Quantum Entanglement and the Bell-CHSH Inequality

From the EPR Paradox to Experimental Violation via Photon Polarization

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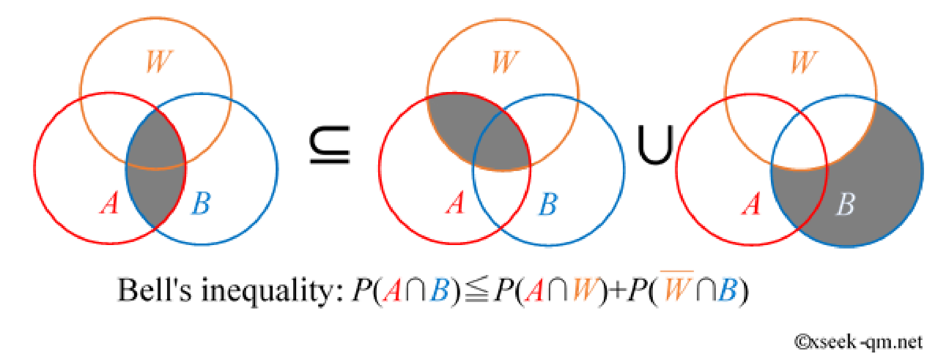
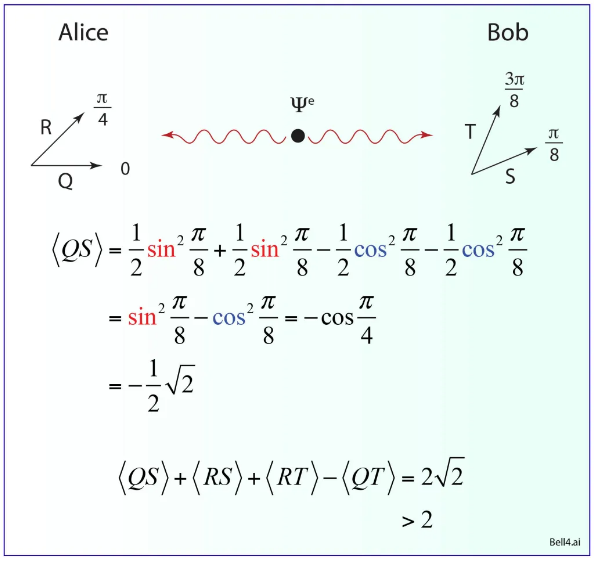
Abstract

Quantum entanglement using photon polarization to derive the Bell-CHSH inequality under the assumptions of Local Hidden Variables (LHV), quantum mechanical violation of the inequality for rotated measurement bases, and the resulting philosophical implications for the nature of reality will be discussed.





1 Formalism and The EPR Paradox

1.1 The Physical System: Photon Polarization

While photons are massless spin-1 bosons, the subspace of their polarization states is isomorphic to a spin-1/2 system.

We shall define our measurement basis in the "Z-direction" (lab frame) as Linear Polarization:

- **Vertical Polarization (V):** Analogous to spin-up $|\uparrow_z\rangle$.
- **Horizontal Polarization (H):** Analogous to spin-down $|\downarrow_z\rangle$.

The computational basis states are:

$$|V\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |H\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

1.1.1 Basis Rotation

If a second observer measures polarization at an angle θ relative to the vertical, the new basis states $|\theta\rangle$ and $|\theta_\perp\rangle$ are:

$$|\theta\rangle = \cos\theta\,|V\rangle + \sin\theta\,|H\rangle \tag{2}$$

$$|\theta_\perp\rangle = -\sin\theta\,|V\rangle + \cos\theta\,|H\rangle \tag{3}$$

Note: In the SU(2) isomorphism, a spatial rotation of polarization by θ corresponds to a rotation on the Poincaré sphere (Bloch sphere) by 2θ.

1.2 The Entangled Singlet State

We consider a source (e.g., Type-II Spontaneous Parametric Down-Conversion) that emits a pair of photons to spacelike separated observers, Alice (A) and Bob (B), in the rotationally invariant singlet state $|\Psi^-\rangle$:

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|V\rangle_A \otimes |H\rangle_B - |H\rangle_A \otimes |V\rangle_B\right) \tag{4}$$

This state exhibits perfect anti-correlation. If Alice measures V, Bob *must* measure H, regardless of the distance separating them.

