# Computing Mondshein Sequence in Linear Time

Hemant Saxena

April 3, 2014

University of Waterloo
CS762 - Write Project

# 1 Background

We have seen before in lecture notes that canonical ordering is a useful tool for graph drawing, we used it to prove some planar graph properties e.g. Splitting to three trees, Visibility representation. A quick recall about the canonical ordering: *Its a vertex order $v_1, v_2, ..., v_n$ of a triangulated planar graph where, $v_1 v_2 v_n$ is an outer face and all other vertices have at least two predecessors and at least one successor.* Canonical order exist for any 3-connected planar graph [4], in these notes we extend the canonical ordering for non-planar graphs and make them applicable for an arbitrary 3-connected graphs.

The idea of canonical orderings was given much before, in 1971 by Lee F. Mondshein at M.I.T. in his PhD-thesis [5]. Mondshein proposed a sequence that generalizes canonical orderings to non-planar graphs. Mondshein's sequence was later, in 1988 were independently found by Cheriyan and Maheshwari [2] under the concept of *non-separating ear decompositions.* Complexity of calculating Mondshein sequences, is an intriguing question. Mondshein himself gave an algorithm with running time of $O(m^2)$. Cheriyan in his work achieved a running time of $O(nm)$ by using Tutte's theorem that proves the existence of non-separating cycles in 3-connected graphs [8]. The challenge of achieving a sub-quadratic time for calculating Mondshein sequences was still open. The work by Jens M. Schmidt [7] presents the first algorithm that computes a Mondshein sequence in $O(m)$ time and space, and this will be the major focus of these notes. The motivation for computing Mondshein sequence in sub-quadratic time stems around three main applications of it, that can now be solved in linear-time. First, computing three independent spanning trees in a 3-connected graph in linear time. Second, linear time processing of the output-sensitive data structure by Di Battista et. al. [3] that reports three internally disjoint paths between any given vertex pair. Third, a simple linear-time planarity testing.

# 2 Ear Decomposition and Mondshein sequence

This section will give an overview of ear-decomposition and Mondshein-sequences.

## 2.1 Ear Decomposition

**Definition 1.** An *ear decomposition* of a 2-connected graph $G = (V, E)$ is a decomposition $G = (P_0, P_1, ...P_K)$ that partitions E, where $P_0$ is a cycle and $P_i, 1 \leq i \leq K$ is a path with only its two distinct end vertices in common with $(P_0 \cup P_1 \cup ... \cup P_{i-1})$. Each $P_i$ is called an *ear.*

Now we will define some terms associated with an ear decomposition. An *ear* is called *short* if it is an edge and *long* otherwise. For any $i$, let $G_i = (P_0 \cup P_1 \cup ... \cup P_i)$ and $\overline{V_i} := V - V(G_i)$, and $\overline{G_i}$ is a graph induced by $\overline{V_i}$. Note that, $\overline{G_i}$ does not necessarily contain all edges in $E - E(G_i)$, there can be short ears which have both the end points in $G_i$. For a path $P$ and two of its vertices $x$ and $y$, $P[x, y]$ be the sub-path in P from $x$ to $y$. A path with endpoints $v$ and $w$ is called a *vw-path.* A vertex $x$ in a *vw-path* is called an *inner*

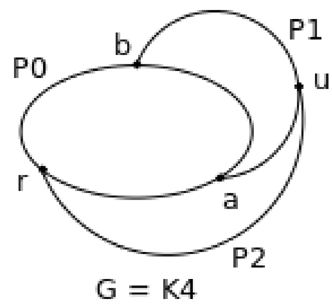


Figure 1: An example of ear decomposition of a $K_4$ containing 3 ears.

