UNIT-V
DYNAMIC ANALYSIS

Dynamics is a special branch of mechanics where inertia of accelerating masses must be considered in the force-deflection relationships. In order to describe motion of the mass system, a component with distributed mass is approximated by a finite number of mass points. Knowledge of certain principles of dynamics is essential to the formulation of these equations.

Every structure is associated with certain frequencies and mode shapes of free vibration (without continuous application of load), based on the distribution of mass and stiffness in the structure. Any time-dependent external load acting on the structure, whose frequency matches with the natural frequencies of the structure, causes resonance and produces large displacements leading to failure of the structure. Calculation of natural frequencies and mode shapes is therefore very important.

Consider i^{th} mass m_i of a system of connected rigid bodies and the force components F_j (j = 1,2,..6) acting upon it in three-dimensional space. If the mass m_i is in equilibrium at rest, then $\Sigma F_i = 0$.

If mass m_j is not in equilibrium, it will accelerate in accordance with Newton's second law i.e., $F_j = m_i \ddot{u}_j$

The force $(-m_i.\ddot{u}_j)$ is called the reversed effective force or inertia force. According to **D'Alembert's principle**, the net external force and the inertia force together keep the body in a state of '*fictitious equilibrium*' i.e., $\Sigma(F_j - m\,\ddot{u}_j) = 0$.

If the displacement of the mass m_i is represented by δu_j (j = 1,2,..6), then the virtual work done by these force components on the mass m_i in equilibrium is given by

$$\delta W_i = \sum F_j \,.\, \delta u_j = 0.$$

D'Alembert's principle rewritten in the form,

$\delta W_i = \sum F_j \, . \, \delta u_j - \sum (m \, \ddot{u}_j) \, . \, \delta u_j = 0$ is a statement of ***virtual work for a system in motion***.

For a simple spring of stiffness 'k' and a lumped mass 'm' under steady state undamped condition of oscillation without external force, the force equilibrium condition of the system is given by

$$k \, u(t) + m \, \ddot{u}(t) = 0,$$

where, $F_i = -k \, u(t)$ is the reactive elastic force applied to the mass.

Displacement in vibration is a simple harmonic motion and can be represented by a sinusoidal function of time as

$$u(t) = u \sin \omega t$$

where, ω is the frequency of vibration in radians/sec

It is more often expressed in 'f' cycles/sec or Hertz (Hz) where $\omega = 2\pi f$

Then, velocity $\dot{u}(t) = -\omega u \cos \omega t$

and acceleration $\ddot{u}(t) = -\omega^2 u \sin \omega t = -\omega^2 u(t)$

$\therefore$ $K.u(t) + m \, \ddot{u}(t) = (k - \omega^2 m) \, u(t) = 0$

In general, for a system with 'n' degrees of freedom, stiffness 'k' and mass 'm' are represented by stiffness matrix [K] and mass matrix [M] respectively.

Then, $([K] - \omega^2 [M]) \{u\} = \{0\}$

or $([M]^{-1}[K] - \omega^2 [I]) \{u\} = \{0\}$

Here, [M] is the mass matrix of the entire structure and is of the same order, say n × n, as the stiffness matrix [K]. This is also obtained by assembling element mass matrices in a manner exactly identical to assembling element stiffness matrices. The mass matrix is obtained by two different approaches, as explained subsequently.

This is a typical eigenvalue problem, with ω^2 as eigenvalues and {u} as eigenvectors. A structure with 'n' DOF will therefore have 'n' eigenvalues and 'n' eigenvectors. Some eigenvalues may be repeated and some eigenvalues may be complex, in pairs. The equation can be represented in the standard form, $[A]\{x\}_i = \lambda_i \{x\}_i$. In dynamic analysis, ω_i indicates i^{th} natural frequency and $\{x\}_i$ indicates i^{th} natural mode of vibration. A natural mode is a ***qualitative*** plot of nodal displacements. In every natural mode of vibration, all the points on the component will reach their maximum values at the same time and will pass through zero displacements at the same time. Thus, in a particular mode, all the points of a component will vibrate with the same frequency and their relative displacements are indicated by the components of the corresponding eigenvector. These relative (or proportional) displacements at different points on structure remain same at every time instant for undamped free vibration (Ref. Fig. 8.1). Hence, without loss of generality, $\{u(t)\}$ can be written as $\{u\}$.

