# Unit -3

# LAPLACE TRANSFORMS

## LAPLACE TRANSFORM

***Definition:***

**Let f(t) be a function of t, defined $\forall$ t≥0. If the integral**

**$\infty$ $^{-st}$ f(t) dt exists, then it is called the Laplace Transform of**

$_0$? $e$

**f(t ) and it is denoted by L{f(t)} or f(s).**

**Here s is parameter, real or complex.L is called Laplace Transform operator.**

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

## *Def: Piece-wise Continuous Function:*

**Afunction is said to be piece-wise continuous (or) Sectionally Continuous) over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided into a finite number of sub intervals, in each of which f(t) is continuous and both right and left hand limits at every end point if the sub intervals.**

## Def:Functions of Exponential Order:

**A function f(t) is said to be of exponential order as t →∞ if**

$$\lim_{t\to\infty}(e)^{-at} f(t) = finite\ quantity$$

***(or)***

**If for a given positive integer T, ∃ a positive number M Such that** $|f(t)| < Me^{at} \quad \forall\, t \geq T,$

**Sufficient Conditions for existence of Laplace Transform are 1)**

**f(t) is Piece-wise Continuous Function in [a, b] where a>0, 2)**

**f(t) is of Exponential Order function.**

## Linear Property:

**Theorem:** If $c_1$, $c_2$ are constants and $f_1$, $f_2$ are functions of t, then

$L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$

**Proof:** The definition of Laplace Transform is

$$L[f(t)]=\int_0^\infty e^{-st} f(t)\, dt \text{ -----(1)}$$

By definition

$$L[c_1 f_1(t) + c_2 f_2(t)]=\int_0^\infty e^{-st} [c_1 f_1(t)+ c_2 f_2(t)]\, dt$$

$$=\int_0^\infty e^{-st} c_1 f_1(t)\, dt + \int_0^\infty e^{-st} c_2 f_2(t)\, dt$$

$$= c_1\int_0^\infty e^{-st} f_1(t)\, dt + c_2\int_0^\infty e^{-st} f_2(t)\, dt$$

$=c_1 L[f_1(t)] + c_2 L[f(t)]$

## Laplace Transform (L.T) of some Standard Functions:

**1)Show that L{1}=** $\frac{1}{s}$

**Solution: By definition of L.T L[f(t)]= f(t)** $\int_0^\infty e^{-st}$ **dt----------(1)**

**Put f(t)=1 o.b.s** **L[1] =** $\int_0^\infty e^{-st}$ **.1. dt**

$-1 = =$ ___ **(0-1) =** $\left[ \underline{e^{-st}} \right]^\infty$

**1/s**

$-s \quad s$ **0**

**2) L[c] = L[c.1] =c. L[1]= c.(1/s) = c/s**

**3) Show that** $L[e^{at}] = \frac{1}{s-a}$

**Solution: By definition of L.T ,** $L[f(t)] = \int_0^\infty e^{-st} f(t)$ **dt----------(1)**

**L[f(t**

$e^{at}] = \int_0^\infty e^{-st} e^{at}$ **dt**

$= \int_0^\infty e^{-(s-a)T}$ **dt**

$= \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty \quad (e^{-\infty} = 0)$

$= \frac{1}{s-a}$

Put f(t) = $e^{at}$ o.b.s in (1) L[

Note: L[$e^{-at}] = \frac{1}{s+a}$

4) Show that L[ Cos at]= $\frac{s}{s^2+a^2}$ and L[ Sin at] = =$\frac{a}{s^2+a^2}$

Solution: W.k.t $e^{i\theta}$ = cos $\theta$ + i sin$\theta$

$e^{iat}$= cos at + i sin at

L[$e^{iat}$] = L[cos at + i sin at ]

L[cos at + i sin at]= L[$e^{iat}$]

$$=\frac{1}{s-ia} \qquad (L\,[e^{at}]=\frac{1}{s-a})$$

$$=\frac{s+ia}{(s-ia)(s+ia)}$$

$$=\frac{s+ia}{s^2+a^2}$$

$$=\frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

Equte real and imaginary parts we get

L[ Cos at]= $\frac{s}{s^2+a^2}$ and L[ Sin at] = =$\frac{a}{s^2+a^2}$

5) Find L [ Sin hat ]

Solution: L [ Sin hat ] = L [$\frac{e^{at}-e^{-at}}{2}$] = ½ [ L {$e^{at}$} –L {$e^{-at}$} ]

= ½ [ $\frac{1}{s-a} - \frac{1}{s+a}$ ]

= ½ [ $\frac{s+a-s+a}{s^2-a^2}$ ]

= $\frac{a}{s^2-a^2}$

6) **Find L [ Cos hat ]**

Solution: L [ Cos hat ] = L [$\frac{e^{at}+e^{-at}}{2}$] = ½ [ L {$e^{at}$} +L {$e^{-at}$} ]

= ½ [ $\frac{1}{s-a} + \frac{1}{s+a}$ ]

= ½ [ $\frac{s+a+s-a}{s^2-a^2}$ ] = = $\frac{s}{s^2-a^2}$

7) **Show that** (i) ) L [$t^n$] = $\rho$(n+1)/$s^{n+1}$, n>-1

(ii) L [$t^n$] = n!/$s^{n+1}$, n is +ve integer

**Solution: : By definition of L.T**

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \text{--------}(1)$$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n dt \qquad \text{put } st = x \quad \text{i.e } t = x/s$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} \qquad dt = \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x}\, x^n dx$$

