CS6109 – GRAPH THEORY

Module – 4

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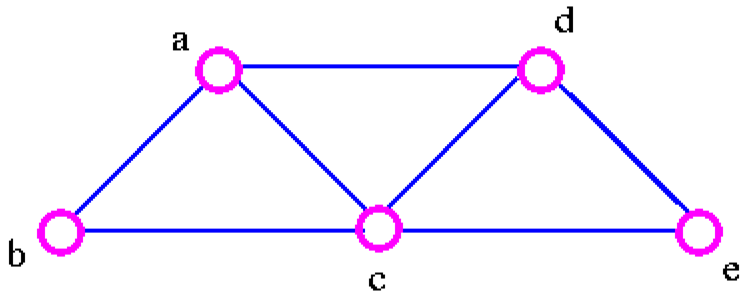
Module - 4

- Spanning Tree
- Fundamental Circuits
- Cut Sets
- Cut Sets – Properties
- Connectivity - Separability
- Network Flows
- 1 – Isomorphism
- 2 – Isomorphism
- Related Theorems

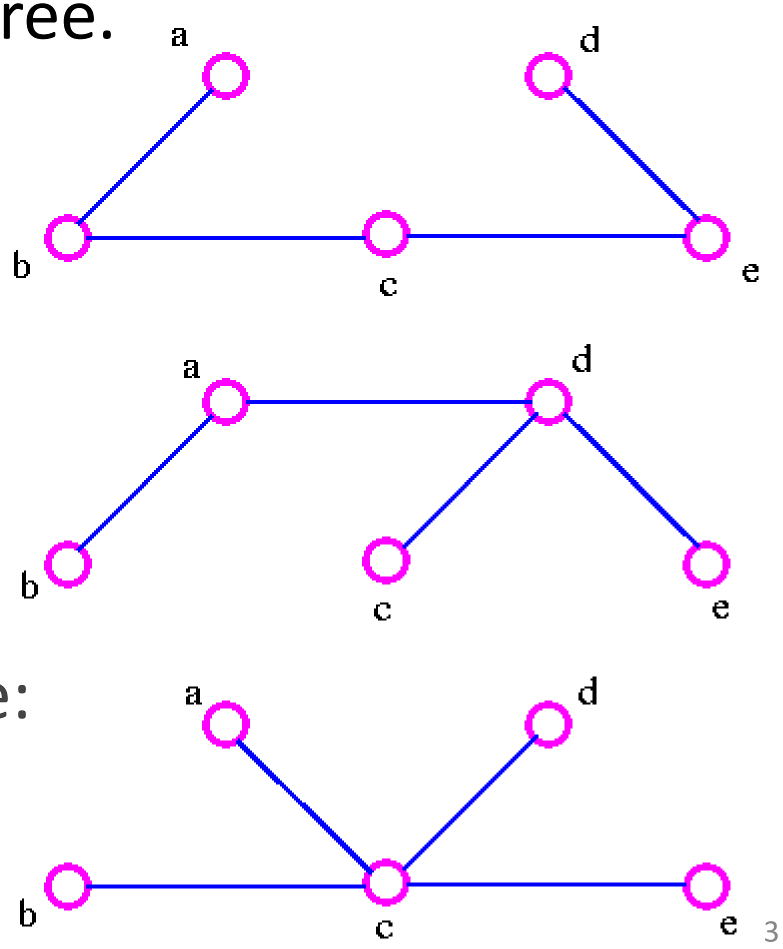
Spanning Tree

- A tree *T* is said to be a *spanning tree* of a connected graph *G* if *T* is a subgraph of *G* and *T* contains all vertices of *G*.
- Let G be a connected graph. A spanning tree in G is a subgraph of G that includes all the vertices of G and is also a tree.
- The edges of the trees are called branches.



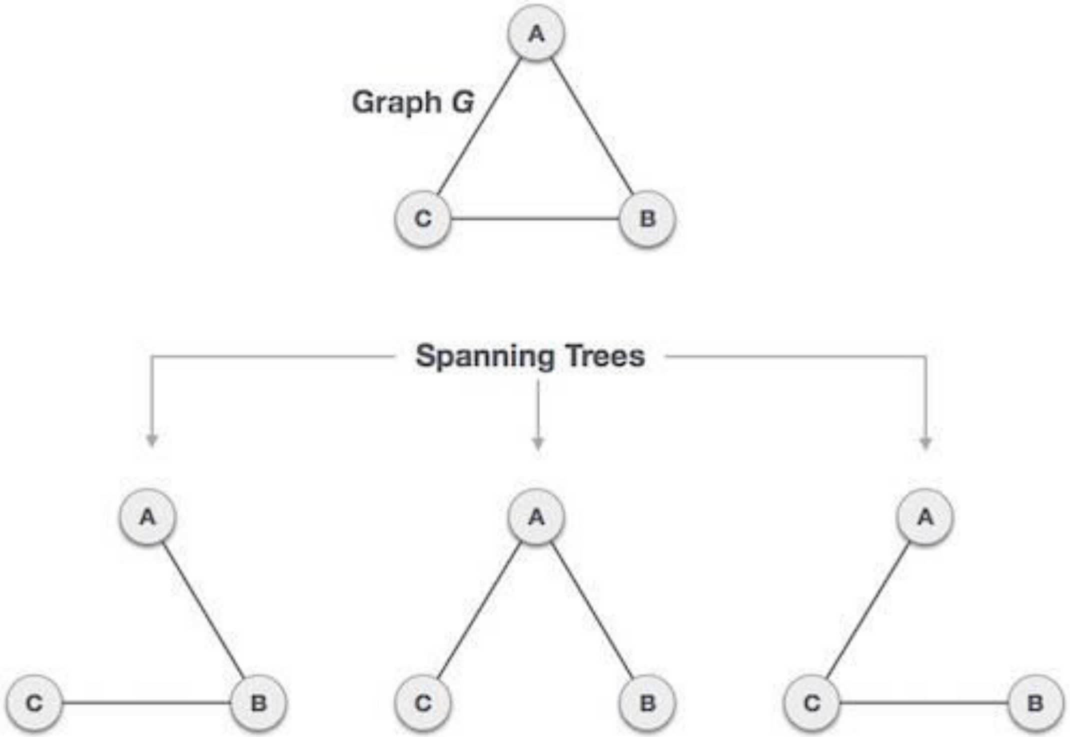
Graph *G*

The three spanning trees G are:



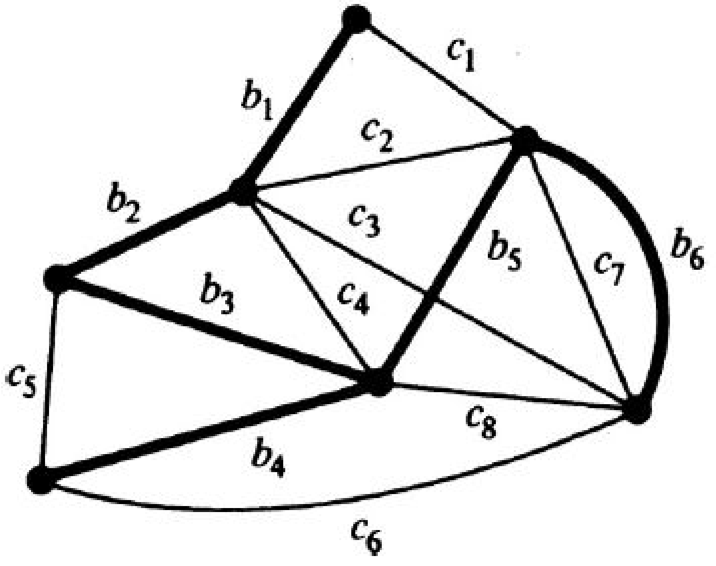
Spanning Tree

Graph G

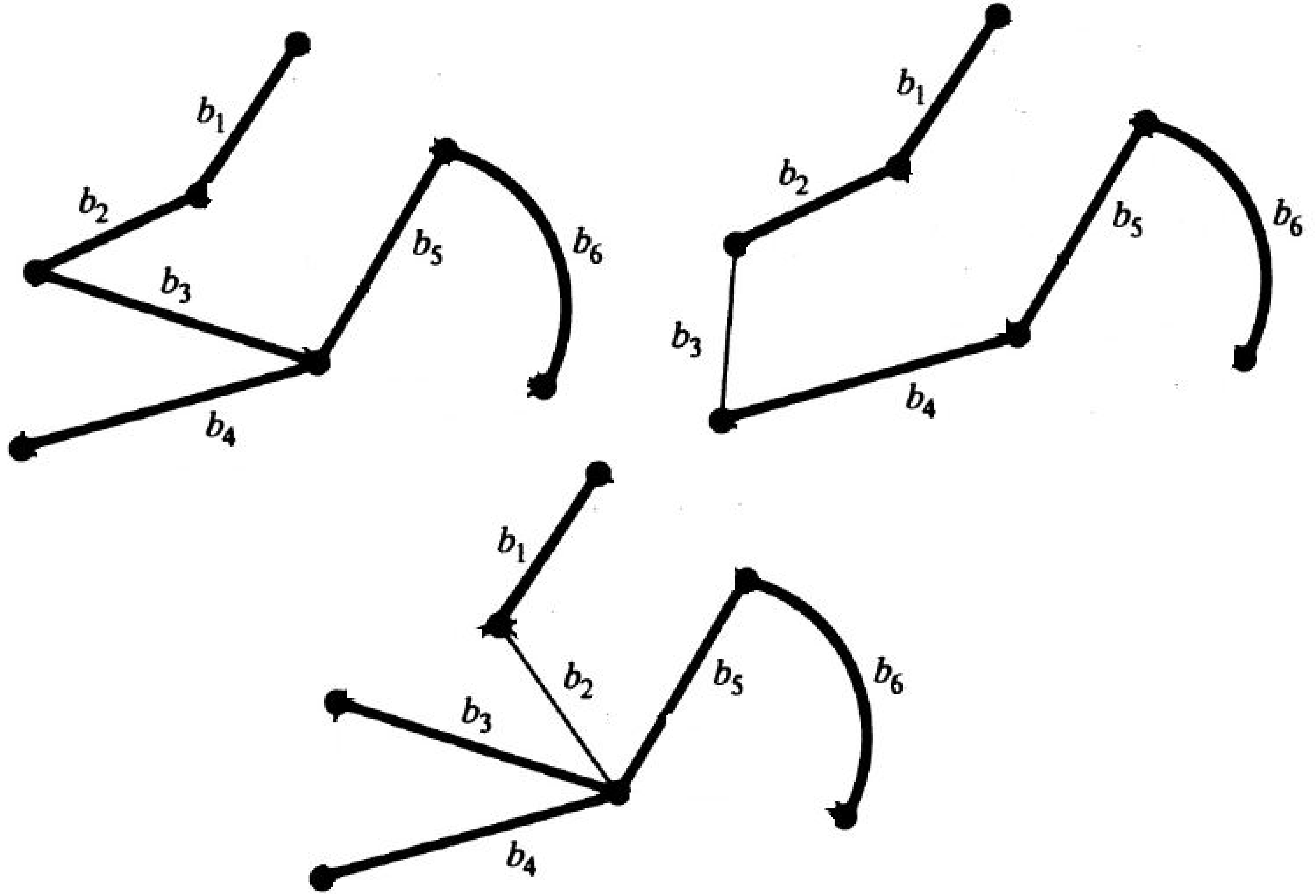


Spanning Tree

Graph *G*



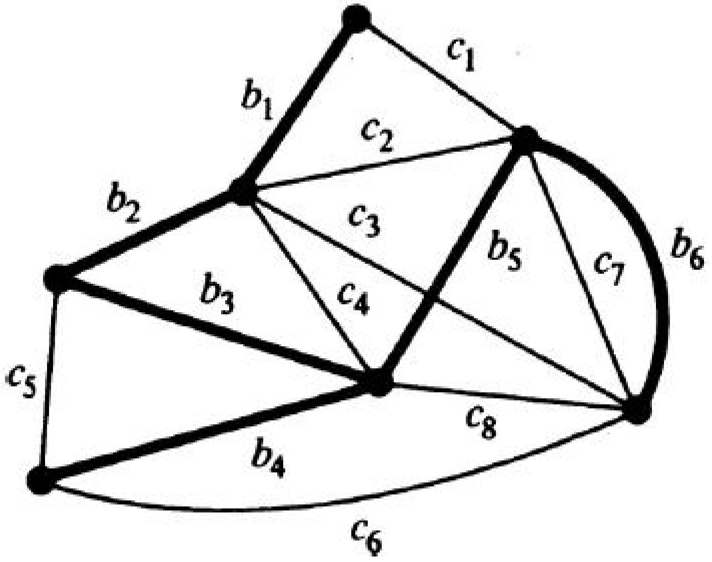
Spanning trees G are:



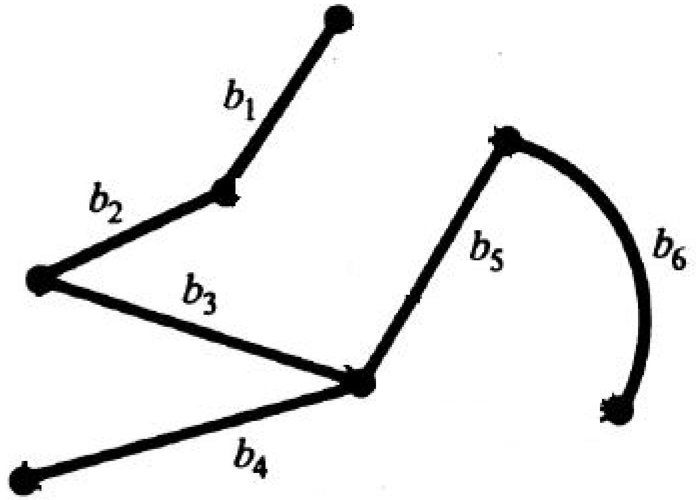
Spanning Tree - Branch of tree

An edge in a spanning tree T is called a branch of T.

