CS6109 – GRAPH THEORY

Module – 6

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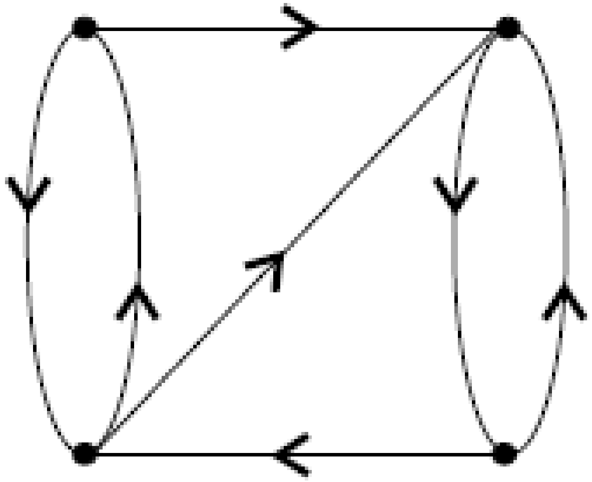
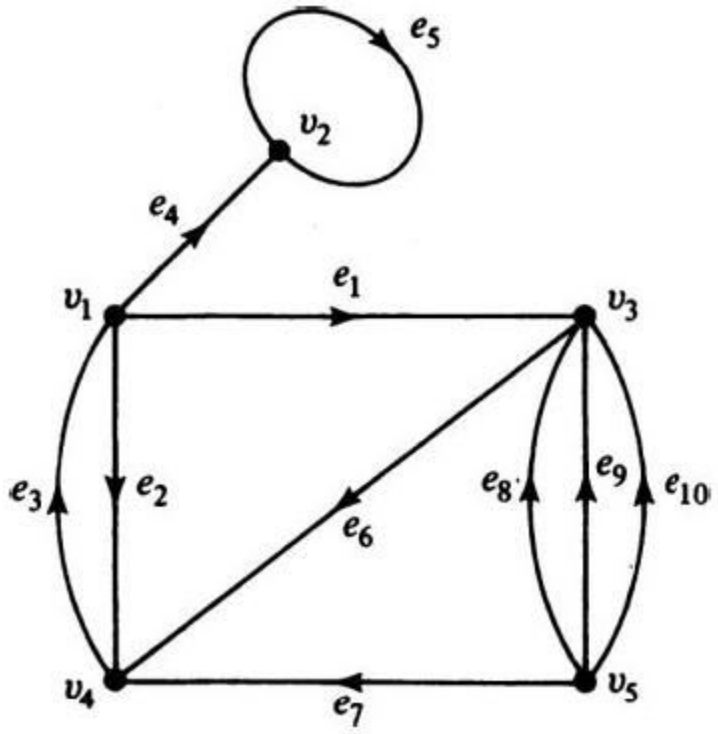
Module - 6

- Digraph
- Properties
- Euler Digraph
- Tournament graph
- Applications

Digraphs (Directed graphs)

- A digraph D is a pair (V, A), where V is a nonempty set whose elements are called the vertices and A is the subset of the set of ordered pairs of distinct elements of V.

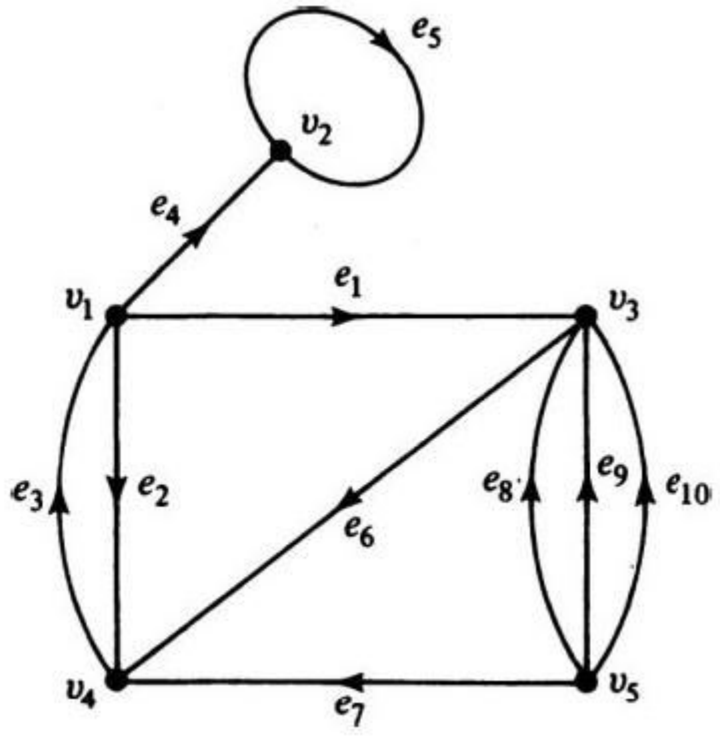




Directed graph with 5 vertices and 10 edges.