*vertex* if $x \notin \{v, w\}$. For simplicity we assume that every vertex in $P_0$ is an inner vertex. *inner(P)* denotes a set of inner vertices of an ear P. For an edge $e \in G$, $birth(e)$ is the index $i$ where $e$ is born, i.e. $i$ such that $P_i$ contains $e$. Similarly, for a vertex $v \in G$, $birth(v)$ is the index $i$ of the first path $P_i$ that contains $v$, in other words, $P_{birth(v)}$ is the ear containing $v$ as an inner vertex.

**Definition 2.** Let $D = (P_0, P_1, ...P_K)$ be an ear decomposition of G. D is defined as a *non-separating* ear decomposition if, for all $i, 0 \leq i \leq k$, $\overline{G_i}$ is connected and each internal vertex of ear $P_i$ has a neighbor in $\overline{G_i}$.

We will now see how canonical ordering for a plane graph can be expressed using a non-separating ear decomposition.

**Definition 3.** Canonical ordering (another definition). Let G be a 3-connected plane graph having edges $v_1v_2$ and $v_2v_n$ on its outer face. A *canonical ordering* with respect to $v_1v_2$ and $v_2v_n$ is an ear decomposition D of G such that:

1. $v_1v_2 \in P_0$,
2. $P_{birth(v_n)}$ is the last long ear, that contains $v_n$ as its only inner vertex and does not contain $v_2v_n$, and
3. D is non-separating.

*How is the above definition similar to old definition (Def. 8.1 in lecture notes) of canonical ordering?* Recall, old definition (Def 8.1 in lecture notes) also had $v_1, v_2, v_n$ fixed on the outer triangular face. Edge $v_1v_2$ was first fixed on the outer face and we incrementally formed G by adding $v_{k+1}$ to $G_k$ at each step. Vertex $v_{k+1}$ was adjacent to two or more outer-face vertices of $G_k$, in other words $G_k$ was connected to $\overline{G_k}$, hence property 3 makes sense. Vertex $v_n$ was added at last, one can imagine arc $v_1v_nv_i$, $i \notin \{1, 2, n\}$ as a last long ear where $v_n$ is an inner vertex. The difference (between Def 3 and Def 8.1 in lecture notes) lies in property 2, remember that in lecture we saw canonical ordering for a triangulated graphs, but Def. 3 is for any 3-connected graph, therefore vertex $v_n$ should have degree at least three. Hence, long ear $P_{birth(v_n)}$ cannot contain $v_2v_n$; $v_2v_n$ should be a short ear. It is also evident that adding vertex $v_3$ forms a cycle $v_1v_2v_3$, think of it as $P_0$ in the above definition.

## 2.2 Mondshein sequence

Now we will see how can Definition 3 be extended for non-planar graphs. Notice that Definition 3 uses planarity only at one place: we assumed edges $v_1v_2$ and $v_2v_n$ are on the outer face. By dropping this assumption we can generalize Definition 3 for non-planar graphs, all we need is that $v_1v_2$ and $v_2v_n$ should be edges in graph G.

In 1971, Mondshein used a similar definition to define a $(2,1)-$*sequence* [[5] Def. 2.2.1], but it was in the notation of a special vertex ordering. For conciseness, we will stick to the ear-decomposition based definition in these notes, which is similar to the one given by Cheriyan [2].

**Definition 4.** ([5], [2]). Let G be a graph with an edge $v_2v_n$. A *Mondshein sequence avoiding* $v_2v_n$ is an ear decomposition D of G such that:

1. $v_2 \in P_0$
2. $P_{birth(v_n)}$ is the last long ear, that contains $v_n$ as its only inner vertex and does not contain $v_2v_n$, and
3. D is non-separating.

In other words the edge $v_2v_n$ is added last in D as a short ear, right after the last long ear $P_{birth(v_n)}$ has been added, and as a direct consequence of property (2) and (3), G must have a minimum degree 3 to have a Mondshein sequence. This has also been explained after Def. 3. Moreover, Mondshein in his thesis [5] proved that every 3-connected graph has a Mondshein sequence. *If D satisfies property (1) and (2) it is said to avoid* $v_2v_n$.

