1

The eigenvalue problem for the angular momentum operator $\hat{L}^2$ is formulated as follows:

$$\hat{L}^2 Y(\theta,\phi) = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] Y(\theta,\phi) = \hbar^2 \lambda Y(\theta,\phi). \tag{1}$$

We require $Y(\theta,\phi)$ to be a simultaneous eigenfunction of $\hat{L}_z$. Using separation of variables $Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$, we get:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + \lambda\Theta = 0. \tag{2}$$

By changing the variable $\xi = \cos\theta$, $F(\xi) = \Theta(\theta)$, the equation becomes:

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dF}{d\xi}\right] - \frac{m^2}{1-\xi^2}F + \lambda F = 0. \tag{3}$$

For the particular case $m = 0$, this simplifies to the **Legendre equation**:

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dF}{d\xi}\right] + \lambda F = 0. \tag{4}$$

Show that the regular solution of this equation exists only when $\lambda = l(l+1)$, where $l = 0, 1, 2, \ldots$, and that the solutions correspond to the Legendre polynomials $P_l(\cos\theta)$.

Question 2

A particle in a spherically symmetric potential is described by the wavefunction:

$$\Psi(x,y,z) = C(xy + yz + zx)e^{-\alpha r^2}, \qquad r^2 = x^2 + y^2 + z^2. \tag{5}$$

1. What is the probability that a measurement of the square of the angular momentum yields 0?
2. What is the probability that it yields $6\hbar^2$?
3. If the value of l is found to be 2, what are the **relative probabilities** (no need to normalize) for $m = 2, 1, 0, -1, -2$?

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