Graph *G*



Spanning trees G are:



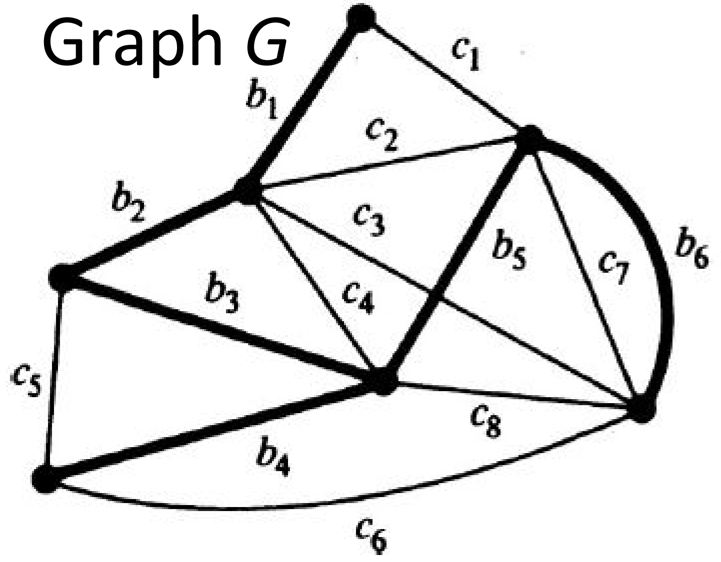
Spanning Tree - Chord

An edge of G that is not in a given spanning tree is called a chord.

Note :

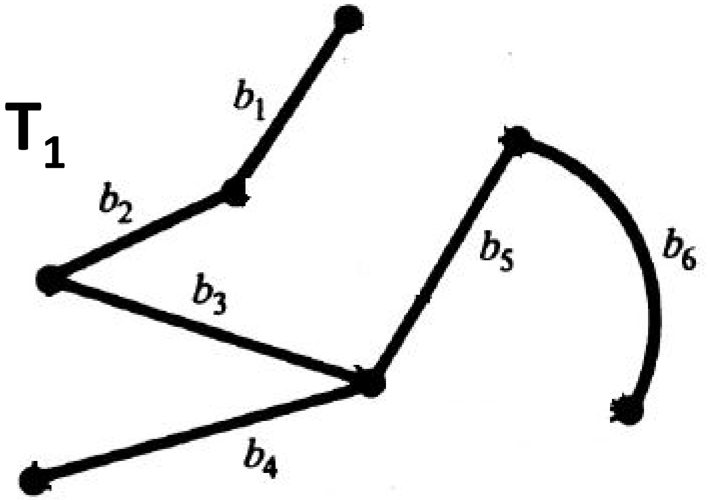
1. The branches and chords are defined only with respect to a given spanning tree.
2. An edge that is a branch of one spanning tree T_1 (in a graph G) may be chord, with respect to another spanning tree T_2.

Graph *G*

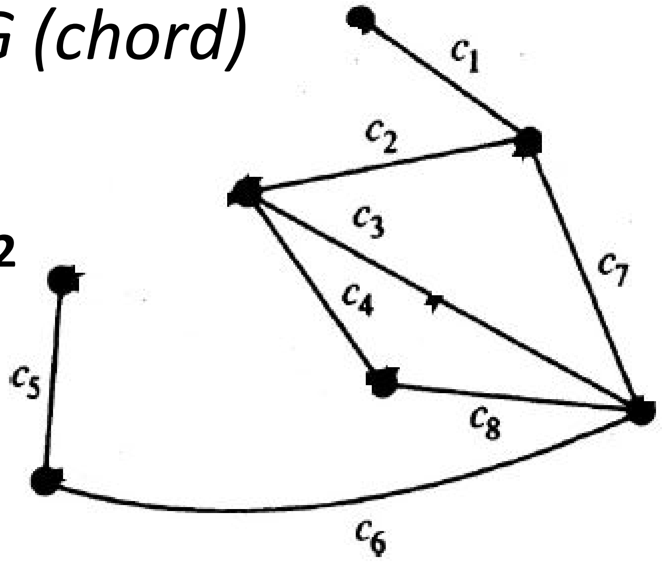


Spanning Tree T of *G (chord)*

T_1

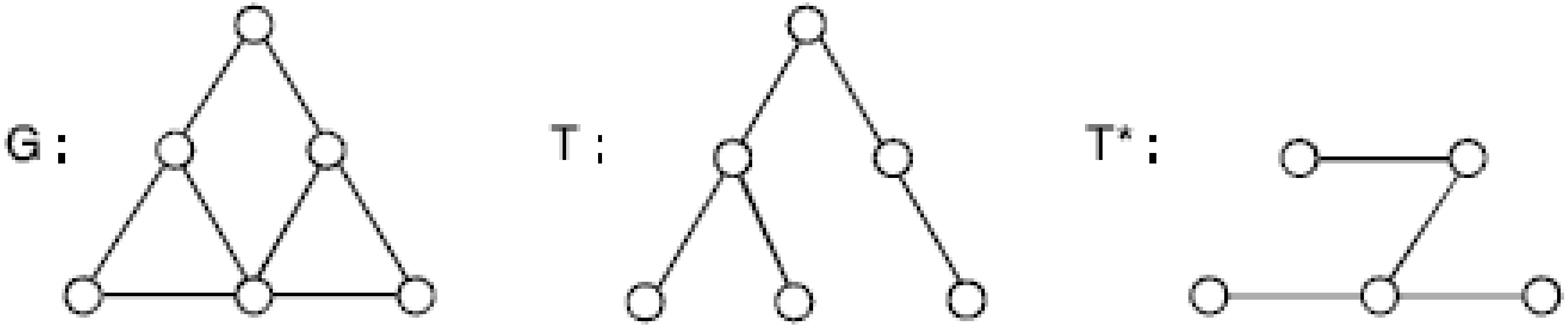


T_2



Spanning Tree - Cotree

- The cotree T* of a spanning tree T in a connected graph G is the spanning subgraph of G containing exactly those edges of G which are not in T.
- The edges of G which are not in T* are called its twigs.



Spanning Tree – Forest, Spanning Forest

Forest

- A collection to tree in a graph G is called a forest in the graph.

Spanning Forest

- A forest that contains every vertex of a graph *G* such that two vertices are in the same tree of the forest when there is a path in *G* between these two vertices.
- In other words, a spanning forest of a graph *G* is a collection of exactly one spanning tree from each of its connected components.

Spanning Tree – Rank, Nullity

Rank

- If in a graph G there are total n vertices and k compo- nents then the rank, generally denoted by r, is defined as
  $r = n - k$.

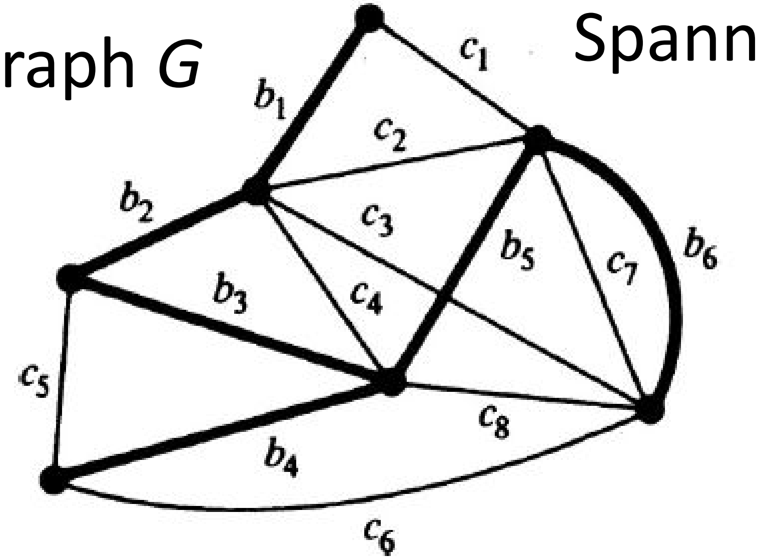
Nullity

- If in a graph G there are total n vertices, k components and e edges then the nullity of G, generally denoted by μ, is defined as $\mu = e - n + k$

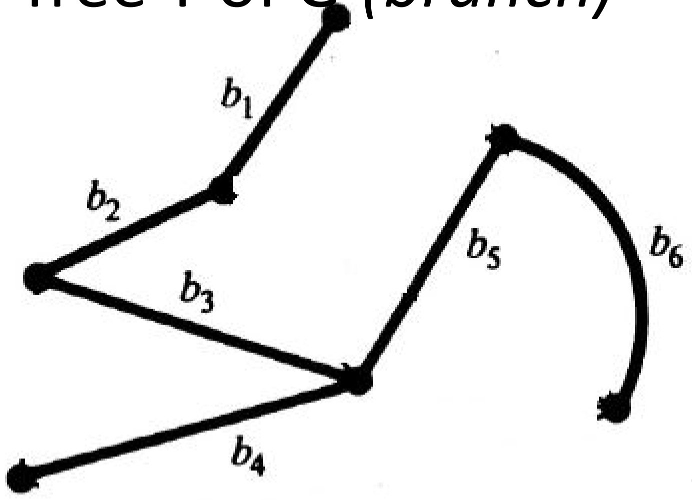
THEOREM 3-11: Every connected graph has at least one spanning tree.

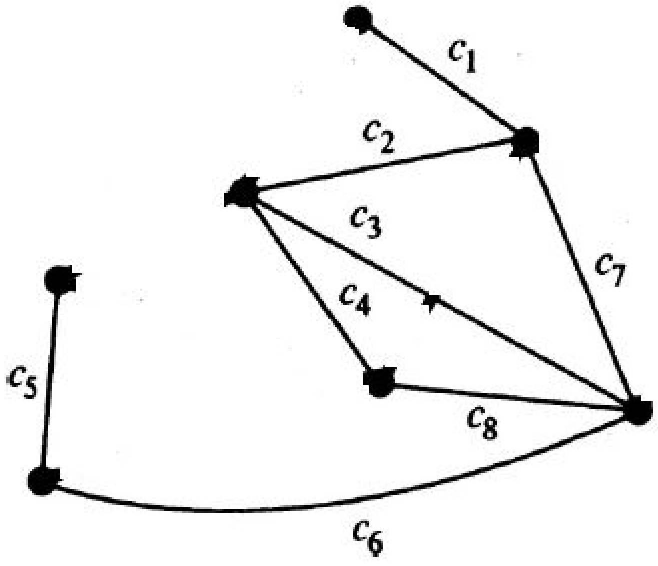
- An edge in a spanning tree *T* is called a *branch* of *T*.
- An edge of *G* that is not in a given spanning tree *T* is called a *chord*.
- An edges b_1, b_2, b_3, b_4, b_5, and b_6 are branches of the spanning tree.
- An edges c_1, c_2, c_3, c_4, c_5, c_6, c_7, and c_8 are chords.
- It must be kept in mind that branches and chords are defined only with respect to a given spanning tree.
- An edge that is a branch of one spanning tree T_1 (in a graph *G*) may be a chord with respect to another spanning tree T_2.

