30/09/21

TUTORIAL - 1

GRAPH THEORY -

Hemanth. N.
2019503519

2.8) Prove that a simple graph with n vertices must be connected if it has more than $(n-1)(n-2)/2$ edges.

First Part: To Prove,

a simple graph with n vertices & k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Let no of vertices in each of the k component of a graph G be $n_1, n_2, n_3, \ldots n_k$. Thus we have,

$$n_1 + n_2 + \ldots + n_k = n.$$

$$n_i \geq n,$$

Proof of the thm. follows by $\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$

Max no of edges in ith component of G is $\frac{1}{2} n_i(n_i - 1)$. Max no of edges in G is $\leq \frac{1}{2}(n-k)(n-k+1)$ — ①

By ① &, we can say if k=1, the graph is connected, and hence, will have only one connected component, thus simple graph.

∴ No of edges $= \frac{(n-1)(n)}{2}$ (substitute k=1 in ①)

we see that $\frac{n(n-1)}{2} > \frac{(n-1)(n-2)}{2}$

(∵ $n > n-2$) ②

∴ we say, simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

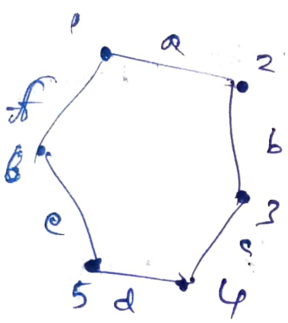
2.26) Euler Line, which is also a Hamiltonian circuit.

Hamiltonian graph - each vertex is visited exactly once except the starting & ending vertex.

Euler Line - vertex may be repeated, but each edge should be visited exactly once without repitition.

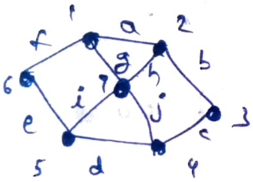
So if a cycle, with each deg of vertex being 2, is Hamiltonian and Euler line. Also, if complement of that cycle is taken, say c', Any subgraph of c' has all vertices of even degree.

∴ C ∪ C' is also both Hamiltonian and Euler Line.



Euler Line : 1-a-2-b-3-c-4-d-5-e-6-f-1

Hamiltonian Circuit : 1-a-2-b-3-c-4-d-5-e-6-f-1



- Hamiltonian cycle: Includes every vertex of G.
- Euler Line → Vertex (every) of G is of even degree

✓ Both satisfy

(*) These graphs are cyclic graphs and each vertex has degree 2.

→ n vertices & n edges - forms a cycle with all edges included in path.

(*) Basically, if a simple connected graph itself contains a path, then that path must be both Hamiltonian & Eulerian.

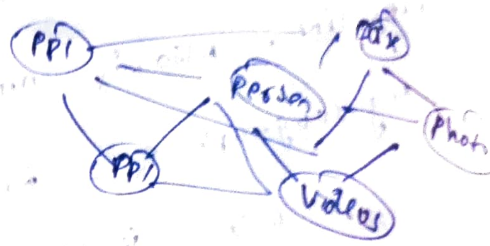
1.3) Situations that can be represented by means of Graph:

(i) Social networks:

Graphs b/w diff peoples, places, things interacted.

→ Basically everything (people, Photos, etc..) is vertice or node

→ Any connection or relationship is an edge.



(ii) Structure of websites containing many pages can be represented using a directed graph.

Each page is a vertex, and link b/w pages is edge.

(iii) Graphs model molecule structure for computer processing.

Atoms - vertices, Bonds are edges. Computers analyse and visualize this.

(iv) Tracing Covid contacts using Graph database algorithm, analysing complex relational networks.

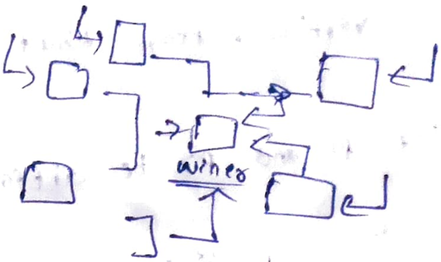
People (Covid contacts) - vertices or nodes

& connections are the edges. Complete graph may be formed.

(v) A special graph, Tree - May be used for generating parse Tree while designing compilers for programming languages, in the syntax analysis stage.

(vi) Representing the qualification scenario of cricket or football matches and teams.

Teams are the vertices and their path for finals is the edge.



(vii) Graphs can be used in Game Theory, where a connected, acyclic graph is used to represent the sequence of game.

