# CS6109 – GRAPH THEORY

## Module – 2

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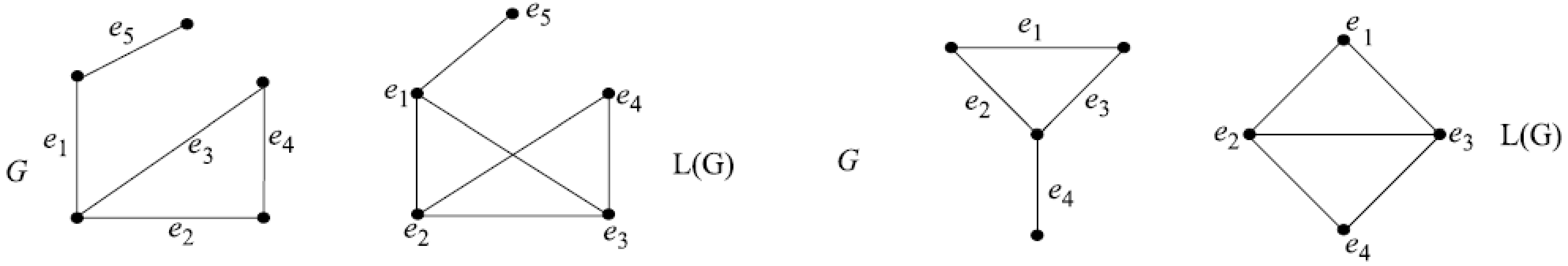
# Module - 2

- Edge Graphs and Traversability
- Eccentricity Sequences and Sets
- Isometry

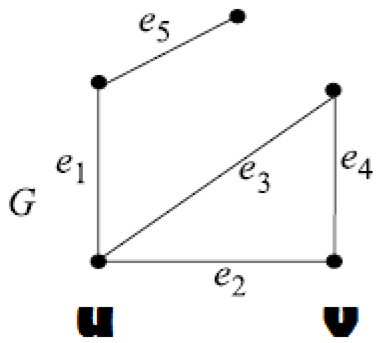
# Edge Graphs

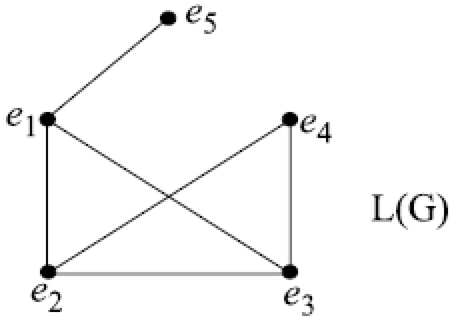
- Definition: Let $G(V, E)$ be a graph with $V = \{v_1, v_2, . . ., v_n\}$ and $E = \{e_1, e_2, . . . , e_m\}$.
- The edge graph $L(G)$ of $G$ has the vertex set $E$ and two vertices $e_i$ and $e_j$ are adjacent in $L(G)$ if and only if the corresponding edges $e_i$ and $e_j$ of $G$ are adjacent in $G$.
- $L(G)$ is the edge graph of $G$. A graph $G$ is an edge graph if it is isomorphic to the edge graph $L(H)$ of some graph $H$.

