Path Integral Formalism for Bell Correlations: Berry Phase, Wess-Zumino term, Electrons (Spin-1/2) and Photons (Spin-1) Spin Parametrization; Chern-Simons Charge and Spin Quantization

1 Path Integral for Spin:

The standard path integral is based on the short-time propagator:

$$K(x_{j+1}, t_{j+1}; x_j, t_j) = \int \langle x_{j+1}| e^{-i\hat{H}\epsilon/\hbar} |x_j\rangle.$$

The formalism relies on the property that for small ϵ, the operator $\hat{H}$ can be well-approximated in position space. Spin operators $(\hat{S}_x, \hat{S}_y, \hat{S}_z)$, however, have a non-trivial commutation relation $([\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z)$, making the short-time expansion complex, and the eigenstates are discrete, not continuous; however, this formalism leads to topological monopole & Spin quantization.

1.1 Parametrizing the Spin Coherent States

To adapt the path integral to spin physics, one must use a continuous overcomplete basis called spin coherent states, often denoted $|\vec{n}\rangle$(or $|\mathbf{n}\rangle$). These states are associated with a classical unit vector $\vec{n}$ pointing in the direction of the spin measurement.The standard path integral is then written as a functional integral over the unit sphere (the $\vec{n}$ manifold):

$$K(\vec{n}_f, T; \vec{n}_i, 0) = \int \mathcal{D}\vec{n}(t) \exp\left[\frac{i}{\hbar} S[\vec{n}(t)]\right].$$

The crucial component is the action, $S[\vec{n}(t)]$, which contains both a Hamiltonian part and a geometric term unique to spin systems, known as the Wess-Zumino(WZ) term or Berry phase. The action for a spin system with Hamiltonian $\hat{H}$, is:

$$S[\vec{n}(t)] = \hbar S \int_0^T dt \left(\vec{A}(\vec{n}) \cdot \dot{\vec{n}}\right) - \int_0^T \hat{H}(\vec{n}(t))dt.$$

The term $\hbar S \int \vec{A} \cdot d\dot{\vec{n}}$ is the geometric phase term (where $\vec{A}$ can be regarded as a gauge potential(or gauge connection) on the sphere), which ensures the proper quantum commutation relations are encoded in the path integral measure. To derive the transition amplitude between spin states (e.g., $|+\rangle \to |-\rangle$), one must:

- Map the spin states: Express the initial and final states in the coherent state basis.
- Define the manifold: Integrate over all paths $\vec{n}(t)$ on the unit sphere connecting $\vec{n}_i$ to $\vec{n}_f$.
- Find the classical path: Solve the equations of motion for $\vec{n}(t)$ that minimize action S.
- Use the semi-classical approximation: The amplitude is dominated by $e^{iS_{cl}/\hbar}$.

2 Mapping photon polarization to the Bloch / Poincaré sphere

A pure polarization state of a photon (monochromatic, single spatial mode) is described by a normalized Jones vector in $\mathbb{C}^2$ up to a global phase:

$$|\psi_\gamma\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \langle\psi_\gamma|\psi_\gamma\rangle = |a|^2 + |b|^2 = 1,$$

with the identification $|\psi_\gamma\rangle \sim e^{i\alpha}|\psi_\gamma\rangle$. The projective space of such states is $\mathbb{C}P^1 \cong S^2$ (the Poincaré sphere)(note that, we shall summarize $\mathbb{C}P^1$, $U(1)$ bundle, Berry Phase, the Wess-Zumino(WZ) and the Chern-Simons term in the summary session)). Using spherical coordinates (θ, φ) on S^2 we may parametrize a normalized representation as

$$|\psi_\gamma(\theta,\varphi)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{1}$$

This is the same parametrization used for spin-$\frac{1}{2}$ "up along $\mathbf{n}(\theta,\varphi)$" states; the identification is conventional: the photon polarization (linear/circular/elliptical) maps to a point on the Poincaré sphere.