# 3 Computing a Mondshein Sequence

As mentioned in Section 1, Mondshein himself gave an $O(m^2)$ algorithm to compute his sequence [5], later Cheriyan gave an $O(mn)$ time algorithm [2]. The main focus of these notes is the linear time algorithm proposed by Schmidt [7] to compute a Mondshein sequence. At the core of Schimdt's algorithm lies a classical construction of 3-connected graphs proposed by Barnette and Grunbaum [1] and Tutte [[9], Thms. 12.64 and 12.65]. Before describing the Schimidt's algorithm, we will first see what are *BG-Operations*.

## 3.1 BG-Operations

**Definition 5.** The following operations on simple graphs are BG-operations (See Fig 2):

1. *vertex-vertex-addition*: Add an edge between two distinct non-adjacent vertices.
2. *edge-vertex-addition*: Subdivide an edge $ab$, $a \neq b$, by a vertex $v$ and add an edge $vw$ where $w \notin \{a, b\}$.

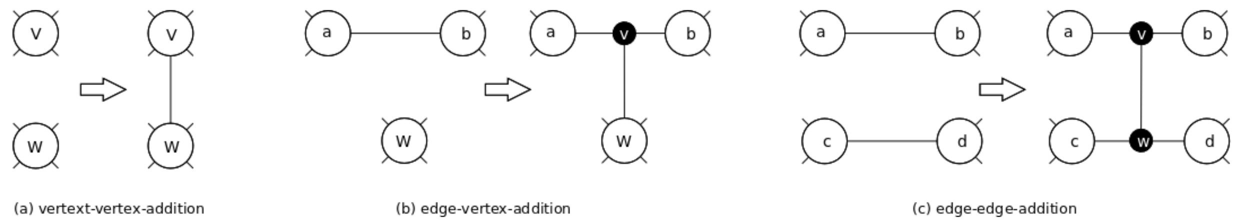


Figure 2: BG Operations as described in Def. 5.

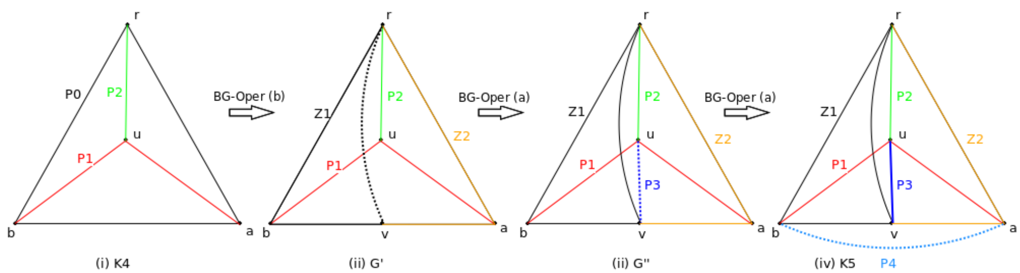


Figure 3: Obtaining $K_5$ from $K_4$ using BG-operations. Pi and Zi shows the ear decomposition. Dotted line denotes the newly added edge.

3. *edge-edge-addition*: Subdivide two distinct edges by vertices $v$ and $w$, respectively, and add the edge $vw$.

Barnette and Grunbaum, and Tutte also gave the following Theorem:

**Theorem 1.** *([1], [9]). A graph is 3-connected if and only if it can be constructed from $K_4$ using the BG-operations.*

It can be seen from Theorem 1 that applying BG-operation on 3-connected graphs keeps them simple and 3-connected. Figure 3 gives an example of obtaining $K_5$ from $K_4$ using BG-operations. A sequence of BG-operations that construct G from $K_4$ is known as *BG-sequence*. Theorem 6 and 52 in [6] shows that BG-sequence of a 3-connected graph can be computed in time $O(m)$.