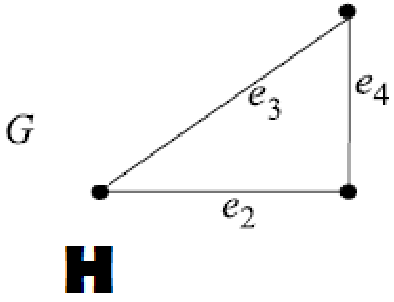


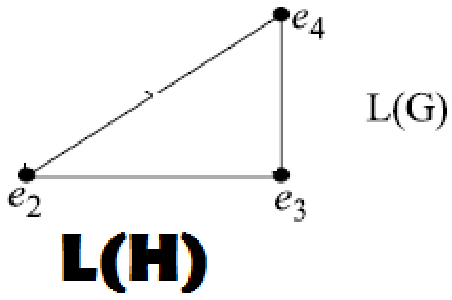
# Observations Edge Graphs

1. A graph G is connected if and only if L(G) is connected.
2. If H is a subgraph of G, then L(H) is a subgraph of L(G).
3. The edges incident at a vertex of G form a maximal complete subgraph of L(G).
4. In G, if e = uv is an edge, then the degree of e in L(G) is the number of edges of G adjacent to e in G. Clearly, $d_{L(G)}(e) = d_G(u)+d_G(v)-2$.
5. For n > 1, $L^n(G) = L(L^{n-1}(G))$ and $L^0(G) = G$.



**Theorem 9.1** The number of edges $m'$ in $L(G)$ when $G$ has degree sequence $[d_i]_1^n$ is given by

$$m' = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$

**Proof** Let $[d_i]_1^n$ be the degree sequence of the graph $G$ and let $L(G)$ the edge graph of $G$, have $m'$ edges.

As the degree of the vertex $v_i$ in $G$ is $d_i$, there are $d_i$ edges incident on $v_i$. From these $d_i$ edges, any two are adjacent at $v_i$ in $G$. Hence the number of edges contributed by $v_i$ to $L(G)$ is $\binom{d_i}{2}$.

Thus, $$m' = \sum_{i=1}^{n} \binom{d_i}{2} = \sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2} = \frac{1}{2}\sum_{i=1}^{n}(d_i^2 - d_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \frac{1}{2}\sum_{i=1}^{n} d_i$$

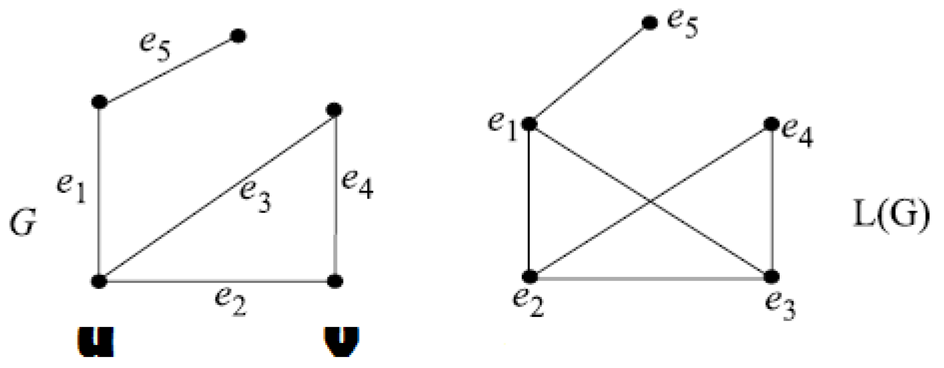
$$= \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$

**Theorem 9.1** The number of edges $m'$ in $L(G)$ when $G$ has degree sequence $[d_i]_1^n$ is given by

$$m' = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$

Proof:

We know that $m = \frac{1}{2}\left(\sum_{i=1}^{n} d_i\right)$




Therefore $m' = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - \frac{1}{2}\left(\sum_{i=1}^{n} d_i\right)$