In-degree/Out-degree

- The number of edges incident out of a vertex v_i is called the *out-degree* (or *out-valence* or *outward demidegree*) of v_i and is written $d+(v_i)$.
- The number of edges incident into v_i is called the *in-degree* (or *invalence* or *inward demidegree*) of v_1 and is written as $d-(v_i)$.

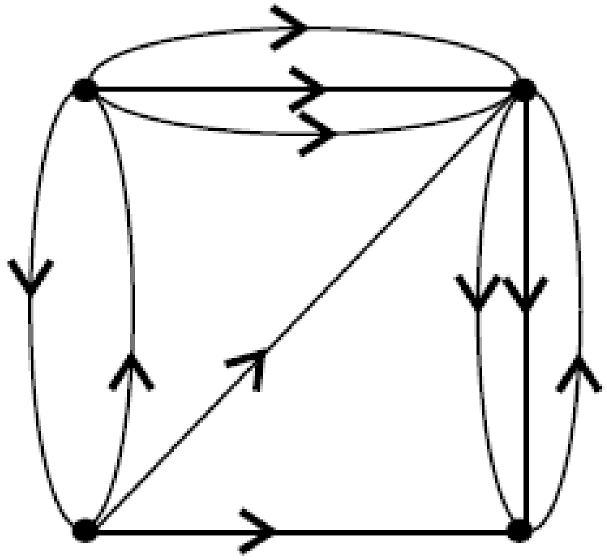


Directed graph with 5 vertices and 10 edges.

$$d+(v_1) = 3, \quad d-(v_1) = 1,$$
$$d+(v_2) = 1, \quad d-(v_2) = 2,$$
$$d+(v_5) = 4, \quad d-(v_5) = 0.$$

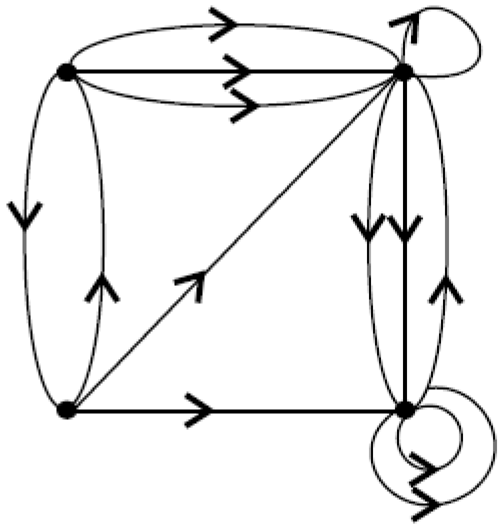
Multidigraphs

- A multidigraph D is a pair (V, A), where V is a nonempty set of vertices and A is a multiset of arcs, which is a multisubset of the set of ordered pairs of distinct elements of V.
- The number of times an arc occurs in D is called its multiplicity and arcs with multiplicity greater than one are called multiple arcs of D.



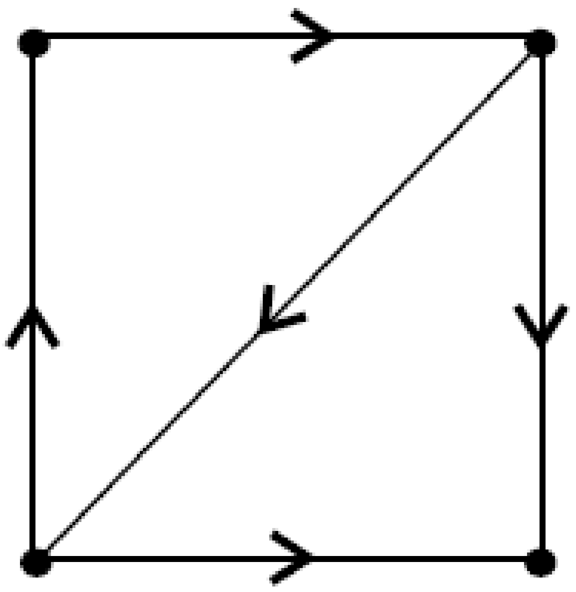
General Digraphs

- A general digraph D is a pair (V, A), where V is a nonempty set of vertices, and A is a multiset of arcs, which is a multisubset of the cartesian product of V with itself.
- An arc of the form uu is called a loop of D and arcs which are not loops are called proper arcs of D.
- The number of times an arc occurs is called its multiplicity. A loop with multiplicity greater than one is called a multiple loop.