Graph *G*



Spanning Tree T of *G (branch)*



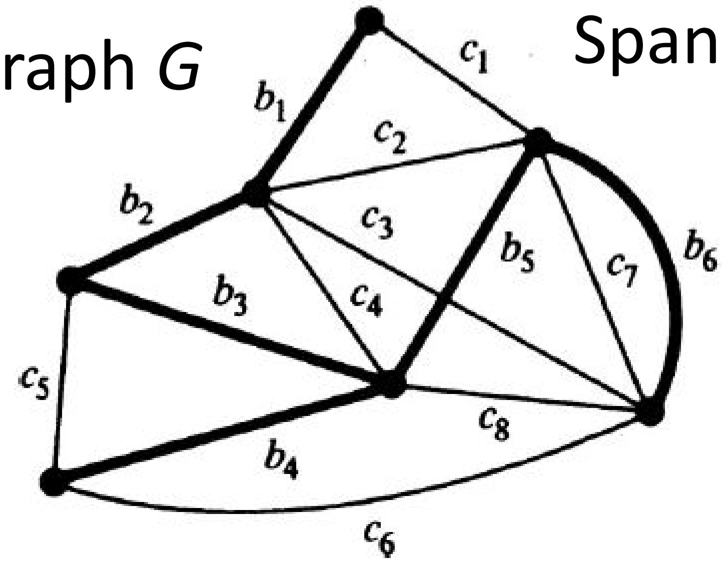


Spanning Tree T of *G (chord)*

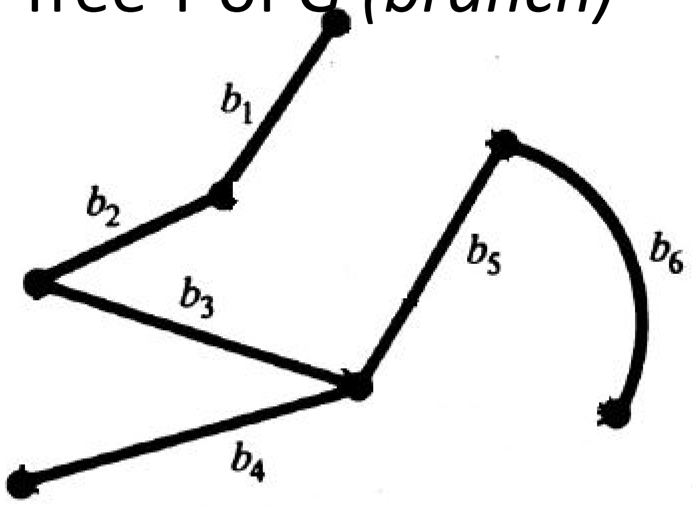
THEOREM 3-11: Every connected graph has at least one spanning tree.

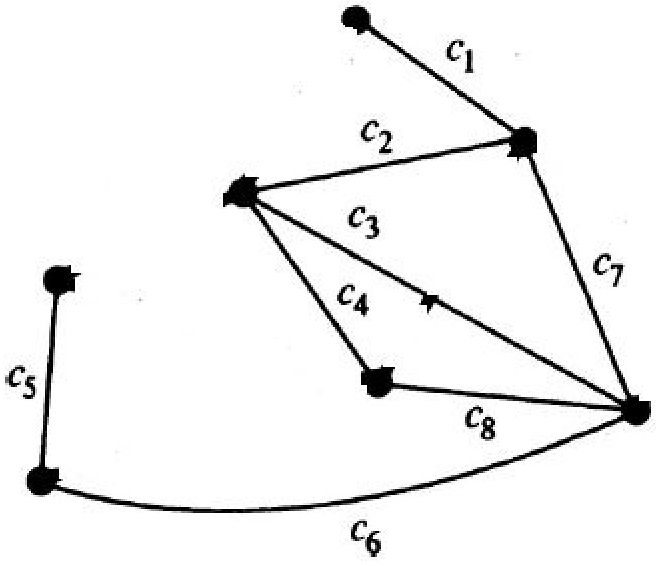
- It is sometimes convenient to consider a connected graph *G* as a union of two sub-graphs, *T* and ; that is, $T \cup \bar{T} = G$.
- where *T* is a spanning tree, and $\bar{T}$ is the complement of T in *G*. Since the sub-graph is the collection of chords, it is quite appropriately referred to as the *chord set* (or *tie set* or *cotree*) of *T*. From the definition, and from Theorem 3-3(A tree with *n* vertices has *n* − 1 edges.), the following theorem is evident.

Graph *G*



Spanning Tree T of *G (branch)*



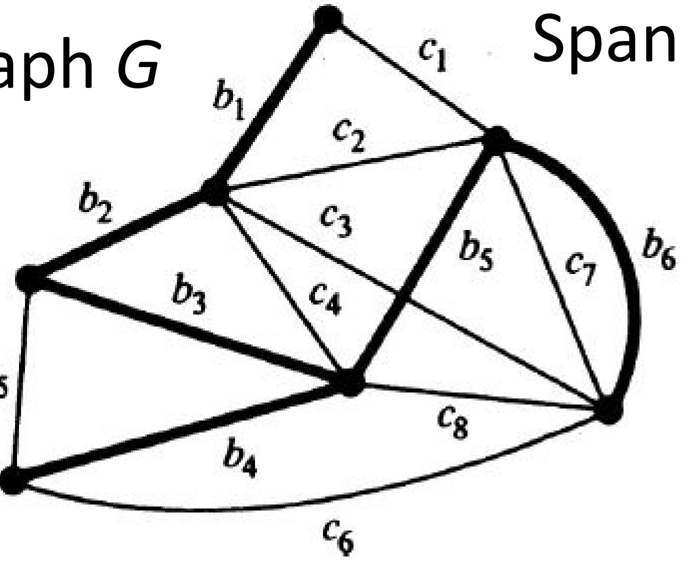


Spanning Tree T of *G (chord)*

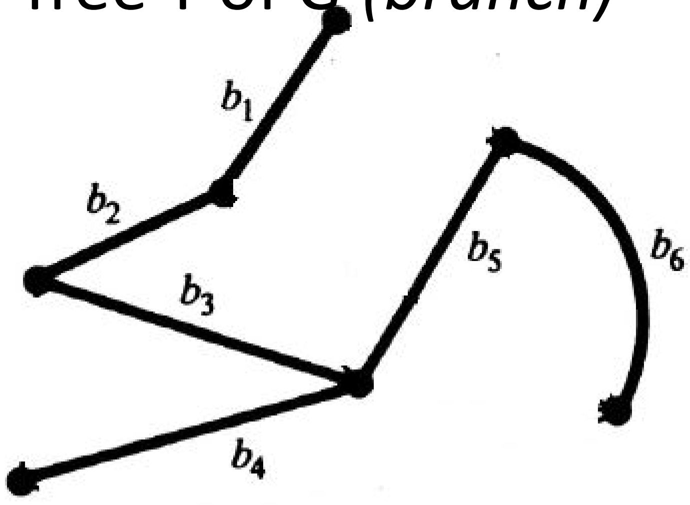
THEOREM 3-12: With respect to any of its spanning trees, a connected graph of *n* vertices and *e* edges has *n* − 1 tree branches and *e* − *n* + 1 chords.

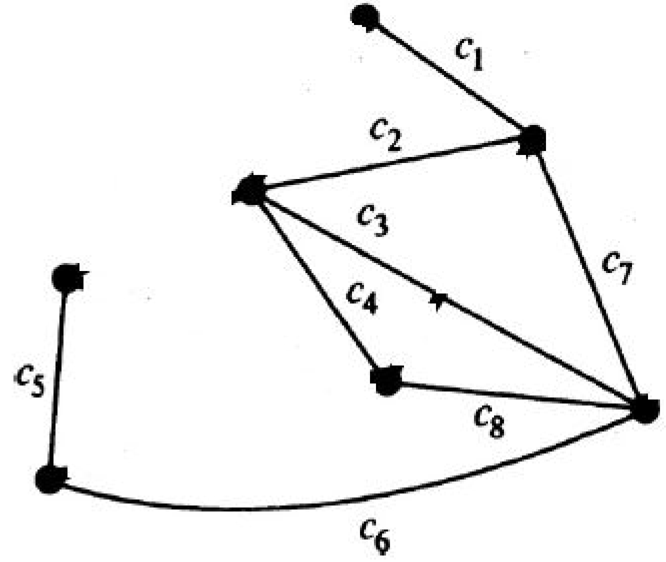
- n = 7
- e = 14 (6+8) (6 branch, 8 chord)
- Branch = n – 1 ➔ 7 – 1 ➔ 6 branch
- Chord = e – n + 1 ➔ 14 – 7 + 1 ➔ 8 chord

Graph *G*



Spanning Tree T of *G (branch)*

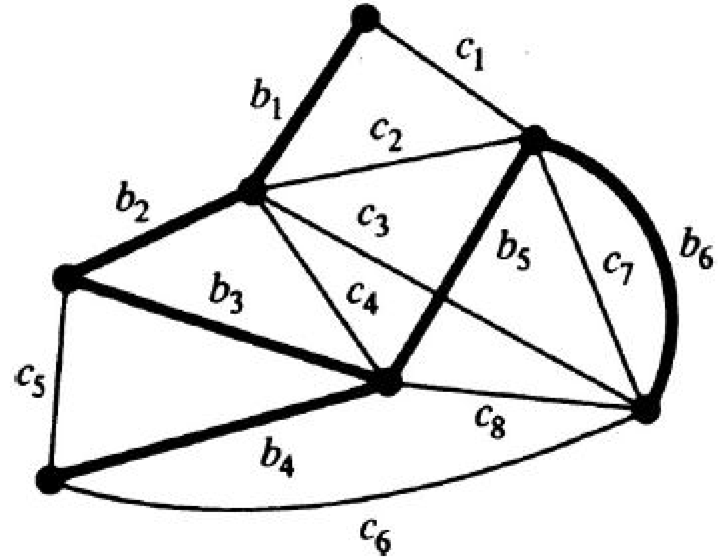


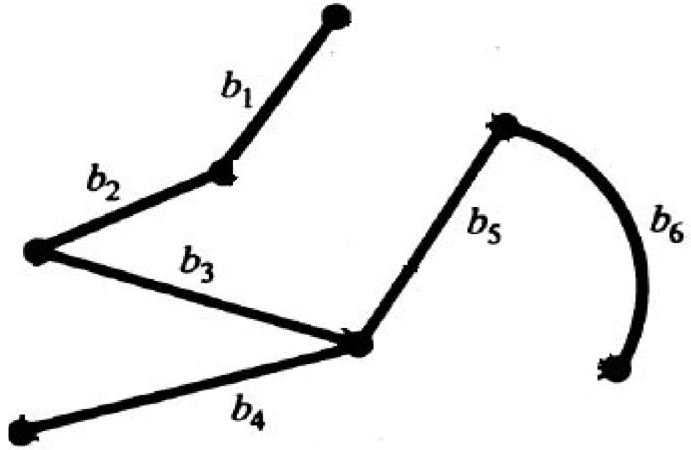


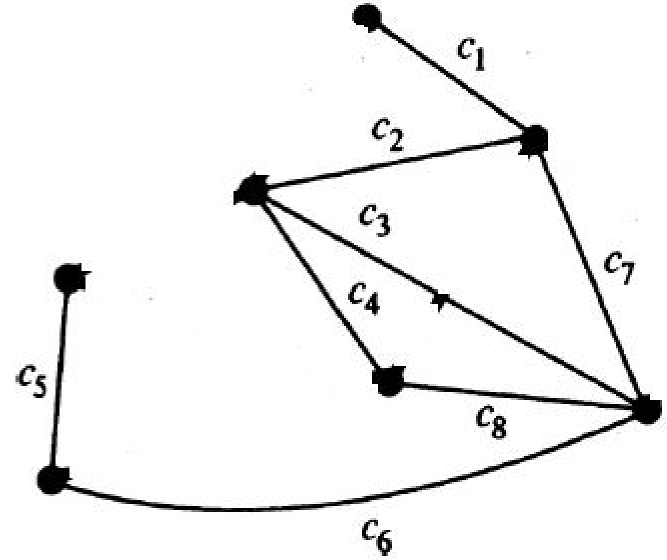
Spanning Tree T of *G (chord)*

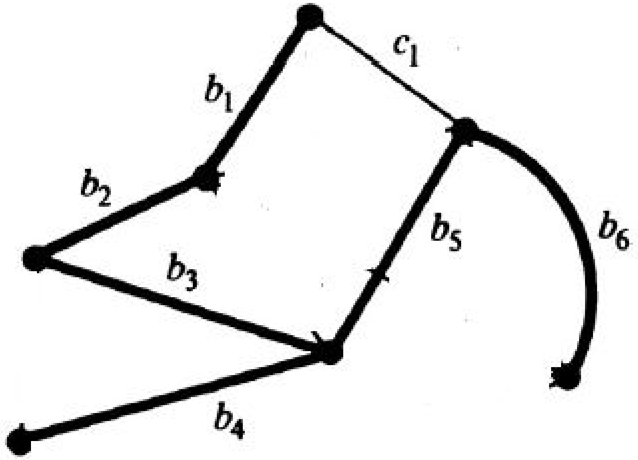
Fundamental Circuit

- Adding any one chord to *T* will create exactly one circuit. Such a circuit, formed by adding a chord to a spanning tree, is called *a fundamental circuit*.