$m' = ½(1^2 + 2^2 + 2^2 + 2^2 + 3^2) – ½( 1 + 2 + 2 + 2 + 3)$

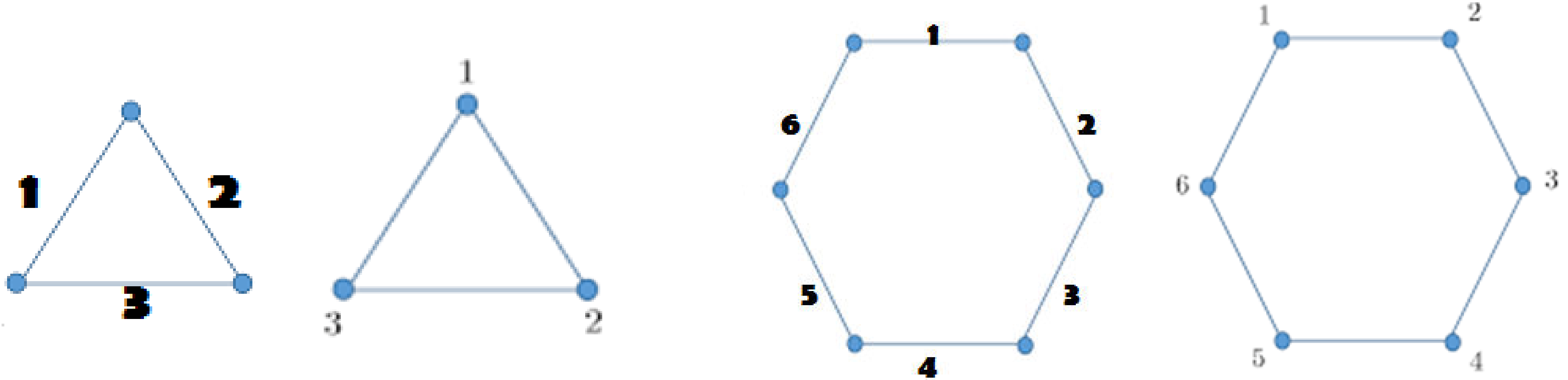
$= 11 – 5$

$= 6$ (Number of edges in L(G).

**Theorem 9.2** The edge graph of a graph $G$ is a path if and only if $G$ is a path.

**Proof** Let $G$ be a graph with $n$ vertices. Assume $G$ is a path $P_n$. Then $L(G)$ is the path $P_{n-1}$ with $n-1$ vertices.

Conversely, let $L(G)$ be a path. Then no vertex of $G$ has degree greater than two. For, if $G$ has a vertex $v$ of degree greater than two, the edges incident to $v$ form a complete subgraph of $L(G)$ with at least three vertices. Therefore $G$ is either a cycle or a path. But $G$ cannot be a cycle, since the edge graph of a cycle is a cycle.



**Theorem 9.3** A connected graph is isomorphic to its edge graph if and only if it is a cycle.

**Proof** Let $G$ be a connected graph with $n$ vertices, $m$ edges and with degree sequence $[d_i]_1^n$. Let $L(G)$ be the edge graph of $G$. The number of vertices in $L(G)$ is $m$. The number of edges $m'$ in $L(G)$ is given by

$$m' = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$

Clearly, $L(G)$ is connected and $L(C_n) = C_n$.

Conversely, let $G \cong L(G)$.

Then $G$ and $L(G)$ have the same number of vertices and edges.

So, $n = m$ and $m = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$

$$m = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$

$$m + m = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right)$$

$$\left(\sum_{i=1}^{n} d_i^2\right) = 4m$$

Therefore, $n = m$ and $\sum_{i=1}^{n} d_i^2 = 4m.$

Thus, variance

$$\{[d_i]\} = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2$$

$$\left[\text{Because } Var = \frac{1}{N}\sum_i f_i x_i^2 - \left(\frac{1}{N}\sum_i f_i x_i\right)^2 \text{ and we have } f_i = 1\right]$$

$$\left(\sum_{i=1}^{n} d_i^2\right) = 4m$$

$$= \frac{1}{n}4m - \frac{1}{n^2}(2m)^2 = \frac{4m}{m} - \frac{4m^2}{m^2} = 4 - 4 = 0.$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2 = \frac{1}{n^2}\left(\sum_{i=1}^{n} d_i\right)^2$$