Vertex → Each decision point & edge is the path he chooses, after taking a decision at the vertex.

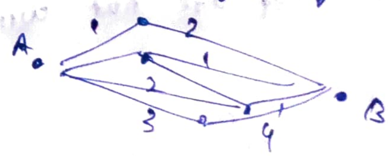
(viii) In electrical and electronic chip design, network may be considered for solving complex circuits,

each component may be vertex, edges represent connection b/w pair of components.

(simple design)

(ix) Shortest path between two cities, where the whole map may be represented as a ~~tree~~ graph, with cities being the vertex and their path b/w cities is represented by edges.

[Find the shortest path b/w A and B]



(x) Graph theory may be used in medical field as well, with biological analysis, Regulatory networks. If all the components are represented in a graph as vertices, it will be a complex graph, with using graph isomorphism for matching two components.

2.1b) In a graph G, let $P1$ and $P2$ be 2 diff paths b/w 2 given vertices. Prove that $P1 \oplus P2$ is a circuit or set of circuits in G.

Let $P1$ and $P2$ be different paths, and a and b be 2 vertices.

Now, $P1 \oplus P2$ has all vertices of $P1 \cup P2$ and all edges which are $P1$ or $P2$ but not in both $P1$ and $P2$.

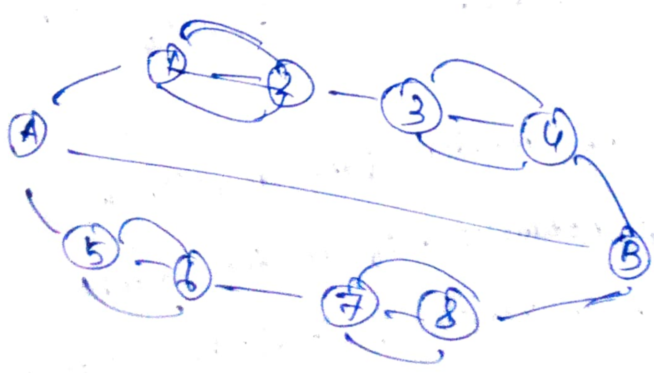
[Basically $P1 \oplus P_2$ of  and  will be ]

$P1 \oplus P_2$ - contains all vertices of $P1 \cup P2$, and edges that are in $P1$ or $P2$, but not in both $P1$ and $P2$.

$P1, P_2$ are different paths $\Rightarrow$ atleast 1 edge in each path will not be common.

Edges that are not common will meet at or before the end point b, this means that a circuit must be there.

$\therefore$ $P1 \oplus P2$, where $P1$ & $P2$ are diff paths, will form a circuit via edges that are not common b/w $P1$ and P_2.



Thus, $P1 \oplus P_2$ will from atleast one circuit

1.8) Decanting Problem. A - 8, B - 5, C - 3 (capacity)

a + b + c = 8 - given

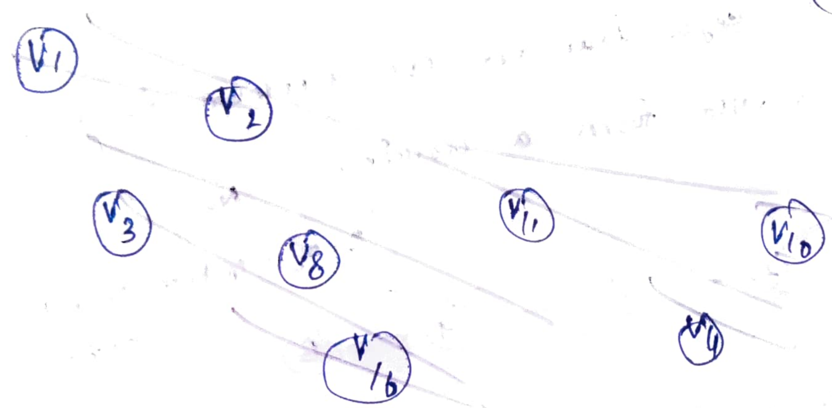
Basic Algorithm:

If A is empty
    Fill A from reserve
else
    if B is empty
        Transfer A to B
else Empty B from Reserve

Available states

v_1 (0,5,3), v_3 (5,0,3),
v_2 (8,0,0), v_4 (6,0,2), v_5 (7,0,1),
v_6 (3,5,0), v_7 (2,5,1), v_8 (1,5,2),
v_9 (5,3,0), v_{10} (6,2,0), v_{11} (7,1,0),
v_{12} (4,4,0), v_{13} (1,4,3), v_{14} (2,3,3),
v_{15} (3,2,3).

