CS6109 – GRAPH THEORY

Module – 5

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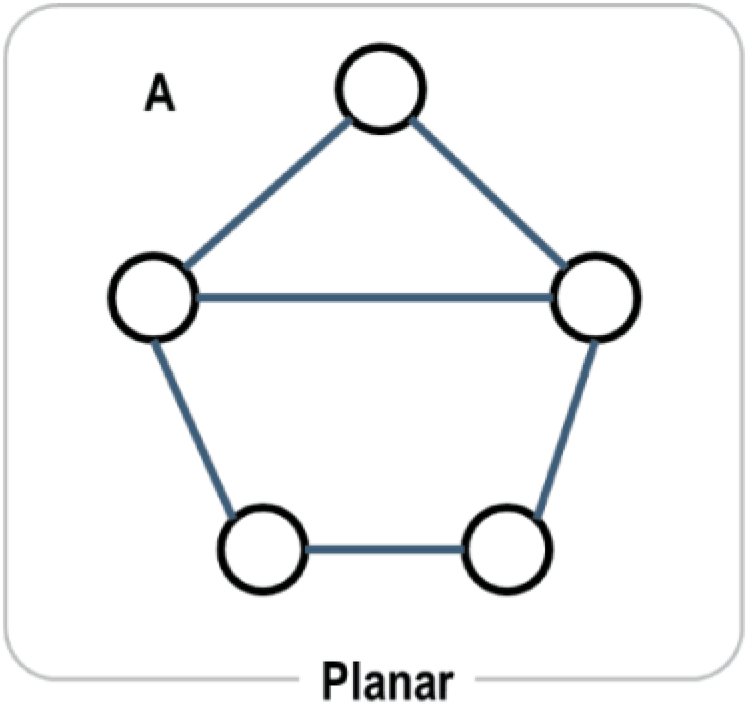
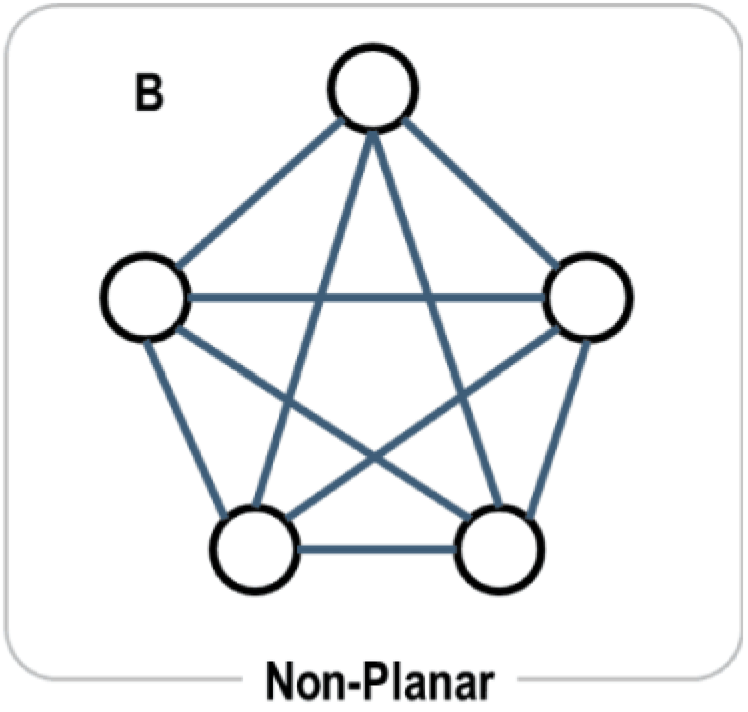
Module - 5

- Planar Graph
- Representation
- Detection of planarity
- Dual Graph
- Related Theorems

Planar Graph

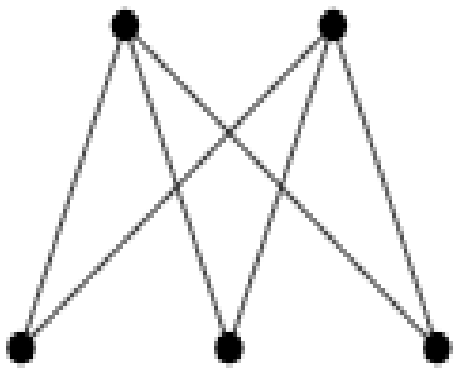
- A graph is said to be planar if it can be drawn in a plane so that no edge cross.
- When a connected graph can be drawn without any edges crossing, it is called ***planar***. When a planar graph is drawn in this way, it divides the plane into regions called ***faces***.