1.3 The EPR Argument (1935)

Einstein, Podolsky, and Rosen argued that Quantum Mechanics (QM) is an **incomplete** theory based on two premises:

1. **Locality:** Measurement choices made by Alice at event x_A cannot influence the result at Bob's event x_B if $(x_A - x_B)^2 < 0$ (spacelike separation).
2. **Realism:** If the value of a physical quantity can be predicted with certainty without disturbing the system, there exists an element of physical reality corresponding to this quantity.

The Hidden Variable Hypothesis: To satisfy Local Realism, there must exist a parameter λ (the "hidden variable") established at the source, such that the outcomes are pre-determined functions of λ.

2 Derivation of the Bell-CHSH Inequality

We utilize the CHSH (Clauser-Horne-Shimony-Holt) framework, which is experimentally more accessible than Bell's original 1964 inequality.

2.1 Definitions and Assumptions

Let the measurement results be binary. We assign values $+1$ (e.g., passes polarizer) and -1 (absorbed/reflected):

- Alice's result: $A(\alpha, \lambda) = \pm 1$
- Bob's result: $B(\beta, \lambda) = \pm 1$

Here, α and β are the angles of the polarizers chosen by Alice and Bob.

Assumption (Locality): A depends only on α and λ, not on β. B depends only on β and λ.

Let $\rho(\lambda)$ be the probability density of the hidden variables, satisfying $\int \rho(\lambda) d\lambda = 1$. The expectation value (Correlation Coefficient) is:

$$E(\alpha, \beta) = \int A(\alpha, \lambda) B(\beta, \lambda) \rho(\lambda) d\lambda \tag{5}$$

2.2 The CHSH Inequality Derivation

Consider the quantity S, a linear combination of correlations for four different setting pairs $(\alpha, \beta), (\alpha, \beta'), (\alpha', \beta), (\alpha', \beta')$:

$$S = E(\alpha, \beta) - E(\alpha, \beta') + E(\alpha', \beta) + E(\alpha', \beta') \tag{6}$$

Substituting the integral definitions:

$$S = \int \left[A(\alpha, \lambda)B(\beta, \lambda) - A(\alpha, \lambda)B(\beta', \lambda) + A(\alpha', \lambda)B(\beta, \lambda) + A(\alpha', \lambda)B(\beta', \lambda)\right] \rho(\lambda) d\lambda \tag{7}$$

We factorize the integrand. Let $Q(\lambda)$ be the term inside the brackets:

$$Q(\lambda) = A(\alpha, \lambda)\big[B(\beta, \lambda) - B(\beta', \lambda)\big] + A(\alpha', \lambda)\big[B(\beta, \lambda) + B(\beta', \lambda)\big] \tag{8}$$

Since $B(\dots) = \pm 1$, the terms involving Bob can only take specific values:

- If $B(\beta) = B(\beta')$, then $[B(\beta) - B(\beta')] = 0$ and $[B(\beta) + B(\beta')] = \pm 2$.
- If $B(\beta) \neq B(\beta')$, then $[B(\beta) - B(\beta')] = \pm 2$ and $[B(\beta) + B(\beta')] = 0$.

Since $A(\dots) = \pm 1$, it follows that for any specific realization λ:

$$Q(\lambda) = \pm 2 \tag{9}$$

Therefore, the absolute value of the average is bounded by the average of the absolute maxima:

$$|S_{LHV}| = \left| \int Q(\lambda)\rho(\lambda) d\lambda \right| \leq \int |Q(\lambda)| \rho(\lambda) d\lambda = \int 2\rho(\lambda) d\lambda = 2 \tag{10}$$

The Bell-CHSH Inequality:

$$|E(\alpha, \beta) - E(\alpha, \beta') + E(\alpha', \beta) + E(\alpha', \beta')| \leq 2 \tag{11}$$

Any theory satisfying Local Realism *must* obey this inequality.

3 Quantum Violation and Implications

3.1 Detailed Derivation of the Quantum Correlation Function

To calculate the correlation $E_{QM}(\alpha, \beta)$, we evaluate the joint probabilities of Alice and Bob obtaining specific outcomes (+1 or -1) when measuring in bases rotated by angles α and β respectively.

