UNIT 1

SIMPLE STRESSES & STRAINS

Course Objectives:

- To understand the nature of stresses induced in material under different loads.

Course Outcomes:

- Determine the simple stresses and strains when members are subjected to axial loads.

Simple Stresses and Strains

Expressions for stresses and strains is derived with the following assumptions:

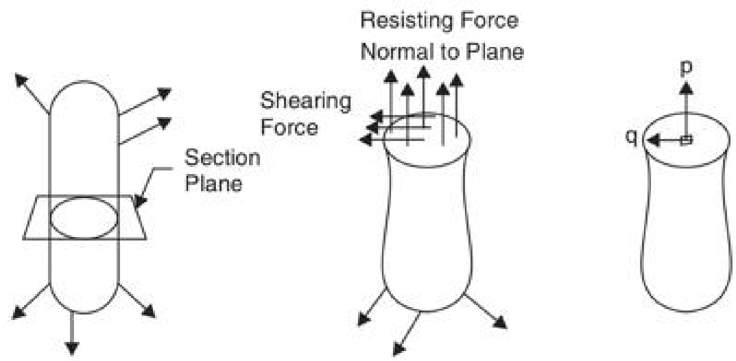
1. For the range of forces applied the material is elastic *i.e.* it can regain its original shape and size, if the applied force is removed.
2. Material is homogeneous *i.e.* every particle of the material possesses identical mechanical properties.
3. Material is isotropic *i.e.* the material possesses identical mechanical property at any point in any direction.

Presenting the typical stress-strain curve for a typical steel, the commonly referred terms like limits of elasticity and proportionality, yield points, ultimate strength and strain hardening are explained.

Linear elastic theory is developed to analyse different types of members subject to axial, shear, thermal and hoop stresses.

MEANING OF STRESS

When a member is subjected to loads it develops resisting forces. To find the resisting forces developed a section plane may be passed through the member and equilibrium of any one part may be considered. Each part is in equilibrium under the action of applied forces and internal resisting forces. The resisting forces may be conveniently split into normal and parallel to the section plane. The resisting force parallel to the plane is called *shearing resistance*. The intensity of resisting force normal to the sectional plane is called *intensity of Normal Stress* (Ref. Fig.).



(a) Members Subject to Forces　　(b) Internal Resistances Developed　　(c)

Fig.

In practice, intensity of stress is called as "stress" only. Mathematically

$$\text{Normal Stress} = p = \lim_{\Delta A \to 0} \frac{\Delta R}{\Delta A}$$

$$= \frac{dR}{dA} \qquad ...(1)$$

where R is normal resisting force.

The intensity of resisting force parallel to the sectional plane is called *Shearing Stress* (q).

$$\text{Shearing Stress} = q = \lim_{\Delta A \to 0} \frac{\Delta Q}{\Delta A} = \frac{dQ}{dA} \qquad ...(2)$$

where Q is Shearing Resistance.

Thus, *stress at any point may be defined as resistance developed per unit area.* From equations (1) and (2), it follows that

$$dR = p\,dA$$

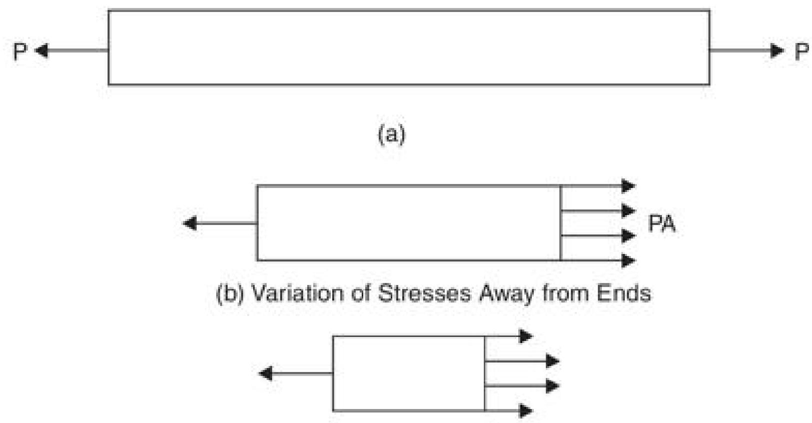
or

$$R = \int p\,dA \qquad \text{...(3a)}$$

and

$$Q = \int q\,dA \qquad \text{...(3b)}$$

At any cross-section, stress developed may or may not be uniform. In a bar of uniform cross-section subject to axial concentrated loads as shown in Fig. 2*a*, the stress is uniform at a section away from the applied loads (Fig. 2*b*); but there is variation of stress at the section near the applied loads (Fig. 2*c*).