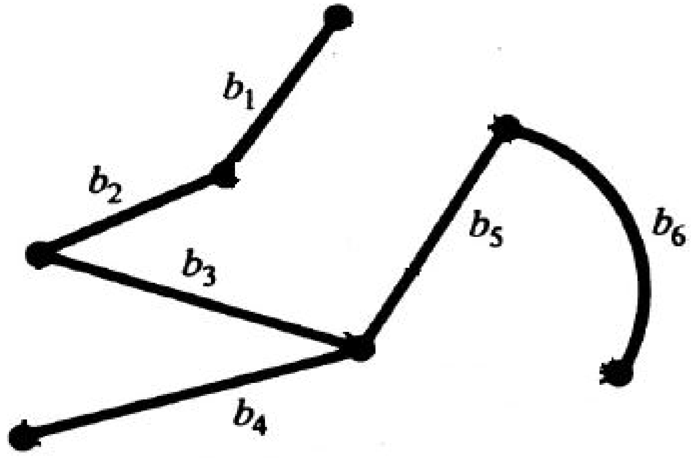


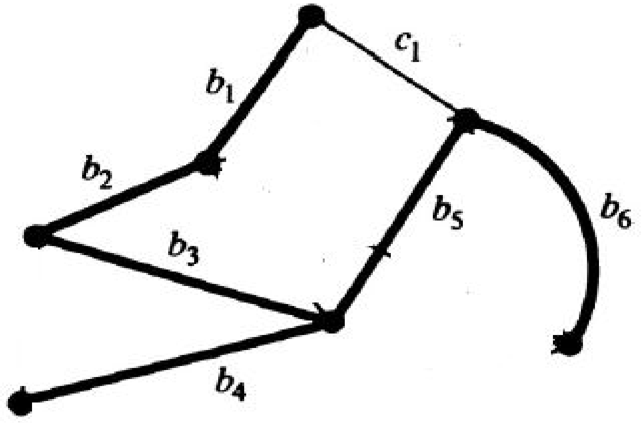


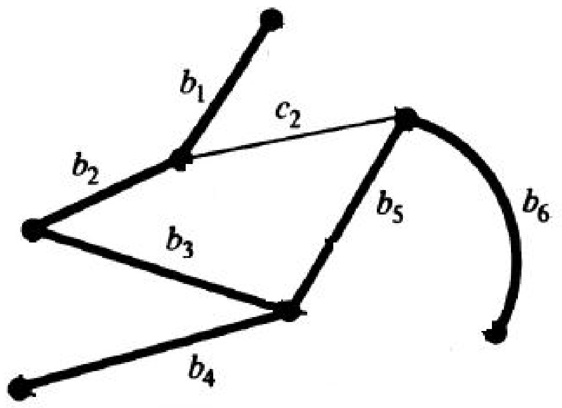


Fundamental Circuit

- Let us look at the tree $\{b_1, b_2, b_3, b_4, b_5, b_6\}$.
- Adding c_1 to it, we get a subgraph $\{b_1, b_2, b_3, b_4, b_5, b_6, c_1\}$, which has one circuit (fundamental circuit), $\{b_1, b_2, b_3, b_4, b_5, c_1\}$. Had we added the chord c_2 (instead of c_1) to the tree, we would have obtained a different fundamental circuit, $\{b_2, b_3, b_5, c_2\}$.

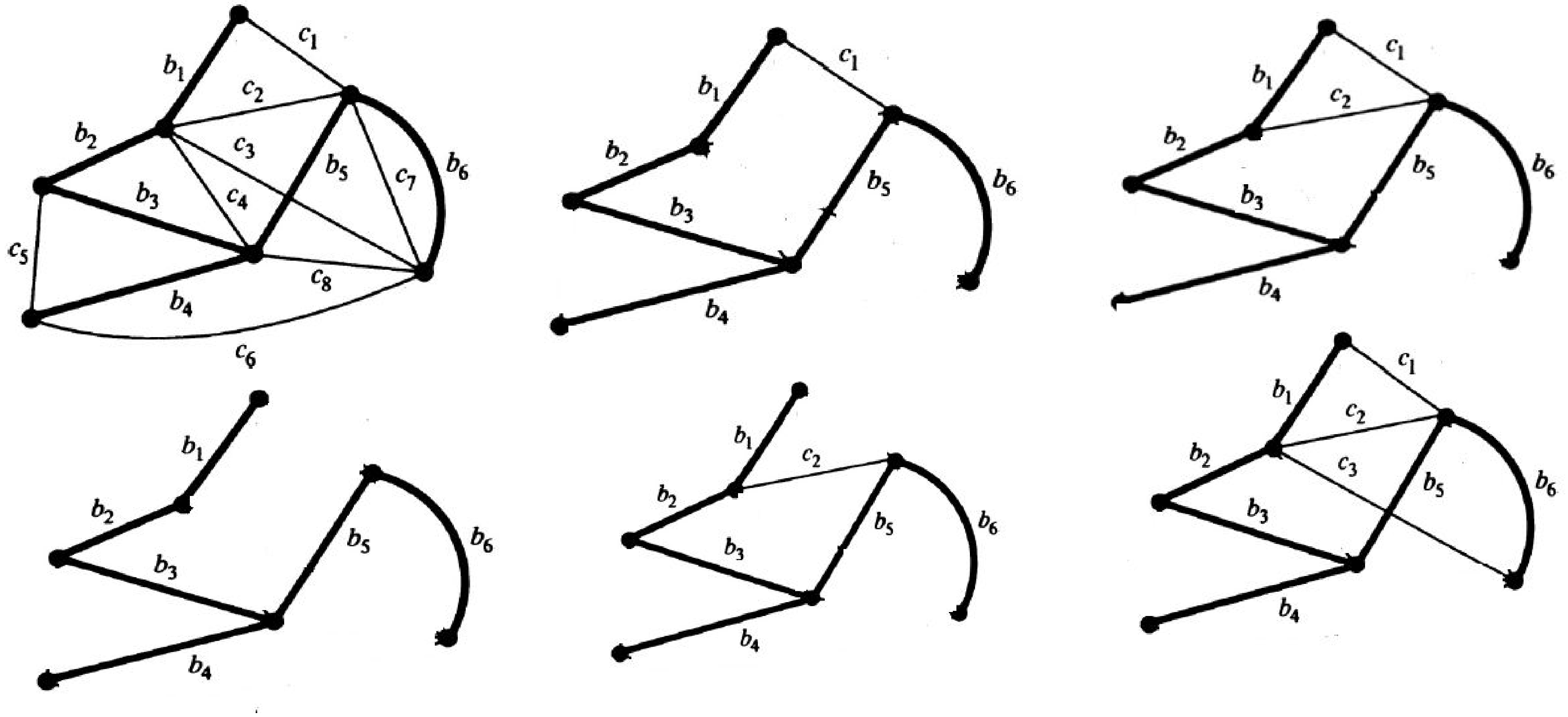




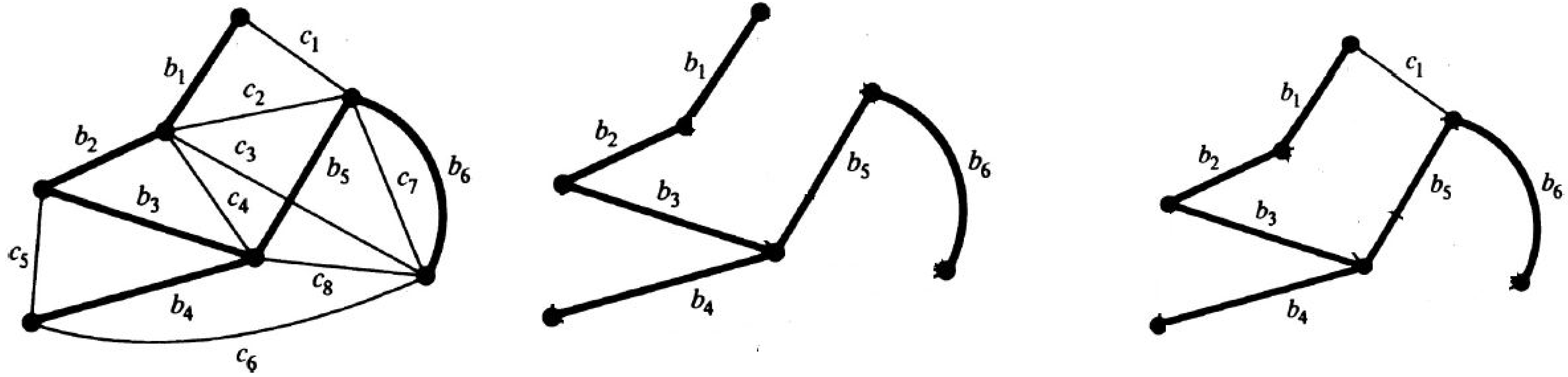


Fundamental Circuit

- Now suppose that we add both chords c_1 and c_2 to the tree. The subgraph $\{b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2\}$, has not only the fundamental circuits we just mentioned, but it has also a third circuit, $\{b_1, c_1, c_2\}$, which is not a fundamental circuit. Although there are 75 circuits in below Figure (enumerated by computer), only eight are fundamental circuits, each formed by one chord (together with the tree branches).



THEOREM 3-13: A connected graph G is a tree if and only if adding an edge between any two vertices in G creates exactly one circuit.



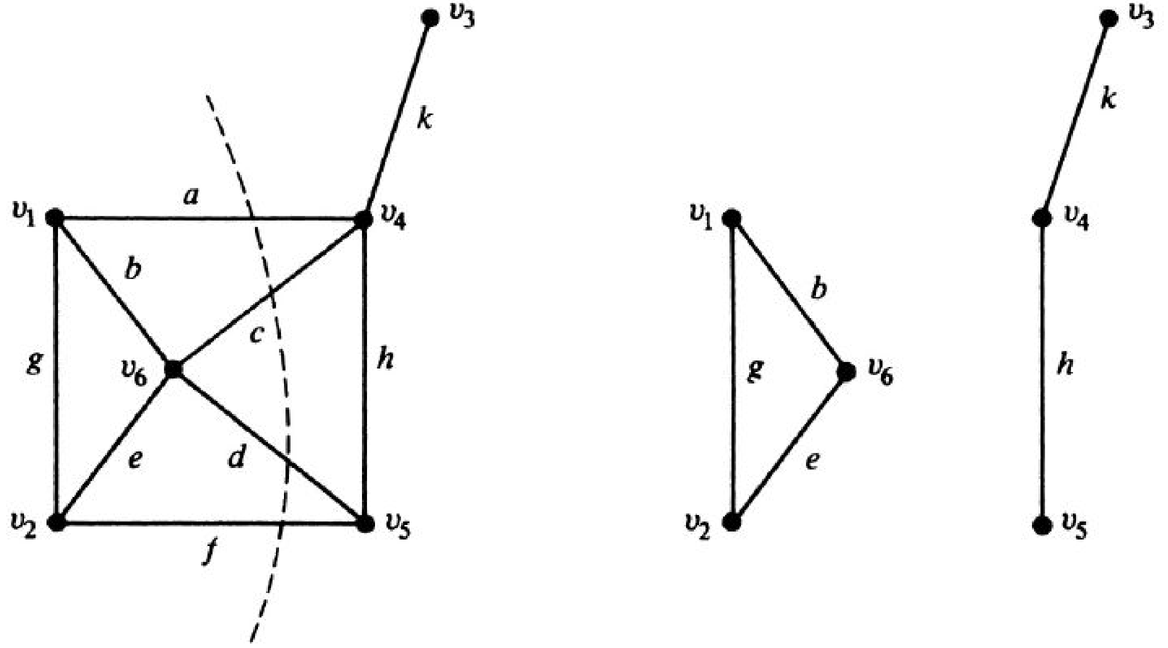
Cut-sets

Cut-sets:

In a connected graph G, a *cut-set* is a set of edges whose removal from *G* leaves *G* disconnected, provided removal of no proper subset of these edges disconnects G.