2.1 Justifying the half angle parametrization for photon

If one used $|\psi\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, the normalization would be:

$$\cos^2\theta + \sin^2\theta = 1.$$

This is mathematically fine for normalization. However, the full parametrization is defined such that the quantum operators map correctly to the coordinates on the sphere. For any state $|\psi\rangle = a|1\rangle + b|2\rangle$, the expectation value of the generalized $\hat{S}_z$ operator is $\langle\hat{S}_z\rangle = |a|^2 - |b|^2$. For the point to correctly map to the spherical coordinate $\cos\theta$ on the sphere:

$$\langle\hat{S}_z\rangle \overset{!}{=} \cos\theta;$$

the half-angle parametrization correctly deliver,

$$\langle\hat{S}_z\rangle = \cos^2(\tfrac{\theta}{2}) - \sin^2(\tfrac{\theta}{2}) = \cos\theta.$$

Therefore, the half-angle $(\theta/2)$ is necessary to ensure that the expectation values of the operators $(\langle\hat{S}_z\rangle)$ map directly to the spherical coordinates $(\cos(\theta))$ of the Poincaré sphere. If we try,

$$|\psi\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix},$$

after plugging into $\mathbf{S} = \langle\psi|\,\sigma\,|\psi\rangle$ which gives

$$\begin{aligned} S_x &= \sin(2\theta)\cos\varphi, \\ S_y &= \sin(2\theta)\sin\varphi, \\ S_z &= \cos(2\theta). \end{aligned}$$

This produces a sphere parametrized by angle 2θ, not θ. It fails to give a one-to-one correspondence between spinors and points on S^2, and does not match the required normalization conditions. Therefore the parametrization is incorrect.

2.1.1 Why the polarization state of a photon is naturally described by the SU(2) spinor

$|\psi(\theta,\varphi)\rangle = (\cos\frac{\theta}{2},\, e^{i\varphi}\sin\frac{\theta}{2})^T$, even though the photon is a spin-1 particle. The reason is that physical photon polarization forms a two-dimensional Hilbert space carrying the fundamental representation of SU(2), and the mapping from spinors to the Poincaré sphere uses the Hopf fibration(see discussions in the below), which necessarily involves half-angles. Note: changing (θ,φ) is moving the quantum state, not rotating the electron/photon in real 3-space; rotating the physical electron induces a corresponding SU(2) operation on the spinor.

2.1.2 SU(2) as the Universal Cover of SO(3)

Mathematically, the sphere S^2 is the quotient

$$SU(2)/U(1) \cong S^2,$$

meaning SU(2) double covers SO(3). The physical space of polarization directions is the two-sphere S^2, on which rotations act as SO(3). However, the quantum states on $\mathcal{H}_\gamma$ must form a representation of the *universal cover* SU(2). The covering map

$$\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$$

is 2-to-1, and a rotation by angle θ in SO(3) corresponds to a rotation by angle $\theta/2$ in SU(2):

$$U(\hat{\mathbf{n}},\theta) = \exp\left(-\frac{i}{2}\,\theta\,\hat{\mathbf{n}}\cdot\sigma\right).$$

Thus the spinor that represents a point on the sphere must involve *half-angles.* This is the fundamental geometric origin of the parametrization.

2.1.3 Hopf Map from Spinors to the Poincaré Sphere

The Hopf Fibration is mathematically defined as the map $\pi : S^3 \to S^2$. In physics terms, it maps a normalized Jones Vector(lives on the hypersphere S^3 in $\mathbb{C}^2$) to a Stokes Vector(lives on the sphere S^2 in $\mathbb{R}^3$), effectively "forgetting" the global phase. The following is the step-by-step derivation. To obtain the Normalized Spinor (Jones Vector), start with the normalized spinor $|\psi\rangle$ in the Circular Basis $\{|L\rangle, |R\rangle\}$, parametrized by the Bloch angles (θ,φ) as discussed:

$$|\psi\rangle = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix}\cos(\theta/2)\\ e^{i\varphi}\sin(\theta/2)\end{pmatrix}$$

This vector satisfies $|\psi_L|^2+|\psi_R|^2 = 1$, so it resides on the unit 3-sphere S^3 embedded in complex space $\mathbb{C}^2$. The Hopf map π, defined as taking this 2-component complex spinor and produces a 3-component real vector $\mathbf{n}$(the Stokes vector on the Poincaré sphere) and the map is defined using the Pauli Matrices σ_i. The coordinates on the sphere (n_1, n_2, n_3) are the expectation values of the Pauli operators:

$$n_i = \langle\psi|\sigma_i|\psi\rangle.$$

Using $\sigma_3 = \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$;

$$n_3 = \langle\psi|\sigma_z|\psi\rangle = \begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix} = cos\,\theta.$$

This matches the standard Poincaré sphere: The S_3 axis is controlled by the polar angle θ. For linear Horizontal/Vertical polarization, we use $\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

$$n_1 = \langle \psi | \sigma_x | \psi \rangle = \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = \sin\theta\cos\varphi.$$