$$= \frac{1}{s^{n+1}}\ \rho(n+1), \quad \text{for } (n+1) > 0$$

$$L[t^n] = \rho(n+1)/s^{n+1}, \quad n > -1$$

$$L[t^n] = n!/s^{n+1}, \quad n \text{ is +ve integer}$$

# FORMULAE

1) $L\{1\} = \frac{1}{s}$

2) $L\{c\} = \frac{c}{s}$

3) $L[e^{at}] = \frac{1}{s-a}$, $L[e^{-at}] = \frac{1}{s+a}$

$s$

4) L[ Cos at]= $\overline{s^2+a^2}$

$a$

5) L[ Sin at] $\overline{s^2+a^2}$ =

$a$

$\overline{s^2-a^2}$

6) L[ Sin hat] $s$ =

$\overline{s^2-a^2}$

7) L[ Cos hat]=

8) $L(t^n)=\rho(n+1)/s^{n+1}$, n>-1

9) $L(t^n)=$ n!$/s^{n+1}$, n is +ve integer

## PROBLEMS

1.Find the Laplace Transformation (L.T) of $t^2 + 2t + 3$

Solution: L $[t^2 + 2t + 3]$ = L$[t^2] + 2L[t] + L[3]$

$= \frac{!}{s^3} + 2. \frac{}{s^2} + \frac{}{s}$ 2 1 3

2. Find $L[t^{\frac{5}{2}} + 4]$

Solution: $L[t^{\frac{5}{2}} + 4] = L[t^{\frac{5}{2}}] + L[4]$

$$= \frac{\rho(\frac{7}{2})}{s^{7/2}} + \frac{4}{s}$$

3. Find $L[e^{3t} + 3e^{-2t}]$

Solution: $L[e^{3t} + 3e^{-2t}] = L[e^{3t}] + 3L[e^{-2t}]$

$$= \frac{1}{s-3} + 3\frac{1}{s+2}$$

4. Find $L[\text{Sin } 3t + Cos^2 2t]$

Solution: $L[\text{Sin } 3t + Cos^2 2t] = L[\text{Sin } 3t] + L[Cos^2 2t]$

$$= \frac{3}{s^2+9} + L[\frac{1 + Cos\,4t}{2}]$$

$$= \frac{3}{s^2+9} + \frac{1}{2}\{ L[1] + L[\text{Cos } 4t] \}$$

$$= \frac{3}{s^2+9} + \frac{1}{2}[\frac{1}{s} + \frac{s}{s^2+16}]$$

5. Find $L[f(t)]$ if $f(t) = 0, \quad 0 < t < 2$

$= 3, \quad t > 2$

Solution: By definition of L.T

$\infty$ $^{-st}$ f(t) dt

L[f(t)]= $_0$ ∫ $e$

= $\int_0^2 e^{-st}$ $\infty$ $^{-st}$ f(t) dt

f(t) dt + $_2$∫ $e$

= 0 + ∫$_2$ $e^{-st}$ .3. dt

= 3 $\left(\frac{e^{-st}}{-s}\right)^{\infty}$

$\infty$

2

$e_{-\infty}$= 0

$e{-2s}$

= 3

$s$

# First shifting Theorem (F.S.T):

If L[f(t)]=f (s) then L[$e^{at}$ f(t)]= f(s-a)

Proof : By definition of L.T

L[f(t)]= $\int_0^{\infty} e^{-st}$ f(t) dt = f(s)--------(1)

L[$e^{at}$f(t)]= $\int_0^{\infty} e^{-st}\ e^{at}$ f(t)dt

= $\int_0^{\infty} e^{-(s-a)t}$ f(t) dt Put s-a=p = $\int_0^{\infty} e^{-pt}$ f(t) dt

= f(p) = f(s-a)

Note: L[$e^{-at}$ f(t)] = f(s+a)

# Problems:

1) Find L[t³ $e^{-3t}$]

Solution : let f(t) = t³

L[ f(t)] = L[ t³] $=\frac{3!}{s^{3+1}} = \frac{6}{s^4}$ = f(s)

By F.S.T , L[$e^{-at}$ f(t)] = f(s+a) a=3 L[$e^{-3t}$ f(t)] = f(s+3)

L[$e^{-3t}$ t³] = $\frac{6}{(s+3)^4}$

2) Find L $[e^{-t}(3\sin 2t - 5\cosh 2t)]$

Solution : Let f(t) = (3 sin 2t – 5 cosh 2t) L

[f(t)] = L[(3 sin 2t – 5 cosh 2t)]

$$= 3\frac{2}{s^2+4} - 5\frac{s}{s^2-4} = f(s)$$

By F.S.T , $L[e^{-at} f(t)] = f(s+a)$ a=1

$L[e^{-1t} f(t)] = f(s+1)$

$$= \frac{6}{(s+1)^2+4} - \frac{5(s+1)}{(s+1)^2-4}$$

$$L[e^{-t}(3\sin 2t - 5\cosh 2t)] = \frac{6}{s^2+2s+5} - \frac{5s+5}{s^2+2s-3}$$

## Second Shifting Theorem (S.S.T)

STATEMENT:- If L[f(t)]=f(s) and g(t)=f(t-a), t>a

= 0, t<a then $L\{g(t)\}=e^{-as} f(s)$

PROOF:- By definition of L.T

L[f(t)]= $\int_0^\infty e^{-st}$ f(t) dt = f(s)--------(1)

$\int_0^\infty e^{-st}$ L[g(t)]= g(t) dt = $= \int_0^u e^{-st}$ g(t)

dt $+ \int_a^\infty e^{-st}$ g(t) dt

$= 0 + \int_a^\infty e^{-st}$ f(t -a) dt put t-a=x $= \int_0^\infty e^{-s(a+x)}$ f(x) dx

t=a+x

$= e^{-as} \int_0^\infty e^{-sx}$ f(x) dx dt=dx, (x=0 to ∞)

$= e^{-as}$ f(s)

**Example :**

Find Laplace Transform of g( t ) = cos( t - $\frac{2\pi}{3}$ ) , if t > $\frac{2\pi}{3}$

= 0, if t < $\frac{2\pi}{3}$

Solution: Let f ( t ) = cos t , a = $\frac{2\pi}{3}$

f ( t-a ) = cos ( t-a ) f ( t - $\frac{2\pi}{3}$ ) = cos ( t - $\frac{2\pi}{3}$ )

L [f(t)] = L [ cos t ]= $\frac{s}{s^2+1}$ = f(s)

By S.S.T  L [g(t)] = $e^{-as}$ f(s)

= $(e^{-\frac{2\pi}{3}s})\frac{s}{s^2+1}$

## Change of scale property:

If L[f(t)] = f(s) then L [f(at)] = $\frac{1}{a}$ f( $\frac{s}{a}$ )

