Equivalence between Heisenberg and Schrödinger formulations of Quantum Mechanics

A compact but explicit demonstration showing how the Heisenberg (matrix) formulation and the Schrödinger (wavefunction / differential equation) formulation are just two different pictures of the same quantum dynamics.

1. define the two pictures and the unitary map between them,
2. derive the Heisenberg equation from the Schrödinger time evolution operator, and
3. show the converse: from Heisenberg matrix time dependence (phase factors in an energy basis) one recovers the Schrödinger equation for state vectors.

All steps use only the unitary time-evolution operator $U(t)$ generated by the Hamiltonian H.

1. Two pictures and the unitary time-evolution operator

Let H be the Hamiltonian (possibly time-dependent; I'll first treat the time-independent case and note the generalization). The time-evolution operator $U(t, t_0)$ satisfies

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = H(t)\, U(t, t_0), \qquad U(t_0, t_0) = \mathbb{I}.$$

For time-independent H this gives $U(t, t_0) = \exp\big(-\frac{i}{\hbar} H(t - t_0)\big)$.

Two common pictures are:

- **Schrödinger picture (S):** Operators are time-independent (unless they have explicit time dependence) and states carry all the time dependence:

  $$A_S = \text{fixed}, \qquad |\psi_S(t)\rangle = U(t, t_0)\, |\psi_S(t_0)\rangle\,.$$

  States satisfy the Schrödinger equation

  $$i\hbar \frac{\partial}{\partial t} |\psi_S(t)\rangle = H(t)\, |\psi_S(t)\rangle\,.$$

- **Heisenberg picture (H):** States are time-independent (frozen at t_0) and operators carry the time dependence:

  $$|\psi_H\rangle = |\psi_S(t_0)\rangle \quad \text{(constant)}, \qquad A_H(t) = U^\dagger(t, t_0)\, A_S\, U(t, t_0).$$

The physical predictions—expectation values and transition amplitudes—must be identical in both pictures. Explicitly,

$$\langle \psi_S(t)|A_S|\psi_S(t)\rangle = \langle \psi_H|A_H(t)|\psi_H\rangle.$$

This unitary transformation is the precise map between the two formulations.

2. Derive Heisenberg equation from Schrödinger time evolution

Differentiate $A_H(t) = U^\dagger(t,t_0)A_S U(t,t_0)$ with respect to t:

$$\frac{d}{dt}A_H(t) = \frac{dU^\dagger}{dt}A_S U + U^\dagger A_S \frac{dU}{dt}.$$

Use $i\hbar\frac{dU}{dt} = HU$ and its adjoint $-i\hbar\frac{dU^\dagger}{dt} = U^\dagger H$. Then

$$\frac{d}{dt}A_H(t) = \frac{1}{i\hbar}\big(U^\dagger H A_S U - U^\dagger A_S H U\big) = \frac{1}{i\hbar}\big(H_H(t)A_H(t) - A_H(t)H_H(t)\big),$$

where $H_H(t) = U^\dagger H U$ is the Hamiltonian in the Heisenberg picture (for a time-independent Hamiltonian $H_H(t) = H$). Thus

$$\boxed{\frac{d}{dt}A_H(t) = \frac{1}{i\hbar}\,[\,A_H(t),\,H_H(t)\,]\;+\;\big(\partial_t A\big)_H},$$

where $(\partial_t A)_H = U^\dagger(\partial_t A_S)U$ accounts for any explicit time dependence of A. This is **Heisenberg's equation of motion**. It is algebraically equivalent to Schrödinger evolution because it was derived from the Schrödinger time evolution operator.

3. Recover Schrödinger evolution from Heisenberg (matrix) mechanics

Heisenberg's original (matrix) approach gives the time dependence of operator matrix elements. To show equivalence in the other direction, pick an energy eigenbasis of the Hamiltonian H (assume discrete nondegenerate spectrum for clarity; degeneracies are handled similarly). Let

$$H\,|n\rangle = E_n\,|n\rangle\,.$$

Work in the Schrödinger picture where basis vectors $|n\rangle$ are chosen at t_0. The matrix element of an operator A in the Heisenberg picture is

$$A^H_{mn}(t) \equiv \langle m|A_H(t)|n\rangle = \langle m|U^\dagger(t,t_0)A_S U(t,t_0)|n\rangle.$$

For time-independent H, $U(t,t_0) = e^{-iH(t-t_0)/\hbar}$, so

$$A^H_{mn}(t) = e^{iE_m(t-t_0)/\hbar}\,\langle m|A_S|n\rangle\,e^{-iE_n(t-t_0)/\hbar} = e^{i(E_m-E_n)(t-t_0)/\hbar}\,A^S_{mn}.$$

Thus the matrix elements evolve by **phase factors** $e^{i(E_m-E_n)t/\hbar}$. This is precisely the Heisenberg/matrix mechanics statement: in the energy basis, operator matrix elements acquire those phase/time factors.

