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TUTORIAL - 2

GRAPH THEORY

Hemanth, N.
2019503519

3.b) Show that in a tree, diameter is not necessarily equal to twice its radius. Under what conditions does this hold?

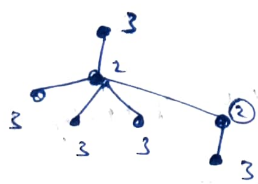
* Eccentricity of the centre of a tree is defined as the radius of the tree.
* Diameter is defined as the length of the longest path in tree.

This is an example tree, where radius and diameter both are equal to 1.

[Radius (max dist. from vertex to other) is 1
Diameter (Max dist b/w 2 vertices) is 1]

③ Tree → Radius is the

↳ Diameter is 5 (6-1), since 6 vertices are there



[Radius is 2
Diameter is 3]

If the graph is disconnected, ⇒ Radius is infinity

If u, v are vertices of G such that d(u,v) is Diam of G, w be central vertex → e(w) = rad G.

⇒ No vertex is at distance greater than rad G from w.

Thus, d(u,w) + d(v,w) ≤ 2 rad G. & also by triangle ineq

d(u,v) ≤ d(u,w) + d(v,w) ⇒ $\boxed{\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G}$

∴ This proves that Diam is not always equal to twice the radius.

3.13) Prove that a pendant edge in a connected graph G is contained in every spanning tree of G.

Proof:

First, we need to prove that, edge e of connected graph will be a bridge, iff e belongs to every spanning tree of G.

we can prove by contraposition,

Suppose that e does not belong to every spanning tree of G.
Let T be a spanning tree that does not contain e.
Then T is a spanning subgraph of $G-e$...

∵ u and v → vertices of $G-e$, then there is a unique u-v path in T. This is also u-v path in $G-e$. ∴ $G-e$ is connected.
If e is not bridge, $G-e$ is connected, $G-e$ has spanning tree T.
∵ $V(H) = V(G-e)$, T is spanning tree of G, spanning tree won't have e.

∴ ~~we~~ By contraposition, we say e is definitely a part of spanning tree T.

∴ If we remove ~~disconnect~~ e from graph, the graph is disconnected.
By corollary,
∴ We say that, a pendant edge is always in the spanning tree of G.

3.15) Nullity of a complete graph of n vertices.

Nullity will be no of chords in the graph

Nullity of a complete graph:

By definition, we know, every complete graph is a connected graph.

Nullity is given by, $e-n+k$,

if $k=1 \Rightarrow$ it is connected.

e → no of edges in G
k → no. of components
n → no of vertices

Nullity of a complete graph is $e-n+1$

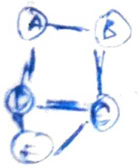
3.16) Show that a Hamiltonian path is a spanning tree.

A Hamiltonian circuit in a graph G, is a closed path that traverses every vertex in G exactly once.

Hamiltonian path - each vertex traversed. Thus, all vertices of G is contained in Hamiltonian path. (ie) any vertex not traversed twice. Thus, graph with no cycle.

// spanning Tree → Graph with no cycle. //

∴ we arrive at a conclusion that Hamiltonian Path is a spanning tree.



ABCDE → Hamiltonian Graph

[Spanning Tree → Possible ABCDE]

4.5) what is the edge connectivity of the complete graph of n vertices?

Edge connectivity is the minimum size of disconnecting set, and is denoted as $k'(G)$.

$\boxed{\text{complete graphs} \rightarrow \text{edge connectivity is } n-1.}$

we can prove by induction,

* connected graph with single vertex → 0 or more edges
* Hypothesis: holds good for $< n$.

we remove edges until we no longer have a connected graph, but 2 smaller connected graphs.

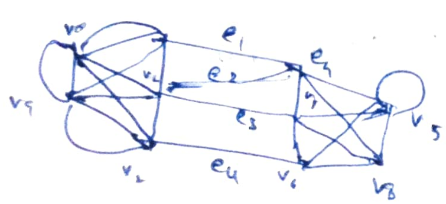
say, 1st first one has k vertices, so other has $(n-k)$ vertices.

∴ atleast $(k-1)$ and $(n-k-1)$ edges, for total of $(n-2)$.

we had to remove atleast one edge to disconnect the graph. ∴ minimum → $(n-1)$.

∴ $\boxed{\text{edge connectivity of complete graph is } n-1}$

4.23) Graph with Edge con. = 4, Ver. con = 3 and Degree of each $G \geq 5$.



vertex connectivity is 3 (∵ removal of v_1, v_2, v_3, and v_4, v_5, v_6 disconnect

edge connectivity is 4.

$\therefore S = \{e_1, e_2, e_3, e_4\}$ is ...

Degree ...