3.1.1 Basis Definitions

Let the vertical and horizontal polarization states be:

$$|V\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |H\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{12}$$

Alice measures in a basis rotated by angle α:

$$|+\rangle_\alpha = \cos\alpha\,|V\rangle + \sin\alpha\,|H\rangle \tag{13}$$
$$|-\rangle_\alpha = -\sin\alpha\,|V\rangle + \cos\alpha\,|H\rangle \tag{14}$$

Bob measures in a basis rotated by angle β:

$$|+\rangle_\beta = \cos\beta\,|V\rangle + \sin\beta\,|H\rangle \tag{15}$$
$$|-\rangle_\beta = -\sin\beta\,|V\rangle + \cos\beta\,|H\rangle \tag{16}$$

Here, outcome $+1$ corresponds to passing the polarizer (parallel), and -1 corresponds to absorption/reflection (perpendicular).

3.1.2 The State Vector

The system is in the singlet state $|\Psi^-\rangle$:

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|V\rangle_A \otimes |H\rangle_B - |H\rangle_A \otimes |V\rangle_B\right) \tag{17}$$

3.1.3 Calculating Probability Amplitudes

We calculate the probability amplitude $\mathcal{A}_{++}$ for both Alice and Bob measuring $+1$.

$$\mathcal{A}_{++} = \left({}_A\langle +|_\alpha \otimes {}_B\langle +|_\beta\right)\left|\Psi^-\right\rangle \tag{18}$$

Expanding the bra vector:

$$\langle +|_\alpha \langle +|_\beta = (\cos\alpha\,\langle V| + \sin\alpha\,\langle H|)_A \otimes (\cos\beta\,\langle V| + \sin\beta\,\langle H|)_B \tag{19}$$

Applying this to the singlet $\frac{1}{\sqrt{2}}(|VH\rangle - |HV\rangle)$:

- The term $\langle VV|$ yields 0 (orthogonal to both singlet terms).
- The term $\langle HH|$ yields 0.
- The term $\langle VH|$ picks up the first part of the singlet: $\langle V|V\rangle\,\langle H|H\rangle = 1$.
- The term $\langle HV|$ picks up the second part of the singlet: $\langle H|H\rangle\,\langle V|V\rangle = 1$.

Thus:

$$\mathcal{A}_{++} = \frac{1}{\sqrt{2}} [(\cos\alpha \sin\beta) \times (1) - (\sin\alpha \cos\beta) \times (1)] \tag{20}$$

$$= \frac{1}{\sqrt{2}} (\sin\beta \cos\alpha - \cos\beta \sin\alpha) \tag{21}$$

Using the trigonometric identity $\sin(x - y) = \sin x \cos y - \cos x \sin y$:

$$\mathcal{A}_{++} = \frac{1}{\sqrt{2}} \sin(\beta - \alpha) = -\frac{1}{\sqrt{2}} \sin(\alpha - \beta) \tag{22}$$

The probability P_{++} is the square modulus of the amplitude. Let $\theta = \alpha - \beta$.

$$P_{++} = |\mathcal{A}_{++}|^2 = \frac{1}{2} \sin^2(\theta) \tag{23}$$

By symmetry, the probability of both measuring -1 (both perpendicular) is identical:

$$P_{--} = \frac{1}{2} \sin^2(\theta) \tag{24}$$

3.1.4 Calculating Cross-Probabilities ($+-$ and $-+$)

Now we calculate the amplitude for Alice measuring $+1$ and Bob measuring -1:

$$\mathcal{A}_{+-} = \left\langle +_\alpha -_\beta \middle| \Psi^- \right\rangle \tag{25}$$

Recalling

$$\langle +|_\alpha = -\sin\alpha \langle V| + \cos\alpha \langle H|$$

and

$$\langle -|_\beta = -\sin\beta \langle V| + \cos\beta \langle H|$$

$$\mathcal{A}_{+-} = \frac{1}{\sqrt{2}} [(\cos\alpha \cos\beta) - (\sin\alpha)(-\sin\beta)] \tag{26}$$

$$= \frac{1}{\sqrt{2}} (\cos\alpha \cos\beta + \sin\alpha \sin\beta) \tag{27}$$

Using the identity $\cos(x - y) = \cos x \cos y + \sin x \sin y$:

$$\mathcal{A}_{+-} = \frac{1}{\sqrt{2}} \cos(\alpha - \beta) = \frac{1}{\sqrt{2}} \cos(\theta) \tag{28}$$

