Scattering from a Delta Function Potential

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Consider a one-dimensional problem of a particle with mass m and energy $E > 0$ scattering off a delta function potential, $V(x) = \alpha\delta(x)$ at $x = 0$. The time-independent Schrödinger equation is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \alpha\delta(x)\psi(x) = E\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = \frac{2m\alpha}{\hbar^2}\delta(x)\psi(x)$$

where the wave number k is defined as $k = \sqrt{\frac{2mE}{\hbar^2}}$.

The wave function is split into two regions:

- **Region I ($x < 0$):** The incoming particle from the left and the reflected particle.
$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$$
- **Region II ($x > 0$):** The transmitted particle moving to the right.
$$\psi_{II}(x) = Ce^{ikx}$$

Goal is to find the reflection, $R = \left|\frac{B}{A}\right|^2$ & transmission coefficient $T = \left|\frac{C}{A}\right|^2$.

1 Matching Conditions at $x = 0$

1.1 Continuity of the Wave Function

The wave function $\psi(x)$ must be continuous at $x = 0$.

$$\psi_I(0) = \psi_{II}(0)$$

$$\Rightarrow A + B = C \tag{1}$$

1.2 Discontinuity in the wave function's 1^{st} Derivative

The potential is singular at x=0, therefore, the first derivative of the wave function has a discontinuity at $x = 0$. But the Schrodinger equation is second order in derivative. We therefore have to smooth out this singularity by integrating the Schrödinger equation from $-\epsilon$ to $+\epsilon$ and taking the limit as $\epsilon \to 0$.

$$\int_{-\epsilon}^{+\epsilon} \left(\frac{d^2\psi(x)}{dx^2} \right) dx + \int_{-\epsilon}^{+\epsilon} \left(k^2\psi(x) \right) dx = \int_{-\epsilon}^{+\epsilon} \left(\frac{2m\alpha}{\hbar^2}\delta(x)\psi(x) \right) dx$$

The second term on the left vanishes as the integration interval shrinks. The first term is a total derivative, and the right side uses the property of the delta function.

$$\left[\frac{d\psi}{dx} \right]_{-\epsilon}^{+\epsilon} = \frac{2m\alpha}{\hbar^2}\psi(0)$$

In the limit, we get the jump condition:

$$\psi'_{II}(0) - \psi'_{I}(0) = \frac{2m\alpha}{\hbar^2}\psi(0)$$

We calculate the derivatives:

$$\psi'_I(x) = ikAe^{ikx} - ikBe^{-ikx} \implies \psi'_I(0) = ik(A - B)$$

$$\psi'_{II}(x) = ikCe^{ikx} \implies \psi'_{II}(0) = ikC$$

Substituting these into the jump condition:

$$ik(C - A + B) = \frac{2m\alpha}{\hbar^2}(A + B) \tag{2}$$

2 Derivation of Coefficients

We solve the system of equations (1) and (2) to find the ratios $\frac{B}{A}$ and $\frac{C}{A}$. Substitute $C = A + B$ from (1) into (2):

$$ik((A + B) - A + B) = \frac{2m\alpha}{\hbar^2}(A + B)$$

$$B\left(2ik - \frac{2m\alpha}{\hbar^2}\right) = A\left(\frac{2m\alpha}{\hbar^2}\right)$$

$$\frac{B}{A} = \frac{\frac{2m\alpha}{\hbar^2}}{2ik - \frac{2m\alpha}{\hbar^2}} = \frac{m\alpha}{ik\hbar^2 - m\alpha}$$

Now, we find $\frac{C}{A}$ using equation (1):

$$\frac{C}{A} = 1 + \frac{B}{A} = 1 + \frac{m\alpha}{ik\hbar^2 - m\alpha} = \frac{ik\hbar^2}{ik\hbar^2 - m\alpha}$$

3 Reflection and Transmission Coefficients

The reflection and transmission coefficients are calculated by taking the magnitude squared of the ratios of the amplitudes.

3.1 Reflection Coefficient, R

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{m\alpha}{ik\hbar^2 - m\alpha}\right|^2 = \frac{(m\alpha)^2}{(m\alpha)^2 + (k\hbar^2)^2}$$

3.2 Transmission Coefficient, T

$$T = \left|\frac{C}{A}\right|^2 = \left|\frac{ik\hbar^2}{ik\hbar^2 - m\alpha}\right|^2 = \frac{(k\hbar^2)^2}{(m\alpha)^2 + (k\hbar^2)^2}$$

Finally, confirm that $R + T = 1$, which implies conservation of probability.

$$R + T = \frac{(m\alpha)^2 + (k\hbar^2)^2}{(m\alpha)^2 + (k\hbar^2)^2} = 1$$

4 Condition for the Derivative Jump at $x = 0$

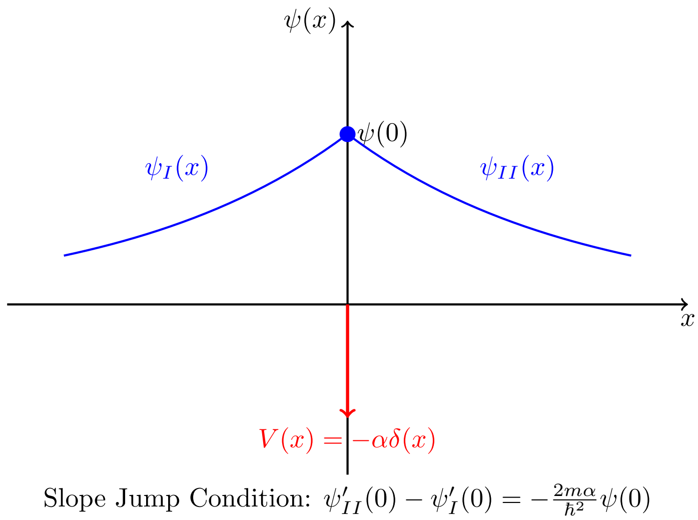

Figure 1: Diagram illustrating the behavior of derivative of wave function $\psi(x)$ scattered by an attractive delta function potential(note the sign difference in the text). The wave function itself is continuous at $x = 0$, but its first derivative is discontinuous. The different slopes of the wave function immediately to the left ($\psi'_I(0)$) and right ($\psi'_{II}(0)$) create a "kink" at the location of the potential.