Analytical Formulation for Gravitational Wave Event Rates

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Last updated: June 2024

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1 Overview

This document presents the analytical framework for calculating gravitational-wave (GW) event rates from compact-binary coalescences (CBCs), encompassing binary black holes (BBH), binary neutron stars (BNS), and neutron star–black hole binaries (NSBH). The formulation combines cosmic merger-rate densities with detector sensitivity and selection effects to predict annual detection rates. Practical implementations using the `ler` package are referenced throughout.

2 Introduction

The annual rate of detectable (unlensed) GW events, $\Delta N_{\rm U}^{\rm obs}/\Delta t$, represents the expected number of observed CBC mergers per year for a given detector network. It is obtained by combining the total intrinsic merger rate in the detector frame, $\Delta N_{\rm U}/\Delta t$, with the population-averaged probability of detection $P(\text{obs})$:

$$\frac{\Delta N_{\rm U}^{\rm obs}}{\Delta t} = \frac{\Delta N_{\rm U}}{\Delta t} \times P(\text{obs})\,. \tag{1}$$

3 Parameter-Marginalized Event Rate

Let the full parameter vector be $\vec{\theta} = \{\vec{\theta}_{\rm int}, \vec{\theta}_{\rm ext}\}$, where intrinsic parameters describe the source-frame properties of the binary, and extrinsic parameters describe its configuration relative to the detector network. Explicitly,

$$\begin{aligned} \vec{\theta} = \{&z_s, m_1, m_2, a_1, a_2, \theta_1, \theta_2, \phi_{12}, \phi_{JL}, \\ &\iota, \phi, \psi, \text{RA}, \text{Dec}, t_c\}. \end{aligned} \tag{2}$$

Given a joint prior $P(\vec{\theta})$ and conditional detection probability $P(\text{obs} \mid \vec{\theta})$, the population-averaged detection probability is

$$P(\text{obs}) = \int P(\text{obs} \mid \vec{\theta})\, P(\vec{\theta})\, d\vec{\theta}. \tag{3}$$

The detectable event rate therefore becomes

$$\frac{\Delta N_{\rm U}^{\rm obs}}{\Delta t} = \frac{\Delta N_{\rm U}}{\Delta t} \int P(\text{obs} \mid \vec{\theta})\, P(\vec{\theta})\, d\vec{\theta}. \tag{4}$$

4 Redshift Distribution and Intrinsic Merger Rates

The `ler` framework parameterizes source distance using redshift z_s and assumes z_s is statistically independent of other source parameters. The redshift probability density is proportional to the detector-frame merger-rate density and the comoving volume element:

$$\begin{aligned} P(z_s) &\propto \frac{d^2N}{dt\, dV_c} \frac{dV_c}{dz_s} \\ &\propto \frac{d^2N}{d\tau\, dV_c} \frac{d\tau}{dt} \frac{dV_c}{dz_s} \\ &\propto \frac{R_{\rm U}(z_s)}{1+z_s} \frac{dV_c}{dz_s}. \end{aligned} \tag{5}$$