(a)

(b) Variation of Stresses Away from Ends

(c) Variation of Stresses Near Ends

Fig. 2

Similarly stress near the hole or at fillets will not be uniform as shown in Figs. 3 and 4. It is very common that at some points in such regions maximum stress will be as high as 2 to 4 times the average stresses.

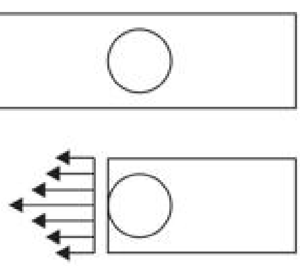


Fig. 3. Stresses in a Plate with a Hole

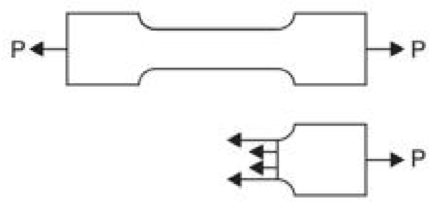


Fig. 4

UNIT OF STRESS

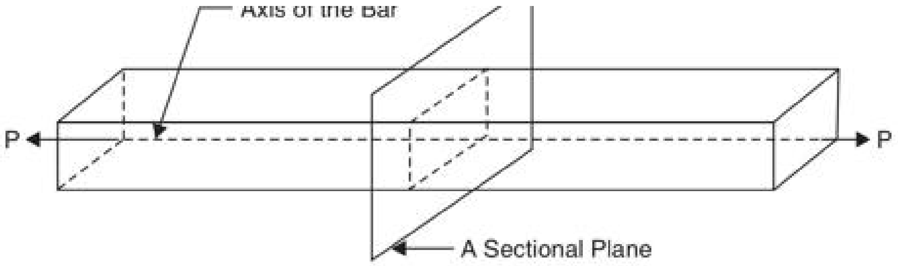
When Newton is taken as unit of force and millimetre as unit of area, unit of stress will be N/mm^2. The other derived units used in practice are kN/mm^2, N/m^2, kN/m^2 or MN/m^2. A stress of one N/m^2 is known as Pascal and is represented by Pa.

Hence, $\quad 1 \text{ MPa} = 1 \text{ MN/m}^2 = 1 \times 10^6 \text{ N/(1000 mm)}^2 = 1 \text{ N/mm}^2.$

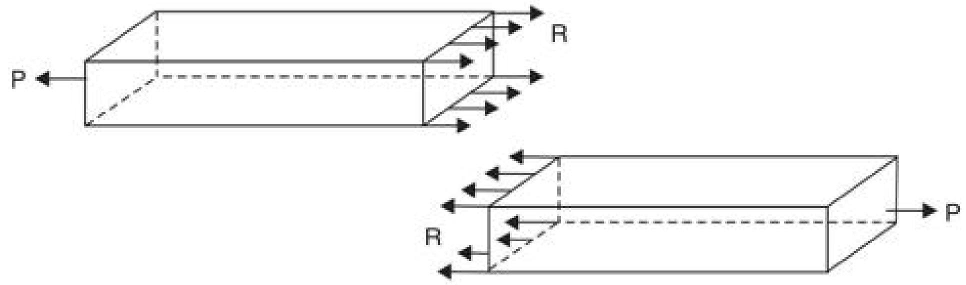
Thus one Mega Pascal is equal to 1 N/mm^2. In most of the standard codes published unit of stress has been used as Mega Pascal (MPa or N/mm^2).

AXIAL STRESS

Consider a bar subjected to force P as shown in Fig. 5. To maintain the equilibrium the end forces applied must be the same, say P.



(a) Bar Subjected to Pulls



(b) Resisting Force Developed

Fig. 5. Tensile Stresses

The resisting forces acting on a section are shown in Fig. 5*b*. Now since the stresses are uniform

$$R = \int pdA = p \int dA = pA \qquad ...(4)$$

where A is the cross-sectional area.