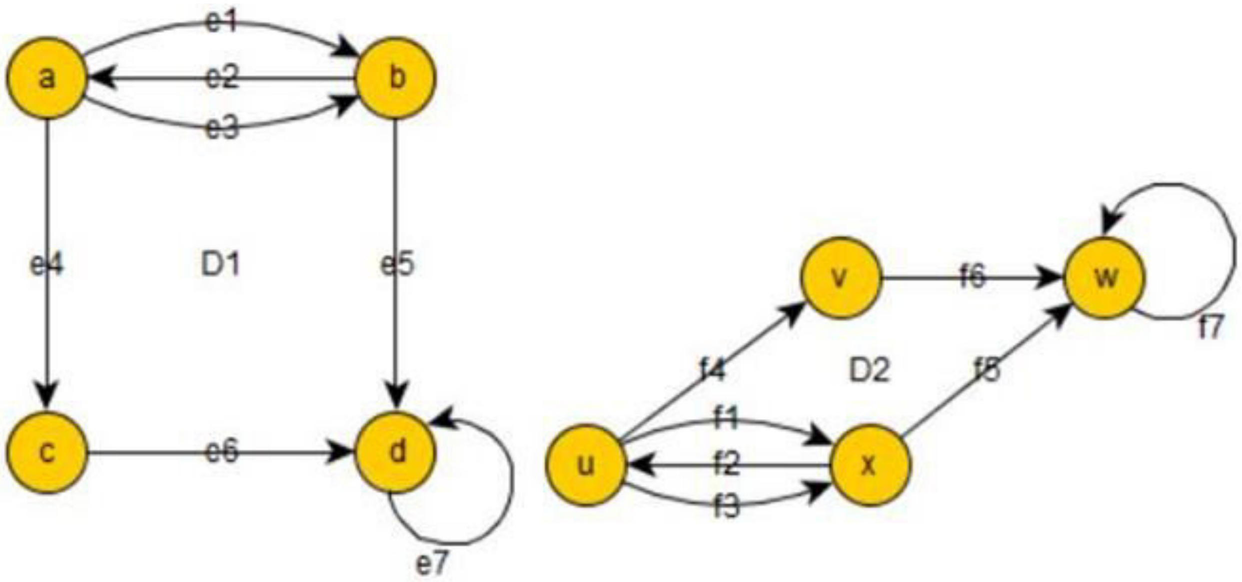
Oriented Graph

- A digraph containing no symmetric pair of arcs is called an oriented graph.

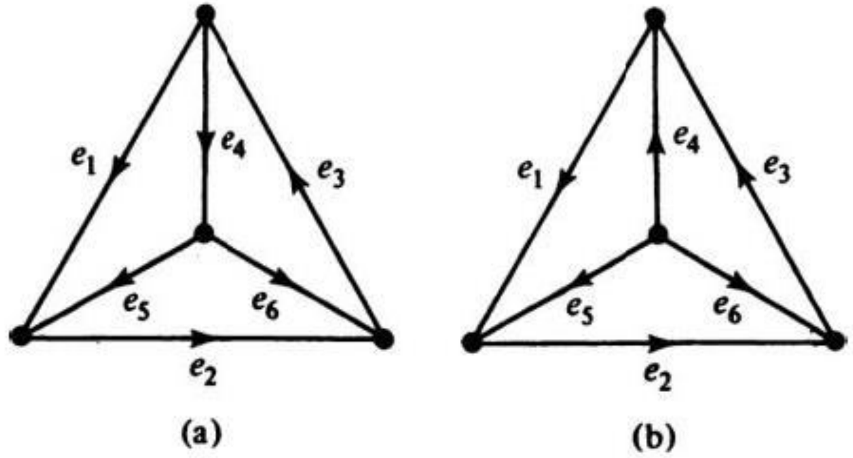


Isomorphic Digraphs

- Isomorphic graphs were defined such that they have identical behavior in terms of graph properties.
- In other words, if their labels are removed, two isomorphic graphs are indistinguishable.



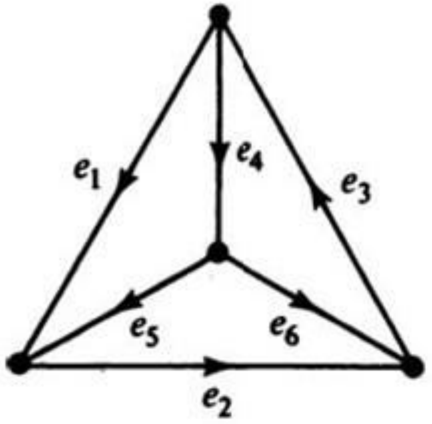
Two isomorphic digraphs.



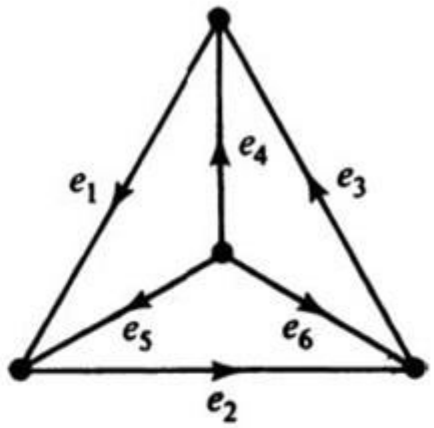
Two nonisomorphic digraphs.