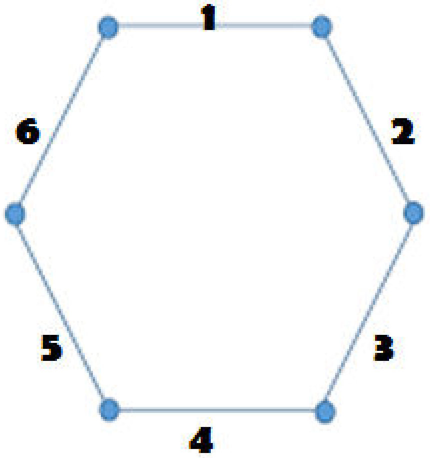
$$m = \frac{1}{2}\left(\sum_{i=1}^{n} d_i\right)$$

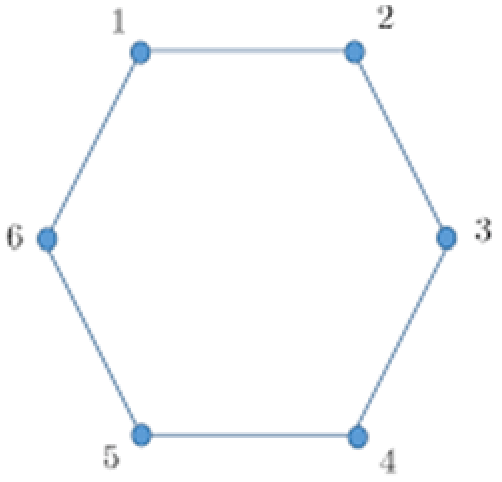
$$\left(\sum_{i=1}^{n} d_i\right) = 2m$$

Therefore the $d_i$'s are equal and $G$ is regular of degree $d$, say.

So $nd = 2m$ implies that $d = \frac{2m}{n} = \frac{2m}{m} = 2.$

Thus $G$ is a 2-regular connected graph, that is, $C_n$.

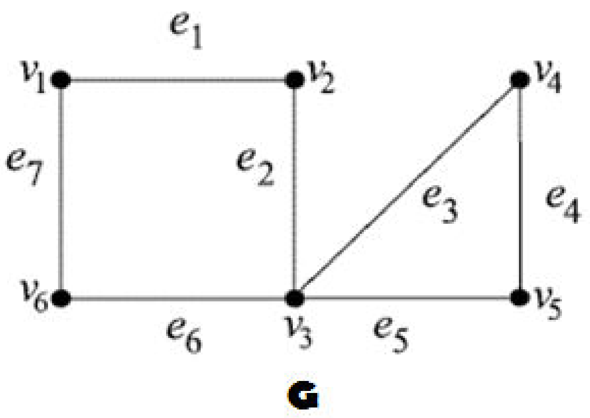


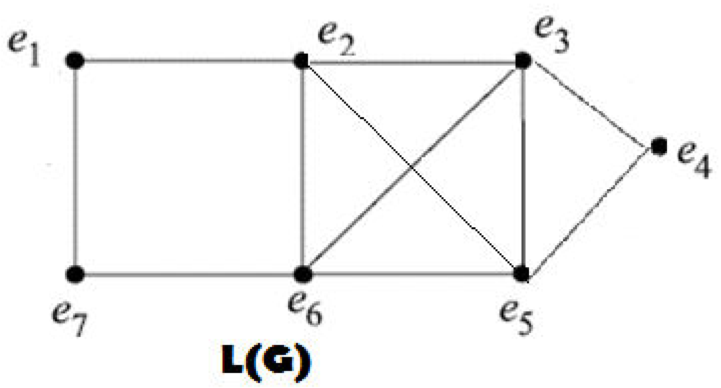



# Edge Graphs and Traversability

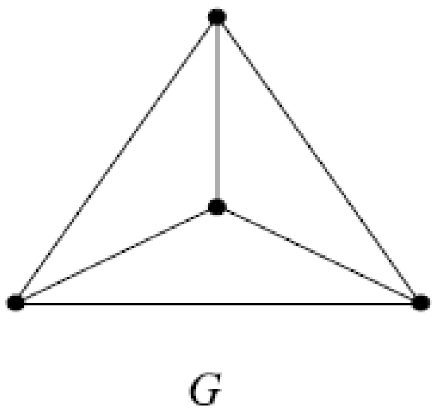
**Theorem 9.10** If $G$ is Eulerian, then $L(G)$ is both Eulerian and Hamiltonian.