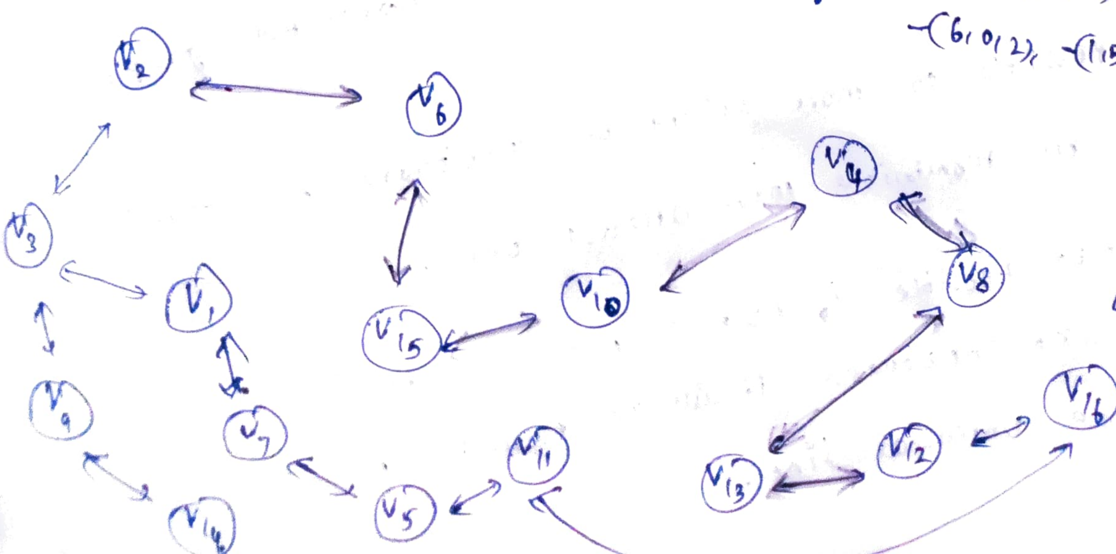
Take starting state (8,0,0) & ending state - (4,4,0).



Path for the soln is

$V_2 - V_6 - V_5 - V_{10} - V_4 - V_8 - V_{13} - V_{12}$

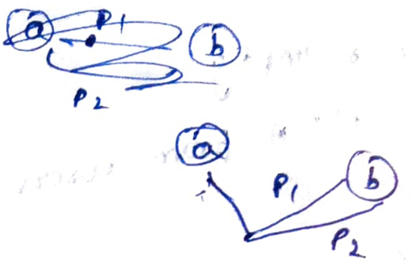
Path is: (8,0,0) - (3,5,0) - (3,2,3) - (6,2,0) - (6,0,2) - (1,5,2) - (1,4,3) - (4,4,0)



Pencil line is solution

2.18) If intersection of 2 paths is a disconnected graph, show union of 2 paths has atleast one circuit.

Let P_1 and P_2 be 2 Paths such that $P_1 \cap P_2$ is disconnected.



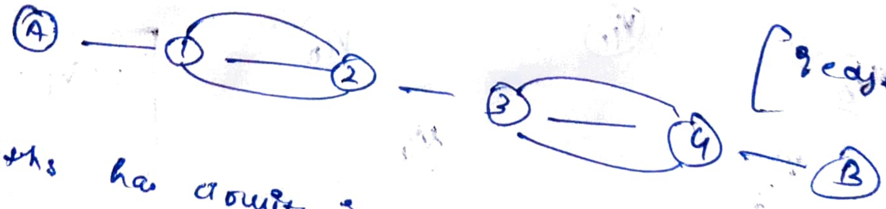
$\Downarrow$

There exists no path b/w a and b in $P_1 \cap P_2$.

Since, a, b, are in $P_1 \cap P_2$, $\Rightarrow$ a and b will be in P_1 and P_2.

But no path b/w in $P_1 \cap P_2$ connecting a and b, paths in P_1 and P_2 connecting a and b are edge-disjoint.

Union of P_1 and P_2 has edges that are not common to both P_1 and P_2, hence it should form a circuit.



Now set of edges in $P_1, P_2 \Rightarrow$ Mutually exclusive

Union of 2 Paths has circuit if intersection is disconnected graph.

2.20) Is it possible to move knight on chessboard, such that it completes all permissible move atleast once.

This will be possible if $n \geq 5$ in $n \times n$ chessboard. So for 8×8 chessboard it will be possible, since it contains a universal line.