Simple Digraphs

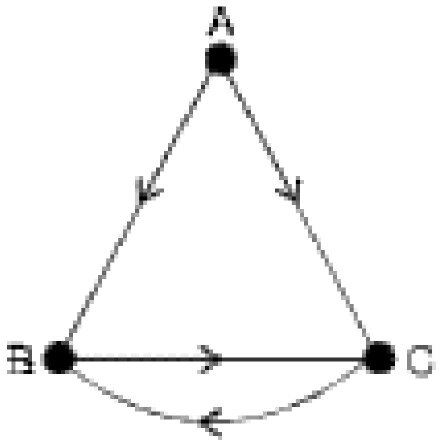
- A digraph that has no self-loop or parallel edges is called a simple digraph.



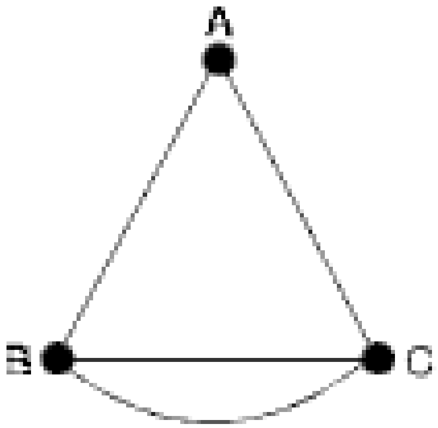
Simple Digraph



Simple Digraph



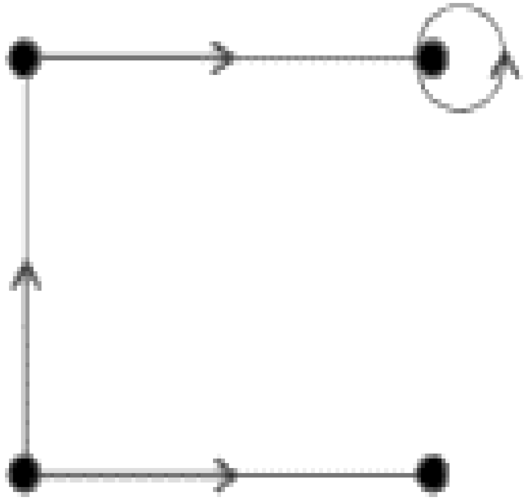
Simple Digraph



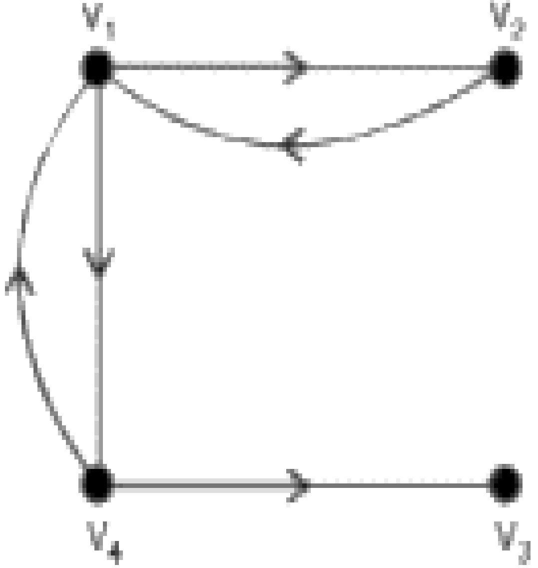
Not Simple Digraph

Asymmetric Digraphs

- Digraphs that have at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called *asymmetric* or *antisymmetric*.