## 3.2 Algorithm

We will first give an overview of the algorithm and later go into the detailed cases. The algorithm starts with computing a Mondshein sequence of $K_4$, which is easy. Then using Theorem 1 we construct our G in a step by step manner from $K_4$, this can be done in linear time, as BG-sequence can be computed in linear time. Our claim is that, each BG-operation in the BG-sequence of $G'$ to G, modifies the Mondshein sequence in a well defined way given by Lemma 2. In other words there are finite number of cases to which each BG-operation can be mapped while transforming G' to G, and each of those cases have finite number of additions to the Mondshein sequence of G'. This gives us a Mondshein sequence of G from the Mondshein sequence of G'.

**Example 1.** For example, Figure 3 explains the changes in Mondshein sequence when obtaining $K_5$ from $K_4$. Mondshein sequence of $K_4$ is [P0, P1, P2] (follow the color coding). After applying $BG - Oper(b)$ on $K_4$, which maps to case 2aiii of Lemma 2 (defined later), we get G' and mondshein sequence is [Z1, Z2, P1, P2]. After $BG - Oper(a)$ on G', which maps to case 1 of Lemma 2, we get G" and Mondshein sequence is [Z1, Z2, P1, P2, P3]. In the final operation $BG - Oper(a)$ which again maps to case 1 of Lemma 2, we get $K_5$ and final Mondshein sequence is [Z1, Z2, P1, P2, P3, P4].

Now we will define some notations for describing the modifications. Let $\Gamma$ be a single BG-operation, that adds edge $vw$, if applicable, as described in Def. 5. Whenever we consider edges $ab$ and $cd$, without loss of generality (w.l.o.g) we assume that $birth(a) \leq birth(b), birth(c) \leq birth(d)$ and $birth(d) \leq birth(b)$. Let set $S \subseteq \{av, vb, vw, cw, wd\}$ be the set of new edges added after operation $\Gamma$, see figure 2.

The following Lemma describes a detailed scheme of modifying Mondshein sequence for each possible scenario of applying $\Gamma$ on G.

**Lemma 2.** *There is a Mondshein sequence $D' = (P'_0, P'_1, ..., P'_{k+1})$ of G' avoiding ru (respectively, rv or rw if applicable) that can be obtained from $D = (P_0, P_1, ..., P_k)$ of G by performing the following four modifications:*

*M1) replacing the long ear $P_{birth(b)}$ with $P'_{b1}, P'_{b2}, P'_{b3}$ in order and if applicable. Each $P'_{bi}$ consists of edges in $P_{birth(b)} \cup S$.*

*M2) if $P_{birth(cd)}$ is a long ear and $birth(d) < birth(b)$, subdivide cd with w and replace $P_{birth(cd)}$ with the long ear $P'_{birth(cwd)}$.*

*M3) if $P_{birth(ab)}$ is short, delete or replace $P_{birth(ab)}$ with an edge in {av,vb,vw}; if $P_{birth(cd)}$ is short, delete or replace $P_{birth(cd)}$ with an edge in {cw,wd}.*

*M4) adding vw as new last ear, if applicable.*

*In particular, each $\Gamma$ lies in one of the following cases and defines the construction of D' from D, also shown in Fig. 4 :*

1. *$\Gamma$ is a vertex-vertex-addition: use M4.*
2. *$\Gamma$ is an edge-vertex-addition: This case will depend whether b is an inner vertex or not*
   (a) *w.l.o.g. birth(b) = birth(ab), i.e. b is an inner-vertex. Let a' and b' be the endpoints of $P_{birth(b)}$ such that a' is closer to a than to b. Let $P'_{avb}$ be the path obtained by subdividing ab in $P_{birth(b)}$ with v. This will have three cases depending upon w's position:*
      i. *$w \notin G_{birth(b)}$ i.e. $birth(w) > birth(b)$*
      *Obtain $P'_{avb}$ from $P_{birth(b)}$ and add ear vw at the end.*