Considering the equilibrium of a cut piece of the bar, we get

$$P = R \qquad ...(5)$$

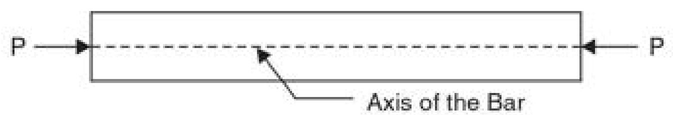
From equations (4) and (5), we get

$$P = pA$$

$$p = P/A$$

Thus, in case of axial load 'P' the stress developed is equal to the load per unit area. Under this type of normal stresses the bar is being extended. Such stress which is causing extension of the bar is called tensile stress.

A bar subjected to two equal forces pushing the bar is shown in Fig. 6. It causes shortening of the bar. *Such forces which are causing shortening, are known as compressive forces and corresponding stresses as compressive stresses.*



(a) Bar Subjected to Compressive Forces

(b) Resisting Force Developed

Fig.6. Compressive Stresses

Now $R = \int pdA = p \int dA$ (as stress is assumed uniform)

For equilibrium of the piece of the bar

$$P = R = pA$$

or $$p = \frac{P}{A} \text{ as in equation 6}$$

Thus, whether it is tensile or compressive, the stress developed in a bar subjected to axial forces, is equal to load per unit area.

STRAIN

No material is perfectly rigid. Under the action of forces a rubber undergoes changes in shape and size. This phenomenon is very well known to all since in case of rubber, even for small forces deformations are quite large. Actually all materials including steel, cast iron, brass, concrete, etc. undergo similar deformation when loaded. But the deformations are very small and hence we cannot see them with naked eye. There are instruments like extensometer, electric strain gauges which can measure extension of magnitude 1/100th, 1/1000th of a millimetre. There are machines like universal testing machines in which bars of different materials can be subjected to accurately known forces of magnitude as high as 1000 kN. The studies have shown that the bars extend under tensile force and shorten under compressive forces as shown in Fig. 8.7. *The change in length per unit length is known as linear strain.* Thus,

$$\text{Linear Strain} = \frac{\text{Change in Length}}{\text{Original Length}}$$

$$e = \frac{\Delta}{L} \qquad ...(7)$$

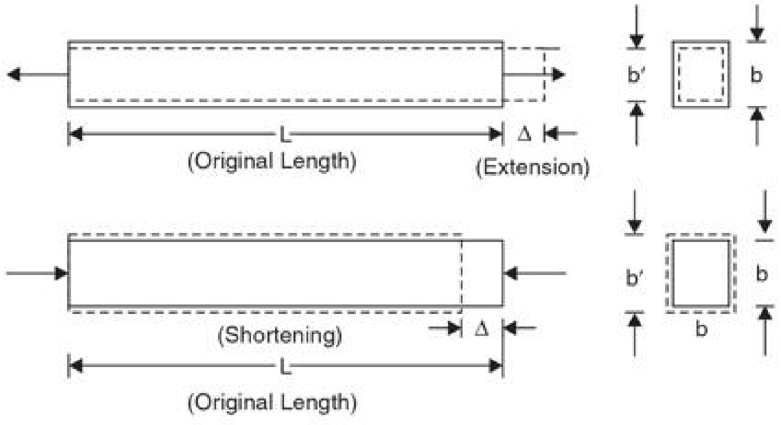


Fig. 7

When changes in longitudinal direction is taking place changes in lateral direction also take place. The nature of these changes in lateral direction are exactly opposite to that of changes in longitudinal direction *i.e.*, if extension is taking place in longitudinal direction, the shortening of lateral dimension takes place and if shortening is taking place in longitudinal direction extension takes place in lateral directions (See Fig. 7). *The lateral strain may be defined as changes in the lateral dimension per unit lateral dimension.* Thus,

$$\text{Lateral Strain} = \frac{\text{Change in Lateral Dimension}}{\text{Original Lateral Dimension}}$$

$$= \frac{b' - b}{b} = \frac{\delta b}{b} \qquad ...(8)$$

STRESS-STRAIN RELATION

The stress-strain relation of any material is obtained by conducting tension test in the laboratories on standard specimen. Different materials behave differently and their behaviour in tension and in compression differ slightly.

Behaviour in Tension

Mild steel. Figure 8 shows a typical tensile test specimen of mild steel. Its ends are gripped into universal testing machine. Extensometer is fitted to test specimen which measures extension over the length L_1, shown in Fig. 8. The length over which extension is mesured is called *gauge length.* The load is applied gradually and at regular interval of loads extension is measured. After certain load, extension increases at faster rate and the capacity of extensometer to measure extension comes to an end and, hence, it is removed before this stage is reached and extension is measured from scale nachine. Load is increased gradually till the specimen breaks.

