1 Quantizing the momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Can't start with $\langle \frac{dx}{dt} \rangle$ because in the standard formulation(Schrödinger picture) of quantum mechanics, the position operator, $\hat{x}$, doesn't change with time. Therefore, $\frac{d\hat{x}}{dt} = 0$, and so is its expectation value. We must start with $\frac{d\langle x \rangle}{dt}$ which describes how the average position (the expectation value) of the particle evolves over time.

- $\frac{d\langle x \rangle}{dt}$: means "the time derivative of the expectation value of position". We first calculate the average position of the particle, $\langle x \rangle$, which is a number that changes over time. Then, calculate how that number changes with time. This tells you the velocity of the center of the wave packet.

- $\langle \frac{dx}{dt} \rangle$: means "the expectation value of the velocity operator". Define a velocity operator, $\hat{v} = \frac{d\hat{x}}{dt}$, by taking the time derivative of the position operator itself. Then, calculate the average value of that velocity operator.

- Why the Distinction Matters?
  The core of the issue lies in the Schrödinger picture, which is the most common way quantum mechanics is taught and used. In this picture: Operators are static. Fundamental operators like position ($\hat{x}$) and momentum ($\hat{p}$) are considered constant in time. They represent the measurement one could make, but they don't evolve. State vectors (wavefunctions) evolve. The state of the system, described by the wavefunction $\Psi(x,t)$, is what changes over time according to the Schrödinger equation.

- The Problem with $\langle \frac{dx}{dt} \rangle$
  The position operator $\hat{x}$ is time-independent in the Schrödinger picture, its time derivative is simply zero: $\frac{d\hat{x}}{dt} = 0 \Rightarrow \langle \frac{d\hat{x}}{dt} \rangle = \langle 0 \rangle = 0$ also.

- The Power of $\frac{d\langle x \rangle}{dt}$
  This expression, on the other hand, is the key to dynamics. The expectation value of position, $\langle x \rangle$, does change with time because the wavefunction $\Psi(t)$ changes,
  $$\langle x \rangle(t) = \int \Psi^*(x,t)\hat{x}\Psi(x,t)dx.$$
  Because Ψ depends on t, the whole integral depends on t. When you take the time derivative of this entire expression and use the Schrödinger equation to substitute for $\frac{\partial \Psi}{\partial t}$, you correctly arrive at Ehrenfest's theorem:
  $$\frac{d\langle x \rangle}{dt} = \frac{1}{m}\langle \hat{p} \rangle.$$
  This is a profound result! It shows that the rate of change of the average position is related to the average momentum. It's the quantum mechanical analogue of the classical definition of velocity, $v = p/m$. This is the correct starting point because it correctly captures how the observable properties

of the quantum system evolve, even though the operators themselves are static.

- The quantization rule for momentum, which gives us its operator form, comes directly from combining the definition of an expectation value with the time-dependent Schrödinger equation.
Step 1: Start with the Time Derivative of the Position Expectation Value: the rate of change of the average position, $\frac{d\langle x\rangle}{dt}$.
The definition of the expectation value of position, $\langle x\rangle$, is:

$$\langle x\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)x\Psi(x,t)dx$$

$$\frac{d\langle x\rangle}{dt} = \int_{-\infty}^{\infty} \left(\frac{\partial \Psi^*}{\partial t}x\Psi + \Psi^* x\frac{\partial \Psi}{\partial t}\right) dx$$

Step 2: Bring in the Schrödinger Equation

$$i\hbar\frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi$$

We also need the complex conjugate of this equation to find $\frac{\partial \Psi^*}{\partial t}$:

$$-i\hbar\frac{\partial \Psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V\Psi^*\right) \implies \frac{\partial \Psi^*}{\partial t} = -\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V\Psi^*\right)$$

Step 3: Substitute the Schrödinger Equation into the Derivative.
This looks messy, but things will simplify nicely.

$$\frac{d\langle x\rangle}{dt} = \int \left[\left(-\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V\Psi^*\right)\right)x\Psi + \Psi^* x\left(\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi\right)\right)\right] dx$$

Since V and x are just functions of x, they commute ($Vx = xV$),

$$\int \left(-\frac{1}{i\hbar}V\Psi^* x\Psi + \frac{1}{i\hbar}\Psi^* xV\Psi\right) dx = 0$$

The potential term cancels out perfectly! This is crucial $\Rightarrow$ result will be universal and not depend on the specific potential the particle is in.

$$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\frac{-\hbar^2}{2m}\int \left[-\left(\frac{\partial^2 \Psi^*}{\partial x^2}\right)x\Psi + \Psi^* x\left(\frac{\partial^2 \Psi}{\partial x^2}\right)\right] dx$$

Step 4: Integration by Parts
Integration by parts and apply it twice, the expression simplifies dramatically to:

$$\frac{d\langle x\rangle}{dt} = \frac{1}{m}\int \Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi dx$$

Step 5: Identify the Momentum Operator
1. Classical Mechanics: Relationship between velocity and momentum is $v = p/m$.
2. Quantum Result: $m\frac{d\langle x\rangle}{dt} = \int \Psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\Psi dx$.
From Ehrenfest's theorem, that the quantum expectation values should behave like classical variables. Equate the expectation value of momentum, $\langle p\rangle$, with $m\frac{d\langle x\rangle}{dt}$:

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt}$$

Comparing our expressions, the expectation value of momentum is:

$$\langle p\rangle = \int \Psi^* \underbrace{\left(-i\hbar\frac{\partial}{\partial x}\right)}_{\text{This must be } \hat{p}} \Psi dx$$

For this to match the general formula for an expectation value,

$$\langle \hat{A}\rangle = \int \Psi^* \hat{A}\Psi dx,$$

the object sandwiched between Ψ^* and Ψ must be the momentum operator, $\hat{p}$. The quantization rule for momentum in the position representation:

$$\boxed{\hat{p} = -i\hbar\frac{\partial}{\partial x}}$$

This operator, when it acts on the wavefunction, gives us information about the momentum of the particle.