***NOTE***: L [f( $\frac{t}{a}$ )] = a f(as)

**Example**: If L [f(t)] = $\frac{9s^2-12s+15}{(s-1)^3}$ then find L [f(3t)]

**Solution:** Given $\frac{9s^2-12s+15}{(s-1)^3}$ = f(s)

L[f(t)] = by Change of scale property, L [f(at)] = $\frac{1}{a}f(\frac{s}{a})$

L [f(3t)] = $\frac{1}{3}f(\frac{s}{3})$

$$= \frac{1}{3}\left[\frac{9(\frac{s}{3})^2-12\left(\frac{s}{3}\right)+15}{(\frac{s}{3}-1)^3}\right]$$

$$= \frac{1}{3}\left[\frac{s^2-4s+15}{(s-3)^3/27}\right]$$

$$= \frac{9(s^2-4s+15)}{(s-3)^3}$$

## *Laplace transformof the derivative of f(t)*

- If f(t)is continousfor all t ≥ 0 and f ′(t) is piecewisecontinous, then

L{f ′(t)} exists,provided $\lim e^{-st}f(t) = 0$ and

$L\{f'(t)\} = sL\{f(t)\} - f(0) = s\bar{f}(s) - f(0)$

$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots f^{n-1}(0)$

**Example** Derivelaplace transformof sin at

Let f(t) sinat thenf'(t) = a cosat and f''( t) -a sinat

Also f(0) = 0, f' (0) = a from this also f''(0) = 0, also from this

By derivative formula,

L[f''(t)] = $s^2$ L[f(t)] – s f(0) – f'(0)-------(1)

L{-$a^2$sinat} $s^2$ L(sin at)–a

$(-a^2)$ L(Sin at) + a = $s^2$ L(sin at) a =

$(s^2 + a^2)$ L(sin at)

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

## Laplace transform of the integration of f(t)

If L[f(t)]=f(s) then $L[\int_0^t f(t)dt] = \frac{f(s)}{s}$

## Example:

Find L.T. of $\int_0^t \sin at\, dt$ Solution:

Let

f(t) =

sin at

$L[f(t)] = L[\sin at] = \frac{a}{s^2+a^2}$

$\int_0^t f(t)dt] = \frac{f(s)}{s}$

= f(s)

L[

$\int_0^t \sin at\, dt = \frac{1}{s}\left(\frac{a}{s^2+a^2}\right)$

**Multiplication by t :**

**If L[f(t)]=f(s) then L[t f(t)] = -** $\frac{d}{ds}$ **[ f(s)]**

**L[$t^2$ f(t)] =** $(-1)^2 \frac{d^2}{ds^2}$ **[f(s)]**

**L[$t^n$ f(t)] =** $(-1)^n \frac{d^n}{ds^n}$ **[f(s)]**

**Example :** Find L[t $sin^2 t$]

Solution: Let f(t) = $sin^2 t$

let L[f(t)] = L[ $sin^2 t$] = L[$\frac{1-\cos 2t}{2}$]

$\frac{1}{2}$ ( L[1] – L[ Cos 2t] ) = $\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \frac{2}{s(s^2+4)}$ = f(s)

By theorem L[t f(t)]

$$= -\frac{d}{ds}[\, f(s)]$$

$$= -\frac{d}{ds}\left[\frac{2}{s(s^2+4)}\right]$$

$$= -2\left[\frac{-1}{\{s(s^2+4)\}^2}\right]\frac{d}{ds}(s(s^2+4))$$

$$= \left[\frac{2}{\{s(s^2+4)\}^2}\right]\frac{d}{ds}(s^3+4s)$$

$$= \left[\frac{2}{\{s(s^2+4)\}^2}\right](3s^2+4)$$

$$= \frac{6s^2+8}{s^2(s^2+4)^2}$$

*Division by t:*

If L[f(t)]=f(s) then L[ $\frac{f(t)}{t}$] = $\int_s^{\infty} f(s)ds$ , provided $\lim_{t\to 0}\frac{f(t)}{t}$ exists.

**Problems:** (1) Find L[ $\frac{e^{-3t} - e^{-4t}}{t}$ ]

Solution: Let f(t) = $e^{-3t} - e^{-4t}$

L[f(t)] = L[$e^{-3t} - e^{-4t}$] = $\frac{1}{S+3}$ - $\frac{1}{S+4}$ = f(s) w.k.t

, L[ $\frac{f(t)}{t}$] = $\int_s^{\infty} f(s)ds$ ,

L[ $\frac{e^{-3t} - e^{-4t}}{t}$] = $\int_s^{\infty} \left(\frac{1}{S+3} - \frac{1}{S+4}\right) ds$

$= \left[\log (s+3) - \log (s+4)\right]_s^{\infty}$

$= \left[\log \left(\frac{s+3}{S+4}\right)\right]_s^{\infty} = \left[\log \frac{s\left(1+\frac{3}{s}\right)}{s\left(1+\frac{4}{s}\right)}\right]_s^{\infty}$

= log 1 - log ( $\frac{s+3}{S+4}$)

= 0 - log ( $\frac{s+3}{S+4}$) = log ( $\frac{s+4}{S+3}$)

(2). Find L.T of $\frac{\cos at - \cos bt}{t}$

Solution: Let f(t) = cos at – cos bt

L[f(t)] = L[cos at – cos bt]

$$f(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

w.k.t , $L[\frac{f(t)}{t}] = \int_s^\infty f(s)ds$

$$\frac{\cos at - \cos bt}{t}] = \int_s^\infty \left( \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds \; L[$$

∞

1

$^2 + a^2) - \log(s^2 + b^2)]$

= [ log ($s$

2

∞ s

1 = ss 2 2 + + a b 2 2

) log ( 2

s

$$= \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2}\right)$$

## Evaluation of integrals by Laplace transforms:

(1). Using L.T. Evaluate $\int_0^\infty \left[ \frac{e^{-t} - e^{-2t}}{t} \right] dt$

Solution: First we will find $L\left[\frac{e^{-t} - e^{-2t}}{t}\right]$ let

$f(t) = e^{-t} - e^{-2t}$

$L[f(t)] = L[e^{-t} - e^{-2t}]$

$= \frac{1}{S+1} - \frac{1}{S+2} = f(s)$

w.k.t, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds$,

$L\left[\frac{e^{-t} - e^{-2t}}{t}\right] = \int_s^\infty \left(\frac{1}{S+1} - \frac{1}{S+2}\right) ds$