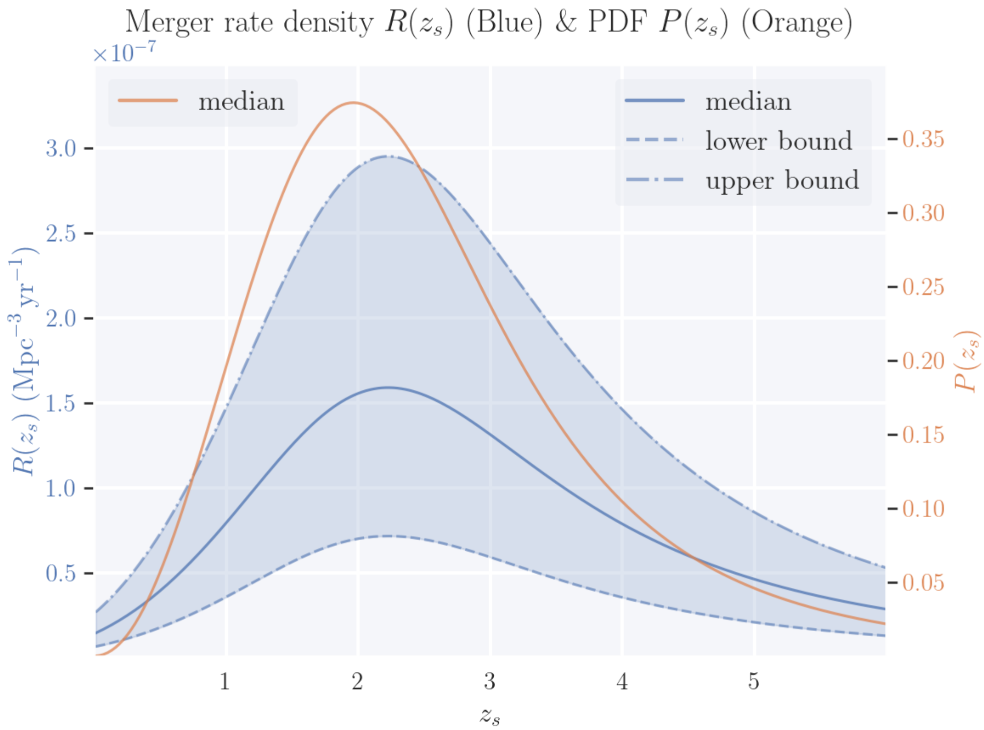

Figure 1: Redshift evolution of the merger-rate density $R(z_s)$ (left axis) and normalized redshift probability density $P(z_s)$ (right axis) for BBH mergers. The model follows a Madau–Dickinson–like prescription including metallicity and time-delay effects.

Here $R_{\rm U}(z_s)$ is the source-frame merger-rate density per comoving volume, and the factor $(1+z_s)^{-1}$ accounts for cosmological time dilation.

Normalizing the distribution introduces $\mathcal{N}_{\rm U}$, the total intrinsic merger rate per detector-frame year:

$$P(z_s) = \frac{1}{\mathcal{N}_{\rm U}} \frac{R_{\rm U}(z_s)}{1+z_s} \frac{dV_c}{dz_s}, \tag{6}$$

with

$$\mathcal{N}_{\rm U} = \int_{z_{\rm min}}^{z_{\rm max}} \frac{R_{\rm U}(z_s)}{1+z_s} \frac{dV_c}{dz_s}\, dz_s. \tag{7}$$

The observed event rate can therefore be written compactly as

$$\frac{\Delta N_{\rm U}^{\rm obs}}{\Delta t} = \mathcal{N}_{\rm U} \left\langle P(\text{obs} \mid \vec{\theta}) \right\rangle_{\vec{\theta}\sim P(\vec{\theta})}. \tag{8}$$

5 Detection Criterion and SNR Modeling

The conditional detection probability is defined through an SNR threshold $\rho_{\rm th}$:

$$P(\text{obs} \mid \vec{\theta}) \equiv P_{\rm det}(\vec{\theta}, \rho_{\rm th}) = \Theta\Big[\rho_{\rm obs}(\vec{\theta}) - \rho_{\rm th}\Big], \tag{9}$$

where Θ is the Heaviside step function.

In practice, the observed SNR $\rho_{\rm obs}$ is treated as a stochastic variable derived from the optimal SNR $\rho_{\rm opt}$. The `ler` and `gwsnr` packages adopt a non-central chi-squared model by default, following modern population-analysis practice.

Remark. The detection-probability formalism is generic and applies to any signal characterized by an SNR-like detection statistic, including electromagnetic transients.

6 Simulation Results

All results shown below are generated using the default configuration of the `ler` package.

6.1 Simulation Settings

A flat ΛCDM cosmology is assumed with $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_m = 0.3$, and $\Omega_\Lambda = 0.7$.

Waveforms are generated using the IMRPhenomXPHM approximant with a sampling frequency of 2048 Hz. The detection threshold is $\rho_{\rm th} = 10$, and a 100% duty cycle is assumed.

6.2 GW Source Parameter Priors

Table 1: Prior distributions for GW source parameters.