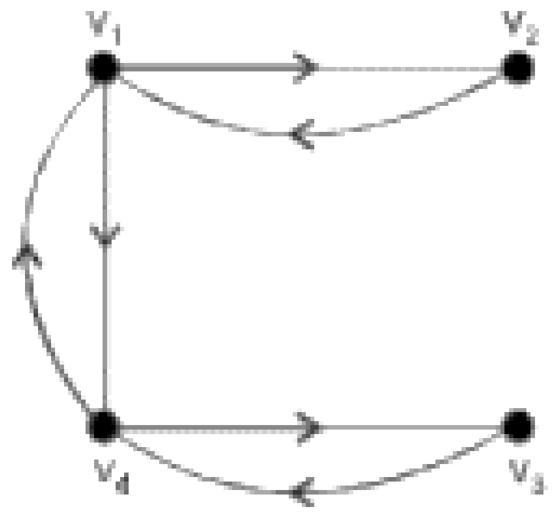
Asymmetric Graph



Not Asymmetric Graph

Symmetric Digraphs

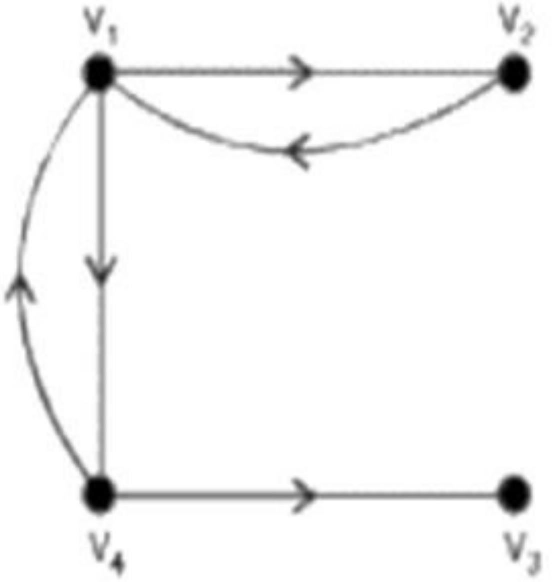
- Digraphs in which for every edge (*a, b*) (i.e., from vertex *a* to *b*) there is also an edge (*b, a*).



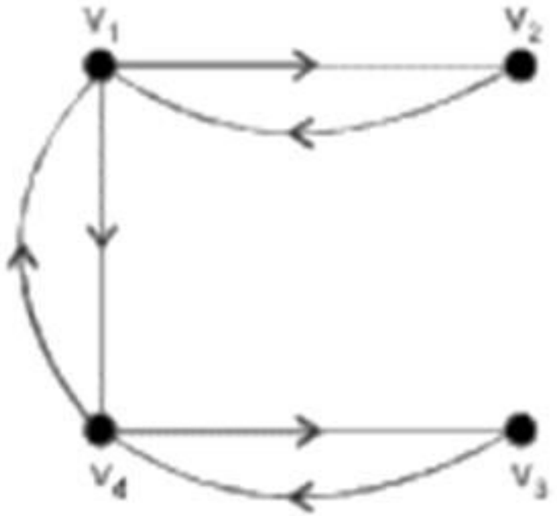
Symmetric Digraphs

Simple Symmetric/Asymmetric Digraphs

- A digraph that is both simple and symmetric is called a *simple symmetric digraph*.
- Similarly, a digraph that is both simple and asymmetric is *simple asymmetric*.



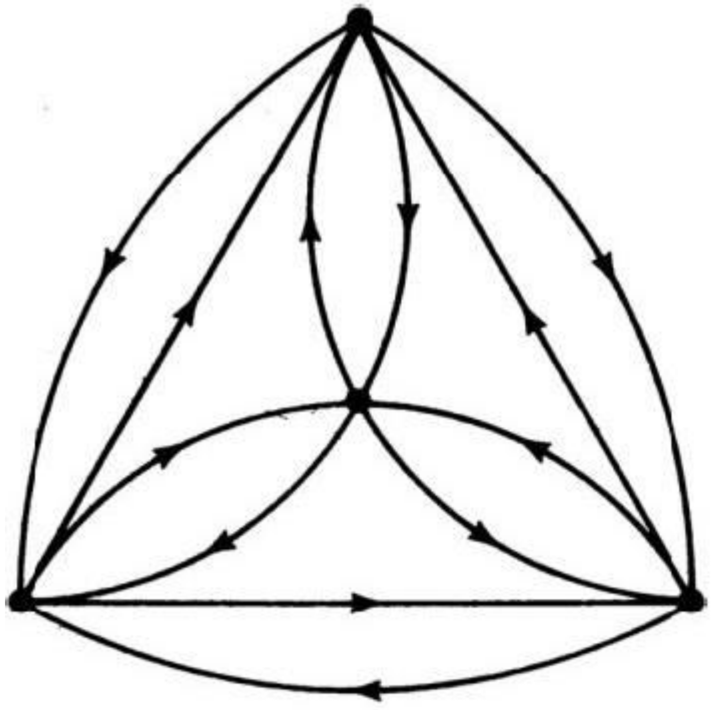
Simple Asymmetric Digraph



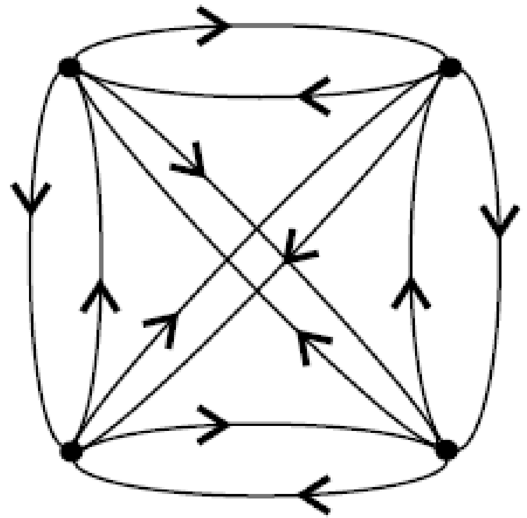
Simple Symmetric Digraph

Complete Digraphs

- Digraphs in which for every edge (*a, b*) (i.e., from vertex *a* to *b*) there is also an edge (*b, a*).



