CS6109 – GRAPH THEORY

Module – 3

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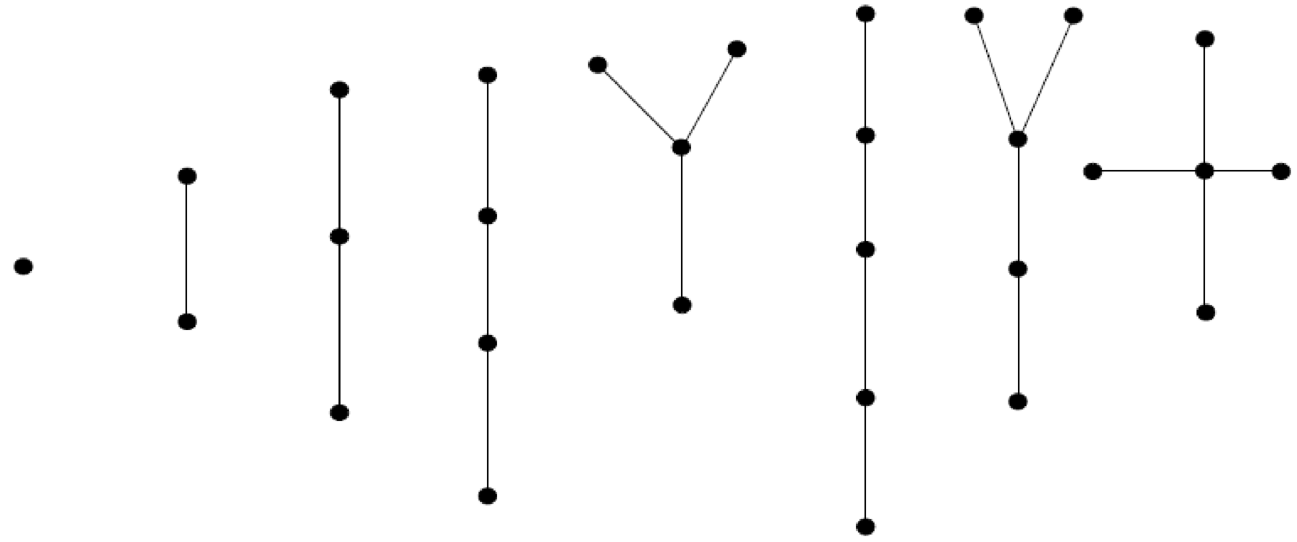
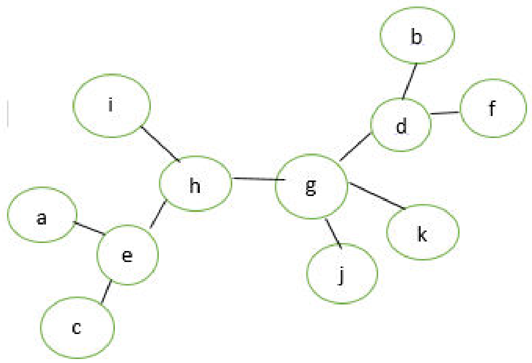
Module - 3

- Trees
- Properties
- Distance and Centres
- Types
- Rooted and Binary Tree
- Tree Enumeration
- Labeled Tree
- Unlabeled Tree

Trees

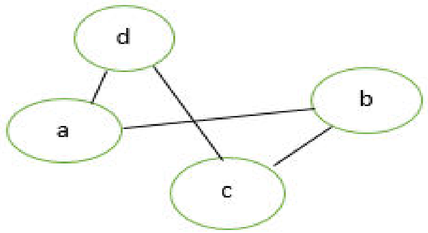
- A tree is a connected graph without any circuits.
- Definition: A graph having no cycles is said to be acyclic. A forest is an acyclic graph.
- Definition: A tree is a connected graph without any cycles, or a tree is a connected acyclic graph. The edges of a tree are called branches. It follows immediately from the definition that a tree has to be a simple graph (because self-loops and parallel edges both form cycles).





PROPERTIES OF TREES

1. There is one and only one path between every pair of vertices in a tree T.
2. In a graph G there is one and only one path between every pair of vertices, G is a tree.
3. A tree with *n* vertices has *n*-1 edges.
4. Any connected graph with *n* vertices has *n*-1 edges is a tree.
5. A graph is a tree if and only if it is minimally connected.
6. A graph G with *n* vertices has *n*-1 edges and no circuits are connected.

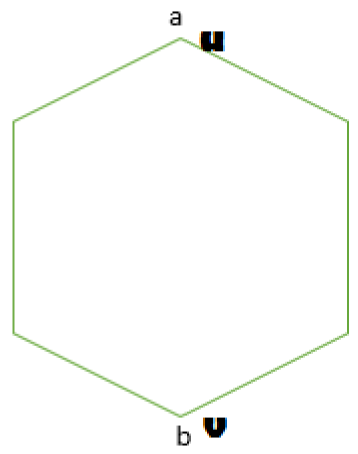


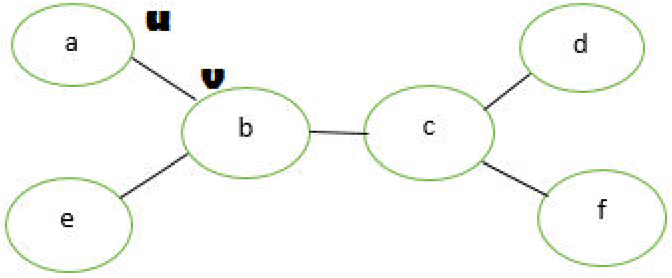
Graph G but not Tree

Theorem 4.1: A graph is a tree if and only if there is exactly one path between every pair of its vertices.