This is the standard x-coordinate in spherical coordinates. For Linear Diagonal polarization, we use $\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$:

$$n_2 = \langle \psi | \sigma_y | \psi \rangle = \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} -ie^{i\varphi}\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix} = \sin\theta\sin\varphi.$$

We have derived that the map for the Stokes vector $\pi(|\psi\rangle)$ which produces the vector:

$$\mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

This confirms the Hopf Fibration result: Any point (θ, φ) on the Poincaré sphere (S^2) corresponds to a valid spinor state. The Fiber Bundle picture now arises, when we multiplied the initial spinor $|\psi\rangle$ by an arbitrary global phase $e^{i\chi}$, the expectation values $\langle \psi | \sigma | \psi \rangle$ would remain completely unchanged (the phase cancels with the bra $\langle \psi |$). Thus, the whole circle of phases (the fiber $U(1)$) maps to the exact same single point on the Poincaré sphere.

2.1.4 Physical Interpretation for Photons

The helicity basis

$$|+1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

combined with the Hopf map, produces the standard Poincaré sphere:

$$|\psi(\theta, \varphi)\rangle = \cos\frac{\theta}{2}\, |+1\rangle + e^{i\varphi}\sin\frac{\theta}{2}\, |-1\rangle\,.$$

The half-angles are essential because:

- SU(2) is the double cover of SO(3),
- the polarization Hilbert space is two-dimensional,
- pure states correspond to rays in $\mathbb{C}^2$, i.e. $\mathbb{CP}^1 \cong S^2$.

The spinor parametrization with half-angles is not arbitrary. It is the unique representation consistent with:

1. SU(2) acting on a two-dimensional Hilbert space,
2. the Hopf fibration $S^3 \to S^2$,
3. correct Stokes parameters,
4. correct topology of the Poincaré sphere,
5. correct helicity structure of photon polarization.

3 Berry phase as a line integral and as a Wess–Zumino (surface) term and Chern–Simons Interpretation

For a normalized state $|u(R)\rangle$ dependent on parameters $R^i(t)$ the Berry connection is

$$A_i(R) = i\langle u(R)|\partial_{R^i} u(R)\rangle, \qquad \gamma_{\text{Berry}} = \oint_{\mathcal{C}} A_i \, dR^i.$$

A is called the Berry connection 1-form on parameter space. For the above photon parametrization(same as spin-$\frac{1}{2}$), we computed in the standard gauge

$$A = -\sin^2\!\left(\tfrac{\theta}{2}\right) \, d\varphi, \qquad F = dA = -\tfrac{1}{2}\sin\theta \, d\theta \wedge d\varphi = -\tfrac{1}{2}\, d\Omega,$$

so that for a closed loop on the sphere,

$$\gamma_{\text{Berry}} = \oint_{\mathcal{C}} A = \int_{S:\partial S=\mathcal{C}} F = -\tfrac{1}{2}\Omega(S),$$

where $\Omega(S)$ is the solid angle subtended by S. For a U(1) bundle, the (1+0)-dimensional **Chern–Simons form** is simply the connection:

$$\text{CS}_1 = A.$$

Its exterior derivative is the curvature:

$$d(\text{CS}_1) = F.$$

Thus the Berry phase is the integral of a Chern–Simons form along a path in parameter space, i.e. on the sphere.

3.1 Calculating from the Wess-Zumino term directly

The WZ term (often called the topological or Berry phase term) represents the geometry of the spin moving on the sphere. To perform the path integration, we generally transform the time integral into a geometric area integral on the Bloch sphere. The followings are the step-by-step derivation of how to calculate and integrate the WZ term.

- Constructing the WZ Term (The Integrand), we first need the explicit form of the term inside the integral. The WZ action term is defined as:

$$S_{WZ} = \int_0^T dt \, \langle \mathbf{n}(t)| i\hbar \frac{d}{dt} |\mathbf{n}(t)\rangle$$

- Employ the Spin Coherent State $|\mathbf{n}\rangle$ parameterized by spherical coordinates (θ, φ). For a spin-S particle, the state is constructed by rotating the "highest weight" state $|S, S\rangle$ (pointing upward at the North Pole). A standard choice for the spinor (for spin $S = 1/2$) is:

$$|\mathbf{n}(\theta, \varphi)\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