Q.6) Prove that complement of a cutset does not contain a spanning tree. and prove that the complement of a spanning tree does not contain a cut-set.

Proof:

If possible, suppose that a cut-set is contained in the complement of a spanning tree.

W.k.T, cut set and any spanning tree must have atleast one edge in common. This common edge must belong to the complement of spanning tree.

Thus, this edge belongs to the spanning tree and also its complement. This is a contradiction.

Hence, the complement of a spanning tree does not contain a cut-set.

lly, we can say complement of cutset does not contain a spanning tree, by interchanging cut set & spanning tree in above proof. since, cut set & spanning tree must have one edge in common.

3.17) Prove that any circuit in a graph G, must have atleast one edge in common with the chord set.

Proof:

We know that, Chord set is the complement of a spanning Tree.

Suppose, on the contrary, that a circuit has no common edge with the complement of a spanning tree. [set of all Chords of a tree is called complement of tree].

Now, this would mean that circuit is wholly contained in the spanning tree. This contradicts the fact that the tree is acyclic. Hence, our assumption is wrong.

Hence, a circuit, has atleast one edge in common with complement of a spanning tree.

3.20) Show that distance between two spanning trees as defined is a metric.

Let G be a connected graph and T_1 and T_2 be 2 spanning Trees.

No of edges in G in one of trees is not in other is called Distance b/w T_1 and T_2. $\boxed{d(T_1, T_2)}$

(i.e) If u, v, w are 3 vertices of a connected graph, then,

$d(u,v) \geq$ and $d(u,v) = 0$ iff $u = v$.

$d(u,v) = d(v,w)$

$d(u,v) \leq d(u,w)$

[From the relation, we need to prove, distance in a graph is a metric.]

Dist b/w trees T_1, T_2 is $d(T_1, T_2)$

(i) For any tree, $d(T,T) = 0$, as long as all costs are positive, then for any ~~[illegible]~~

$T_1 != T_2$, $d(T_1, T_2) > 0$, $\therefore$ non-negativity is satisfied

(ii) As long as cost of removing a vertex is same as cost of adding a vertex, for any

T_1, T_2, $d(T_1, T_2) = d(T_2, T_1)$. $\therefore$ symmetry is satisfied

(iii) For any 3 trees, say T_1, T_2, T_3,

$d(T_1, T_3) \leq d(T_1, T_2) + d(T_2, T_3)$, [Because one could edit T_1 to T_3, by first editing it to T_2]

[$\therefore$ Triangle equality is satisfied]

$\boxed{\text{All the above criteria define a metric, Thus } d(T_1, T_2) \text{ is a metric}}$

~~Elaborating (ii)~~,

~~[illegible]~~

(2) ...

4.(2) Prove that a non-separable graph has nullity 1, iff graph is circuit.

{No separation nodes ⇒ Non-separable.}

Nullity is $e-n+K$

[Nullity is no of chords]

Circuit → Path start and end with same Point

[Spanning Tree's complement, chord → same edges]

We will say the graph is non-separable & forms a circuit, if they have atleast 2 common vertices.

[No 2 graphs G_1, G_2 each containing atleast one arc, which form G if a vertex of one is made to coalesce with other.]

W.K.T, a forest G is a graph of Nullity 0.

∴ G has circuit. Suppose G contained other arcs beside, remove one of these, the nullity remains 1, as circuit is still present, contrary to the assumption.

There are no vertices in G, as G contains no isolated vertices. Hence G is a circuit, with nullity 1.

4.(9) Graph with n vertices & vertex connectivity K must have atleast $Kn/2$ edges.

If m is no of edges in a vertex connectivity K ... graph

$\delta(G) \leq 3m/n$

$\Downarrow$

$K \leq \delta(G) \leq \frac{2m}{n}$ [By whitney's inequality]

[inequality relating connectivity, edge con, & min degree]

$\Rightarrow$ $\boxed{m \geq Kn/2}$

[w.k.t $K \leq \lambda(G) < \delta(G) \leq deg\, x_i$]

Deleting neighbour of x_i disconnect G, so $\boxed{K(G) \leq K = \lambda(G)}$

4.8) Prove that in a non-separable graph G, set of edges incident on each vertex G is a cut-set.

wkt,

there must be no separation nodes in non-separable graph,

we assume that,

set of edges incident on vertex $V_1, V_2, \ldots$ is not a cut-set.

By definition, cut-set is minimal set of edges in a connected graph, whose removal reduces the rank of graph by one.

By our assumption,

$\therefore$ ~~the~~ edges does not destroy the paths b/w 2 set of vertices, $\therefore$ thus we can say our assumption is wrong.

$\therefore$ Removal of edge in that graph, will break the graph, thus,

Non-separable graph G, all set of edges incident on vertex G is a circuit.