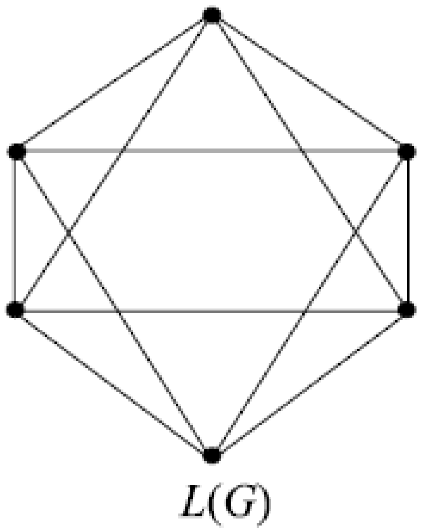
**Proof** Let $G$ be Eulerian and let $\{e_1, e_2, \ldots, e_m\}$ be the edge sequence of an Euler line in $G$. Let the edge $e_i$ in $G$ be represented by the vertex $v_i$ in $L(G)$, $1 \le i \le m$. Then $v_1 v_2 \ldots v_m v_1$ is a Hamiltonian cycle of $L(G)$. Now, if $e = u_i u_j \in E(G)$ and the vertex $v$ in $L(G)$ represents the edge $e$, then $d_{L(G)}(v) = d_G(u_i) + d_G(u_j) - 2$, which is obviously even and greater than or equal to two, since both $d_G(u_i)$ and $d_G(u_j)$ are even (and $\ge 2$). Thus in $L(G)$ every vertex is of even degree ($\ge 2$). Hence $L(G)$ is Eulerian.



# Edge Graphs and Traversability

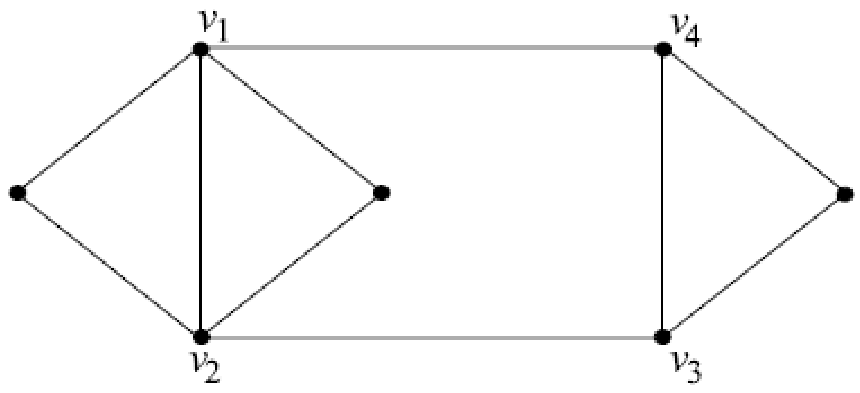



L(G) is both Eulerian and Hamiltonian, but G is not Eulerian.

# Dominating walk

- A dominating walk of a graph G is a closed walk W in G (which can be just a single vertex) such that every edge of G not in W is incident with W. For example, the walk $v_1$ $v_2$ $v_3$ $v_4$ in the graph is the dominating walk.



**Corollary 9.1** The edge graph of a Hamiltonian graph is Hamiltonian.

**Proof** Let $G$ be a Hamiltonian graph with Hamiltonian cycle $C$. Then $C$ is a dominating walk of $G$, and hence, $L(G)$ is Hamiltonian.