$= \left[\log(s+1) - \log(s+2)\right]_s^\infty = \log\left(\frac{s+1}{S+2}\right)\Big|_s^\infty$

$= \log$

s

$= 0 \left( \frac{s(1+\frac{1}{s})}{s(1+\frac{2}{s})} \right) = \log 1 - \log \left( \frac{s+1}{s+2} \right)$

$- \log \left( \frac{s+1}{s+2} \right) = \log \left( \frac{s+2}{s+1} \right)$

therefore, $L\left[ \frac{e^{-t} - e^{-2t}}{t} \right] = \log \left( \frac{s+2}{s+1} \right)$

The definition of Laplace Transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\left[ \frac{e^{-t} - e^{-2t}}{t} \right] = \int_0^\infty e^{-st} \left[ \frac{e^{-t} - e^{-2t}}{t} \right] dt = \log \left( \frac{s+2}{s+1} \right)$$

Put s=0 on both sides

$$\int_0^\infty 1 \left[ \frac{e^{-t} - e^{-2t}}{t} \right] dt = \log \left( \frac{2}{1} \right) = \log 2$$

**2. Using LT find** $\int_0^\infty \left( \frac{\cos at - \cos bt}{t} \right) dt$

**Solution: First we find** $L\left[ \frac{\cos at - \cos bt}{t} \right]$

**: Let f(t) = cos at – cos bt**

**L[f(t)] = L [cos at – cos bt] f(s)**

$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

w.k.t , L [ $\frac{f(t)}{t}$] = $\int_s^{\infty} f(s)ds$

L [ $\frac{\cos at - \cos bt}{t}$ ] = $\int_s^{\infty} \left( \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right) ds$

$= \frac{1}{2}\left[ \log (s^2+a^2) - \log (s^2+b^2) \right]_s^{\infty}$

$= \frac{1}{2} \log \left( \frac{s^2+a^2}{s^2+b^2} \right)_s^{\infty}$

$= \frac{1}{2} \log \left( \frac{s^2+b^2}{s^2+a^2} \right)$

By definition of LT, $\int_0^{\infty} e^{-st} \left( \frac{\cos at - \cos bt}{t} \right) dt = \frac{1}{2} \log \left( \frac{s^2+b^2}{s^2+a^2} \right)$

Put s=0 o.b.s $\int_0^{\infty} \left( \frac{\cos at - \cos bt}{t} \right) dt = \frac{1}{2} \log \left( \frac{b^2}{a^2} \right)$

$= \log\sqrt{\left( \frac{b^2}{a^2} \right)} = \log (b/a)$

3. ST $\int_0^{\infty} \left( \frac{\cos 5t - \cos 3t}{t} \right) dt = \log (3/5)$ Note: put a=5, b=3 in above problem

# Laplace Transform of Periodic Function:

***Definition :*** A function f(t) is said to be periodic with period T , if

$\forall t$ , f(t+T) = f(t) where T is positive constant.

The least value of T > 0 is called the periodic function of f(t).

Example: sin t $= \sin(2\pi + t) = \sin(4\pi + t) = -----$ Here sint is periodic function with period $2\pi$.

Formula :- If f(t) is periodic function with period T $\forall t$ *then*

$$L[f(t)] = \frac{1}{1-e^{-st}} \int_0^T e^{-st} f(t)\, dt$$

Problem : Find the L. T of the function f(t) = $e^t$, 0< $t$ <5 and f(t)=f(t+5)

$$\frac{1}{1-e^{-s5}} \int_0^5 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s5}} \int_0^5 e^{-st}\, e^t\, dt$$

Solution : Here T=5 $L[f(t) = \frac{1}{1-e^{-5s}} \left[\frac{e^{(1-s)t}}{1-s}\right] = \frac{1}{1-e^{-5s}} \left[\frac{e^{5(1-s)}}{1-s}\right]$

## *The unit step function or Heaviside's unit function :*

**It is denoted by u(t-a) or H(t-a) and is defined as H(t-a) = 0, t<a**

**=1, t>a** **L.T. of unit step function:**

**Prove that** $L[H(t-a)] = \frac{e^{-as}}{s}$

***Solution :*** L[H(t-a)] = $\int_0^\infty e^{-st}$ H(t-a) dt

$$= \int_0^a e^{-st} \text{H(t-a) dt} + \int_a^\infty e^{-st} \text{H(t-a) dt}$$

$$= \int_0^a 0 + \int_a^\infty e^{-st} .1 \text{ . dt}$$

$$= \left(\frac{e^{-st}}{-s}\right)$$

$$= \left(\frac{e^{-sa}}{s}\right)$$

# ***Inverse Laplace Transform :***

Definition : If f(s) is the Laplace Transform of f(t) then f(t) is called the inverse Laplace Transform of f(s) and is denoted by $L^{-1}[f(s)]$ . i.e., f(t) = $L^{-1}[f(s)]$

$L^{-1}$ is called inverse Laplace Transform operator, but not reciprocal.

Example : If L$[e^{at}] = \frac{1}{s-a}$ then $e^{at} = L^{-1}[\frac{1}{s-a}]$

***Linear Property :***

If $f_1(s)$ and $f_2(s)$ are L.T. of $f_1(t)$ and $f_2(t)$ respectively then

$L^{-1}[c_1 f_1(s) + c_2 f_2(s)] = c_1 L^{-1}[f_1(s)] + c_2 L^{-1}[f_2(s)]$ where $c_1$

, $c_2$ constants.