Proof:

Let G be a graph and let there be exactly one path between every pair of vertices in G. So G is connected. Now G has no cycles, because if G contains a cycle, say between vertices u and v, then there are two distinct paths between u and v, which is a contradiction. Thus G is connected and is without cycles, therefore it is a tree.

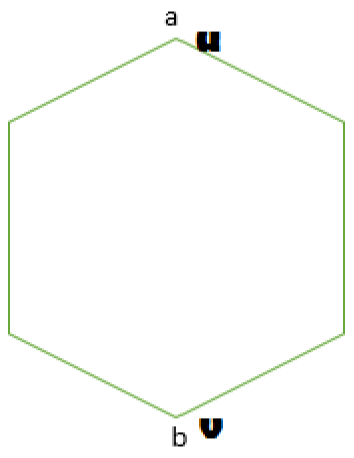


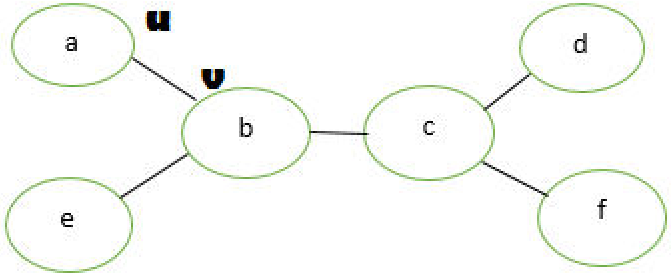


Theorem 4.1: A graph is a tree if and only if there is exactly one path between every pair of its vertices.

Proof:

- Conversely, let G be a tree. Since G is connected, there is at least one path between every pair of vertices in G. Let there be two distinct paths between two vertices u and v of G. The union of these two paths contains a cycle which contradicts the fact that G is a tree. Hence there is exactly one path between every pair of vertices of a tree.





Theorem 4.2: A tree with n vertices has n−1 edges.

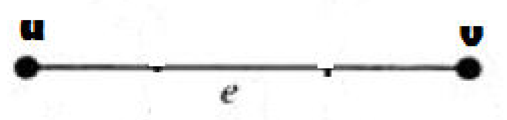
Proof:

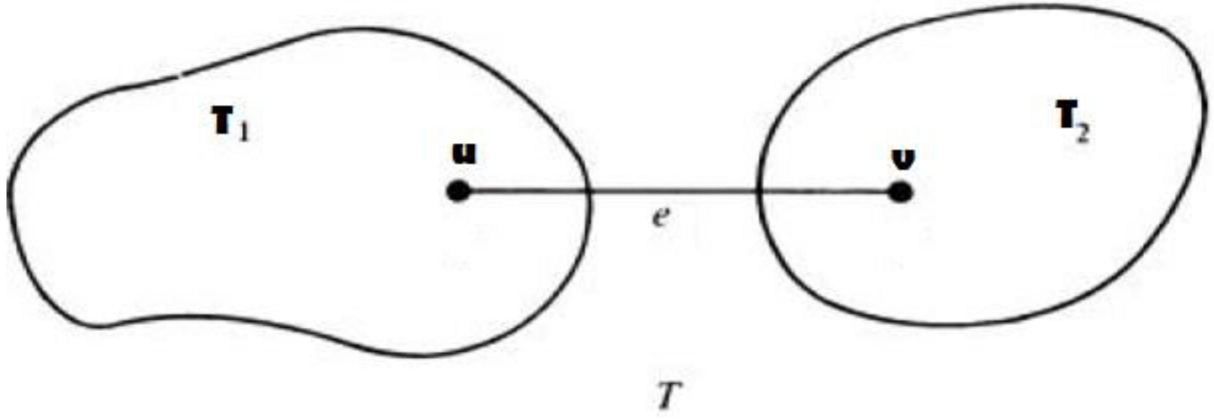
- We prove the result by using induction on n, the number of vertices. The result is obviously true for n = 1, 2 and 3. Let the result be true for all trees with fewer than n vertices.
- Let n be the number of vertices in a tree (T).
  If n=1, then the number of edges=0.
  If n=2 then the number of edges=1.
  If n=3 then the number of edges=2.
- Hence, the statement (or result) is true for n=1, 2, 3.

Theorem 4.2: A tree with n vertices has $n-1$ edges.