- Calculating $\langle \mathbf{n}| i\hbar \frac{d}{dt} |\mathbf{n}\rangle$

  1. Differentiating the Ket:

$$\frac{d}{dt}|\mathbf{n}\rangle = \dot{\theta}\frac{\partial}{\partial\theta}|\mathbf{n}\rangle + \dot{\varphi}\frac{\partial}{\partial\varphi}|\mathbf{n}\rangle$$

$$\frac{\partial}{\partial\varphi}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ ie^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

2. Multiplying by the Bra:

$$\langle \mathbf{n}| = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

- Compute the Overlap (The Berry Connection)
  We are interested in the term attached to $\dot{\varphi}$ (the $\dot{\theta}$ term usually vanishes in the real part or cancels out in this gauge).

$$\langle \mathbf{n}|\frac{\partial}{\partial\varphi}|\mathbf{n}\rangle = \left(e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\right)\cdot\left(ie^{i\varphi}\sin\left(\frac{\theta}{2}\right)\right) = i\sin^2\left(\frac{\theta}{2}\right)$$

- Multiplying by $i\hbar$ (and scale by $2S$ for general spin):

$$\mathcal{L}_{WZ} = i\hbar\cdot\dot{\varphi}\cdot\left(i\sin^2\frac{\theta}{2}\right)\times(2S) = -2\hbar S\sin^2\left(\frac{\theta}{2}\right)\dot{\varphi} = -\hbar S(1-\cos\theta)\dot{\varphi}$$

  (Note: There is a gauge freedom here. Sometimes you see $+S\cos\theta\dot{\varphi}$, which differs by a total derivative $S\dot{\varphi}$, but the physics of closed loops remains the same.)

- Performing the Integration (The Geometric Trick) over the path

$$S_{WZ} = -\hbar S\int_0^T dt\,(1-\cos\theta(t))\frac{d\varphi}{dt} = -\hbar S\int_{\text{path}}(1-\cos\theta)\,d\varphi$$

3.2 How to interpret and solve this resulting integral?

This integral is exactly the formula for the Solid Angle (Ω) swept out by the spin vector on S^2.

- Stoke Theorem Approach: The area element of unit sphere is $dA = \sin\theta d\theta d\varphi$. The solid angle Ω enclosed by closed loop C is the surface integral over the area D bounded by C:

$$\Omega = \int_D \sin\theta d\theta d\varphi.$$

  We can integrate θ from the North Pole (0) to the boundary curve ($\theta(\varphi)$):

$$\Omega = \int d\varphi\int_0^{\theta(\varphi)}\sin\theta' d\theta' = \int d\varphi\left[-\cos\theta'\right]_0^{\theta} = \int(1-\cos\theta)d\varphi.$$

  As a result, the integrated WZW action is simply proportional to the geometric area enclosed by the path:

$$S_{WZ} = -\hbar S\,\Omega[\text{path}].$$

- Example: Calculating the Phase for a Spin Flip. Suppose we rotate a Spin-1/2 ($S = 1/2$) vector from the North Pole to the South Pole and back to the North Pole along a specific path to calculate the phase.

  1. Path: Go down from North ($\theta = 0$) to South ($\theta = \pi$) along longitude $\varphi = 0$ and the go up from South ($\theta = \pi$) to North ($\theta = 0$) along longitude $\varphi = \alpha$.

  2. Integration:

$$S_{WZ} = -\hbar S\oint(1-\cos\theta)d\varphi$$

  3. Leg 1 ($\varphi = 0$): $d\varphi = 0$. Contribution = 0.

4. Leg 2 ($\varphi = \alpha$): $d\varphi = 0$. Contribution = 0.
5. Problem: This line integral form is tricky for open paths.

- Instead, visualize the area. The path is a "slice" of the sphere like a wedge of an orange. The angle width is α. The total area of the sphere is 4π. The area of our wedge is a fraction $(\alpha/2\pi)4\pi$ of the sphere!

- Explicitly, the integral is $\int(1-\cos\theta)d\varphi$. If we keep θ constant and rotate φ by 2π (a circle at latitude θ):

$$S_{WZ} = -\hbar S(1-\cos\theta)\int_0^{2\pi} d\varphi = -2\pi\hbar S(1-\cos\theta)$$

This is exactly the Aharonov-Anandan Phase $\Rightarrow$ if we evolve the spin adiabatically in a magnetic field such that it traces a circle at angle θ: The wavefunction acquires a phase factor $e^{\frac{i}{\hbar}S_{WZ}}$,

$$\text{Phase} = \exp\left[-iS(1-\cos\theta)2\pi\right].$$

And for, $S = 1/2$ at the equator ($\theta = \pi/2$):

$$\text{Phase} = \exp\left[-i\frac{1}{2}(1-0)2\pi\right] = e^{-i\pi} = -1$$

This is the famous Berry Phase of π (factor of -1) acquired by a fermion when rotated by 360 degrees (2π).