## ***Standard Formulae :***

1 $\Rightarrow$ $L^{-1}[\frac{1}{s}] = 1$

**(1) L**

**(2)** L $[e^{at}] = \frac{1}{s-a}$ **[1] =** $\Rightarrow L^{-1}[\frac{1}{s-a}] = e^{at}$

**(3)** L $[e^{-at}] = \frac{1}{s+a}$ $s$ $\Rightarrow L^{-1}[\frac{1}{s+a}] = e^{-at}$

$a$

**(4)** **L [sin at] =** $\overline{s^2+a^2}$ $\Rightarrow L^{-1}[\frac{1}{s^2+a^2}] = \frac{1}{a}$ **sin at**

**(5)** **L [ Cos** $\frac{s}{s^2+a^2}$ **at** $\Rightarrow L^{-1}[\frac{s}{s^2+a^2}] =$ **cos** **at]=**

**5)** **L [ Sin hat]** $\frac{a}{s^2-a^2}$ $\Rightarrow L^{-1}[\frac{1}{s^2-a^2}] = \frac{1}{a}$ **sinh** **= at**

**6)** **L [ Cos** $\frac{}{s^2-a^2}^{s}$ $\Rightarrow L^{-1}[\frac{s}{s^2-a^2}$ **hat]= ] = cosh at**

$n$

7) $L(t^n)=\rho(n+1)/s^{n+1}$, $n > -1$ $\Rightarrow L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{\rho(n+1)}$

8) $L(t^n)= n!/s^{n+1}$, n is +ve integer $\Rightarrow L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$

**Problems:**

(1) Find $L^{-1}[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}]$

solution : $L^{-1}[\frac{1}{s^2}] + L^{-1}[\frac{1}{s+4}] + L^{-1}[\frac{1}{s^2+4}] + L^{-1}[\frac{s}{s^2-9}]$

$= t + e^{-4t} + \frac{1}{2}\sin 2t + \cosh 3t.$

(2) Find $L^{-1}[\frac{1}{s^2+25}]$

solution : $L^{-1}[\frac{1}{s^2+25}] = L^{-1}[\frac{1}{s^2+5^2}] = \frac{1}{5}\sin 5t$

(3) Find $L^{-1}[\frac{1}{2s-5}]$

solution : $L^{-1}\left[\frac{1}{2s-5}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s-5/2}\right] = \frac{1}{2} e^{\frac{5}{2}t}$ solution :

(4) Find $L^{-1}\left[\frac{2s+1}{s(s+1)}\right]$

$$L^{-1}\left[\frac{2s+1}{s(s+1)}\right] = L^{-1}\left[\frac{s+s+1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s+1} + \frac{1}{s}\right] = e^{-t} + 1$$

(5) Find $L^{-1}\left[\frac{3s-8}{4s^2+25}\right]$

] }

solution: $L^{-1}\left[\frac{3s-8}{4s^2+25}\right] = ¼\, L^{-1}\left[\frac{3s-8}{s^2+25/4}\right]$ $\frac{5}{2}$ t - = ¾ Cos

= ¼ { $3L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right]$ $\frac{5}{2}$ t - - 8 $L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right]$ (¼ x 8 x t

a=5/2 $\frac{2}{}$5 )

Sin $\frac{5}{}$

2 5

= ¾ Cos 4/5 Sin t

2

## FIRST SHIFTING THEOREM OF INVERSE L.T:

If $L^{-1}$[ f(s) ] = f(t) then $L^{-1}$[ f(s-a) ] = $e^{at} f\,t()$

= $e^{at} L^{-1}$[ f(s) ]

**PROOF**: By definition of L.T

L[f(t)]= $\int_0^\infty e^{-st}$ f(t) dt = f(s)--------(1)

f(t)]= $\int_0^\infty e^{-st}\ e^{at}$ f(t)dt

= $\int_0^\infty e^{-(s-a)t}$

**L[$e^{at}$**

**f(t) dt Put s-a=p** = $\int_0^\infty e^{-pt}$ **f(t)**

**dt**

**= f(p) = f(s-a)**

**L[$e^{at}$f(t)]= f(s-a)**

$\Rightarrow$ $L^{-1}$[ f(s-a) ] = $e^{at} f(t)$ **(or)** $L^{-1}$[ f(s-a) ] = = $e^{at} L^{-1}$[ f(s) ]

***Note:*** $L^{-1}$[ f(s+a) ] = = $e^{-at} L^{-1}$[ f(s) ]

**PROBLEMS**

$L^{-1}[\frac{s+3}{(s+3)^2+8^2}]$ **1) Find**

**Solution** $L^{-1}[\frac{s+3}{(s+3)^2+8^2}] = e^{-3t} L^{-1}[\frac{s}{s^2+8^2}$ : ] **by F.S.T**

= $e^{-3t}$

**Cos 8t.** $L^{-1}[\frac{1}{s^2+2s+5}]$

$L^{-1}[\frac{1}{s^2+2s+5}] = L^{-1}[\frac{1}{(s+1)^2+4}] = e^{-t} L^{-1}[\frac{1}{s^2+2^2}] = e^{-t}$

$L^{-1}[\frac{1}{(s+1)^2}]$

$L^{-1}[\frac{1}{(s+1)^2}] = L^{-1}[\frac{1}{(s+1)^2}] = e^{-t} L^{-1}[\frac{1}{s^2}] = e^{-t}$ **t**

**2) Find**

**Solution :** **½ Sin 2t**

**3) Find**

**Solution :**

**4) Find Inverse L.T of** $\frac{s}{(s+3)^2}$

$$L^{-1}[\frac{s}{(s+3)^2}] = L^{-1}[\frac{s+3-3}{(s+3)^2}] = e^{-3t} L^{-1}[\frac{s-3}{s^2}]$$

**Solution :**

$$= e^{-3t} \{ L^{-1}[\frac{1}{s}] - 3 L^{-1}[\frac{1}{s^2}] \} = e^{-3t}(1\text{-}3t)$$

$$L^{-1}[\frac{s+3}{s^2-10s+29}]$$

$$L^{-1}[\frac{s+3}{s^2-10s+29}] = L^{-1}[\frac{s+3}{(s-5)^2+4}] = L^{-1}[\frac{(s-5)+5+3}{(s-5)^2+4}]$$

$$= e^{5t} L^{-1}[\frac{s+8}{s^2+4}$$

$$= e^{5t} \{L^{-1}[\frac{s}{s^2+4}] + 8 L^{-1}[\frac{1}{s^2+4}]\}$$

$$= e^{5t} \{L^{-1}[\frac{s}{s^2+2^2}] + 8 L^{-1}[\frac{1}{s^2+2^2}]\}$$

**5) Find**

**Solution :**

] **(By F.S.T)**

= $e^{5t}$[ Cos 2t + 8 x ½ x Sin 2t ]    a=2

=$e^{5t}$

***SECOND SHIFTING THEOREM:***    [ Cos 2t + 4 Sin 2t ]

If $L^{-1}$[ f(s) ] = f(t) then $L^{-1}$[ $e^{-as}$ f(s) ] = g $(t)$ *where g(t) = f(t-a), t>a*