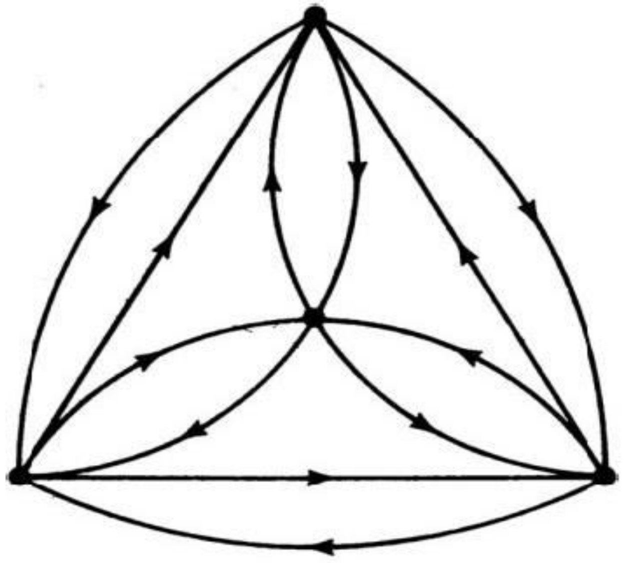
Complete symmetric digraph of four vertices.



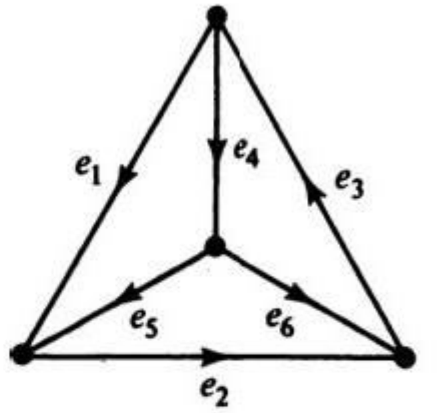
Complete symmetric digraph

Complete Symmetric/Asymmetric Digraphs

- A *complete symmetric digraph* is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex.
- A *complete asymmetric digraph* is an asymmetric digraph in which there is exactly one edge between every pair of vertices.



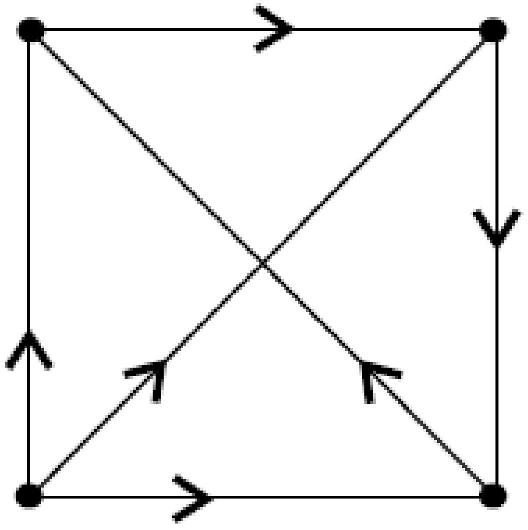
Complete symmetric digraph



Complete asymmetric digraph

Tournament

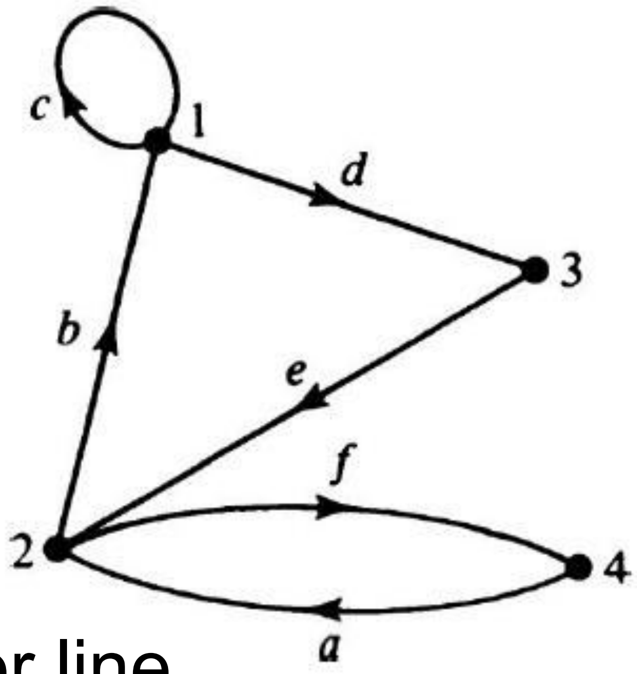
- A complete antisymmetric digraph, or a complete oriented graph is called a tournament. Clearly, a tournament is an orientation of K_n.