Thus, the probabilities for anti-correlated results are:

$$P_{+-} = P_{-+} = \frac{1}{2} \cos^2(\theta) \tag{29}$$

3.1.5 Computing the Correlation Function E

The correlation coefficient is the expectation value of the product of outcomes ($A \cdot B$). The assignment of values in this experiment is that we treat the measurement as a binary variable (it has only two possible values). We arbitrarily but consistently assign:

- $+1$: The photon passes through the polarizer (Detection).
- -1: The photon is absorbed/blocked by the polarizer (No Detection).
- If outcomes match (i.e. $++$ or $--$), product is $+1$.
- If outcomes differ (i.e. $+-$ or $-+$), product is -1.

This mathematical trick allows us to calculate the Correlation Coefficient (E) easily. By averaging these products over many runs:

- If the average is close to $+1$, the particles are Correlated (acting the same).
- If the average is close to -1, the particles are Anti-Correlated (acting opposite).
- If the average is 0, the particles are Uncorrelated (random with respect to each other).

$$
\begin{aligned}
E_{QM}(\theta) &= (P_{++} + P_{--}) \cdot (+1) + (P_{+-} + P_{-+}) \cdot (-1) &(30)\\
&= \left(\frac{1}{2}\sin^2\theta + \frac{1}{2}\sin^2\theta\right) - \left(\frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^2\theta\right) &(31)\\
&= \sin^2\theta - \cos^2\theta &(32)
\end{aligned}
$$

Using the double-angle identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$:

$$E_{QM}(\theta) = -(\cos^2\theta - \sin^2\theta) = -\cos(2\theta) \tag{33}$$

$$E_{QM}(\alpha, \beta) = -\cos(2(\alpha - \beta)) \tag{34}$$

Note: The factor of 2 arises because photon polarization states repeat every $180°$, *unlike spin-1/2 particles which repeat every* $720°$ *(spinors) or map to vectors repeating every* $360°$. *A physical rotation of photon polarization by* $90°$ *leads to an orthogonal state, whereas in spin-1/2, orthogonality is* $180°$

3.2 Numerical Demonstration of Violation

We seek angles that maximize the violation. Let us choose the following polarizer orientations:

Alice's Settings:

$$\alpha = 0°, \quad \alpha' = 45° \tag{35}$$

Bob's Settings:

$$\beta = 22.5^\circ, \quad \beta' = 67.5^\circ \tag{36}$$

We calculate the relative angles $\theta = \alpha - \beta$ and the resulting correlations $E = -\cos(2\theta)$:

1. **Term 1:** $\alpha = 0^\circ, \beta = 22.5^\circ \implies \theta = -22.5^\circ = -\pi/8$

$$E(\alpha, \beta) = -\cos(2 \times -22.5^\circ) = -\cos(-45^\circ) = -\frac{1}{\sqrt{2}} \tag{37}$$

2. **Term 2:** $\alpha = 0^\circ, \beta' = 67.5^\circ \implies \theta = -67.5^\circ = -3\pi/8$

$$E(\alpha, \beta') = -\cos(2 \times -67.5^\circ) = -\cos(-135^\circ) = -(-\frac{1}{\sqrt{2}}) = +\frac{1}{\sqrt{2}} \tag{38}$$

3. **Term 3:** $\alpha' = 45^\circ, \beta = 22.5^\circ \implies \theta = +22.5^\circ$

$$E(\alpha', \beta) = -\cos(2 \times 22.5^\circ) = -\cos(45^\circ) = -\frac{1}{\sqrt{2}} \tag{39}$$

4. **Term 4:** $\alpha' = 45^\circ, \beta' = 67.5^\circ \implies \theta = -22.5^\circ$

$$E(\alpha', \beta') = -\cos(2 \times -22.5^\circ) = -\cos(-45^\circ) = -\frac{1}{\sqrt{2}} \tag{40}$$

Calculating S:

$$S_{QM} = E(\alpha, \beta) - E(\alpha, \beta') + E(\alpha', \beta) + E(\alpha', \beta') \tag{41}$$

$$S_{QM} = \left(-\frac{1}{\sqrt{2}}\right) - \left(+\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) \tag{42}$$

$$S_{QM} = -\frac{4}{\sqrt{2}} = -2\sqrt{2} \approx -2.828 \tag{43}$$

Taking the absolute value:

$$|S_{QM}| = 2\sqrt{2} \nleq 2 \tag{44}$$

This is a maximal violation of the Bell-CHSH inequality (Tsirelson's bound).