We note that the converse of Corollary 9.1 is not true in general. To see this, consider the graph $G$ as shown in Figure 9.16. Clearly $L(G)$ is Hamiltonian but $G$ is not.

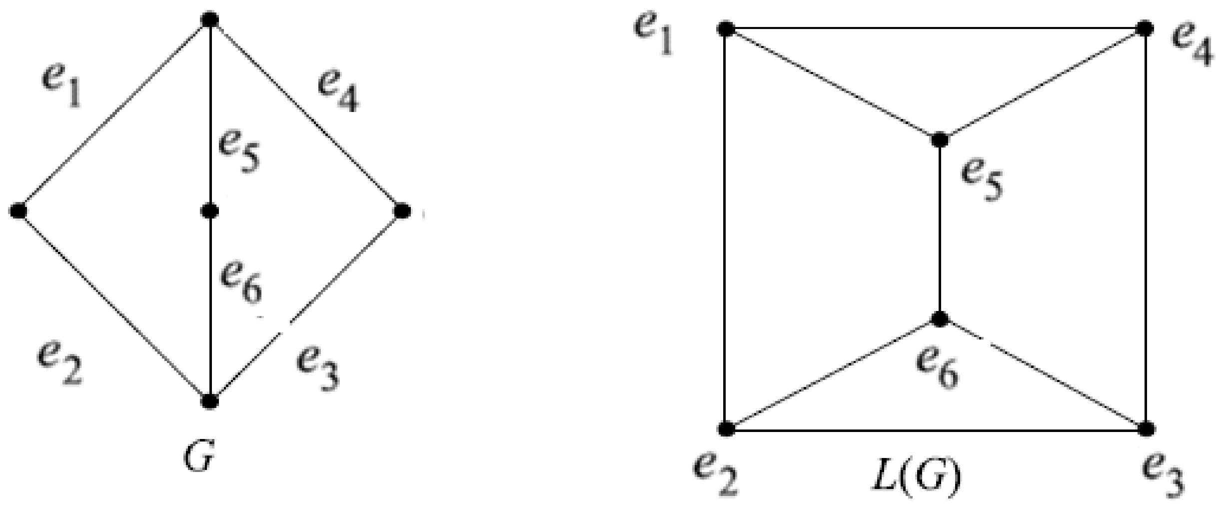



**Fig. 9.16**

# Eccentricity Sequences and Sets

**Eccentricity of a vertex:** Let $G$ be a connected graph. The eccentricity of a vertex $v$ in $G$ is the distance of the vertex $u$ farthest from $v$. It is denoted by $e(v)$. That is, $e(v) = \max\{d(u, v) : u \in V\}$.

The minimum eccentricity is called the *radius* of $G$ and the maximum eccentricity is called the *diameter* of $G$. The radius is denoted by $r$ and diameter by $d$.

Therefore, $r = \min\{e(v) : v \in V\}$ and $d = \max\{e(v) : v \in V\}$.

**Definition:** A positive sequence $[e_i]_1^n$ is called an *eccentricity sequence* if it is an eccentricity sequence of some graph. The graph is said to realise the sequence. A set of positive integers is called an *eccentricity set* if it is an eccentricity set of some graph. The graph is said to realise the set. (The set of distinct eccentricities in a graph is called the eccentricity set of that graph.)

**Theorem 9.18** If $uv$ is an edge of a connected graph $G$, then $|e(u) - e(v)| \leq 1$.

**Proof** Let $w$ be an eccentric vertex of $u$ (i.e., $w$ is the farthest vertex from $u$). Then by the triangle inequality for the metric $d$ (distance), we have

$$d(u, w) \leq d(u, v) + d(v, w)$$

so that $e(u) \leq d(u, v) + d(v, w)$. (9.18.1)

But $u$ and $v$ are adjacent, therefore $d(u, v) = 1$.

Also, $e(v) \geq d(v, w)$ so that $d(v, w) \leq e(v)$.