ii. $w \in G_{birth(b)} - P_{birth(b)}$ *i.e.* $birth(w) < birth(b)$ *and* $w \notin \{a', b'\}$
*Obtain* $P'_{avb}$ *from* $P_{birth(b)}$ *and replace* $P_{birth(b)}$ *with* $Z_1 = a'w - path$ *and* $Z_2 = vb' - path$ *in this order to obtain D'.*

iii. $w \in P_{birth(b)}$ *i.e.* $birth(w) = birth(b)$ *or* $w \in \{a', b'\}$
*Obtain* $Z = P'_{avb}$ *from* $P_{birth(b)}$. *Let* $Z_2 = vw - path$, $Z_1 = Z - Z_2 + vw$. *(If* $P_{birth(b)}$ *is* $P_0$ *then there will be two vw-path, choose the one not containing r as an inner vertex.) Get D' by replacing* $P_{birth(b)}$ *with* $Z_1$ *and* $Z_2$ *in this order, (from now on we will call this process as applying* $M1(Z_1, Z_2, \emptyset)$, *with* $P'_{b1} = Z_1, P'_{b2} = Z_2, P'_{b3} = \emptyset$.*)*

(b) *w.l.o.g. birth(b) < birth(ab) and* $P_{birth(ab)} = ab$, *i.e. b is not an inner vertex. Note that ear* $P_{birth(ab)}$ *cannot be long as a and b have to be born before ab. Again three cases depending upon w's position:*

i. $birth(w) > birth(b)$
*Replace* $P_{birth(ab)} = ab$ *with ear* $av \cup vb$ *and add ear ear vw at the end.*

ii. $birth(w) < birth(b)$*: cases depending upon a's relation with b.*

A. $birth(a) < birth(b)$
*If* $ab = ru$, *add new ear* $wv \cup vu$ *directly after* $P_{birth(u)}$ *and replace* $P_{birth(ru)} = ru$ *with rv. If* $ab \neq ru$, *add new ear* $av \cup vw$ *directly after* $P_{birth(b)-1}$ *and replace* $P_{birth(ab)} = ab$ *with vb.*

B. $birth(a) = birth(b)$
*Let* $Z_1 = av \cup vw \cup P_{birth(b)}[a', a]$ *and* $Z_2 = P_{birth(b)}[a, b']$. *Apply* $M1(Z_1, Z_2, \emptyset)$ *and replace* $P_{birth(ab)} = ab$ *with vb.*

iii. $birth(w) = birth(b)$
*If* $birth(a) = birth(b) > 0$, *let a' and b' be the endpoints of* $P_{birth(b)}$ *such that a' is closer to a than to b. Cases depending upon a's relation with b:*

A. $birth(a) = birth(b) > 0$ *and w lies strictly between either a and a' or b and b' in* $P_{birth(b)}$, *(say w.l.o.g. between b and b')*
*Let* $Z_1 = av \cup vw \cup P_{birth(b)}[a', a] \cup P_{birth(b)}[w, b']$ *and* $Z_2 = P_{birth(b)}[a, w]$. *Apply* $M1(Z_1, Z_2, \emptyset)$ *and replace* $P_{birth(ab)} = ab$ *with vb.*

B. $birth(a) = birth(b) > 0$ *and w lies strictly between a and b in* $P_{birth(b)}$.
*Let* $Z_1 = av \cup vb \cup P_{birth(b)}[a', a] \cup P_{birth(b)}[b, b']$ *and* $Z_2 = P_{birth(b)}[a, b]$. *Apply* $M1(Z_1, Z_2, \emptyset)$ *and replace* $P_{birth(ab)} = ab$ *with vw.*