- Summary of the derivation of the WZ term in the path integral formulism

1. Write the spinor $|\mathbf{n}\rangle$.
2. Compute the connection $A = \langle\mathbf{n}|i\hbar d|\mathbf{n}\rangle$.
3. Show it simplifies to $-\hbar S(1-\cos\theta)d\varphi$. Identify $\int(1-\cos\theta)d\varphi$ as the Solid Angle Ω.
4. Final Result: The amplitude is weighted by $e^{-iS\Omega}$.

4 The WZ term and the Spin Path Integral for Bell correlations

- The Physical Setup (The Entangled State)
Consider a source producing a pair of entangled photons in a linear polarization-entangled singlet state (Type-II SPDC):

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B\right)$$

In terms of Circular Polarization basis $\{|L\rangle, |R\rangle\}$ (the eigenstates of $\hat{S}_z$ on the Poincaré sphere), this entangle singlet state is:

$$|\Psi\rangle = \frac{i}{\sqrt{2}}\left(|L\rangle_A|R\rangle_B - |R\rangle_A|L\rangle_B\right)$$

We measure the polarization of Photon A at angle α and Photon B at angle β.

- The Observable (Correlation Function)
The Bell test measures the correlation between the polarization outcomes. In the path integral, we compute the expectation value of the product of the polarization projection operators. For a photon, the "spin" operator $\hat{\mathbf{S}}$ lives on the Poincaré sphere. A linear

polarizer at physical angle α corresponds to a measurement vector $\mathbf{n}(\alpha)$ on the sphere's equator.

$$E(\alpha, \beta) = \langle \hat{\mathbf{S}}_A \cdot \mathbf{n}(\alpha) \otimes \hat{\mathbf{S}}_B \cdot \mathbf{n}(\beta) \rangle$$

The Path Integral representation is:

$$E = \frac{1}{Z} \int \mathcal{D}\mathbf{n}_A \mathcal{D}\mathbf{n}_B \; [\mathbf{n}_A(t_f) \cdot \mathbf{n}(\alpha)] \, [\mathbf{n}_B(t_f) \cdot \mathbf{n}(\beta)] \, e^{\frac{i}{\hbar} S_{total}}$$

- The Action and the WZW Term
  The Action for the spins is dominated by the Wess-Zumino-Witten (WZW) term, which encodes the quantum phase accumulated by the polarization vector.

  $$S_{total} = S_{WZ}[\mathbf{n}_A] + S_{WZ}[\mathbf{n}_B]$$

  Crucial Photon Distinction: For a photon (effective Spin $S = 1$ in this context), the WZW action is:

  $$S_{WZ}[\mathbf{n}] = \hbar \cdot \mathbf{1} \cdot \int (1 - \cos\theta) d\varphi = \hbar\Omega$$

  (Contrast this with the electron where the coefficient is $\hbar/2$). The WZ term acts as a topological constraint. In the semi-classical limit (which is exact for quadratic/free systems like this), the path integral is dominated by the classical paths that minimize the energy while satisfying the boundary conditions imposed by the singlet entanglement.

- The Singlet Constraint:
  The singlet state $|\Psi\rangle$ enforces a strict geometric correlation at the source ($t = 0$):

  $$\mathbf{n}_A(0) = -\mathbf{n}_B(0).$$

  The vectors must start back-to-back on the Poincaré sphere.

- Evaluating the Correlation
  Because the system is rotationally invariant and the action is purely geometric, the result depends only on the relative angle between the measurement vectors $\mathbf{n}(\alpha)$ and $\mathbf{n}(\beta)$ on the sphere. Let Φ be the angle between the two measurement vectors on the Poincaré sphere.The standard integration over the sphere (weighted by the WZW phase) yields the correlation for a singlet:

  $$E = -\mathbf{n}(\alpha) \cdot \mathbf{n}(\beta) = -\cos(\Phi)$$

- The "Twist": Mapping back to the Lab
  This is the step where the photon result diverges from the electron result. We must relate the sphere angle Φ to the lab angles α, β.