Thus, from (9.18.1) we have

$$e(u) \leq 1 + d(v, w) \text{ so that } e(u) \leq 1 + e(v).$$

Therefore, $e(u) - e(v) \leq 1$. (9.18.2)

Similarly, by considering an eccentric vertex of $v$, we have

$$e(v) - e(u) \leq 1. \tag{9.18.3}$$

From (9.18.2) and (9.18.3) it follows that

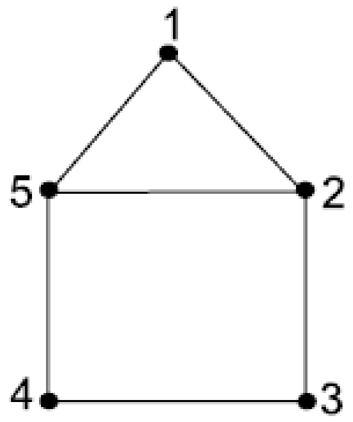
$$|e(u) - e(v)| \leq 1.$$

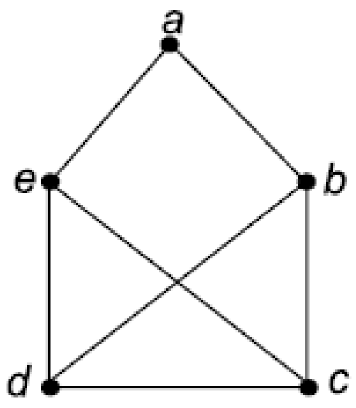
**Note** The above result shows that the eccentricities of two adjacent vertices are either equal or differ by 1 as $|e(u) - e(v)| \leq 1$ gives $|e(u) - e(v)| = 0$ or $|e(u) - e(v)| = 1$.

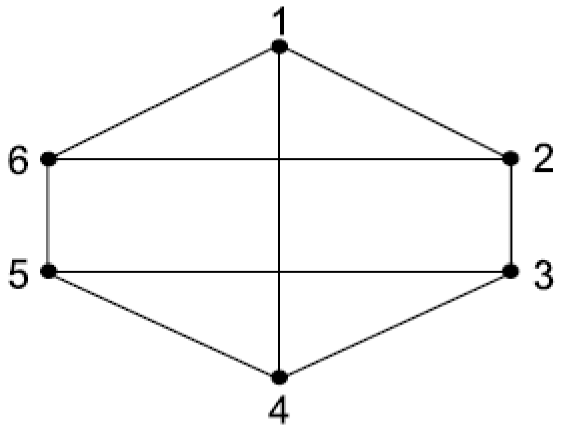
# Isometry

Let $G_1$ and $G_2$ be connected graphs with vertex sets $V_1$ and $V_2$ respectively. Then $G_2$ is said to be *isometric* from $G_1$ if for each $v \in V_1$, there is a one-one map $\phi_v : V_1 \to V_2$ such that $\phi_v$ preserves distances from $v$, that is $d_{G_2}(u,v) = d_{G_1}(\phi_v(v), \phi_v(u))$ for every $u \in V_1$.

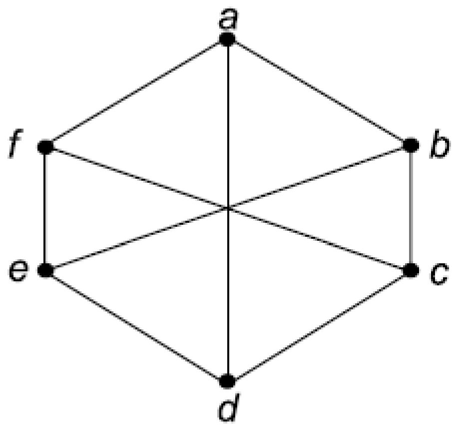
Two graphs $G_1$ and $G_2$ are said to be *isometric* if they are isometric from each other.