Proof:

- Let T be a tree with n vertices and let e be an edge with end vertices u and v. So the only path between u and v is e. Therefore deletion of e from T disconnects T. Now, $T - e$ consists of exactly two components T_1 and T_2 say, and as there were no cycles to begin with, each component is a tree. Let n_1 and n_2 be the number of vertices in T_1 and T_2 respectively, so that $n_1 + n_2 = n$. Also, $n_1 < n$ and $n_2 < n$. Thus, by induction hypothesis, number of edges in T_1 and T_2 are respectively n_1-1 and n_2-1.
- Hence the number of edges in $T = n_1-1+n_2-1+1 = n_1+n_2-1 = n-1$.

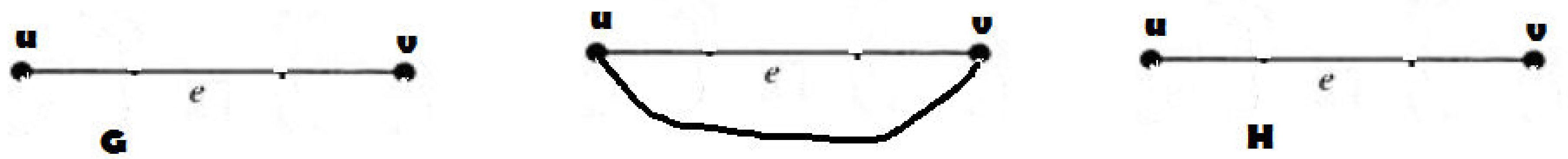




Theorem 4.3: Any connected graph with n vertices and n−1 edges is a tree.

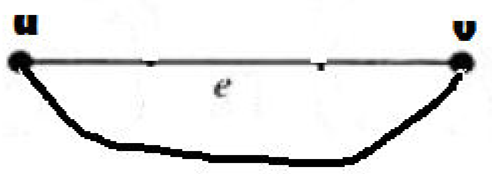
Proof:

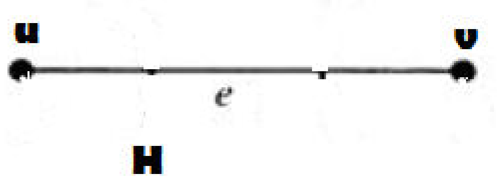
- Let G be a connected graph with n vertices and n −1 edges. We show that G contains no cycles. Assume to the contrary that G contains cycles.
- Remove an edge from a cycle so that the resulting graph is again connected. Continue this process of removing one edge from one cycle at a time till the resulting graph H is a tree. As H has n vertices, so number of edges in H is n−1. Now, the number of edges in G is greater than the number of edges in H. So n−1 > n−1, which is not possible. Hence, G has no cycles and therefore is a tree.



Minimally Connected Graph

- A graph is said to be minimally connected if removal of any one edge from it disconnects the graph. Clearly, a minimally connected graph has no cycles.

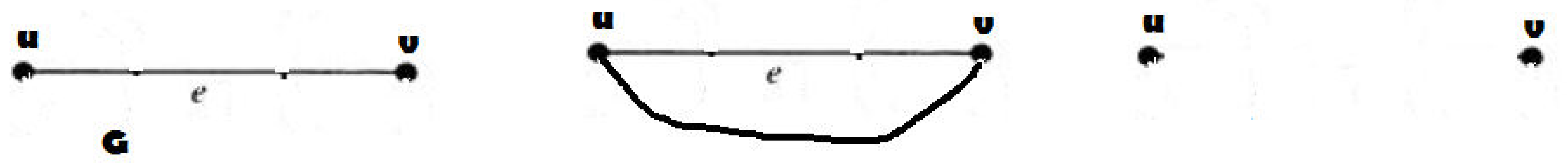




Theorem 4.4: A graph is a tree if and only if it is minimally connected.

Proof:

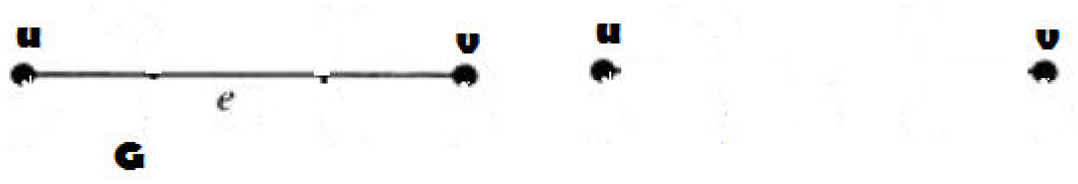
- Let the graph G be minimally connected. Then G has no cycles and therefore is a tree.
- Conversely, let G be a tree. Then G contains no cycles and deletion of any edge from G disconnects the graph.
- Hence G is minimally connected.



Theorem 4.5: A graph G with n vertices, n−1 edges and no cycles is connected.

Proof:

- Let G be a graph without cycles with n vertices and n−1 edges. We have to prove that G is connected.
- Assume that G is disconnected.
- So G consists of two or more components and each component is also without cycles. We assume without loss of generality that G has two components, say G_1 and G_2 (Fig. 4.1(b)). Add an edge e between a vertex u in G_1 and a vertex v in G_2. Since there is no path between u and v in G, adding e did not create a cycle. Thus G U e is a connected graph (tree) of n vertices, having n edges and no cycles. This contradicts the fact that a tree with n vertices has n−1 edges. Hence G is connected.