Tournament

Euler Digraphs

- A digraph D is said to be Eulerian if it contains a closed walk which traverses every arc of D exactly once. Such a walk is called an Euler walk.
- A digraph *G* a closed directed walk (i.e., a directed walk that starts and ends at the same vertex) which traverses every edge of *G* exactly once is called a *directed Euler line*. A digraph containing a directed Euler line is called an *Euler digraph*.

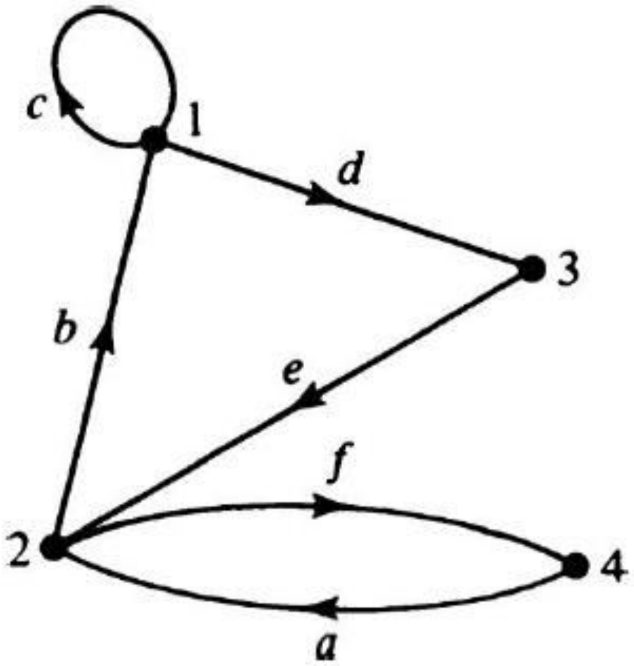


The walk *a b c d e f* is an Euler line.

THEOREM 9-1: A digraph *G* is an Euler digraph if and only if *G* is connected and is balanced [i.e., $d^-(v) = d^+(v)$ for every vertex *v* in *G*].

- $d^+(1) = 2$,
- $d^+(2) = 2$,
- $d^+(3) = 1$,
- $d^+(4) = 1$.

- $d^-(1) = 2$,
- $d^-(2) = 2$,
- $d^-(3) = 1$,
- $d^-(4) = 1$.

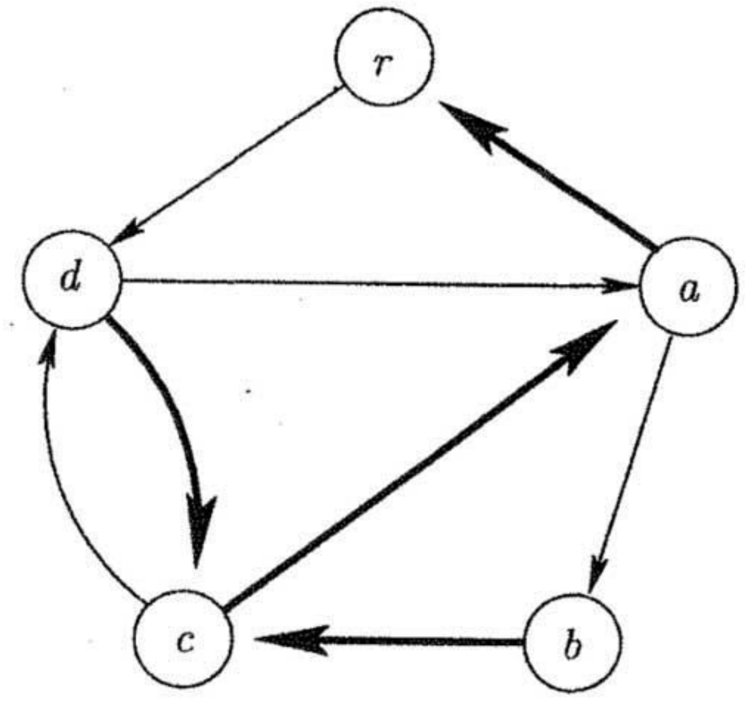


The walk *a b c d e f* is an Euler line.

THEOREM 9-1: A digraph *G* is an Euler digraph if and only if *G* is connected and is balanced [i.e., $d^-(v) = d^+(v)$ for every vertex *v* in *G*].

- $d^+(r) = 1$,
- $d^+(d) = 2$,
- $d^+(c) = 2$,
- $d^+(a) = 2$.
- $d^+(b) = 1$.