Parameter	Unit	Prior	Range	Description
z_s	–	Derived	$[0, 10]$	Source redshift
$m_{1,2}$	$M_\odot$	PowerLaw+Peak / Bimodal	see text	Component masses
$a_{1,2}$	–	Uniform	$[0, 0.99]$	Spin magnitudes
$\theta_{1,2}$	rad	Sine	$[0, \pi]$	Spin tilt angles
RA	rad	Uniform	$[0, 2\pi]$	Right ascension
Dec	rad	Cosine	$[-\pi/2, \pi/2]$	Declination
ι	rad	Sine	$[0, \pi]$	Inclination
ψ	rad	Uniform	$[0, \pi]$	Polarization angle

6.3 Detectable vs Intrinsic Populations

6.4 Rate Estimates

Detector Network	BBH yr^{-1}	BNS yr^{-1}	BBH:BNS
L1–H1–V1 (O4)	292.7	7.4	39.5
CE–ET (3G)	8.87×10^4	1.49×10^5	0.6

Note. O4 rates assume design sensitivity and a 100% duty cycle and should be regarded as optimistic upper limits.

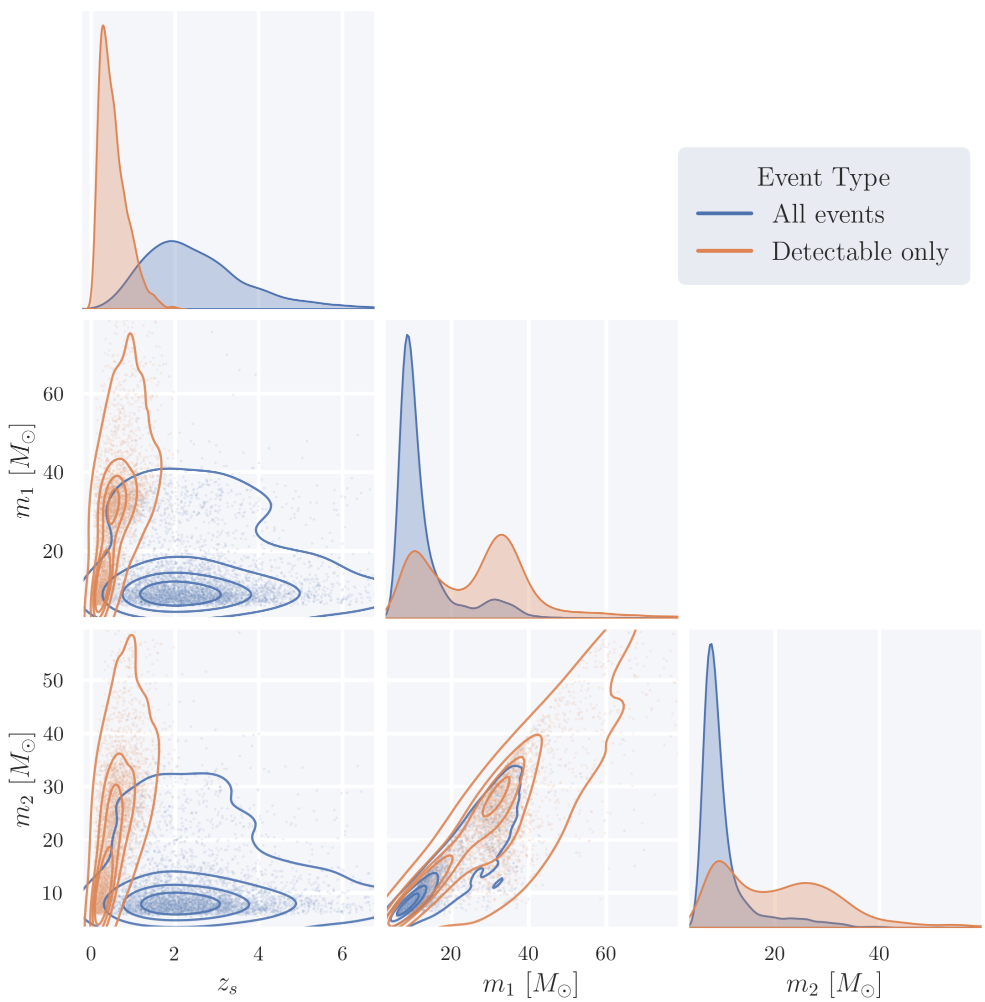

Figure 2: Corner plot comparing intrinsic (blue) and detectable (orange) BBH populations for the LIGO–Virgo–KAGRA O4 network. Detectable events are biased toward lower redshift and higher masses.