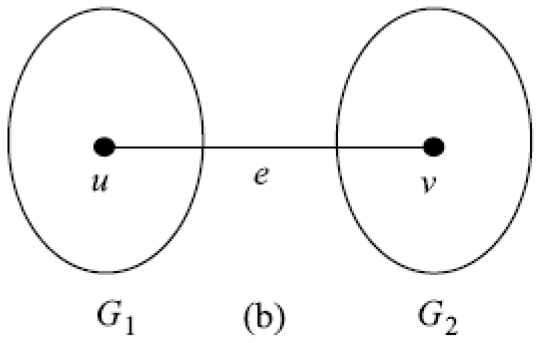
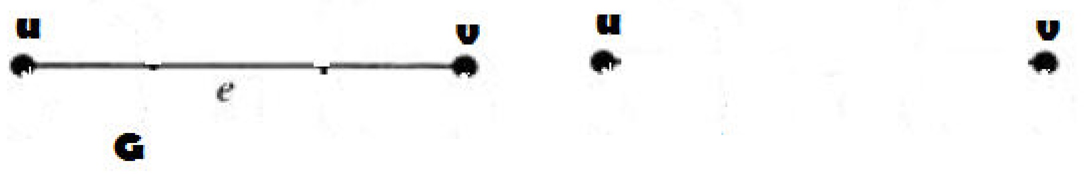


Fig. 4.1(b)

Theorem 4.6: Any tree with at least two vertices has at least two pendant vertices.

Proof:

- Let the number of vertices in a given tree T be n(n >1). So the number of edges in T is n−1. Therefore the degree sum of the tree is 2(n−1). This degree sum is to be divided among the n vertices.
- Since a tree is connected it cannot have a vertex of 0 degree. Each vertex contributes at least 1 to the above sum. Thus there must be at least two vertices of degree exactly 1.



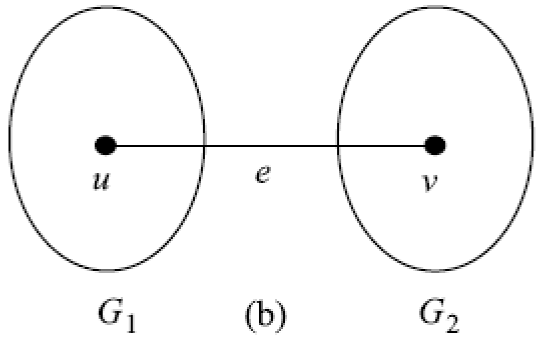
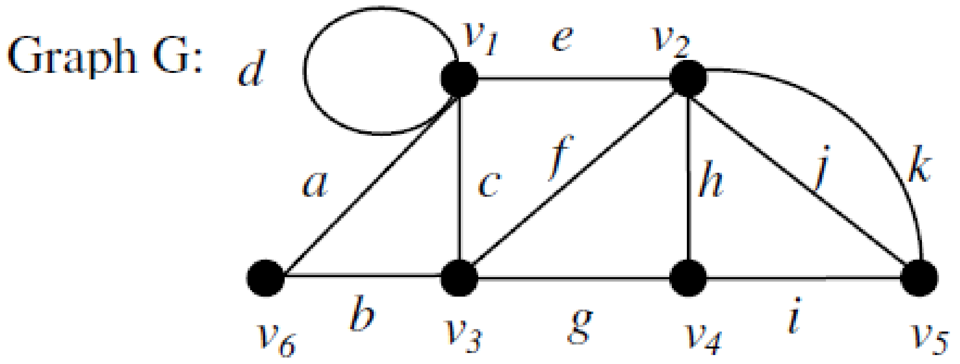


Fig. 4.1(b)

DISTANCE AND CENTERS IN TREE

- In a connected graph G, the distance $d(v_i\ ,\ v_j)$ between two of its vertices v_i and v_j is the length of the shortest path.



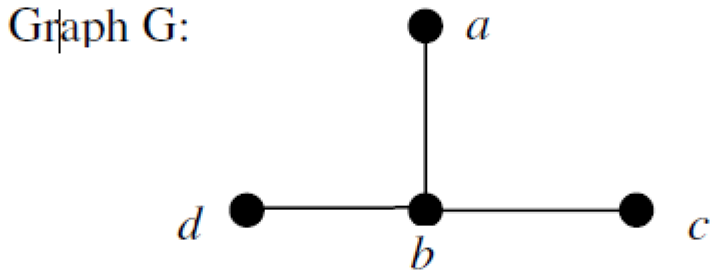
- Paths between vertices v_6 and v_2 are (a, e), (a, c, f), (b, c, e), (b, f), (b, g, h), and (b, g, i, k).
- The shortest paths between vertices v_6 and v_2 are (a, e) and (b, f), each of length two.
- Hence $d(v_6\ , v_2) = 2$