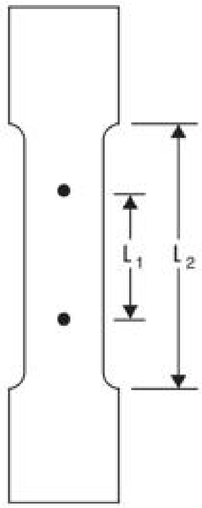


Fig. 8. Tension Test Specimen

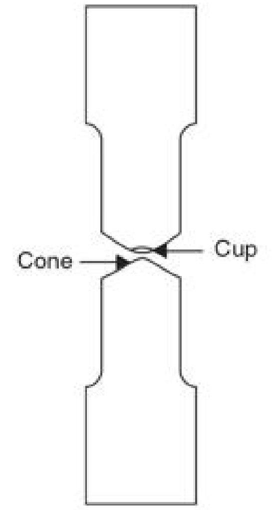


Fig. 9. Tension Test Specimen after Breaking

Load divided by original cross-sectional area is called as nominal stress or simply as stress. Strain is obtained by dividing extensometer readings by gauge length of extensometer (L_1) and by dividing scale readings by grip to grip length of the specimen (L_2). Figure 810 shows stress *vs* strain diagram for the typical mild steel specimen. The following salient points are observed on stress-strain curve:

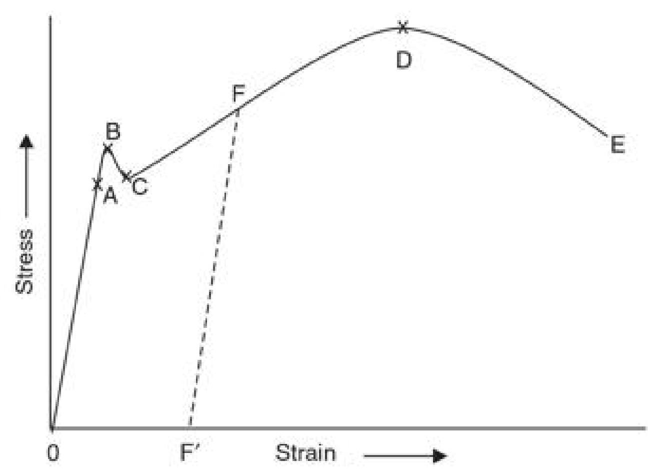


Fig. 10

(*a*) ***Limit of Proportionality (A):*** *It is the limiting value of the stress up to which stress is proportional to strain.*

(*b*) ***Elastic Limit:*** *This is the limiting value of stress up to which if the material is stressed and then released (unloaded) strain disappears completely and the original length is regained.* This point is slightly beyond the limit of proportionality.

(*c*) ***Upper Yield Point (B):*** *This is the stress at which, the load starts reducing and the extension increases.* This phenomenon is called yielding of material. At this stage strain is about 0.125 per cent and stress is about 250 N/mm^2.

(*d*) ***Lower Yield Point (C):*** *At this stage the stress remains same but strain increases for some time.*

(*e*) ***Ultimate Stress (D):*** This is the maximum stress the material can resist. This stress is about 370–400 N/mm^2. At this stage cross-sectional area at a particular section starts reducing very fast (Fig. 8.9). This is called neck formation. After this stage load resisted and hence the stress developed starts reducing.

(*f*) ***Breaking Point (E):*** *The stress at which finally the specimen fails is called breaking point.* At this strain is 20 to 25 per cent.

If unloading is made within elastic limit the original length is regained *i.e.*, the stress-strain curve follows down the loading curve shown in Fig. 8.6. If unloading is made after loading the specimen beyond elastic limit, it follows a straight line parallel to the original straight portion as shown by line *FF′* in Fig. 10. Thus if it is loaded beyond elastic limit and then unloaded a permanent strain (*OF*) is left in the specimen. This is called *permanent* set.

Stress-strain relation in aluminium and high strength steel. In these elastic materials there is no clear cut yield point. The necking takes place at ultimate stress and eventually the breaking point is lower than the ultimate point. The typical stress-strain diagram is shown in Fig. 11. The stress *p* at which if unloading is made there will be 0.2 per cent permanent set is known as 0.2 per cent proof stress and this point is treated as yield point for all practical purposes.