The set of edges {*a*, *c*, *d*, *f*} is a cut-set.

There are many other cutsets, such as {*a*, *b*, *g*} , {*a*, *b*, *e*, *f*} , and {*d*, *h*, *f*}. Edge {*k*} alone is also a cut-set.

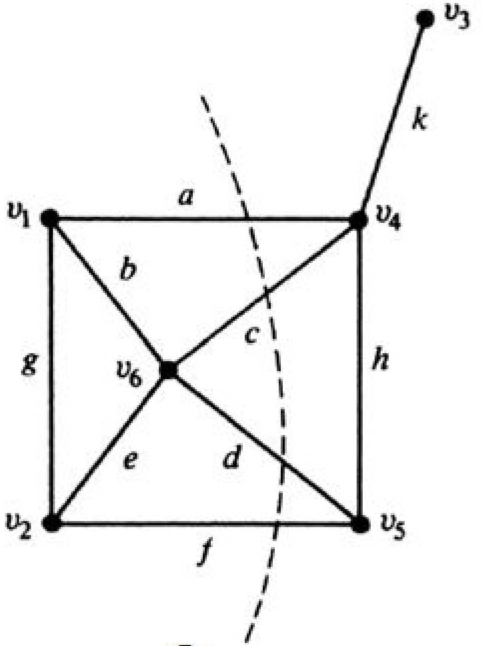


PROPERTIES OF A CUT-SET

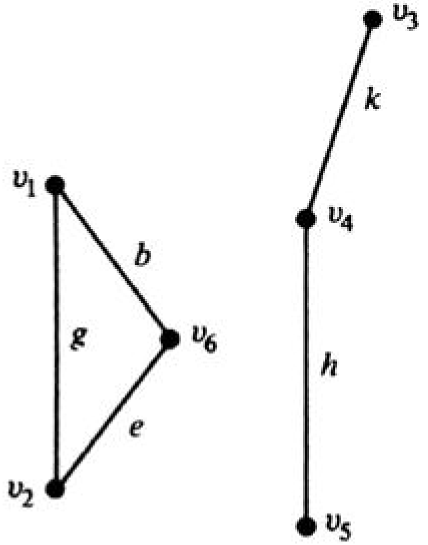
- Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.
- In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.
- Every circuit has an even number of edges in common with any cut set.

THEOREM 4-1: Every cut-set in a connected graph *G* must contain at least one branch of every spanning tree of G.

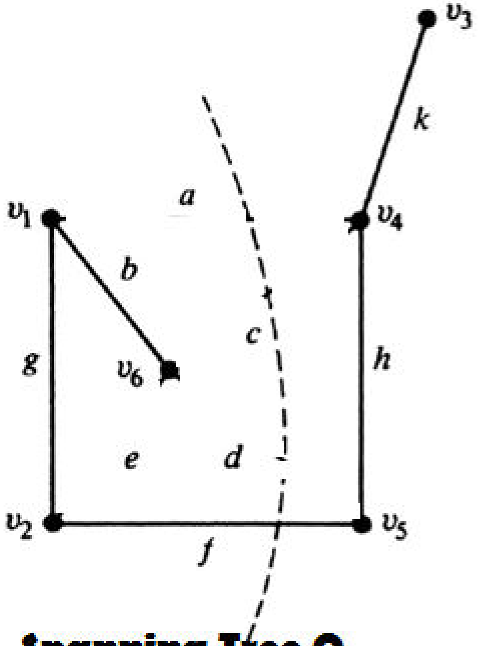
- In a given connected graph *G*, let *Q* be a minimal set of edges containing at least one branch of every spanning tree of *G*.
- Consider *G* − *Q*, the subgraph that remains after removing the edges in *Q* from *G*.
- Since the subgraph *G* − *Q* contains no spanning tree of G, *G* − *Q* is disconnected (one component of which may just consist of an isolated vertex).



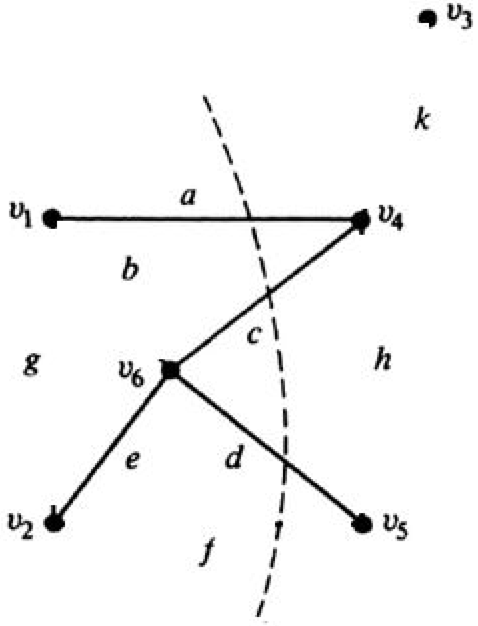
G



cut sets



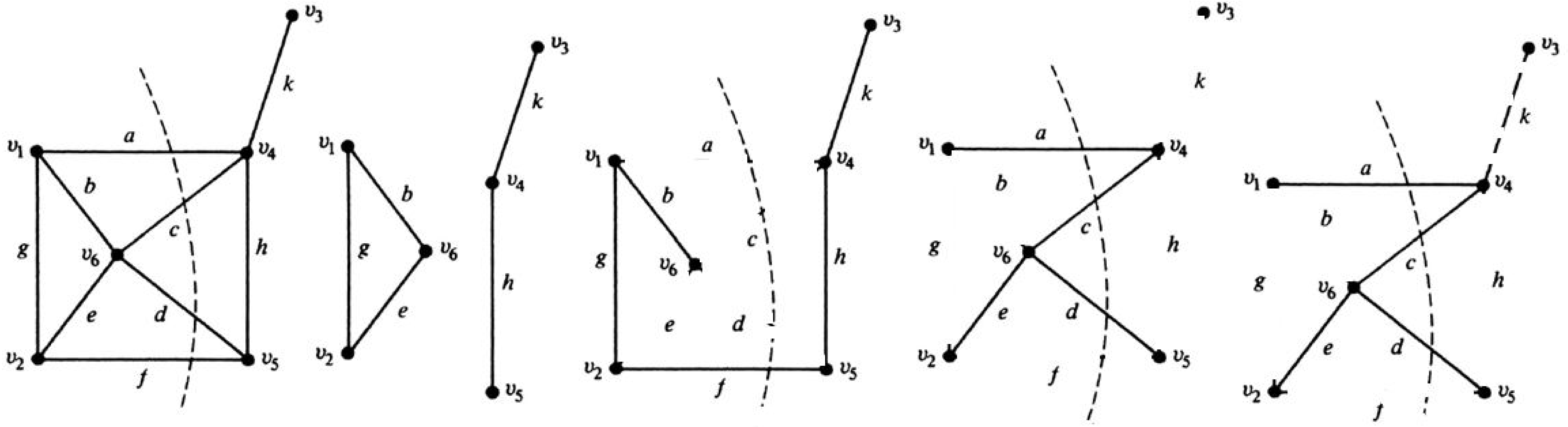
Spanning Tree Q



G - Q

THEOREM 4-1: Every cut-set in a connected graph *G* must contain at least one branch of every spanning tree of G.

- Also, since *Q* is a minimal set of edges with this property, any edge *e* from *Q* returned to *G* − *Q* will create at least one spanning tree.
- Thus the subgraph *G* − *Q* + *e* will be a connected graph.
- Therefore, *Q* is a minimal set of edges whose removal from *G* disconnects G. This, by definition, is a cut-set. Hence

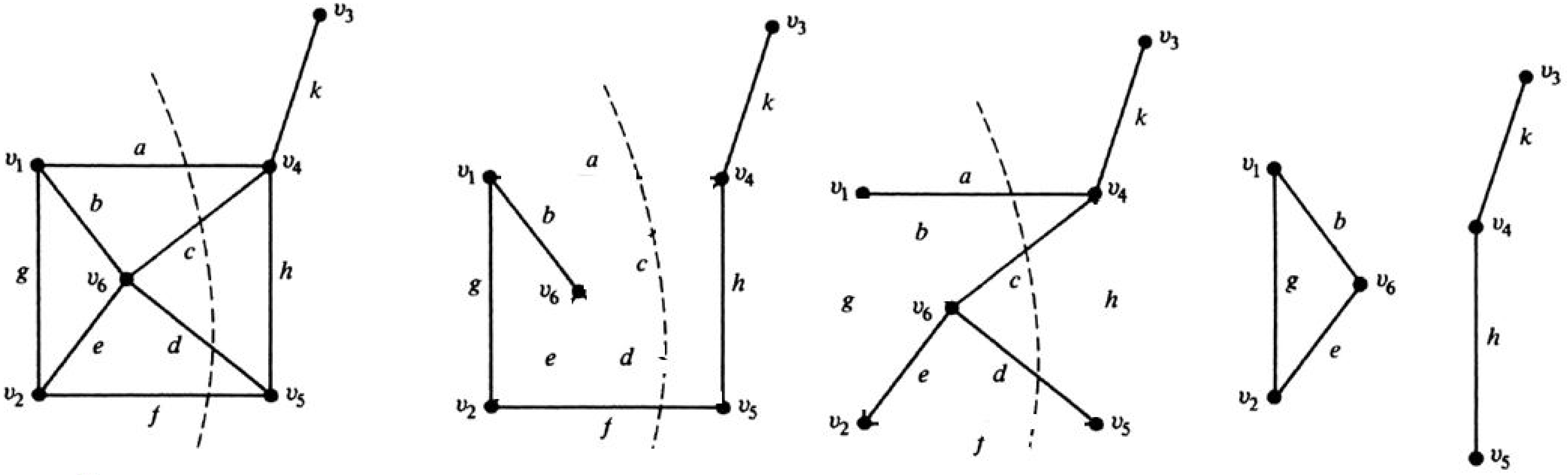


G **Cut Sets** **Spanning Tree Q** **G - Q** **G - Q + e**

PROPERTIES OF A CUT-SET

THEOREM 4-2:

In a connected graph *G*, any minimal set of edges containing at least one branch of every spanning tree of *G* is a cut-set.



G

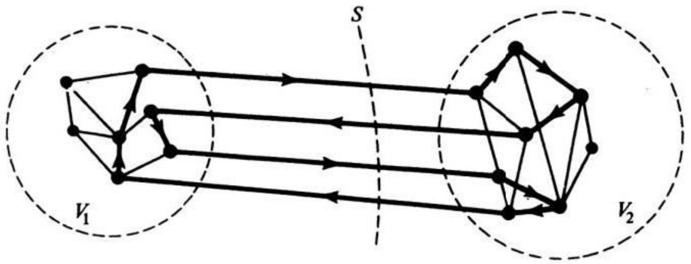
Spanning Tree Q

Cut Sets

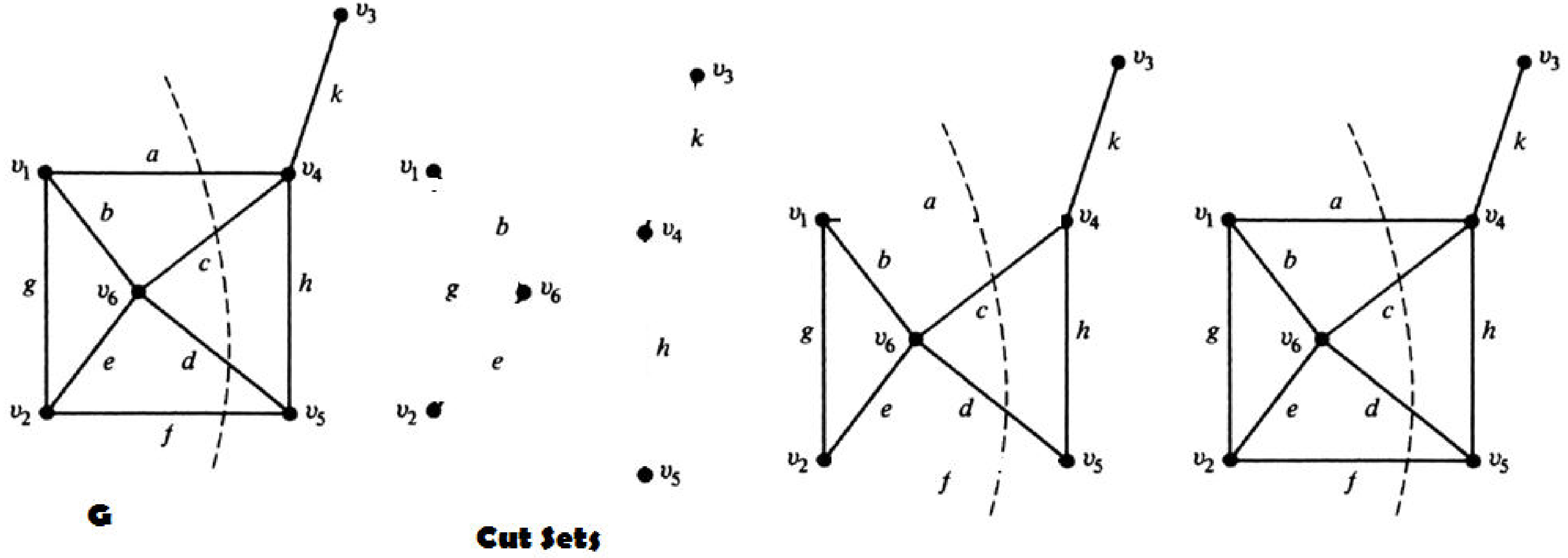
THEOREM 4-3: Every circuit has an even number of edges in common with any cut-set.