*=0,    t<a*

*Proof: By S.S.T of L.T , L [g(t)] =* $e^{-as}$ f(s)  (write proof of SST)

$\Rightarrow L^{-1}$[ $e^{-as}$ f(s) ] = g $t()$

$\Rightarrow L^{-1}$[ $e^{-as}$ f(s) ] = *f(t-a), t>a*

*=0,    t<a* ***Note:***

***We can also written as*** $L^{-1}$[ $e^{-as}$ f(s) ] = *f(t-a)* H(t-a)

**Problem:**

Find $L^{-1}[\frac{e^{-\pi s}}{s^2+1}]$

$L^{-1}[\frac{e^{-}}{s^2+1}] = L^{-1}[e^{-\pi s}\frac{1}{s^2+1}]_{\pi s}$

**Solution:**

Let f(s) = $\frac{1}{s^2+1}$

$L^{-1}$[ f(s)] = $L^{-1}[\frac{1}{s^2+1}]$  = Sin t = f(t)

**by S.S.T** $L^{-1}[\, e^{-as} f(s) \,] = f(t-a),\ t>a$

$= 0, \quad t<a$

*So* $L^{-1}[\, e^{-\pi s} f(s) \,] = f(t-\pi),\ t>\pi$

$= 0, \quad t<\pi$

$L^{-1}[e^{-\pi s} \frac{1}{s^2+1}] = \text{Sin}\,(t-\pi),\ t>\pi = 0,$

$t<\pi$

## Chang of scale property :

If $L^{-1}[f(s)] = f(t)$ then $L^{-1}[f(\frac{s}{a})] = a\, f(at)$

( or ) $L^{-1}[f(as)] = \frac{1}{a} f(\frac{t}{a})$

**Proof** : By the change of scale property,

$$L[f(at)] = \frac{1}{a} f(\frac{s}{a})$$

$$\Rightarrow L^{-1}[f(\frac{s}{a})] = a\, f(at)$$

(or)

$$L^{-1}[f(as)] = \frac{1}{a} f(\frac{t}{a})$$

**Problem(1):** **If** $L^{-1}[\frac{s^2-1}{(s^2+1)^2}]$ **= t cost , then find** $L^{-1}[\frac{9s^2-1}{(9s^2+1)^2}]$

**Solution : Given** $L^{-1}[\frac{s^2-1}{(s^2+1)^2}]$ **= t cost**

**i.e.,** $L^{-1}[f(s)] = f(t)$

**, Here** $f(s) = \frac{s^2-1}{(s^2+1)^2}$ **f(t) = t cost**

**Now** $L^{-1}[\frac{9s^2-1}{(9s^2+1)^2}] = L^{-1}[\frac{(3s)^2-1}{\{(3s)^2+1\}^2}]$

$= L^{-1}[f(3s)]$ **By change of scale property ,**

$= \frac{1}{3} f(\frac{t}{3})$

$L^{-1}[f(as)] = \frac{1}{a} f(\frac{t}{a}) = \frac{1}{3}\frac{t}{3} \cos\frac{t}{3}$ **a = 3**

# Inverse Laplace Transform of partial fractions :

Problems : (1) Find $L^{-1}[\frac{(s^2+1)(s-1)}{s^4}]$

Solution : Given $L^{-1}[\frac{(s^2+1)(s-1)}{s^4}] = L^{-1}[\frac{(s^3-s^2+s-1)}{s^4}]$

$= L^{-1}[\frac{1}{s}] - L^{-1}[\frac{1}{s^2}] + L^{-1}[\frac{1}{s^3}] - L^{-1}[\frac{1}{s^4}]$

$= 1 - t + \frac{1}{2}t^2 - \frac{t^3}{6}$

(2). Find $L^{-1}[\frac{s+5}{s^2-3s+2}]$ $L^{-1}[\log(\frac{s+3}{s+4})]$ Solution : Here $f(s) = \frac{s+5}{s^2-3s+2}$ reduce into partial fractions

$\frac{s+3}{s+4}$

log

$f(s) = \frac{s+5}{s^2-3s+2} = \frac{s+5}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \quad ----(1)$

$\Rightarrow s+5 = A(s-2) + B(s-1)$

put s=1 on both sides $\Rightarrow A = -6$

put s=2 on both sides $\Rightarrow B = 7$

Therefore (1) $\Rightarrow f(s) = \frac{-6}{s-1} + \frac{7}{s-2}$

$L^{-1}[f(s)] = L^{-1}[\frac{-6}{s-1} + \frac{7}{s-2}] = -6e^t + 7e^{2t}$

**Inverse Laplace Transform of derivatives :-**

If $L^{-1}[f(s)] = f(t)$ then $L^{-1}[\frac{d^n}{ds^n} f(s)] = (-1)^n t^n f(t)$

$f'(s) =$

Proof : By theorem of L.T. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$\Rightarrow L^{-1}[\frac{d^n}{ds^n} f(s)] = (-1)^n t^n f(t)$ Note:- $L^{-1}[f'(s)] = - t f(t)$

Problem (1):- Find

Solution : Let $f(s) = \log(\frac{s+3}{s+4}) = \log(s+3) - \log(s+4)$

$s+3 \qquad s+4$

$L^{-1}[f'(s)] = L^{-1}[\frac{1}{s+3} - \frac{1}{s+4}]$

$= e^{-3t} - e^{-4t}$

$L^{-1}[\log(\frac{s+1}{s-1})]$

By theorem, $-t\,f(t) = e^{-3t} - e^{-4t}$ H.W. Find $\frac{e^{-3t} - e^{-}}{-t}$ $4tt$ so, $\frac{e^{t} - e^{-}}{t}$

f(t) = Ans: $L^{-1}[f(s)] =$

[replace 3 by $\Rightarrow L^{-1}[f(s)] = \frac{e^{-4t} - e^{-3t}}{t}$ 1 and 4 by (-1)]

(2) Find $L^{-1}[\frac{s}{(s^2+a^2)^2}]$

Solution: W.K.T $L^{-1}[\frac{1}{(s^2+a^2)}] = \frac{1}{a}\sin at$

i.e $L^{-1}[f(s)] = f(t)$ 1 Let f(s)