Now construct the Schrödinger picture state components in the same energy basis. If the initial state at t_0 has components $\psi_n(t_0) \equiv \langle n|\psi_S(t_0)\rangle$, then the Schrödinger evolution by $U(t,t_0)$ gives

$$\psi_n(t) = \langle n|\psi_S(t)\rangle = \langle n|U(t,t_0)|\psi_S(t_0)\rangle = e^{-iE_n(t-t_0)/\hbar}\,\psi_n(t_0).$$

Differentiate:

$$i\hbar \frac{d}{dt}\psi_n(t) = E_n \psi_n(t),$$

so in vector form $|\psi_S(t)\rangle$ obeys

$$i\hbar \frac{d}{dt} |\psi_S(t)\rangle = H |\psi_S(t)\rangle ,$$

the Schrödinger equation.

So from the Heisenberg matrix time dependence in an energy basis (phase factors), we recover the state evolution of Schrödinger picture. Conversely, from Schrödinger evolution we derived the Heisenberg equation. Both yield identical expectations and transition amplitudes:

$$\langle\psi_S(t)|A_S|\psi_S(t)\rangle = \sum_{m,n} \psi_m^*(t)\, A_{mn}^S\, \psi_n(t) = \sum_{m,n} \psi_m^*(t_0)\, e^{iE_m(t-t_0)/\hbar} A_{mn}^S e^{-iE_n(t-t_0)/\hbar}\, \psi_n(t_0) = \langle\psi_H|A_H(t)|\psi_H\rangle.$$

4. Remarks tying the two formulations more conceptually

- **Same Hilbert space, different bookkeeping.** The Heisenberg picture moves time dependence into operators; the Schrödinger picture moves it into states. Both are related by the unitary transformation $U(t, t_0)$ and predict identical measurable numbers (expectation values, probabilities).

- **Canonical commutation and equations of motion.** The canonical commutation relations $[x, p] = i\hbar$ hold in both pictures. In the Heisenberg picture these commutators plus Heisenberg's equation give operator equations similar to classical Hamilton's equations (but with quantum commutators). In the Schrödinger picture those same commutators constrain operator algebra while states obey the Schrödinger PDE.

- **Interaction (Dirac) picture.** The equivalence generalizes to the interaction picture used in time-dependent perturbation theory: part of the Hamiltonian generates rapid phases (Heisenberg-like evolution) while the remainder acts in the Schrödinger-like way on states.

- **Time-dependent Hamiltonians.** For $H(t)$ noncommuting at different times, the evolution operator is the time-ordered exponential $U(t, t_0) = \mathcal{T} \exp\big(-\frac{i}{\hbar}\int_{t_0}^{t} H(s)\, ds\big)$. The same unitary map yields the Heisenberg equation with $H_H(t) = U^\dagger(t, t_0)H(t)U(t, t_0)$ and identical equivalence.

5. Short, self-contained summary (the core identity)

Define

$$A_H(t) = U^\dagger(t, t_0)A_S U(t, t_0), \qquad |\psi_H\rangle = |\psi_S(t_0)\rangle .$$

Then for all t,

$$\langle\psi_S(t)|A_S|\psi_S(t)\rangle = \langle\psi_H|A_H(t)|\psi_H\rangle,$$

and the differential forms are related by

$$i\hbar\frac{d}{dt}\left|\psi_S(t)\right\rangle = H\ \left|\psi_S(t)\right\rangle \quad \Longleftrightarrow \quad \frac{d}{dt}A_H(t) = \frac{1}{i\hbar}[A_H(t), H_H(t)] + (\partial_t A)_H.$$

These two statements are equivalent representations of the same unitary quantum dynamics.