C. $birth(a) = birth(b) = 0$
*In this case we are at* $P_0$, *consider* $P_0 = P_{ab} \cup P_{bw} \cup P_{wa}$, *one of these path must contain r, say* $P_{ab}$. *Let Z be the union of remaining two paths,* $Z = P_{bw} \cup P_{wa}$. *Let* $P'_0$ *be the cycle 'arbva', and* $Z_2 = Z$ *added directly after* $P'_0$, *and replacing* $P_{birth(ab)} = ab$ *with vw, i.e. connect v to vertex* $j \in \{a, b, w\}$ *that is not an endpoint of Z.*

D. $birth(a) < birth(b)$
*Let b' and b" be the two endpoints of* $P_{birth(b)}$ *such that b' is closer to w than to b. If* $b' \neq a$, $Z_1 = av \cup vw \cup P_{birth(b)}[w, b']$ *and* $Z_2 = P_{birth(b)}[w, b'']$.

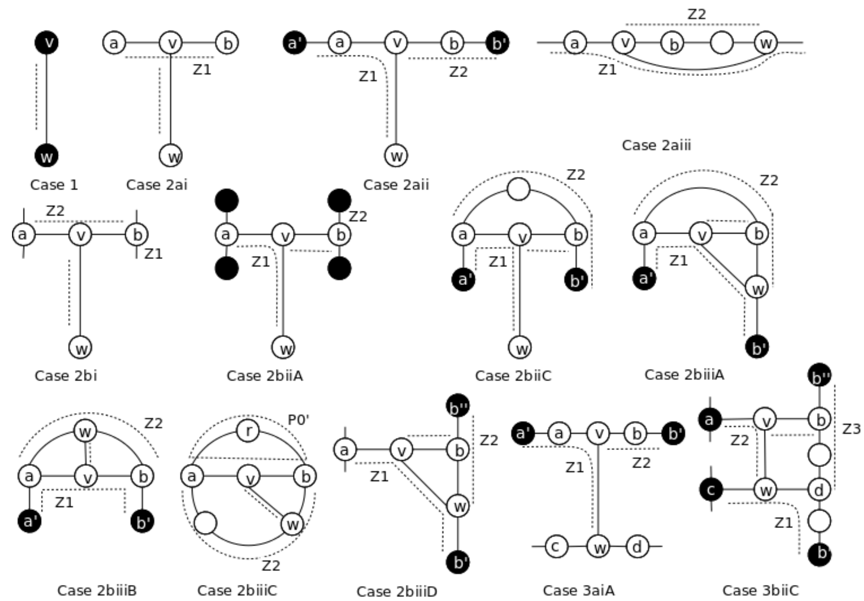


Figure 4: Case 1, 2 and 3 of the Lemma 2. Black vertices are the endpoints of the ears that are contained in $G_{birth(b)}$. Dashed paths are part of the ears in D'.

*If* $b' = a$, $Z_1 = av \cup vb \cup P_{birth(b)}[b', b'']$ *and* $Z_2 = P_{birth(b)}[b, b']$. *Apply* $M1(Z_1, Z_2, \emptyset)$ *and replace* $P_{birth(ab)} = ab$ *with* $vw$.

3. $\Gamma$ *is an edge-edge-addition: It will again depend whether b is an inner vertex or not. (Due to space constraints we will only see 2 of the 13 cases described in [7].)*

*3aiA)* $birth(b) = birth(ab)$ *(inner vertex),* $birth(d) < birth(b)$ *and* $birth(cd) < birth(b)$. *Let a' and b' be the endpoints of* $P_{birth(b)}$ *such that a' is closer to a than to b. Subdivide ab with v. Then* $Z_1 = P_{birth(b)}[a', v] \cup vw$ *and* $Z_2 = P_{birth(b)}[v, b']$. $P_{birth(cd)}$ *can be short or long. If* $P_{birth(cd)} = cd$ *(i.e. short), delete* $P_{birth(cd)}$ *and apply* $M1(cw \cup wd, Z_1, Z_2)$. *If* $P_{birth(cd)}$ *is long, apply M2 and* $M1(Z_1, Z_2, \emptyset)$.