Eccentricity Centers

- The eccentricity E(v) of a vertex v in a graph G is the distance from v to the vertex farthest from v in G; that is,

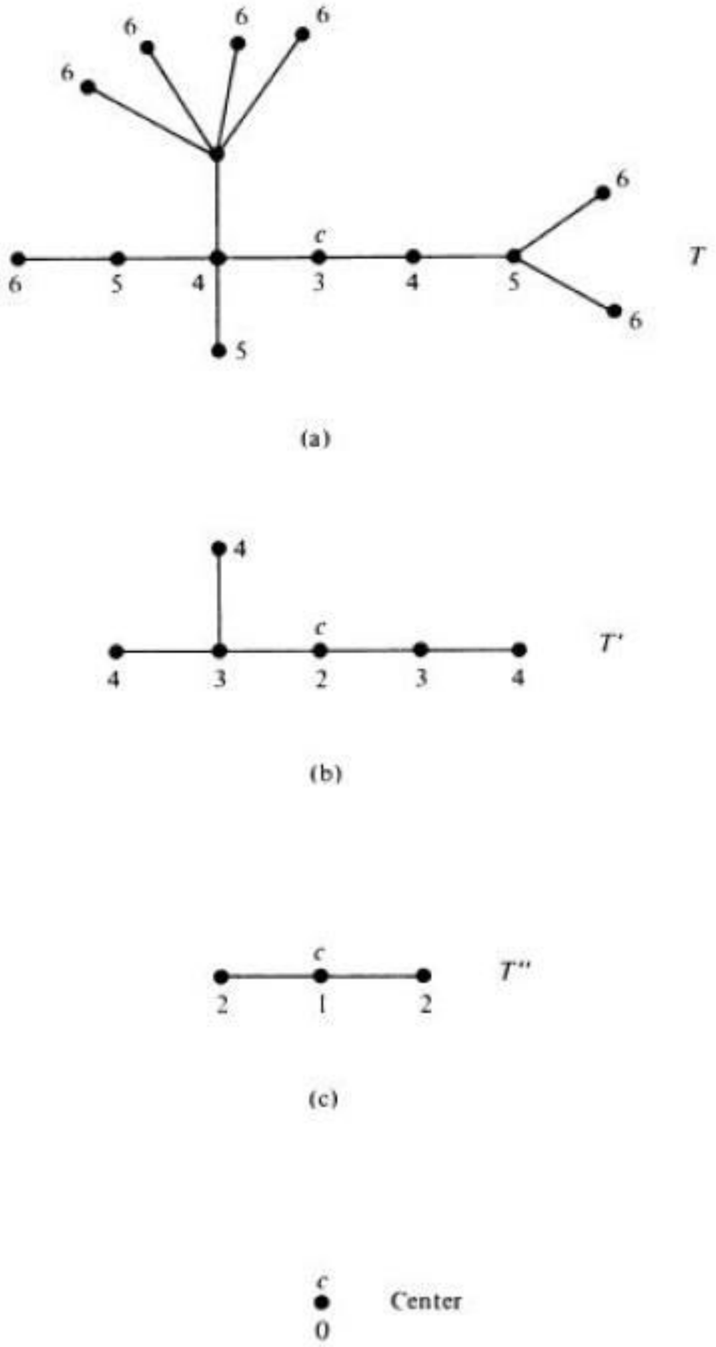
$$E(v) = \max_{v_i \in G} d(v, v_i)$$

- A vertex with minimum eccentricity in graph G is called a center of G



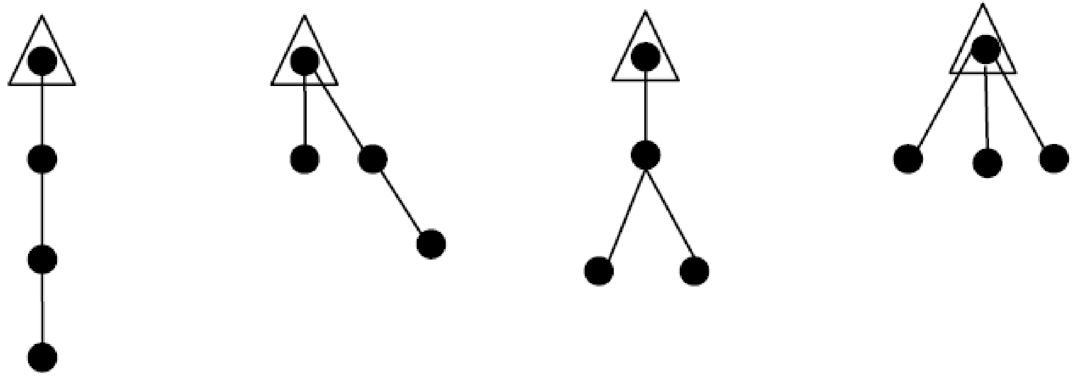
- Distance d(a, b) = 1, d(a, c) =2, d(c, b)=1, and so on.
- Eccentricity E(a) =2, E(b) =1, E(c) =2, and E(d) =2.
- Center of G = A vertex with minimum eccentricity in graph G = b.