$=, \quad f(t) \; \frac{1}{(s^2+a^2)} \quad - = \sin at$

$a$

We have $L^{-1}[f'(s)] = -\,t\,f(t)$

$$L^{-1}\left[\frac{d}{ds}\left(\frac{1}{(s^2+a^2)}\right)\right] = -\,t\,\frac{1}{a}\sin at$$

$$L^{-1}\left[\frac{-2s}{(s^2+a^2)^2}\right] = -\,\frac{t}{a}\sin at$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

**Inverse L.T. of integrals** :-

If $L^{-1}[f(s)] = f(t)$ then $L^{-1}\left[\int_s^\infty f(s)\,ds\right] = \frac{f(t)}{t}$

Proof : We have $L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds$ provided exist

$$\Rightarrow L^{-1}\left[\int_s^\infty f(s)\,ds\right] = \frac{f(t)}{t}$$

**Multiplication by powers of s** :-

If $L^{-1}$[f(s)] = f(t) and f(0) = 0, then $L^{-1}$[s f(s)] = f'(t) Proof :

W.K.T. L[f'(t)] = s L[f(t)] – f(0)

= s f(s) – 0

$\Rightarrow L^{-1}$[s f(s)] = f'(t)

In general we have, $\Rightarrow L^{-1}[s^n f(s)] = f^n(t)$ if = $f^n(0) = 0$

# Problems :

$$\boldsymbol{L^{-1}}[\frac{s^2}{(s^2+a^2)^2}]$$

**(1) Find** $\boldsymbol{L^{-1}}[\frac{s}{(s^2+a^2)^2}] = \boldsymbol{L^{-1}}[\text{s.} \frac{s}{(s^2+a^2)^2}]$

2

**solution :**

$s$

**Let f(s) =**

$L^{-1}[\mathrm{f(s)}] =$ $(s^2+a^2)^2$ f(t) = f'(t) =

$\frac{1}{2a}$ [ sin $L^{-1}[\frac{s}{(s^2+a^2)^2}]$ at + t a cos at ]

We have $L^{-1}$[s f(s)] = f'(t)

$\Rightarrow L^{-1}[\frac{s^2}{(s^2+a^2)^2}] = \frac{1}{2a}$

(2) Find $L^{-1}[\frac{s^2}{(s-1)^4}]$ ( sin at + at cos at )

$\frac{s}{(s-1)^4}$

Solution

: [f(s)] = $L^{-1}[\frac{s}{(s-1)^4}]$

$= L^{-1}[\frac{s-1+1}{(s-1)^4}]$

$= e^t\, L^{-1}[\frac{s+1}{s^4}$

$= e^t\, L^{-1}[\frac{1}{s^3} + \frac{1}{s^4}]$

$= e^t\,(\frac{t^2}{2} + \frac{t^3}{6}) = \mathrm{f(t)}$

Let f(s) = $L_{-1}$

] by F.S.T.

$$e^t\left(\frac{t^2}{2}+\frac{t^3}{6}\right)+e^t\left(t+\frac{t^2}{2}\right)$$

Now $f'(t) = = e^t\left(t + t^2 + \frac{t^3}{6}\right)$

By theorem $L^{-1}[s\, f(s)] = f'(t)$

$L^{-1}\left[s\, \frac{s}{(s-1)^4}\right] = e^t\left(t + t^2 + \frac{t^3}{6}\right)$ <u>Division</u>

<u>by power of S :</u>

<u>Theorem:</u> If $L^{-1} f\, s\, [\ (\ )]\quad (\ )\quad (\ ) = f\ t$, then $L^{-1}\left[\frac{f(s)}{}\right]$

$$s = {}_0 ?^t f\ t\ dt$$

<u>Prof:</u> we have by LT,

L [$\int_0^t f$(t) dt] = $\frac{f(s)}{s}$

$\Rightarrow L{-1}\left[\frac{f(s)}{s}\right] = \int_0^t f(t)\,dt$

$L^{-1}\left[\frac{f(s)}{s^2}\right] = \int_0^t\left[\int_0^t f(t)\,dt\right]dt$

**Note:** ( )

**Problem:**

1) Find $L^{-1}\left[\frac{1}{s(s+3)}\right]$

solution: Let f (s) = $\frac{1}{s+3}$

$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$= f(t)

By theorem , $L^{-1}\left[\frac{1}{s}.f(s)\right] = \int_0^t f(t)dt$

$\Rightarrow L^{-1}\left[\frac{1}{s(s+3)}\right] = \int_0^t e^{-3t}dt = \frac{e^{-3t}}{-3}]_0^t = \frac{1-e^{-3t}}{3}$

2) Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Solution : let f(s) $\frac{1}{s^2+a^2}$ , $L^{-1}$ = [f(s)] = $\frac{1}{a}$sinat = f(t)

By theorem $L^{-1}[\frac{1}{s}$ f(s)] = $\int_0^t f$(t)dt

$\Rightarrow L^{-1}[\frac{1}{s(s^2+a^2)}] = \int_0^t \frac{1}{a} sinat = \frac{1}{a}$ ( - $\frac{cosat}{a}$

= $\frac{1}{a^2}$(1 -cos at )

)

3) Find $L^{-1}[\frac{1}{s^2(s^2+a^2)}]$

$\overline{s^2+a^2}$

1 1 solution : let f(s)

= , f (t) = _ sin at

$a$

by theorem, $L^{-1}[\frac{1}{s^2}$ f(s)] =

$\int_0^t \int_0^t f$(t)dt = ?$_0^t$ [ ?$_0$ $a$ sin at dt ]dt

= $\int_0^t$ $\frac{1}{a^2}$(1- cos at ) dt = $\frac{1}{a^2}$ ( t - $\frac{sin\ at}{a}$ ) $t$

$\underline{1}$

## Convolution : -

If f(t) and g(t) are two functions defined for $t \geq 0$ ,then the convolution of f(t) and g(t) is defined as , f(t) * g(t) = $\int_0^t f(u)\, g(t-u)du$.

f(t) * g(t) can also be written as (f * g)(t). Note:- The convolution operation is commutation

i.e. , ( f * g ) (t) = (g * t) (t)

$\Rightarrow \int_0^t f(u)\, g\,(t-u)du = \int_0^t f(t-u)\, g\,(u)du$

## Convolution theorem :-

If L[f(t)] = f(s) and L[g(t)] = g(s) then L[ f(t) * g(t)] = L[f(t)] . L[g(t)]

(or)

= f(s). g(s)

So, L[( f * g) (t)] = f(s) . g(s)

Corollary :-$L^{-1}$[f(s). g(s)] = (f * g) t

= $\int_0^t f(u)\, g(t-u)du$

= $\int_0^t f(t-u)\, g\,(u)du$.