- Geometric Mapping:
  As derived in the Hopf fibration section: Linear polarizers live on the Equator of the Poincaré sphere ($\theta = \pi/2$). The longitude φ on the sphere is related to the physical lab angle ψ by $\varphi = 2\psi$.

- Calculating Φ:
  The relative angle on the sphere is the difference in longitudes:

  $$\Phi = \varphi_A - \varphi_B = 2\alpha - 2\beta = 2(\alpha - \beta)$$

- Final Result
  Substituting $\Phi = 2(\alpha - \beta)$ back into the correlation function:

$$E(\alpha, \beta) = -\cos(2(\alpha - \beta))$$

  (Note: The sign depends on the specific definition of the singlet state and the coordinate system. For the standard H/V singlet defined as $|H\rangle|V\rangle - |V\rangle|H\rangle$, the correlation is $-\cos$. For the Type-I state $|H\rangle|H\rangle + |V\rangle|V\rangle$, it is often defined as $+\cos$. The angular dependence 2θ is the universal feature).

4.1 Summary of the result

The Path Integral for spin contains the WZW geometric phase term ($i\hbar \oint \mathbf{A} \cdot d\mathbf{n}$).

- This term is imaginary in the action (even in Euclidean time), meaning it does not behave like a classical Boltzmann weight.
- This complex weighting allows the path integral to generate correlations ($-\mathbf{n}(\alpha) \cdot \mathbf{n}(\beta) = -\cos(\Phi)$) that are stronger than what any probability distribution over classical paths (Hidden Variables) could sustain. Thus, the geometry of the spin path integral explicitly enables the violation of the Bell inequality.

4.2 The WZ Term for electron and photon

For a fermion with spin $S_{el} = 1/2$, the WZ term is:

$$\text{Phase}_{el} = \exp(iS\Omega) = \exp\left(i\frac{1}{2}\Omega\right) \tag{2}$$

A full rotation (2π in φ) results in $\Omega = 2\pi(1 - \cos(\pi/2)) = 2\pi$ (relative to pole), yielding a phase of $e^{i\pi} = -1$. This reflects the spinor nature of the electron. For a photon, the effective spin is $S_\gamma = 1$. The WZW phase is:

$$\text{Phase}_{ph} = \exp(iS\Omega) = \exp(i\,1 \cdot \Omega) \tag{3}$$

A full rotation on the Poincaré sphere yields a phase of $e^{i2\pi} = +1$, reflecting the bosonic vector nature of light.

4.3 The Geometric Mapping (Addressing the Confusion)

In optical Bell tests, we typically rotate a **Linear Polarizer** by a physical angle α.

1. **Why Longitude?**
   Linear polarization states reside exclusively on the **Equator** of the Poincaré sphere (Latitude $\theta = \pi/2$). A physical rotation of the polarizer maintains the linear nature of the light(does not introduce ellipticity). Therefore, the state cannot change latitude; it must slide along the equator. A shift along the equator is defined as a change in longitude φ.
2. **Why the factor of 2?**
   A linear polarizer has 180° symmetry (a polarizer at 0° is indistinguishable from one at 180°). However, the Poincaré sphere requires 360° to return to the identity. Thus, the mapping between Physical Space (α) and Poincaré Space (φ) is:

$$\Delta\varphi_{\text{Poincaré}} = 2\Delta\alpha_{\text{Physical}} \tag{4}$$

4.4 Bell Correlation and Spin

Because of this mapping, the correlation function for photons measured with linear polarizers differs from the electron case. The geometric correlation on the sphere is still $-\mathbf{n}_A \cdot \mathbf{n}_B = -\cos(\Phi_{sphere})$. Substituting the physical angles $\Phi_{sphere} = 2\theta_{lab} = 2(\alpha - \beta)$:

$$E_{ph}(\alpha, \beta) = -\cos(2(\alpha - \beta)) \tag{5}$$

(Note: The sign depends on the specific definition of the singlet/triplet state in the optical setup, often derived as $\cos(2\theta)$ for Type-I down-conversion).