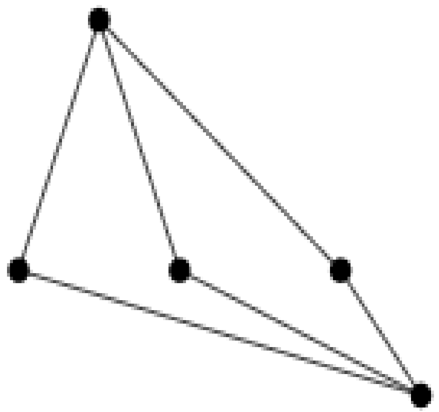




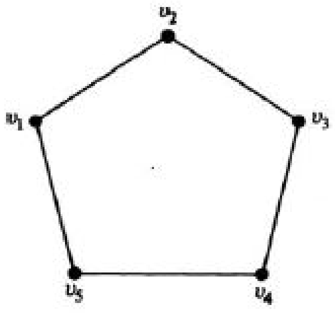
Planar Graph

- A drawing of a geometric representation of a graph on any surface such that no edges intersect is called *embedding*.
- To declare that a graph *G* is nonplanar, we have to show that of all possible geometric representations of *G* none can be embedded in a plane.
- An embedding of a planar graph G on a plane is called a *plane representation* of *G*.





THEOREM 5-1: The complete graph of five vertices is nonplanar.



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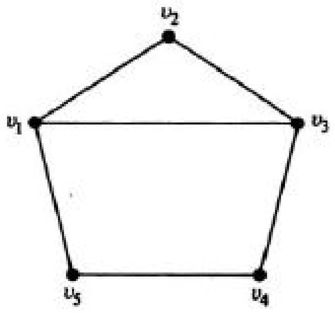

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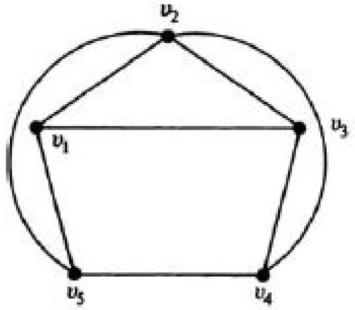

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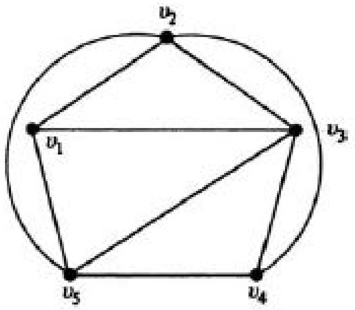

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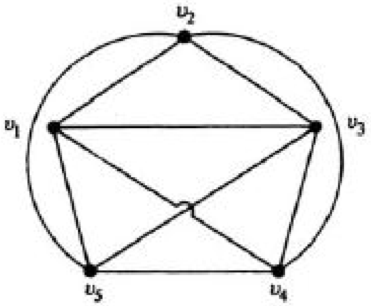
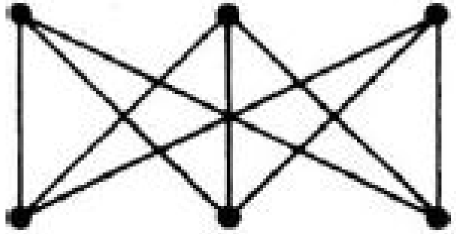


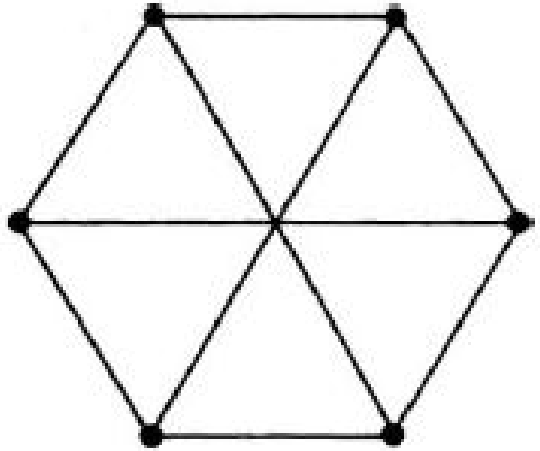
(e)

THEOREM 5-2: Kuratowski's second graph is also nonplanar.

- A regular connected graph with six vertices and nine edges is also nonplanar.