$G_1$




$G_2$

**Theorem 9.23** If $G_1$ and $G_2$ are $k$-regular graphs of order $n$, where $k \geq n-1/2$, then $G_1$ and $G_2$ are isometric.

**Proof** Since $G_1$ is a $k$-regular graph with $k \geq n-1/2$, $d(G_1) \leq 2$.
Let $u \in V(G_1)$ and $v \in V(G_2)$ be any two vertices and define

$$\phi_u : V(G_1) \to V(G_2) \text{ by } \phi_u(u) = v.$$

For $i = 1, 2, \ldots, k$, let $u_i \in N_1(u)$ and $v_i \in N_1(v)$ and define $\phi_u(u_i) = v_i$.

For $i = k+1, \ldots, n-1$, let $u_i \in N_2(u)$ and $v_i \in N_2(v)$ and again let $\phi_u(u_i) = v_i$.

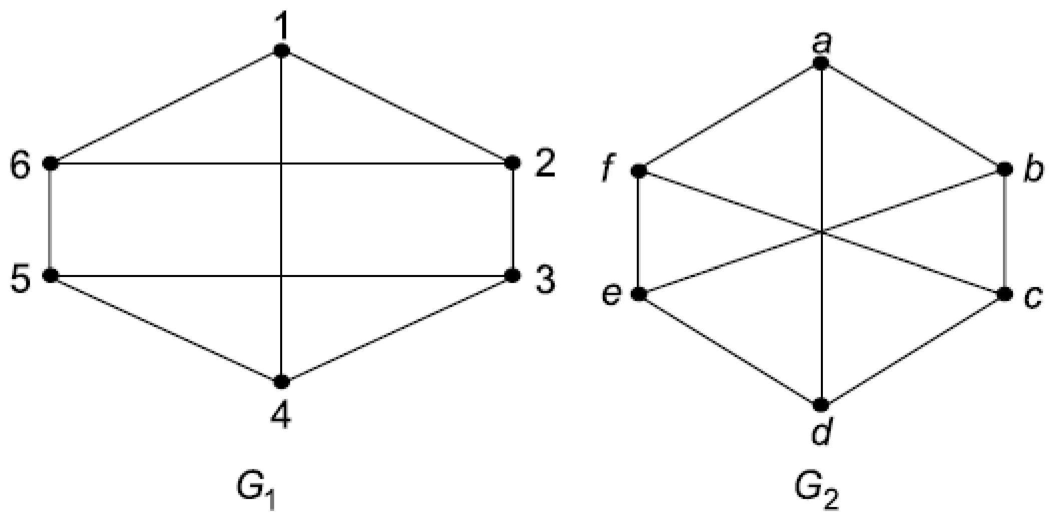
The neighbourhoods are in the appropriate graphs. Then $\phi_u$ is an isometry of $G_2$ from $G_1$ at $u$. Since $u$ and $v$ are arbitrary, it is easily seen that $G_2$ is isometric from $G_1$, and $G_1$ is isometric from $G_2$.

**Theorem 9.24** A necessary condition for two graphs to be isometric is that they have the same degree set and the same eccentricity set.

**Proof** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isometric graphs. As $G_2$ is isometric from $G_1$, let $\phi_v$ be the one-one mapping from $V(G_1)$ to $V(G_2)$. Therefore, $d(v|G_1) = d(\Phi_v(v)|G_2)$. Also $\phi_v$ has the property of preserving distance, therefore $e(v|G_1) = e(\phi_v(v)|G_2)$. So the eccentricity set of $G_1$ is included in the eccentricity set of $G_2$.

Again, as $G_1$ is isometric from $G_2$, therefore, the degree set and eccentricity set of $G_2$ are included respectively in the degree set and eccentricity set of $G_1$.

Hence the degree sets are equal in $G_1$ and $G_2$ and the eccentricity sets are equal in $G_1$ and $G_2$.



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Thank you.