Finding Center of graph

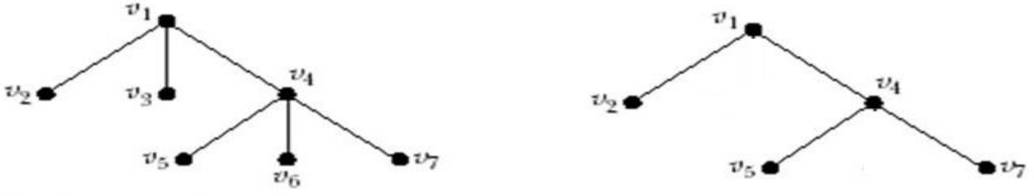


Rooted and Binary Trees

- A tree in which one vertex (called the root) is distinguished from all the others is called a rooted tree.



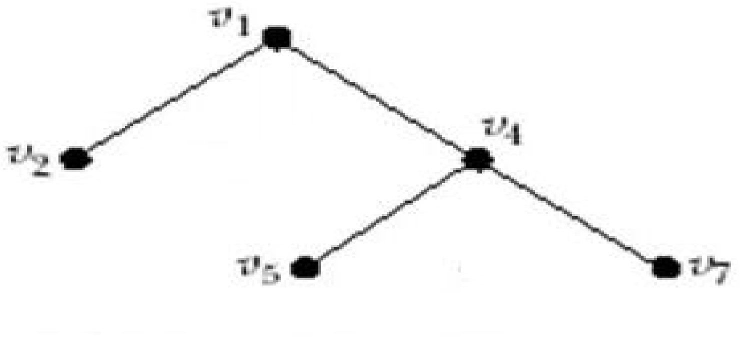
- A binary tree is defined as a tree in which there is exactly one vertex of degree two and each of the remaining vertices is of degree one or three. Obviously, a binary tree has three or more vertices. Since the vertex of degree two is distinct from all other vertices, it serves as a root, and so every binary tree is a rooted tree.



Theorem 4.10: Every binary tree has an odd number of vertices.

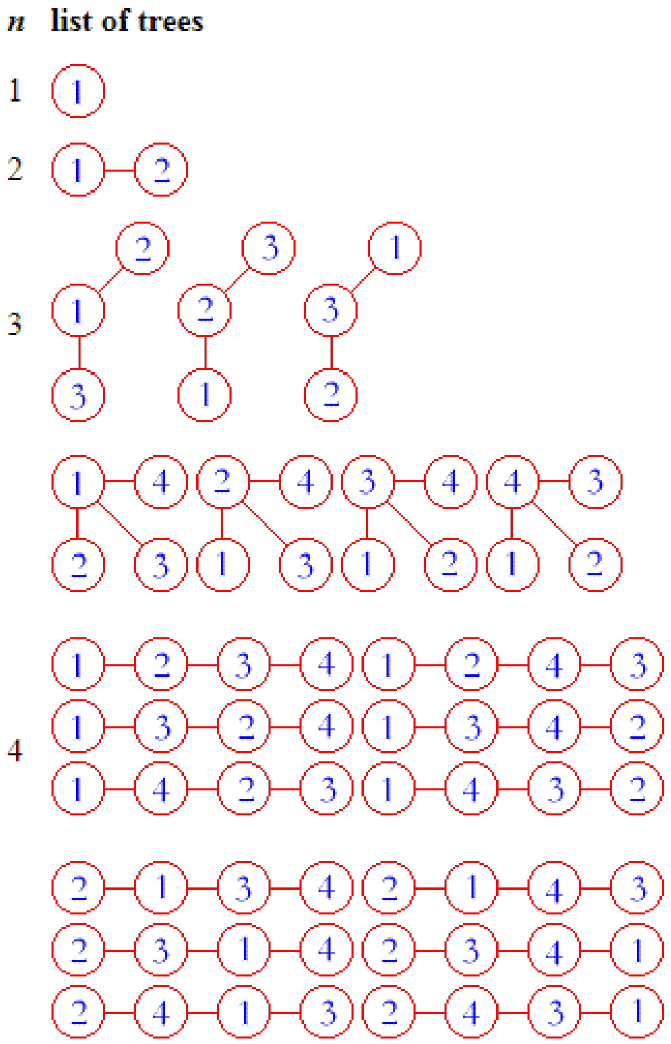
Proof :

- Apart from the root, every vertex in a binary tree is of odd degree.
- We know that there are even number of such odd vertices.
- Therefore when the root (which is of even degree) is added to this number, the total number of vertices is odd.