Proof:

Consider a cut-set S in graph G. Let the removal of S partition the vertices of G into two (mutually exclusive or disjoint) subsets V_1 and V_2. Consider a circuit Γ in G. If all the vertices in Γ are entirely within vertex set V_1 (or V_2), the number of edges common to S and Γ is zero: that is. $N(S \cap \Gamma) = 0$, an even number.

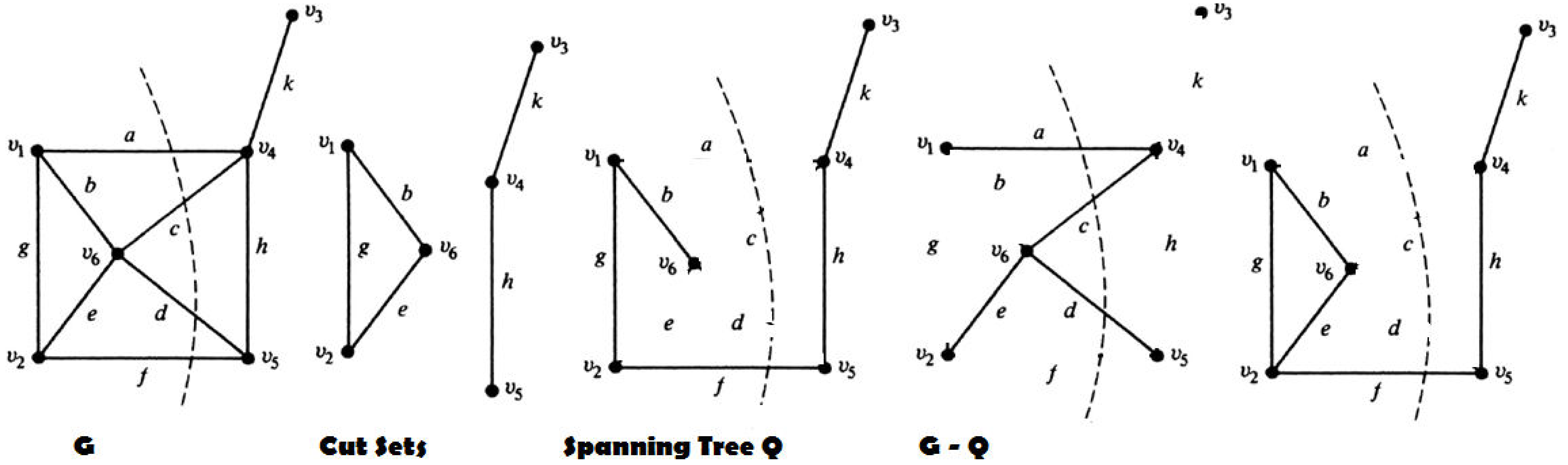


Circuit Γ shown in heavy lines, and is traversed along the direction of the arrows



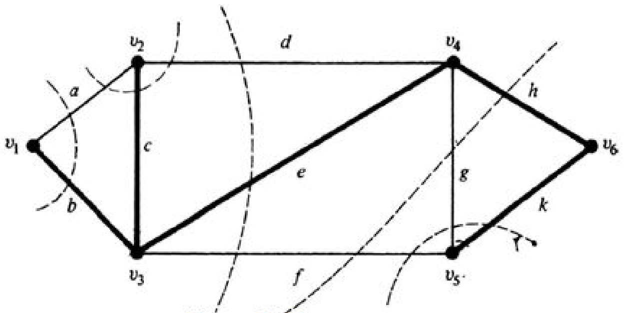
Fundamental Circuit and CUT-SETS

Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T, and let the fundamental circuit made by c_i be called Γ, consisting of k branches $b_1, b_2, \ldots, b_k$ in addition to the chord c_i; that is, $\Gamma = \{c_i, b_1, b_2, \ldots, b_k)$ is a fundamental circuit with respect to T.

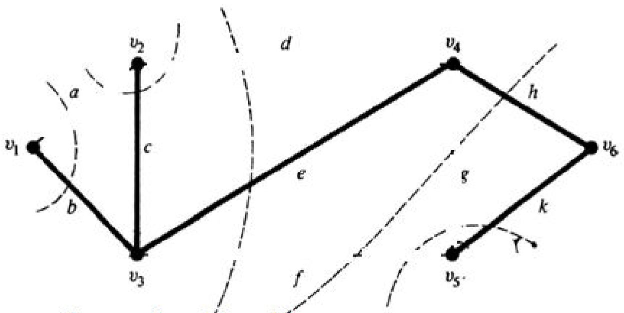


THEOREM 4-5: With respect to a given spanning tree *T*, a chord c_i that determines a fundamental circuit Γ occurs in every fundamental cut-set associated with the branches in Γ and in no other.

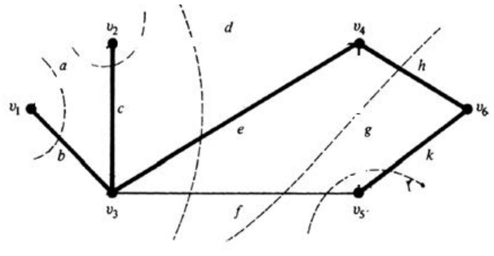
- consider the spanning tree {*b*, *c*, *e*, *h*, *k*}
- The fundamental circuit made by chord *f* is {*f, e, h, k*}.
- The three fundamental cut-sets determined by the three branches *e, h*, and *k* are
  - determined by branch *e*: {*d*, *e*, *f*},
  - determined by branch *h*: {*f*, *g*, *h*},
  - determined by branch *k*: {*f*, *g*, *k*}.
- Chord *f* occurs in each of these three fundamental cut-sets, and there is no other fundamental cut-set that contains *f*. The converse of Theorem 4-5 is also true.



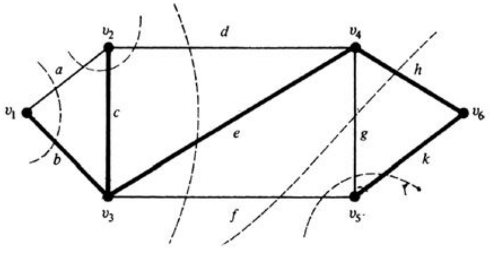
Graph G



Spanning Tree T



Fundamental Circut using f

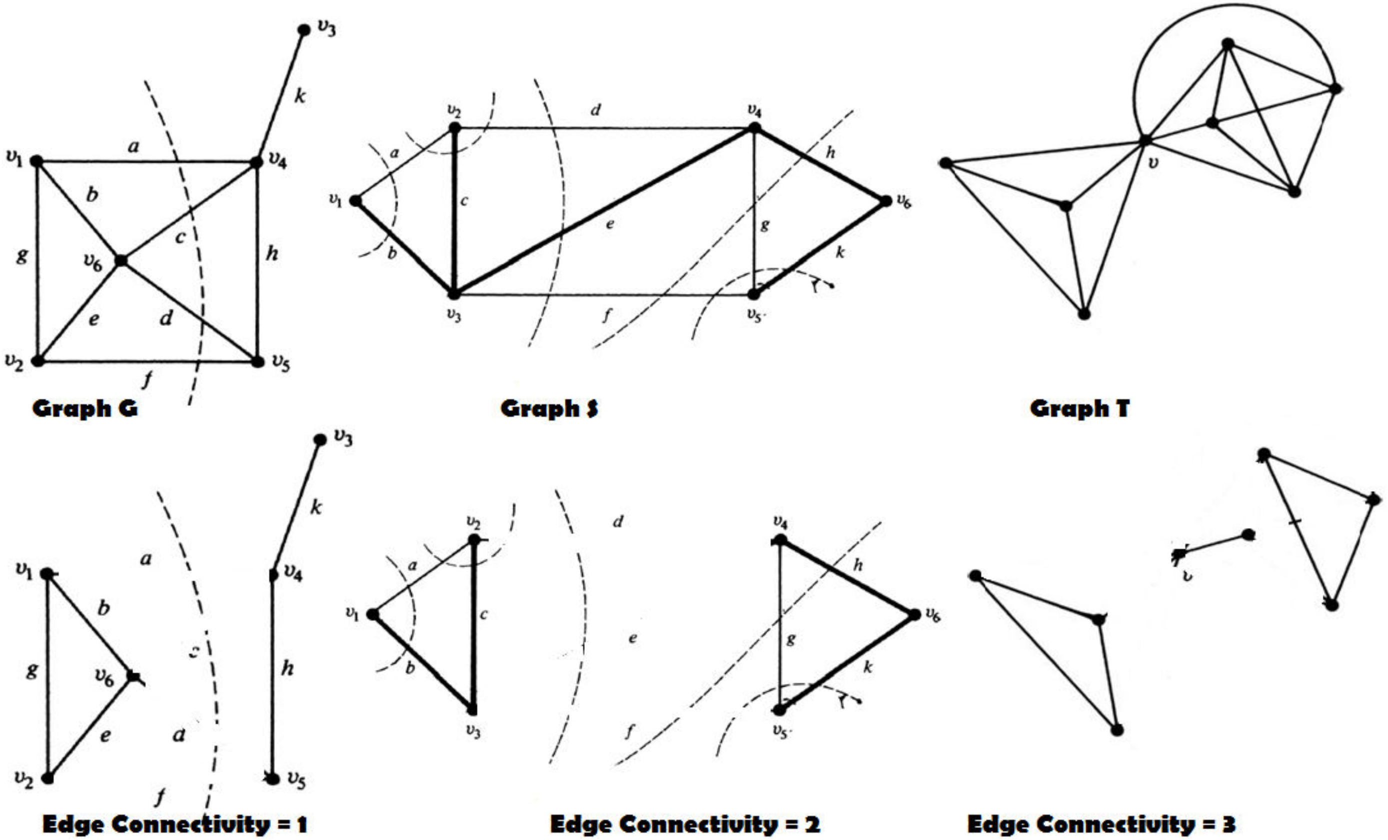


Fundamental cutsets of a graph.

CONNECTIVITY AND SEPARABILITY

Edge Connectivity

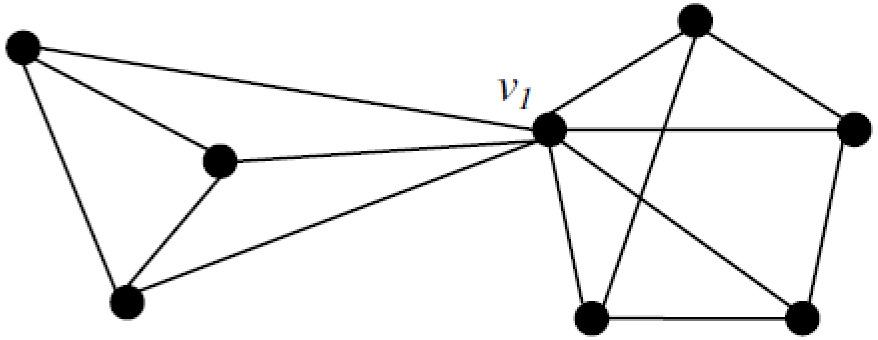
- Each cut-set of a connected graph G consists of certain number of edges.
- The number of edges in the smallest cut-set is defined as the **edge Connectivity of G.**
- The **edge Connectivity** of a connected graph G is defined as the minimum number of edges whose removal reduces the rank of graph by one.
- The edge Connectivity of a tree is one.