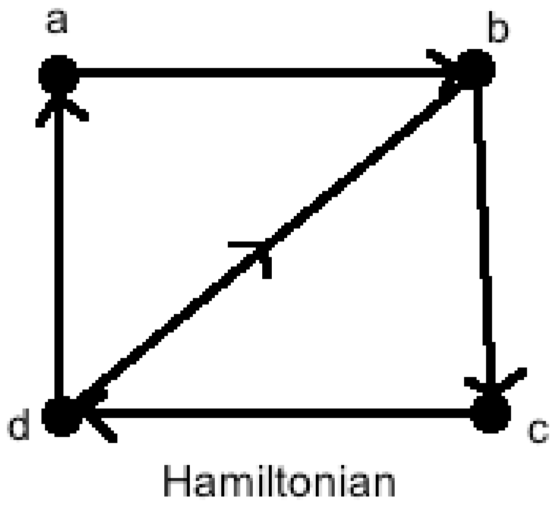
- $d^-(r) = 1$,
- $d^-(d) = 2$,
- $d^-(c) = 2$,
- $d^-(a) = 2$.
- $d^+(b) = 1$.



The walk r→d→a→b→c→d→c→a→r is an Euler line.

Hamiltonian Digraphs

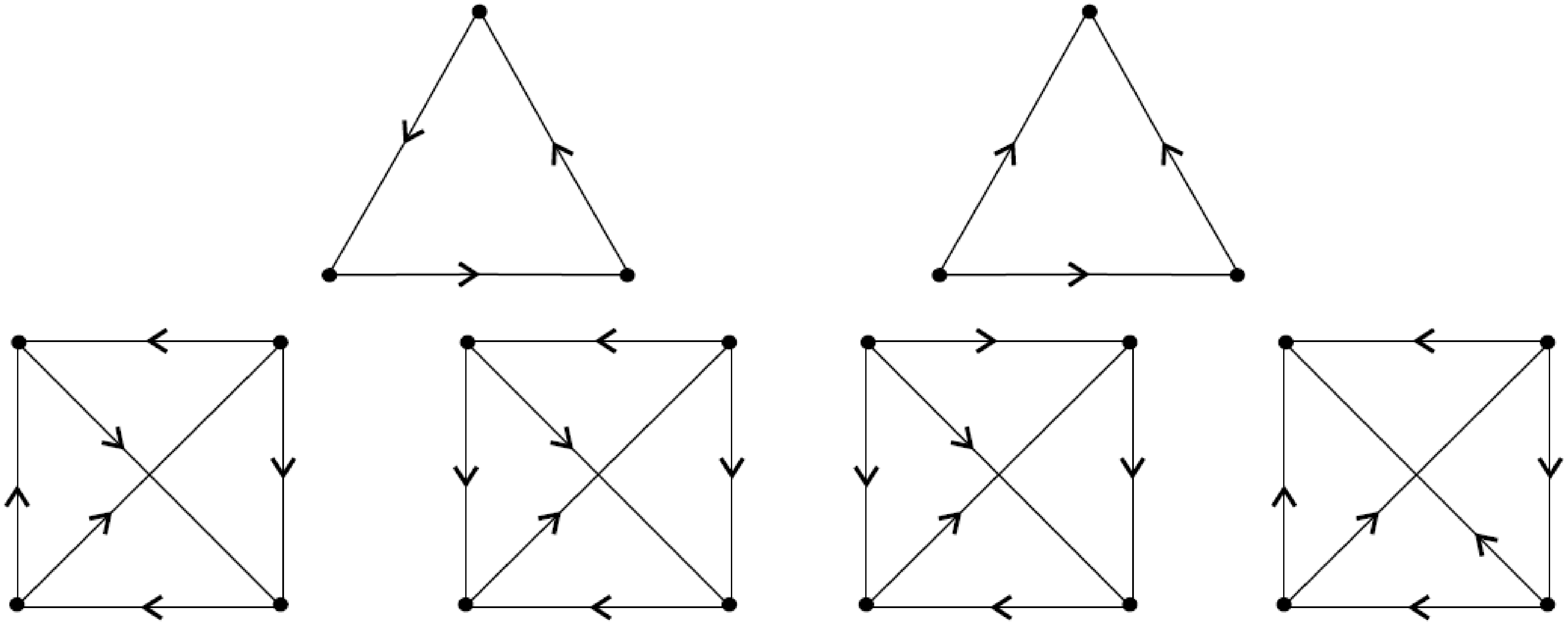
- A connected digraph *D* is **Hamiltonian** if *D* contains a cycle that itself contains all of the vertices in D.
- This cycle is known as a **Hamiltonian Cycle**.



Hamiltonian cycle - *abcda*

Tournaments

- A tournament is an orientation of a complete graph.
- Therefore in a tournament each pair of distinct vertices v_i and v_j is joined by one and only one of the oriented arcs (v_i, v_j) or (v_j, v_i).
- If the arc (v_i, v_j) is in T , then we say v_i dominates v_j and is denoted by $v_i \rightarrow v_j$. The relation of dominance thus defined is a complete, irreflexive, antisymmetric binary relation.



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Thank you.