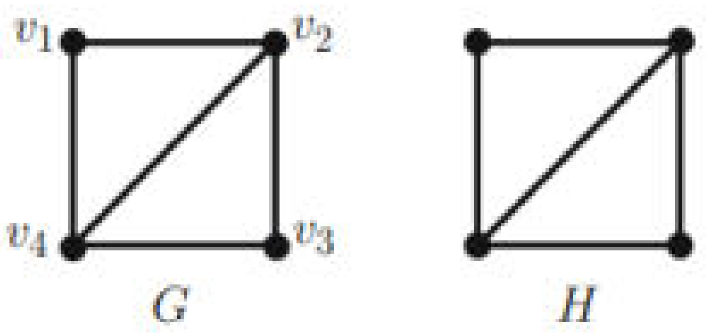
Enumeration of Trees

- Enumeration of trees is concerned with counting how many different trees there are of various kinds on *n* vertices, where *n* is a natural number {1, 2, 3, … , *n*}.
- The problem on the number of labelled trees with a given number of vertices was proposed by Cayley.
- A <u>labelled graph</u> is then a pair (***G***, *X*), where ***G*** is a graph and *X* is a labelling of ***G***. The integers 1, 2, 3, … , *n* are referred to as the <u>lables</u> of ***G***.
- The two labelled graphs $(\boldsymbol{G}_1, X_1)$ and $(\boldsymbol{G}_2, X_2)$ are <u>isomorphic</u> if there exists an isomorphism between $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ which preserves the labelling of the vertices.
- *There are* $\boldsymbol{n^{n-2}}$ *distinst labelled trees on n vertices.*



Labeled trees

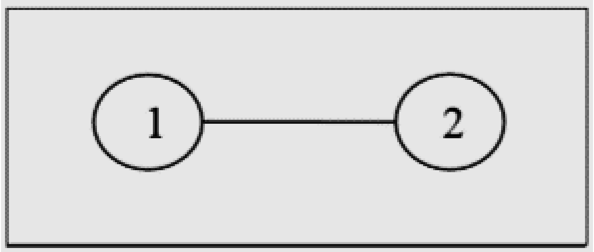
- A graph is said to be labeled, if its n vertices are distinguished from one another by labels such as $v_1, v_2, \ldots, v_n$.