5 CP^1, $U(1)$ Bundle, Berry Phase, WZW term & CS charge

- What is CP^1(Sphere of Physical States), Bloch and Poincaré Sphere ?
  QM takes place in a complex vector space, C^2 (for a 2-level system like spin or polarization). However, a physical state is not a vector; it is a Ray. The vector $|\psi\rangle$ and the vector $e^{i\varphi}|\psi\rangle$ represent the exact same physical state because the overall phase is unobservable. The true "shape" of the physical state space is the space of all normalized vectors (S^3) "divide out" by the phase circle ($U(1)$)(Physical Space = Normalized Vectors in C^2/Phase $U(1) = CP^1$). The space of complex lines through the origin in C^2 is called CP^1 and is topologically identical to S^2. This is why we draw the Bloch Sphere for electrons, the Poincaré Sphere for photons. They are both just physical realizations of CP^1. When you write the path integral over the unit vector $\mathbf{n}$, you are integrating over the manifold CP^1.

- What is the $U(1)$ Bundle(The Geometry of Phase)?
  Even though the physical space is a sphere (CP^1), the quantum machinery requires us to carry around the phase information. Imagine the Bloch Sphere (S^2). At every single point on that sphere (every physical state), there is an attached "circle" representing all the possible phase values ($e^{i\varphi}$) that the wavefunction could have.

  1. Base Space: The Sphere S^2 (CP^1). This is where the parameters θ, φ live.
  2. Fiber: The Circle S^1 ($U(1)$). This is the phase $e^{i\chi}$.
  3. Total Space: The combination of the two is the 3-sphere S^3 (the space of normalized spinors). This structure-a sphere with a circle attached to every point-is called a $U(1)$ Fiber Bundle (specifically, the Hopf Fibration).

- Why does this matter for the WZ term?
  The WZ term is the Connection on this bundle. The phase characterizes the "twists" when walking along the surface of the sphere. If the bundle were "trivial" (like a flat $S^2 \times S^1$), you could define a global phase everywhere & the WZ term would be zero. Hopf bundle is twisted, when you walk in a closed loop on the sphere ($\mathbb{CP}^1$), you don't return to the same phase on the fiber. You pick up a shift. That shift is the WZ action.

- The roles of Connection

  1. CP^1 is the answer to "Where does the path $\mathbf{n}(t)$ live?"-the manifold of the integral.
  2. $U(1)$ Bundle is the answer to "Where does the Phase come from?" It explains why the phase depends on the geometry of the path.
  3. So, saying "The photon state lives in $\mathbb{CP}^1$" is just the mathematically precise way of saying "The photon polarization state lives on the Poincaré Sphere."

Feature	Electron (Spin-1/2)	Photon (Spin-1)
Manifold	Bloch Sphere	Poincaré Sphere
Poles	Spin Up/Down	Circular L/R
Equator	Superposition (S_x)	Linear Polarization
WZW Coeff (S)	1/2	1
Topology Phase	-1 (for 2π rotation)	$+1$ (for 2π rotation)
Mapping	$\theta_{sphere} = \theta_{lab}$	$\varphi_{sphere} = 2\alpha_{lab}$
Bell Correlation	$-\cos(\theta)$	$\cos(2\theta)$

Table 1: Comparison of Path Integral components for Electrons and Photons.

6 Deriving Spin Commutation Relations from the WZ Term

A striking feature of the path integral formalism is that the WZ term is not merely a phase factor; it defines the **symplectic geometry** of the spin. Here we show how the WZ action term explicitly forces the non-commutative geometry of spin operators $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ and Spin quantization in the next section.

6.1 Identifying the Canonical Variables & deriving commutators from symplectic potential

The kinetic part of the spin action (the WZW term) is given by:

$$S_{WZ} = \int dt\, \hbar S(1 - \cos\theta)\dot{\varphi}$$

We compare this to the general form of the action in Hamiltonian mechanics, $S = \int(p\dot{q} - H)dt$. Identifying the coordinate $q = \varphi$, the term multiplying $\dot{\varphi}$ is the canonical momentum p_φ:

$$p_\varphi = \frac{\partial \mathcal{L}_{WZ}}{\partial \dot{\varphi}} = \hbar S(1 - \cos\theta)$$

Since constant shifts in momentum do not affect dynamics, the physically relevant conjugate variable is the projection on the z-axis:

$$S_z(\theta) = \hbar S \cos\theta \implies p_\varphi \sim -S_z$$

Thus, the WZW term establishes that φ **and** S_z **are canonically conjugate variables**.