(a)



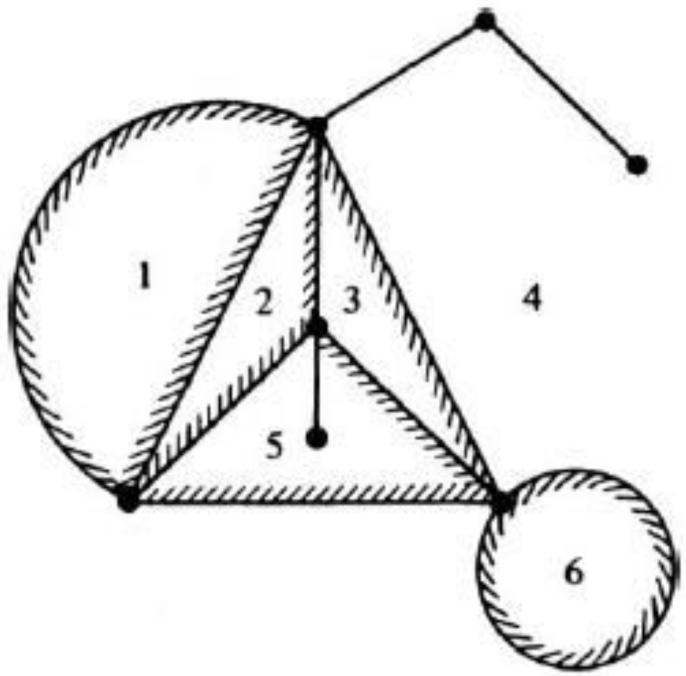
(b)

Representation of planar graph

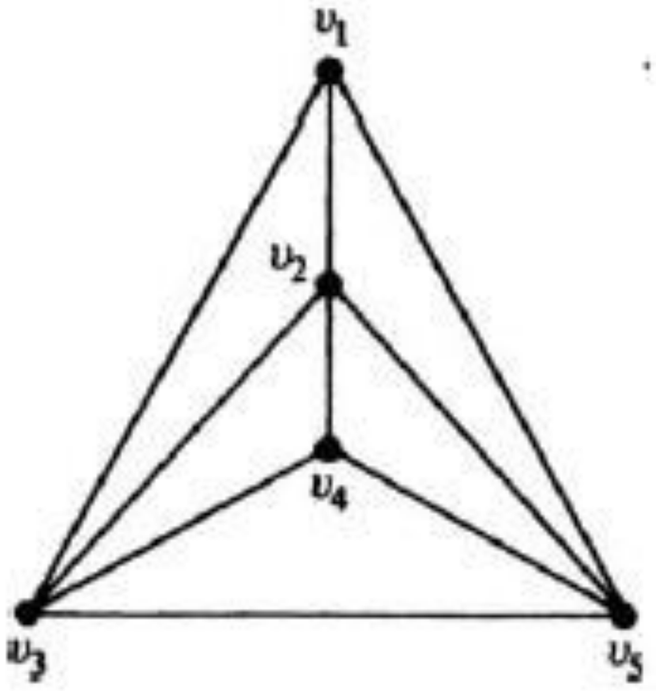
Region of a Graph: Consider a planar graph G=(V,E). A region is defined to be an area of the plane that is bounded by edges and cannot be further subdivided. A planar graph divides the plans into one or more regions. One of these regions will be infinite.

Finite Region: If the area of the region is finite, then that region is called a finite region.

Infinite Region: If the area of the region is infinite, that region is called a infinite region. A planar graph has only one infinite region.



THEOREM 5-3: Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.



Straight-line representation of the graph

THEOREM 5-6: A connected planar graph with n vertices and e edges has $e - n + 2$ regions.
In a connected planar graph with n vertices, m edges and f regions (faces), $n - m + f = 2$.

Proof:
Without loss of generality, assume that the planar graph is simple. Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon (a polygon net).
Let the polygon net representing the given graph consist of f regions. Let k_p be the number of p-sided regions.

Since each edge is on the boundary of exactly two regions,

$3k_3 + 4k_4 + 5k_5 + \ldots + rk_r = 2m$, → (1)

where k_r is the number of polygons with r edges.