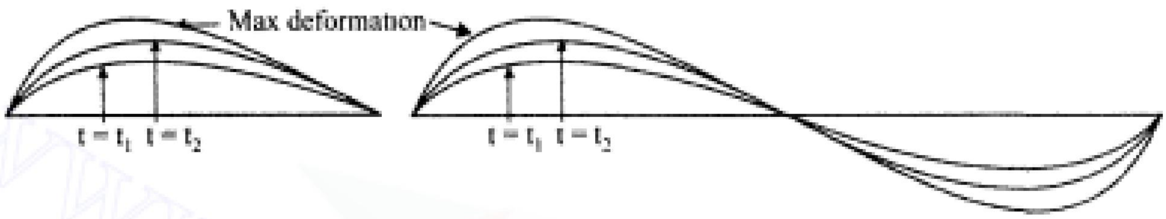


FIGURE 8.1 Mode shape

Since $\{u\} = \{0\}$ forms a trivial solution, the homogeneous system of equations $([A] - \lambda[I])\{u\} = \{0\}$ gives a non-trivial solution only when

$$([A] - \lambda[I]) = \{0\},$$

which implies **Det ([A] – λ[I]) = 0.**

This expression, called ***characteristic equation***, results in n^{th} order polynomial in λ and will therefore have n roots. For each λ_i, the corresponding eigenvector $\{u\}_i$ can be obtained from the n homogeneous equations represented by $([K] - \lambda[M])\{u\} = \{0\}$. The mode shape represented by $\{u(t)\}$ gives relatives values of displacements in various degrees of freedom.

It can also be represented as

$$[A][X] = [X][\Lambda]$$

where, $[A] = [M]^{-1}[K]$

$[X]$ is called the ***modal matrix***, whose i^{th} column represents i^{th} eigenvector $\{x\}_i$

and $[\Lambda]$ is called the ***spectral matrix*** with each diagonal element representing one eigenvalue, corresponding to the eigenvector of that column, and off-diagonal elements equal to zero.

coupling that exists between different nodal displacements. Lumped mass matrices [M] of some elements are given here.

Lumped mass matrix of truss element with 1 translational DOF per node along its local X-axis

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Lumped mass matrix of plane truss element in a 2-D plane with 2 translational DOF per node (Displacements along X and Y coordinate axes)

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Please note that the same lumped mass is considered in each translational degree of freedom (without proportional sharing of mass between them) at each node.

Lumped mass matrix of a beam element in X-Y plane, with its axis along x-axis and with two DOF per node (deflection along Y axis and slope about Z axis) is given below. Lumped mass is not considered in the rotational degrees of freedom.

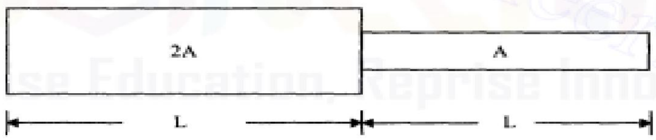
$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Note that lumped mass terms are not included in 2^{nd} and 4^{th} rows, as well as columns corresponding to rotational degrees of freedom.

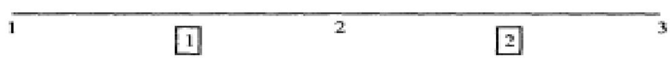
Lumped mass matrix of a CST element with 2 DOF per node. In this case, irrespective of the shape of the element, mass is assumed equally distributed at the three nodes. It is distributed equally in all DOF at each node, without any sharing of mass between different DOF

$$[M]=\frac{\rho AL}{3}\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Example 8.1 : Find the natural frequencies of longitudinal vibrations of the unconstrained stepped shaft of areas A and 2A and of equal lengths (L), as shown below.