## Problems:

(1). Find $L^{-1}[\frac{1}{(s-2)(s^2+1)}]$ by using convolution theorem.

1 1 solution: Let f(s) =

$\overline{s-2}$ , g(s) = $\overline{s^2+1}$

$L^{-1}$ [f(s)] = $L^{-1}[\frac{1}{s-2}] = e^{2t}$ , $L^{-1}$ [g(s)] = $L^{-1}[\frac{1}{s^2+1}]$ = sin t

By convolution theorem ,

$L^{-1}$[f(s). g(s)] = $\int_0^t f(t-u)\, g(u)du$

$\Rightarrow L^{-1}[\frac{1}{(s-2)(s^2+1)}$ = $\int_0^t e^{2(t-u)}$ sin $u\, du$

= $e^{2t} \int_0^t e^{-2u}\, sin\, u\, du$

= $e^{2t}\, [\frac{e^{-2u}}{(-2)^2+1^2}$ (-

2

sin u –

cos u)]

cos t) - $\frac{e^0}{5}$ (-1)]

= $e^{2t}[\frac{e^{-2t}}{5}$ (-2 sin t – $e^{2t}$

2 sin t – cos t) + = $\frac{1}{5}$ (-

5

$= \frac{1}{5}\,[e^{2t}$ - **2 sin t – cos t]**

2) Find $L^{-1}[\frac{1}{s(s^2-a^2)}]$ by convolution theorem

Solution : Let $f(s) = \frac{1}{s}$ , $g(s) = \frac{1}{s^2-a^2}$

$L^{-1}$[f(s)] = $L^{-1}[\frac{1}{s}]$ = 1 = f(t) , $L^{-1}$[g(s)] = $L^{-1}[\frac{1}{s^2-a^2}]$

$= \frac{1}{a}\sinh at$ = g(t) By convolution theorem ,

$L^{-1}$[f(s). g(s)] = $\int_0^t f(t-u)\, g(u) du$

$\Rightarrow L^{-1}[\frac{1}{s(s^2-a^2)}] = \int_0^t 1\ \frac{1}{a} \sinh au\, du$

$= \frac{1}{a}[\frac{\cosh au}{a}]$ , (apply limits o to t)

$= \frac{1}{a^2}$ (cosh at – 1)

## Application of L . T to Ordinary Differential Equations :

**The L . T method is easier , time – saving and excellent tool for solving O.D.Es**

Working rule for finding solution of D . E by L . T:

1) **Write down the given equation and apply L . T O . B . S**
2) **Use the given conditions**
3) **Re arrange the given equation to given transformation of the solution**
4) **Take inverse L.T O. B. S to obtain the desireds obesve Sali stying the given conditions**

**The formulae to be used in this process are:**

$L[f^{1}(t)] = s\,f(s) - f(0)$

$L[f^{11}(t)] = s^2 f(s) - s\,f(0) - f^{1}(0)$

$L[f^{111}(t)] = s^3 f(s) - s^2 f(0) - sf(0) - f^{11}(0)$

**Note : let f(t) = y (t) and f (s) = y (s)** **Problems :**

1) **Solve** $4y^{11} + \pi^2 y = 0$ , $y(0) = 2$ , $y^{1}(0) = 0$

**Solution :** Here y = y (t)

Given D . E $4y^{11}(t) + \pi^2 y(t) = 0$ Let L . T O.B.S

$4L[y^{11}(t)] + \pi^2 L[y(t)] = L[0]$

$\Rightarrow 4[s^2 L(y)] - s\,y(0) - y^1(0)] + \pi^2 L[y] = 0$

$\Rightarrow L[y][4s^2 + \pi^2] - 4s(2) - 0 = 0$

$\Rightarrow L[y] = \frac{8s}{4s^2+\pi^2}$

Let $L^{-1}$ O . B . S, we get y (t) $= 8\, L^{-1}[\frac{s}{4(s^2+\pi^2/4)}]$

$= \frac{8}{4} L^{-1}[\frac{s}{s^2+(\pi^2/2)^2}] = 2.\cos \pi/2t$

$\Rightarrow y(t) = 2.\cos \pi/2t$ is solution of gven D.E

3) Solve $y^{111}+2y^{11} - y^1 - 2y = 0$ with $y(0) = y^1(0) = 0$ , $y^{11}(0) = 6$

**Solution :** given D . E

**Let L . T On Both Sides**

$L[y^{111}] + 2 \ L[y^{11}] - L[y^{1}] - 2 L[y] = 0$

- – – - s y (0)

$y^{1}(0)$ ]

$- s L[y] - y(0) - 2 L[y] = 0$

$\Rightarrow L[y] (s^3 + 2s^2 - s - 2) - 6 = 0$

$\Rightarrow L[y] = \frac{6}{s^3+2s^2-s-2}$

$\Rightarrow s^3 L[y] s^2 y(0) s y^{1}(0) y^{11}(0) + 2[s^2 L[y]$

$L[y] = \frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$ _ _ _ _ _ (1)

6= A (s + 1 ) (s +2 ) + B (s - 1 ) (s + 2) + C (s – 1 ) (s+1) _ _ _ _(2) Put s = 1 in _ _ _ (2) 6 = A (2) (3) ⇒ A = 1

Put s = - 1 in (2)

⇒ 6 = B (-2) (1) ⇒B = - 3

Put s = -2 in (2)

⇒6 = C (-3) (-1) ⇒ C = 2

**Substitute A , B , C in (1)**

$$\Rightarrow L[y] = \frac{1}{S-1} - \frac{3}{S+1} + \frac{2}{S+2}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}\right]$$

$$\Rightarrow y(t) = e^{t} - 3e^{-t} + 2e^{-2t}$$

**is the solution of given D . E**

**HW: Solve the D.E** $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$

**Ans:** $y(t) = \frac{e^{-t}}{3}(\sin t - 2\sin 2t)$