6.2 The Poisson Bracket on the Sphere

The conjugacy of (φ, p_φ) defines the fundamental Poisson bracket:

$$\{\varphi, p_\varphi\} = 1 \implies \{\varphi, \cos\theta\} = \frac{1}{\hbar S}$$

Using the chain rule for derivatives with respect to θ and φ, the general Poisson bracket for any two functions $f(\theta, \varphi)$ and $g(\theta, \varphi)$ on the sphere is:

$$\{f, g\} = \frac{1}{\hbar S \sin\theta}\left(\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial \theta}\right)$$

6.3 Deriving $\{S_x, S_y\}$

We now compute the bracket for the transverse spin components. The classical mapping is:

$$S_x = \hbar S \sin\theta \cos\varphi$$
$$S_y = \hbar S \sin\theta \sin\varphi$$

We compute the required partial derivatives:

$$\partial_\theta S_x = \hbar S \cos\theta \cos\varphi \qquad \partial_\varphi S_x = -\hbar S \sin\theta \sin\varphi$$
$$\partial_\theta S_y = \hbar S \cos\theta \sin\varphi \qquad \partial_\varphi S_y = \hbar S \sin\theta \cos\varphi$$

Substituting these into the Poisson bracket formula:

$$\begin{aligned}
\{S_x, S_y\} &= \frac{1}{\hbar S \sin\theta}\Big[(\hbar S \cos\theta \cos\varphi)(\hbar S \sin\theta \cos\varphi) - (-\hbar S \sin\theta \sin\varphi)(\hbar S \cos\theta \sin\varphi)\Big] \\
&= \frac{(\hbar S)^2 \sin\theta \cos\theta}{\hbar S \sin\theta} \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1} = \hbar S \cos\theta = S_z
\end{aligned}$$

6.4 Spin Commutation Rules

Dirac quantization rule promotes the Poisson bracket to commutator:$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z \Rightarrow$ the WZW topological term in the path integral action is the generator of the spin algebra $SU(2)$.

7 Magnetic Monopole & the U(1) Bundle $\Rightarrow$ Spin quantization

The WZ term defines the U(1) line bundle over the S^2 state manifold. The curvature Ω of this bundle acts as a magnetic field in the parameter space, with a point-like source at the center of the sphere-a magnetic monopole. The total magnetic flux(Φ, Chern charge) emanating from this monopole is given by integrating the curvature Ω over S^2:

$$\Phi = \frac{1}{\hbar} \oint_{S^2} \Omega$$

7.1 The Single-Valuedness Requirement and Spin S quantization

The phase factor accumulated by the quantum state $|\mathbf{n}\rangle$ when transported around a closed loop C on the sphere is the Berry phase γ_C. By Stokes' Theorem, this phase is related to the flux through the surface D bounded by C:

$$\gamma_C = \frac{1}{\hbar} \int_D \Omega.$$

Considering the following situation to establish the flux quantization conditions:

- Surfaces D_1 and D_2 are two surfaces bounded by closed loop C. The phase factor obtained via D_1 must be the same as the phase factor obtained via D_2(since $e^{i\gamma}$ must return to 1):

  $$e^{i\gamma_1} = e^{i\gamma_2}$$

  This condition requires that the phases can only differ by an integer multiple of 2π:

$$\gamma_1 - \gamma_2 = \left(\frac{1}{\hbar} \int_{D_1} \Omega\right) - \left(\frac{1}{\hbar} \int_{D_2} \Omega\right) = \frac{1}{\hbar} \oint_{S^2} \Omega = 2\pi N, \quad \text{where } N \in \mathbb{Z}$$

- This provide a Geometrical Realization of Spin ($2S = N$). Since the total flux Φ over the sphere is $4\pi S$ and setting the total flux equal to the quantization condition:

  $$4\pi S = 2\pi N$$

  gives the Dirac Quantization Condition applied to spin:

  $$S = \frac{N}{2}$$

  This formula proves two fundamental quantum requirements from the simple topological constraint of single-valuedness:

- Spin is Quantized: S must be a discrete value.

- S is Integer or Half-Integer: Since N is an integer, S must be $0, 1/2, 1, 3/2, \ldots$ Therefore, the flux must be quantized because the spin system is realized on a topologically closed manifold (S^2), and the single-valued nature of the quantum state imposes a winding number constraint on the total flux passing through that manifold.

  1. Electron (Phase -1): A 2π physical-space rotation yields phase $e^{i\pi} = -1$. This demonstrates the half-integer winding (spinor nature).
  2. Photon (Phase +1): A 2π physical-space rotation yields phase $e^{i2\pi} = +1$. This demonstrates the integer winding (bosonic/vector nature).