*3biiC)* $birth(b) < birth(ab)$ *and* $P_{birth(ab)} = ab$, $birth(d) = birth(b)$, $birth(a) < birth(b)$ *and* $C \notin P_{birth(b)}$.
*Then* $P_{birth(cd)} = cd$, *have to be a short ear, because* $C \notin P_{birth(b)}$, $birth(d) = birth(b)$ *and* $birth(c) < birth(d)$. *Let b' and b" be the two endpoints of* $P_{birth(b)}$ *such that b' is closer to d than to b on* $P_{birth(b)}$ *(a may be in* $\{b', b'', c\}$*). There can be cases where b=d, and b=d and* $ru \in \{ab, cd\}$, *but due to space constraint we will not go into their details. Consider* $b \neq d$, $Z_1 = P_{birth(b)}[d, b'] \cup cw \cup wd$, $Z_2 = av \cup vw$ *and* $Z_3 = P_{birth(b)}[d, b'']$. *Apply* $M1(Z_1, Z_2, Z_3)$, *delete* $P_{birth(cd)} = cd$ *and replace* $P_{birth(ab)} = ab$ *with* $vb$.

# 4 Proof of Lemma 2

**Completeness** - We will first prove that Lemma 2 covers all the possible cases of BG-operations. Case (1) is simple and cannot have further subdivisions in the main case. Case (2), is an edge-vertex-operation. W.l.o.g we can choose either vertex $a$ or $b$ to look all possible relations with $birth(ab)$, say $b$. Observe, only two possibilities, either $b$ will be an inner vertex of $P_{birth(ab)}$ or not, hence $birth(b) < birth(ab)$ or $birth(b) = birth(ab)$. Third vertex in Case (2) is $w$, it can either be in $P_{birth(b)}$ or in $G_{birth(b)} - P_{birth(b)}$ or in $\overline{G_{birth(b)}}$, and hence three cases under (2a) and (2b). See [7] for completeness of case (3).

**D' is an ear decomposition** - As the newly added ears in all the cases are paths, it is sufficient to show that only the first ear, $P_0'$ in D' is a cycle. The only cases which can modify $P_0$ are (2ai), (2aiii), (2bi), (2biiic), and some cases in (3) (see [7]). All of these cases can subdivide an edge in $P_0$ by a new vertex and replace a path in $P_0$ with a shorter one having the same endpoints. This will still maintain a cycle in D'. Hence D' is an ear decomposition.

**D' avoids $ru$ ($rv$ or $rw$ if applicable)** - Recall Def 4, to prove this we need show that D' satisfy conditions (1) and (2) of Def. 4. To prove condition (1), it is sufficient to consider cases where $P_0$ is different from $P_0'$, i.e. cases (2aiii), (2biiic) and some in case (3). In all the cases, a path of $P_0$ that does not contain $r$ has been modified, therefore $r \in P_0'$ also. This shows condition (1) is satisfied.

To prove condition (2), first consider the vertex-vertex-addition: only a short ear is added at the end, hence satisfied. Now consider a edge-vertex-addition, first consider the case where $v$ does not subdivide $ru$. $P_{birth(u)}$ is the last long ear in D, therefore $birth(u) \geq birth(b)$. If $birth(u) > birth(b)$, that means change is in $P_{birth(b)}$ and $P_{birth(u)}$ is unchanged and is still a last long ear in D'. Thus, condition (2) satisfied for this case.

If $birth(u) = birth(b)$, and as $P_{birth(b)}$ has $b$ as inner vertex, it is long; this implies $b = u$ as $u$ is the only inner vertex of $P_{birth(u)}$. Now it will depend if $a \in P_{birth(b)}$, $a$ must be a neighbor of the inner vertex $b = u$ and $birth(ab) = birth(b)$ follows and also $w \in G_{birth(b)}$, because we are at the last ear. This means we are in case (2aii) or (2aiii). In both these cases $Z_2$ (the last long ear) contains exactly $b = u$ as the inner vertex and does not contain $ru$, hence condition (2) satisfied. If $a \notin P_{birth(b)}$, $birth(a) < birth(b) < birth(ab)$ and again $w \in G_{birth(b)}$, therefore we are in case (2biiA) with $a \neq r$. See that in this case $P_{birth(b)}$ remains unchanged and no long ear is added after $P_{birth(b)}$, hence condition (2) satisfied.