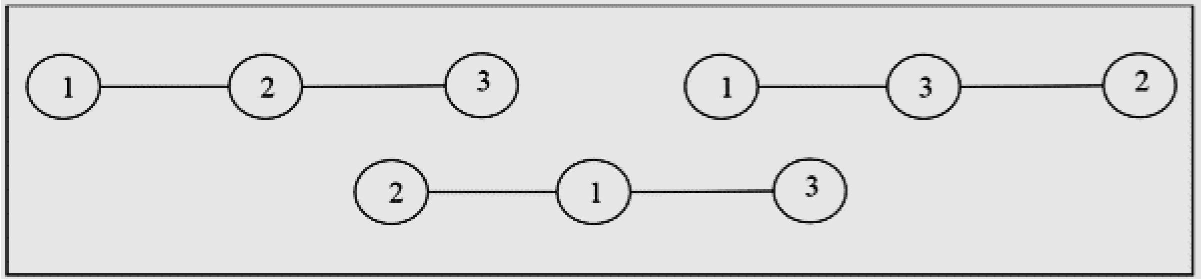


A labeled graph G and an unlabeled graph H

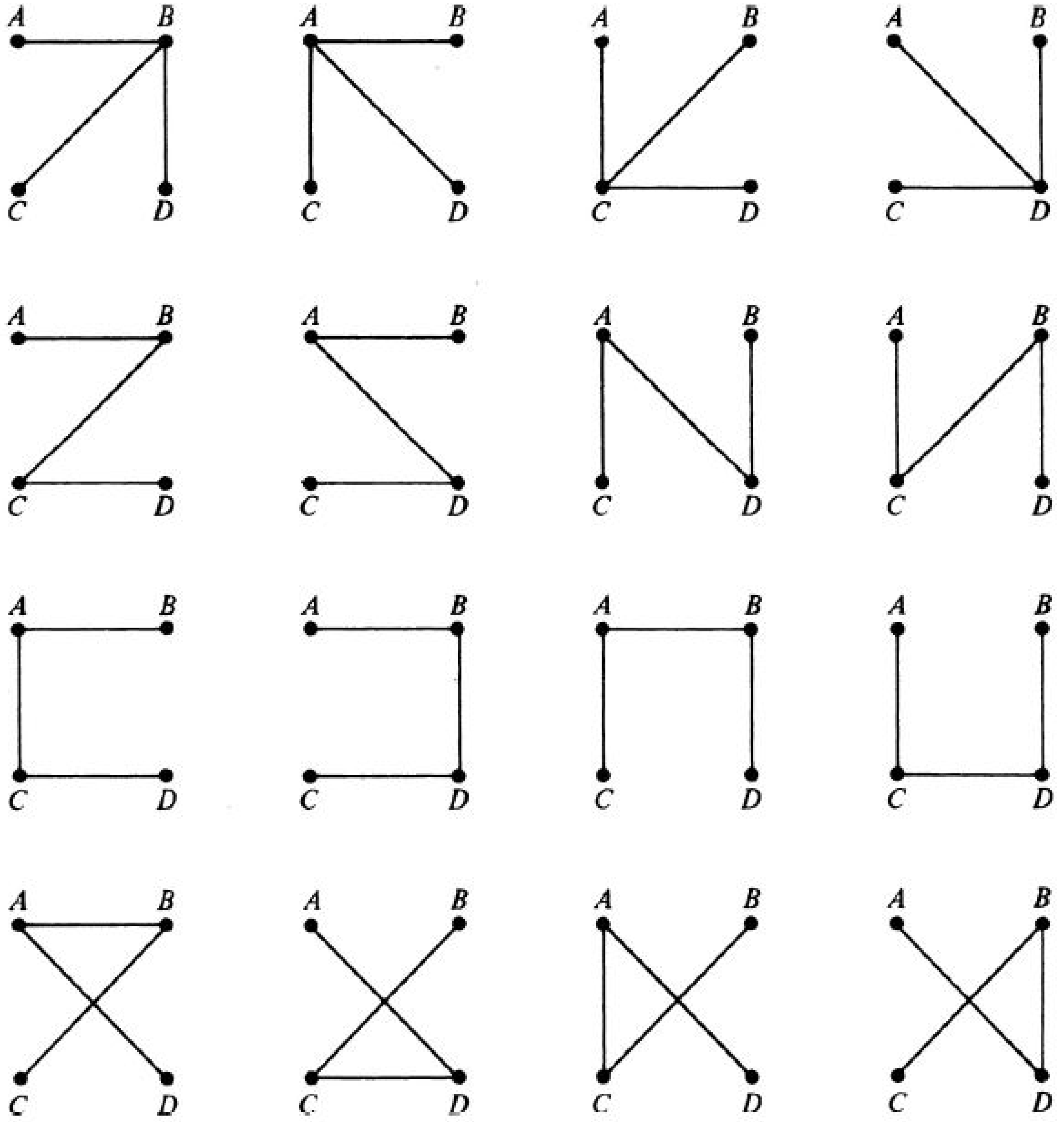
Labeled trees

- labeled tree is a tree the vertices of which are assigned unique numbers from 1 to n.
- We can count such trees for small values of n by hand so as to conjecture a general formula. The number of labeled trees of n number of vertices is n^{n-2} .
- Two labeled trees are isomorphic if their graphs are isomorphic and the corresponding points of the two trees have the same labels.
- Cayley's formula states that there are n^{n-2} trees on n labeled vertices. A classic proof uses prufer sequences, which naturally show a stronger result: the number of trees with vertices 1, 2, ..., n of degrees d_1, d_2, ..., d_n respectively.





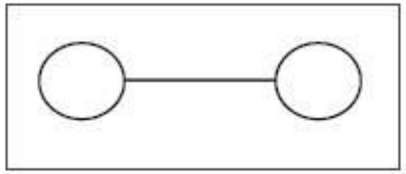
Labeled trees



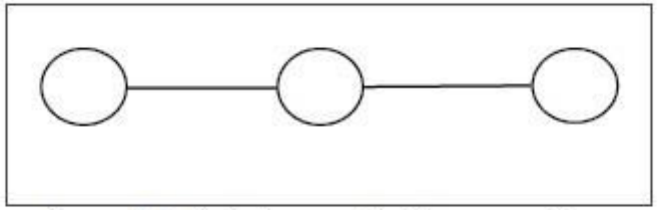
16 trees of four labeled vertices.

Unlabeled trees

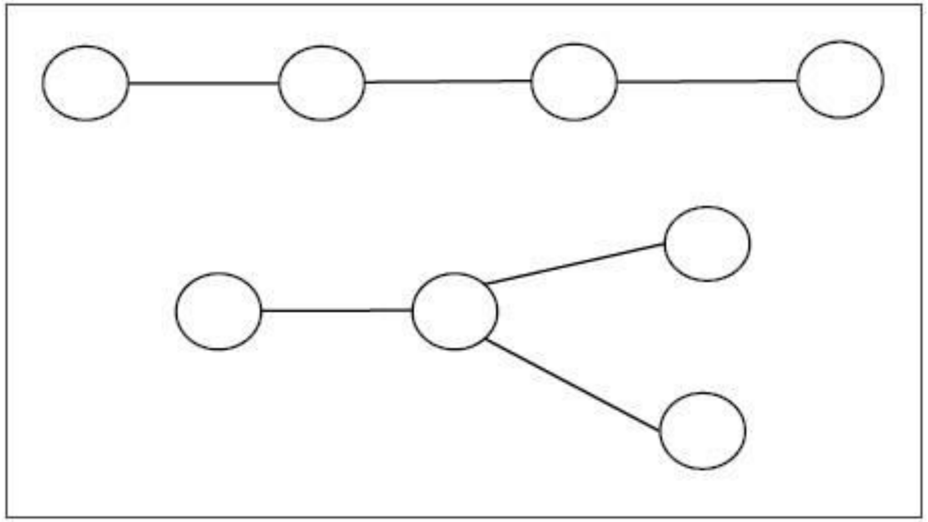
- An unlabeled tree is a tree the vertices of which are not assigned any numbers.
- The number of labeled trees of n number of vertices is $\frac{(2n)!}{(n+1)!n!}$ (nth Catalan number)



An unlabeled tree with two vertices



An unlabeled tree with three vertices



Two possible unlabeled trees with four vertices

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Thank you.