Graph G Graph S Graph T

Edge Connectivity = 1 Edge Connectivity = 2 Edge Connectivity = 3

Edge Connectivity

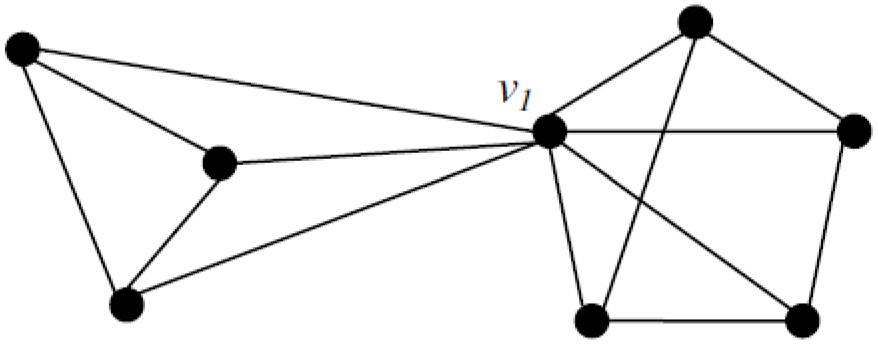


The edge Connectivity of the above graph G is three.

CONNECTIVITY AND SEPARABILITY

Vertex Connectivity

- The **vertex Connectivity** of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.
- The vertex Connectivity of a tree is one.



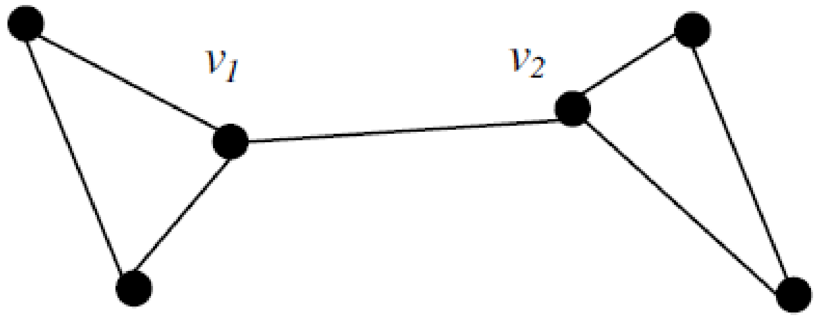
The vertex Connectivity of the above graph G is one.

SEPARABILITY

Separable and non-separable graph

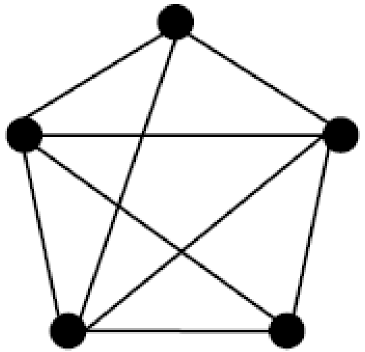
- A connected graph is said to be separable graph if its vertex connectivity is one.
- All other connected graphs are called non-separable graph.

Separable Graph G:



articulation point.

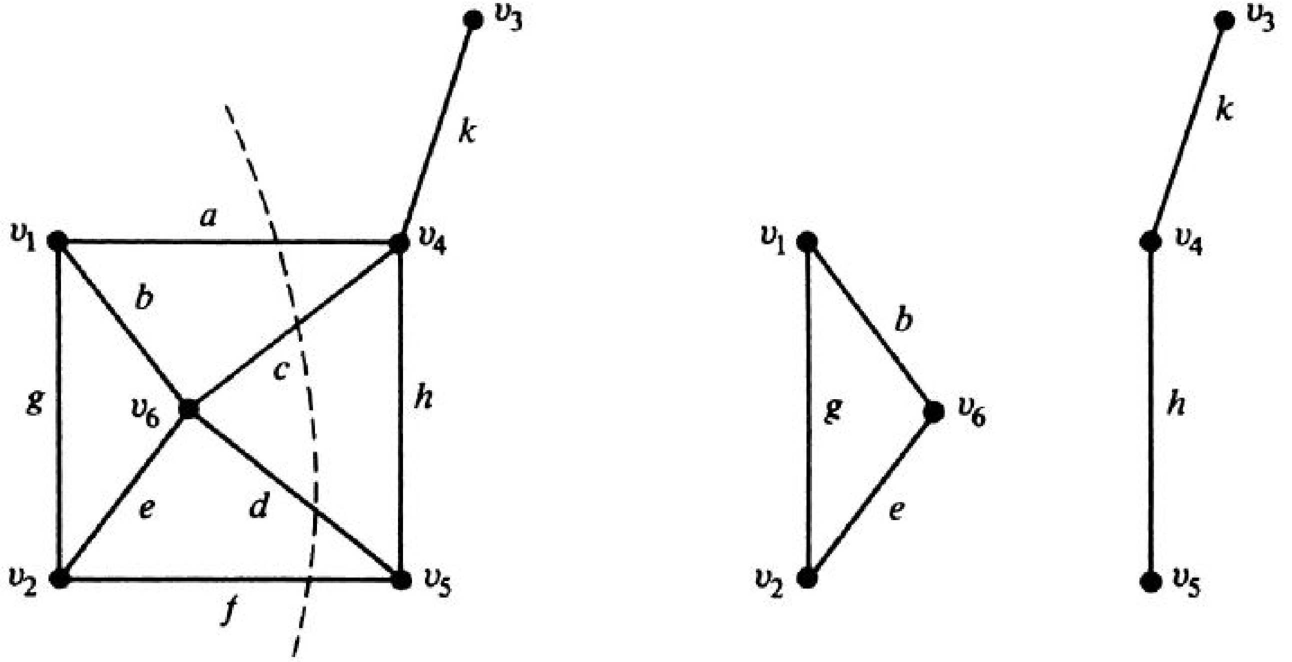
Non-Separable Graph H:



THEOREM 4-8: The edge connectivity of a graph *G* cannot exceed the degree of the vertex with the smallest degree in *G*.

Proof:

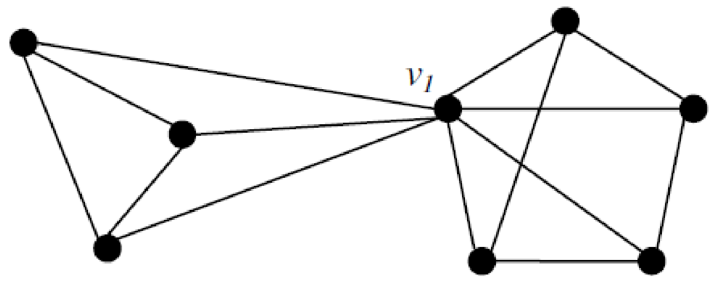
- Let vertex v_i be the vertex with the smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from *G* by removing the $d(v_i)$ edges incident on vertex v_i. Hence the theorem.



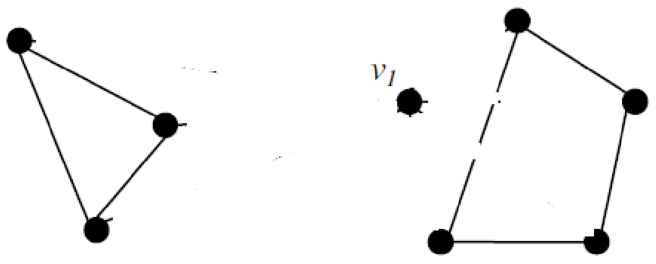
THEOREM 4-9: The vertex connectivity of any graph *G* can never exceed the edge connectivity of *G*.

Proof:

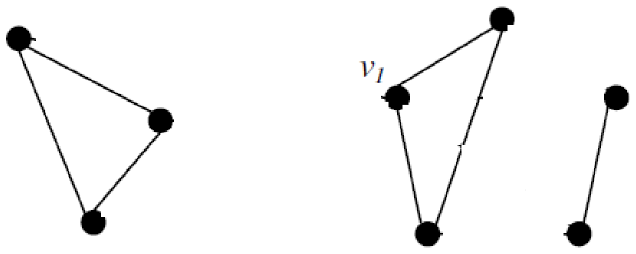
- Let α denote the edge connectivity of *G*. Therefore, there exists a cut-set *S* in *G* with α edges. Let *S* partition the vertices of *G* into subsets V_1 and V_2. By removing at most a vertices from V_1 (or V_2) on which the edges in *S* are incident, we can effect the removal of *S* (together with all other edges incident on these vertices) from G. Hence the theorem.



Graph *G*



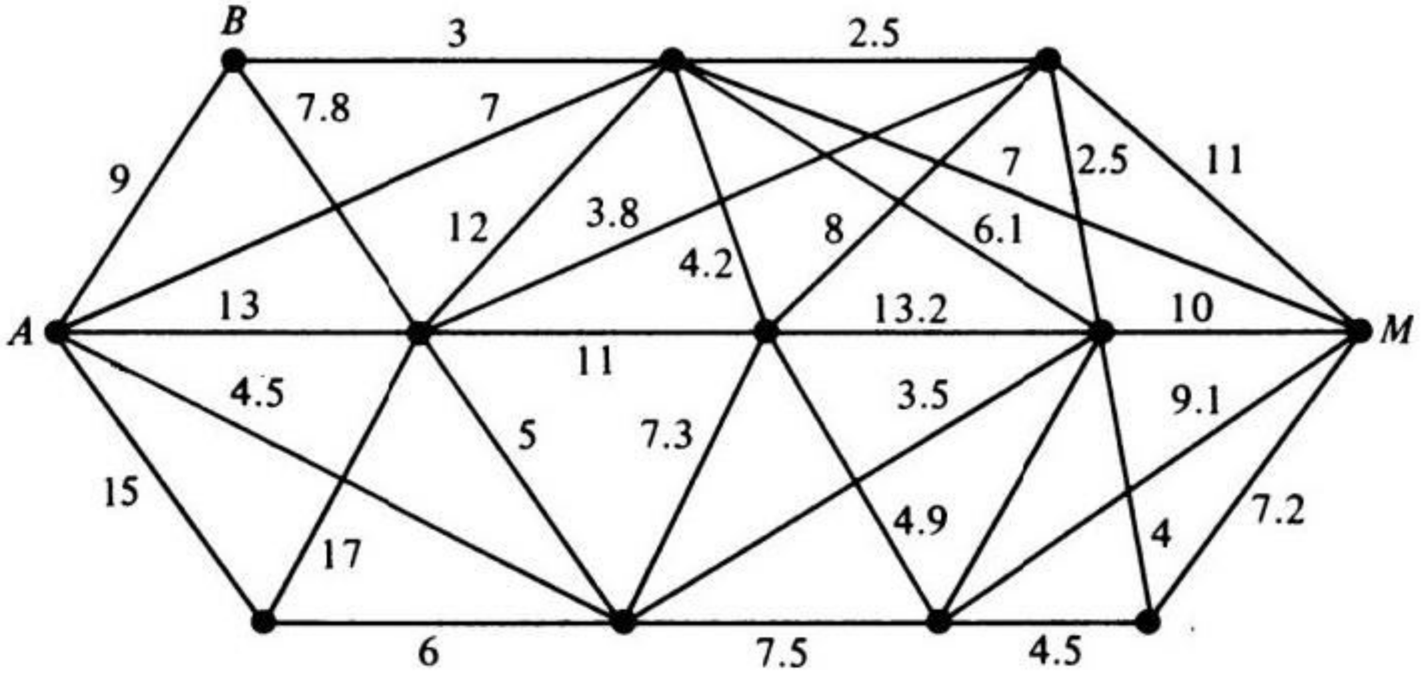
Vertex Connectivity = 1



Edge Connectivity = 3

NETWORK FLOWS

- A flow network (also known as a transportation network) is a graph where each edge has a capacity and each edge receives a flow. The amount of flow on an edge cannot exceed the capacity of the edge.

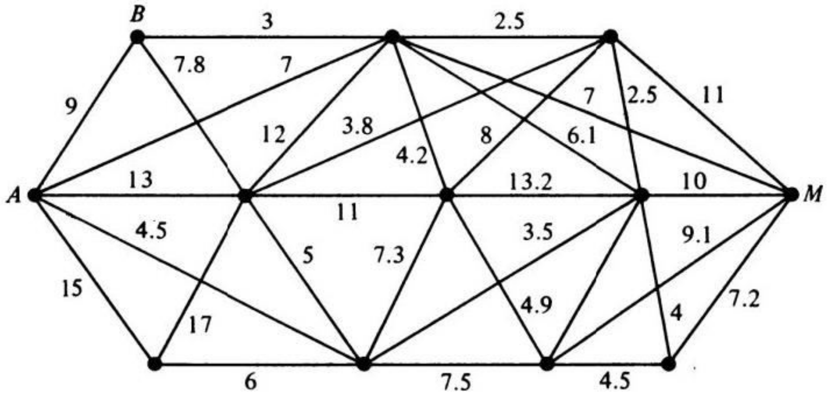


Graph of a flow network.