Now lets assume that $v$ subdivides $ru$. This mean $a = r$ and $b = u$ with $birth(a) < birth(b)$, as $r \in P_0$ and $u \notin P_0$. Also, as D satisfies condition (2), it cannot have $ab = ru$ in its last long hear, hence, is itself a short ear. Also $w \in G_{birth(b)}$, hence we are in case (2biiA) with $ab = ru$. In this ear $wv \cup vb$ is added as last long ear and $av$ is the new avoided edge of D', hence condition (2) satisfied.

See [7] for edge-edge-addition correctness.

**D' is non-separating** - Consider $D' = (P_0', P_1', ..., P_{k+1}')$ and let $z$ be any inner vertex of $P_i'$. It suffices to prove that $z$ has a neighbor in $\overline{G_i'} \neq \emptyset$. First consider the vertex-vertex-addition: it only adds a short ear at the end, hence the configuration of all long ears remains identical in D and D'. Given that D is non-separating, D' is also non-separating.

| | (2ai) | (2aii) | (2aiii) | (2bi) | (2biiA) | (2biiB) | (2biiiA) | (2biiiB) | (2biiiC) | (2biiiD) |
|---|---|---|---|---|---|---|---|---|---|---|
| Neighbor of v | w | b | a or b | w | b | b | a or b | w | $inner(Z) \cap \{a, b, w\}$ | b or w |

Table 1: Neighbors of $v \in inner(P_i')$ in $\overline{G_i'}$

Now consider a edge-vertex-addition. We will see an overview of the proof, due to space constraints. Observe that $z = v$ is the only new inner vertex possible in D', compared to D. It is sufficient to show that $z = v$ has a neighbor in $\overline{G_i'}$. Table 4 shows the neighbors of $z = v$ in $\overline{G_i'}$ for all the sub-cases in case (2).

This ends the proof of Lemma 2. The paper [7] presents the Case (3) and its proof in more detail, and also presents 3 applications which can now be solved in linear time, because Mondshein sequence can be calculated in linear time. Due to space constraints we conclude these notes here.

# References


[1] D. Barnette. On steinitz's theorem concerning convex 3-polytopes and on some properties of planar graphs. In G. Chartrand and S. Kapoor, editors, *The Many Facets of Graph Theory*, volume 110 of *Lecture Notes in Mathematics*, pages 27–40. Springer Berlin Heidelberg, 1969.

[2] J. Cheriyan and S. Maheshwari. Finding nonseparating induced cycles and independent spanning trees in 3-connected graphs. *Journal of Algorithms*, 9(4):507 – 537, 1988.

[3] G. Di Battista, R. Tamassia, and L. Vismara. Output-sensitive reporting of disjoint paths. *Algorithmica*, 23(4):302–340, 1999.

[4] G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16(1):4–32, 1996.

[5] L. Mondshein and M. I. O. T. L. L. LAB. *Combinatorial Ordering and the Geometric Embedding of Graphs*. Defense Technical Information Center, 1971.

[6] J. M. Schmidt. Contractions, removals, and certifying 3-connectivity in linear time. *SIAM J. Comput.*, 42(2):494–535, 2013.

[7] J. M. Schmidt. The mondshein sequence. *CoRR*, abs/1311.0750, 2013.

[8] W. Tutte. How to draw a graph. *Proceedings of the London Mathematical Society*, 3(1):743–767, 1963.

[9] W. Tutte. Connectivity in graphs. In *Mathematical Expositions*, volume 15. University of Toronto Press, 1966.