Also, $k_3 + k_4 + k_5 + \ldots + k_r = f$. → (2)

The sum of all angles subtended at each vertex in the polygon net is $2\pi n$. → (3)

Now, the sum of all interior angles of a p-sided polygon is $\pi(p-2)$ and the sum of the exterior angles is $\pi(p+2)$. The expression in (3) is the total sum of all interior angles of $f - 1$ finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$\pi(3-2)k_3 + \pi(4-2)k_4 + \ldots + \pi(r-2)k_r + 4\pi$

$= \pi[3k_3 + 4k_4 + \ldots + rk_r - 2(k_3 + k_4 + \ldots + k_r)] + 4\pi$

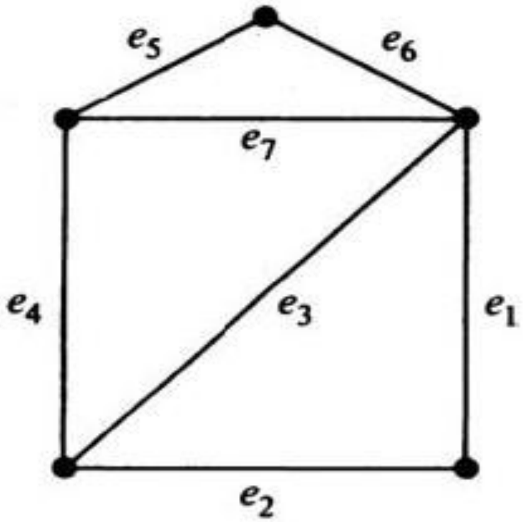
$= \pi(2m - 2f) + 4\pi = 2\pi(m - f + 2)$. → (4)

Equating (3) and (4) we get

$2\pi(m - f + 2) = 2n\pi$, so that $f = m - n + 2$.