THEOREM 4-13: The maximum flow possible between two vertices *a* and *b* in a network is equal to the minimum of the capacities of all cut-sets with respect to *a* and *b*.

Proof:

- Consider any cut-set *S* with respect to vertices *a* and *b* in G. In the sub-graph *G − S* (the sub-graph left after removing *S* from *G)* there is no path between *a* and *b*. Therefore, every path in *G* between *a* and *b* must contain at least one edge of *S*. Thus every flow from *a* to *b* (or from *b* to *a*) must pass through one or more edges of *S*. Hence the total flow rate between these two vertices cannot exceed the capacity of *S*. Since this holds for all cut-sets with respect to *a* and *b*, the flow rate cannot exceed the minimum of their capacities.



Graph of a flow network.

1-ISOMORPHISM

A graph G_1 was 1-Isomorphic to graph G_2 if the blocks of G_1 were isomorphic to the blocks of G_2. Two graphs G_1 and G_2 are said to be 1-Isomorphic if they become isomorphic to each other under repeated application of the following operation.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint sub-graphs.

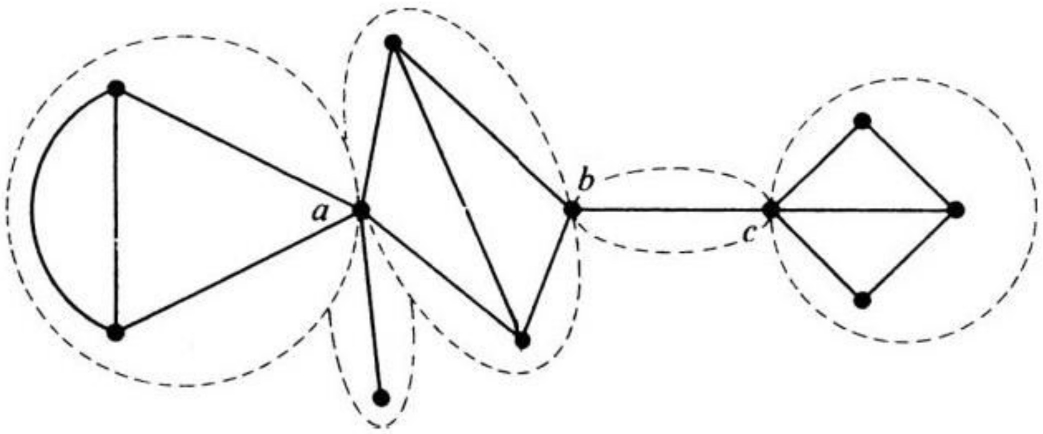


Fig. 4-8 Separable graph with three cut-vertices and five blocks.

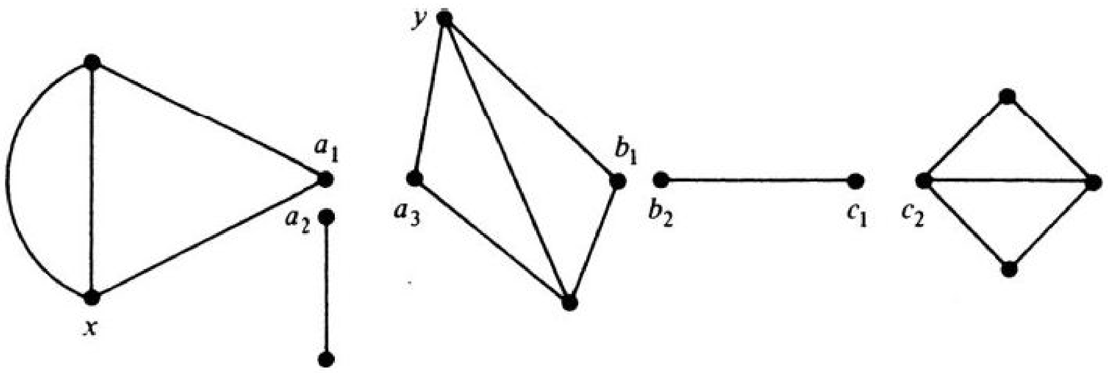


Fig. 4-9 Disconnected graph 1-isomorphic to Fig. 4-8.

THEOREM 4-14: If G_1 and G_2 are two 1-isomorphic graphs, the rank of G_1 equals the rank of G_2 and the nullity of G_1 equals the nullity of G_2.

Proof:

Under operation 1, whenever a cut-vertex in a graph G is "split" in to two vertices, the number of components in G increases by one. Therefore, the rank of G which is number of vertices in G − number of components in G remains invariant under operation 1. Also, since no edges are destroyed or new edges created by operation 1, two 1-isomorphic graphs have the same number of edges. Two graphs with equal rank and with equal numbers of edges must have the same nullity, because nullity = number of edges − rank.

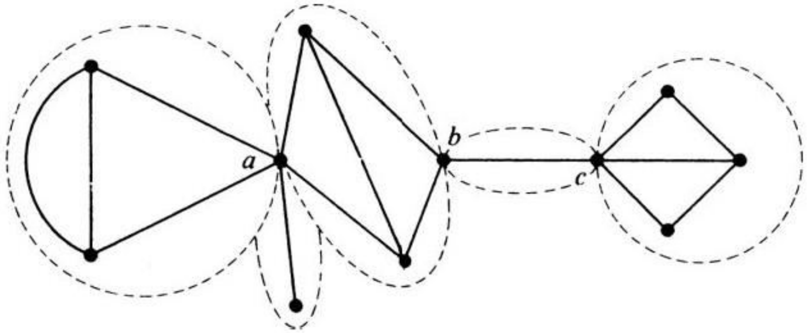


Fig. 4-8 Separable graph with three cut-vertices and five blocks.

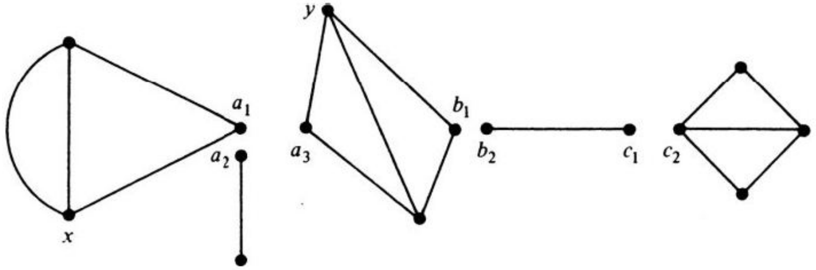


Fig. 4-9 Disconnected graph 1-isomorphic to Fig. 4-8.

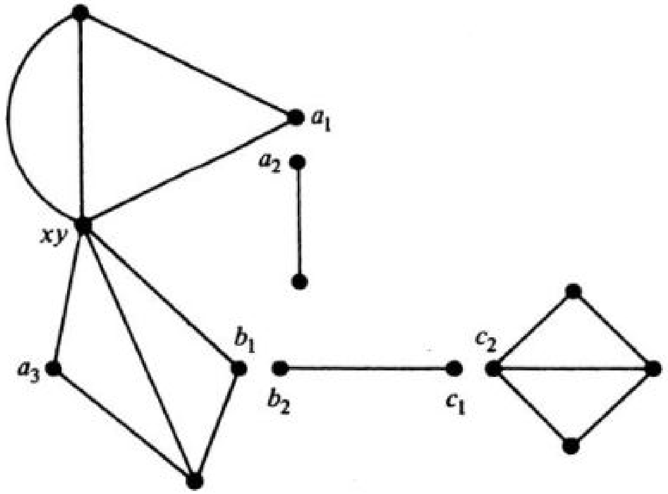


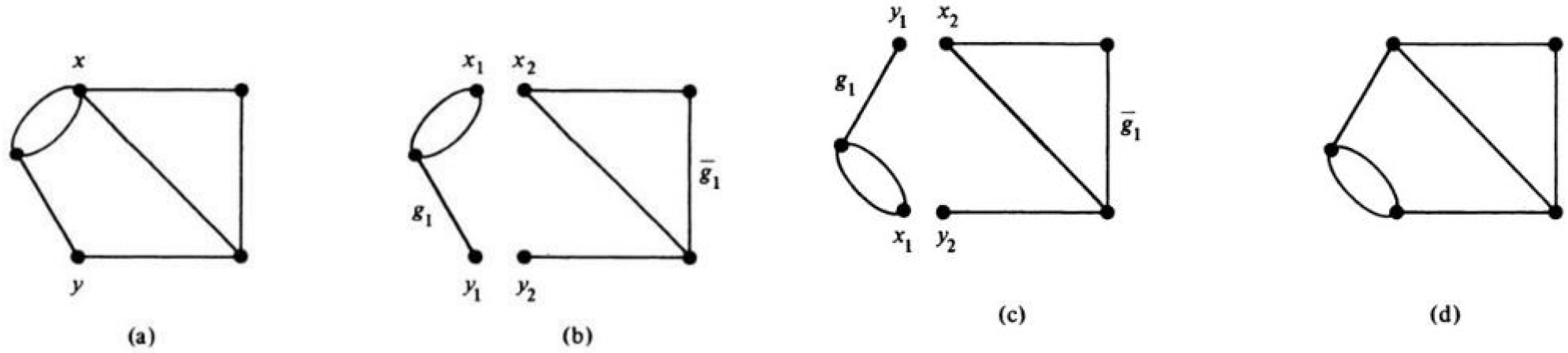
Fig. 4-10 Graph 1-isomorphic to Figs. 4-8 and 4-9.

2-ISOMORPHISM

Two graphs G_1 and G_2 are said to be 2-Isomorphic if they become isomorphic after undergoing operation 1 or operation 2, or both operations any number of times.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint sub-graphs.

Operation 2: "Split" the vertex x into x_1 and x_2 and the vertex y into y_1 and y_2 such that G is split into g_1 and g_2. Let vertices x_1 and y_1 go with g_1 and vertices x_2 and y_2 go with g_2. Now rejoin the graphs g_1 and g_2 by merging x_1 with y_2 and x_2 with y_1.

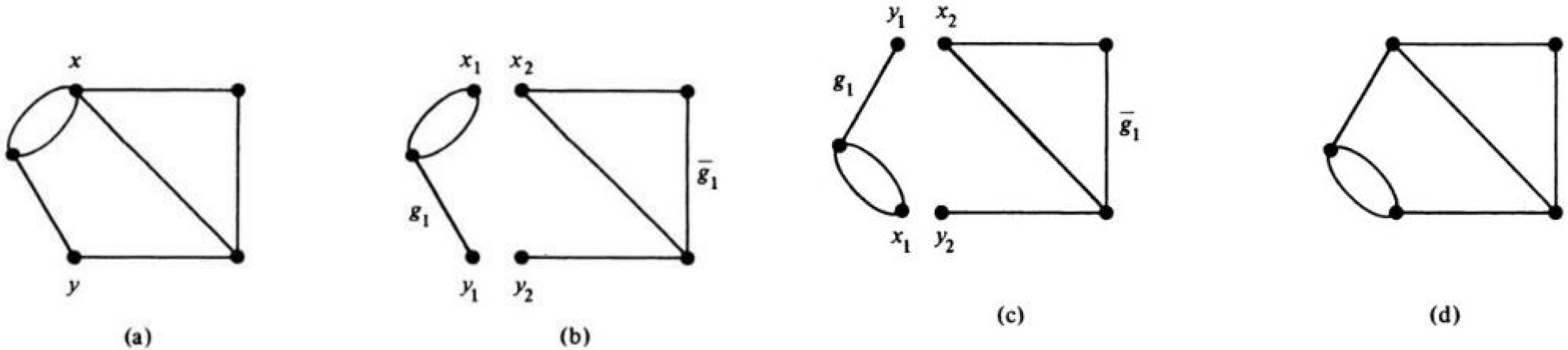


2-isomorphic graphs (a) and (d).

THEOREM 4-15: Two graphs are 2-isomorphic if and only if they have circuit correspondence.

Proof:

The "only if" part has already been shown in the argument preceding the theorem. The "if" part is more involved, and the reader is referred to Whitney's original paper [4-7].As we shall observe in subsequent chapters, the ideas of 2-isomorphism and circuit correspondence play important roles in the theory of contact networks, electrical networks, and in duality of graphs.



2-isomorphic graphs (a) and (d).

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Thank you.