3.3 Philosophical Implications

The experimental verification of this violation forces us to abandon the conjunction of **Locality** and **Realism**. Several interpretations arise:

- **Abandoning Locality (Bohmian Mechanics):** Hidden variables exist (particles have definite trajectories), but the theory is explicitly non-local. A measurement by Alice changes the "Quantum Potential" everywhere instantly, affecting Bob's particle. This preserves Realism but sacrifices Locality.

- **Abandoning Realism (Copenhagen / QBism):** Particles do not possess properties until measured. The wavefunction is not a physical object but a tool for calculating probabilities of outcomes. The "Z-polarization" did not exist prior to the measurement.
  - The Cost: "The moon is not there when nobody looks." Counterfactual definiteness (the ability to ask "what would have happened if I measured X instead of Y") is invalid.
- **Many Worlds Interpretation (Everett):** Locality is preserved, but we abandon the concept of a unique outcome. Both $+1$ and -1 occur in orthogonal branches of the universal wavefunction. Bell's derivation assumes a single outcome per measurement ($A = \pm 1$), which does not hold here(This is sheer nonsense, I enlist here just for fun).
- **Superdeterminism:** Bell's theorem assumes *statistical independence*: that Alice and Bob can choose settings α, β independently of the particle source λ. If the universe is fully deterministic, the source "knew" what settings would be chosen. This closes the "Freedom of Choice" loophole but undermines the scientific method.

4 Wheeler's Delay Choice Experiment(DCE)

Wheeler's delayed-choice experiment is where the decision to observe a photon as a particle or wave is made after the photon has passed through a double-slit setup. This delayed choice challenges classical intuition, suggesting that the photon's behavior (wave or particle) depends on the measurement made later, implying the past is not fixed until observed.

4.1 The Structural Isomorphism between Bell and DCE

- Bell's Theorem (Spatial): Tests correlations between two spacelike separated measurements (A and B). It asks: Do particles have defined properties independent of the distant measurement setting?
- Result: Violation of Local Realism.
- Wheeler's Delayed Choice (Temporal): Tests correlations between an emission event (t_0) and a detection choice (t_1). It asks: Does the system have a defined history independent of the future measurement setting?
- Result: Violation of Non-invasive Measurability (or Macrorealism).
- In Wheeler's DCE before the detector clicks, the photon has no record. It has no "history" in the thermodynamic sense.
- The "Delayed Choice" is simply the system determining which type of irreversible entropy production event will occur.

- History is not "changed"; History is created at the moment of irreversible amplification (measurement). The "past" is a construct we infer from the current low-entropy records.

4.2 How Entropy Refines the Picture

If we introduce Entropy and Information Theory (specifically Landauer's Principle), the "retrocausality" paradox dissolves into a thermodynamic trade-off.

- Information is Physical (Landauer's Principle)
  The "delayed choice" implies we can toggle the past history of the photon. But recording the "which-path" information is a physical process that increases the entropy of the environment.
- To measure "Particle" (Which-Path): You must entangle the photon with a detector (a thermodynamic bath). This correlation creates a record.
- To measure "Wave" (Interference): You must ensure no path information exists in the universe.

4.3 Bell and Wheeler are linked via Contextuality

- Bell proves the universe is not locally real in space; Wheeler (and Leggett-Garg) proves it is not locally real in time.
- Entropy as the Gatekeeper: The "past" is only fixed when information is dissipated into the environment (entropy increase). Until that thermodynamic threshold is crossed, "history" is a superposition.
- Wheeler doesn't change the past; he demonstrates that un-measured time is symmetric, but measured time (entropy) is asymmetric.
- Philosophical Pivot: We must abandon the "Block Universe" (where the past is a fixed brick) in favor of a "Growing Block" or "Crystallizing Universe," where the leading edge of time (the present) crystallizes potentiality into fixed history via thermodynamic irreversibility.