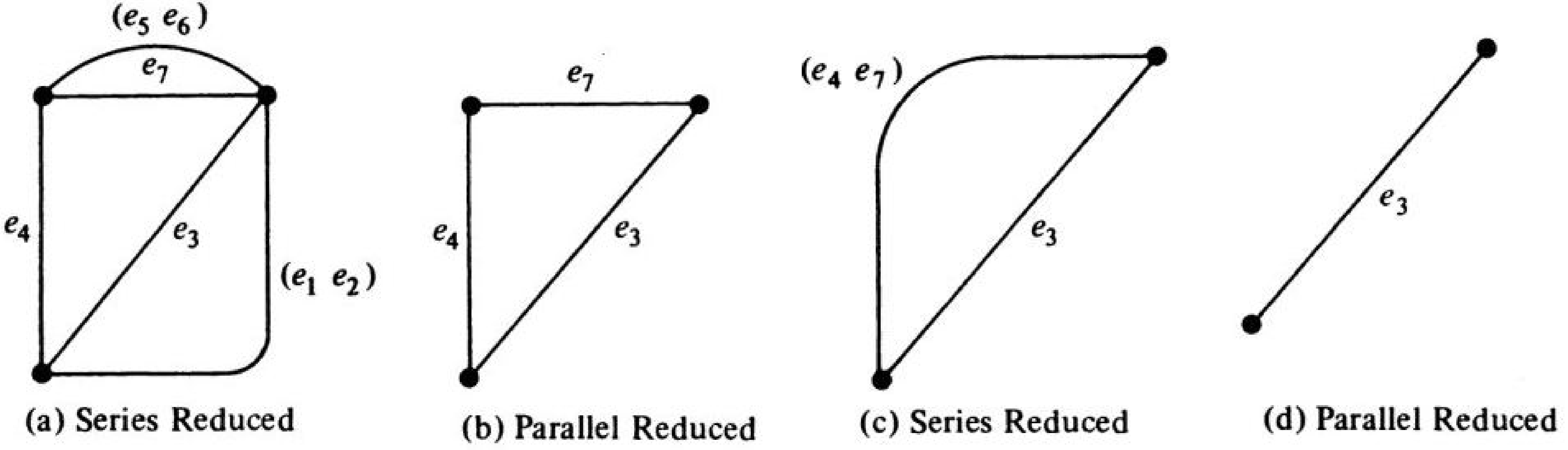
DETECTION OF PLANARITY

Elementary Reduction



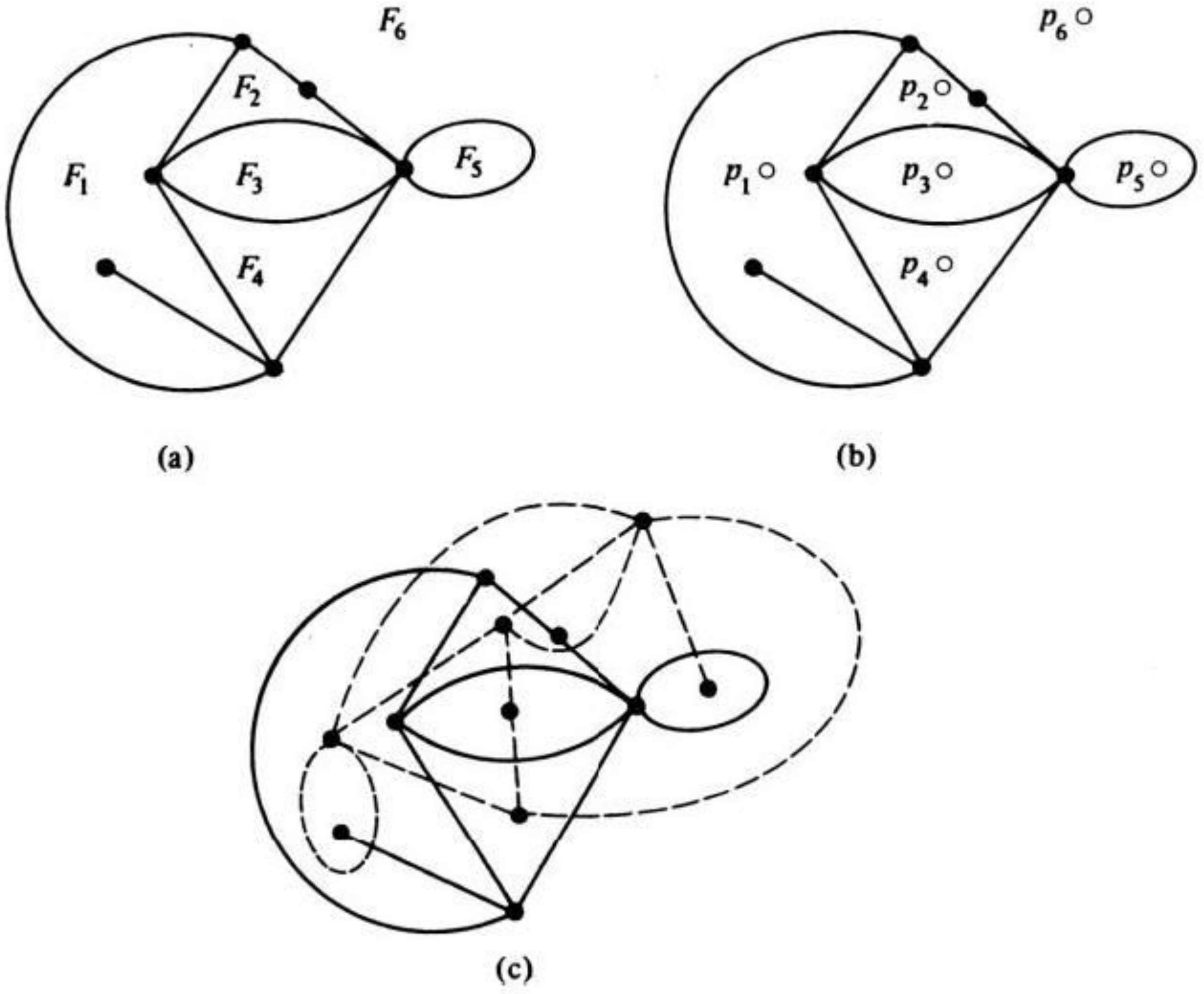
- ***Step 1*:** Since a disconnected graph is planar if and only if each of its components is planar, we need consider only one component at a time. Also, a separable graph is planar if and only if each of its blocks is planar. Therefore, for the given arbitrary graph G, determine the set $G = \{G_1, G_2, \ldots, G_k\}$, where each G_i is a nonseparable block of G. Then we have to test each G_i for planarity.
- ***Step 2*:** Since addition or removal of self-loops does not affect planarity, remove all self-loops.

- ***Step 3*:** Since parallel edges also do not affect planarity, eliminate edges in parallel by removing all but one edge between every pair of vertices.
- ***Step 4*:** Elimination of a vertex of degree two by merging two edges in series does not affect planarity. Therefore, eliminate all edges in series. Repeated application of steps 3 and 4 will usually reduce a graph drastically.



DUAL GRAPH

1. Divide the graph into region or faces.
2. For each face assign points.
3. Connect the points for each edges separating them.



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Thank you.