Solution : Let the finite element model of the shaft be represented by 3 nodes and 2 truss elements (as only longitudinal vibrations are being considered) as shown below.



$$[K]_1 = (2A)\left(\frac{E}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left(\frac{AE}{L}\right)\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix};$$

$$[K]_2 = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M]_2 = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assembling the element stiffness and mass matrices,

$$[K] = \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix};$$

$$[M] = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Eigenvalues of the equation $([K] - \omega^2 [M])\ \{u\} = \{0\}$ are the roots of the characteristic equation represented by

$$\begin{vmatrix} 2AE/L - \omega^2 4\rho\rho AL/ & -2AE/L - \omega^2 2\rho\rho AL/ & 0 \\ 2AE/L - \omega^2 2\rho\rho AL/ & 3AE/L - \omega^2 6\rho\rho AL/ & -1AE/L - \omega^2 \rho AL/6 \\ 0 & -AE/L - \omega^2 \rho AL/6 & AE/L - \omega^2 2\rho\rho AL/ \end{vmatrix} = 0$$

Multiplying all the terms by (L/AE) and substituting $\beta = \frac{\rho L^2 \omega^2}{6E}$

$$\begin{vmatrix} 2(1-2\beta) & -2(1+\beta) & 0 \\ -2(1+\beta) & 3(1-2\beta) & -(1+\beta) \\ 0 & -(1+\beta) & (1-2\beta) \end{vmatrix} = 0$$

or $\quad 18\,\beta\,(\beta - 2)\,(1 - 2\beta) = 0$

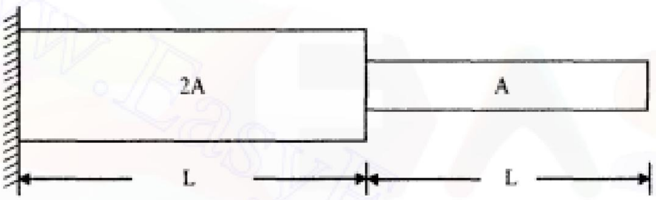
The roots of this equation are $\beta = 0, 2$ or $\frac{1}{2}$ or $\omega^2 = 0, \frac{12E}{\rho L^2}$ or $\frac{3E}{\rho L^2}$

Corresponding eigenvectors are obtained from $([K] - \omega^2 [M])\ \{u\} = \{0\}$ for different values of ω^2 as $[1 \quad 1 \quad 1]^T$ for $\beta = 0$, $[1 \quad 0 \quad -2]^T$ for $\beta = \frac{1}{2}$ and $[1 \ -1 \ \ 1]^T$ for $\beta = 2$.

The first eigenvector implies rigid body motion of the shaft. One component (u_1 in this example) is equated to '1' and other displacement components

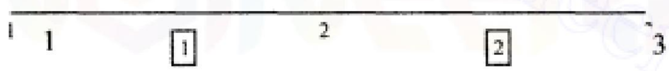
Example 8.2

Find the natural frequencies of longitudinal vibrations of the same stepped shaft of areas A and 2A and of equal lengths (L), when it is constrained at one end, as shown below.



Solution

Let the finite element model of the shaft be represented by 3 nodes and 2 truss elements (as only longitudinal vibrations are being considered) as shown below.



$$[K]_1 = \left(\frac{2AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left(\frac{AE}{L}\right)\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[K]_2 = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Using consistent mass matrix approach

$$[M]_1 = \frac{\rho(2A)L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}; \quad [M]_2 = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assembling the element stiffness and mass matrices,

$$[K] = \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}; \quad [M] = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[M]_1 = \frac{\rho(2A)L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}; \quad [M]_2 = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assembling the element stiffness and mass matrices,

$$[K] = \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}; \quad [M] = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$