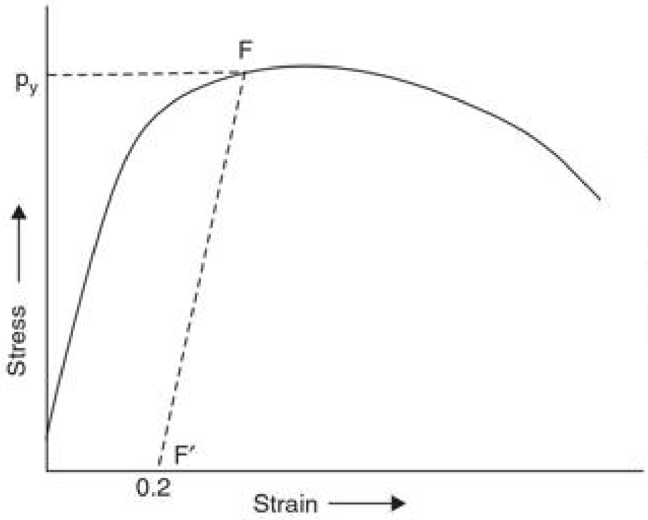


Fig. 11. Stress-Strain Relation in Aluminium and High Strength Steel

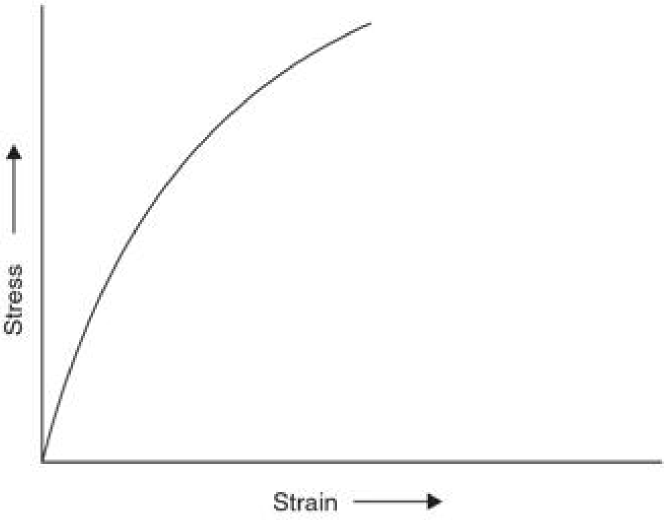


Fig. 12. Stress-Strain Relation for Brittle Material

Stress-strain relation in brittle material. The typical stress-strain relation in a brittle material like cast iron, is shown in Fig. 12.

In these material, there is no appreciable change in rate of strain. There is no yield point and no necking takes place. Ultimate point and breaking point are one and the same. The strain at failure is very small.

Percentage elongation and percentage reduction in area. Percentage elongation and percentage reduction in area are the two terms used to measure the ductility of material.

(*a*) ***Percentage Elongation:*** *It is defined as the ratio of the final extension at rupture to original length expressed, as percentage*. Thus,

$$\text{Percentage Elongation} = \frac{L' - L}{L} \times 100 \qquad ...(9)$$

where L – original length, L'– length at rupture.

The code specify that original length is to be five times the diameter and the portion considered must include neck (whenever it occurs). Usually marking are made on tension rod at every '2.5 *d*' distance and after failure the portion in which necking takes place is considered. In case of ductile material percentage elongation is 20 to 25.

(*b*) ***Percentage Reduction in Area:*** *It is defined as the ratio of maximum changes in the cross-sectional area to original cross-sectional area, expressed as percentage.* Thus,

$$\text{Percentage Reduction in Area} = \frac{A - A'}{A} \times 100 \qquad ...(10)$$

where A–original cross-sectional area, A'–minimum cross-sectional area. In case of ductile material, A' is calculated after measuring the diameter at the neck. For this, the two broken pieces of the specimen are to be kept joining each other properly. For steel, the percentage reduction in area is 60 to 70.

Behaviour of Materials under Compression

As there is chance to bucking (laterally bending) of long specimen, for compression tests short specimens are used. Hence, this test involves measurement of smaller changes in length. It results into lesser accuracy. However precise measurements have shown the following results:

(*a*) In case of ductile materials stress-strain curve follows exactly same path as in tensile test up to and even slightly beyond yield point. For larger values the curves diverge. There will not be necking in case of compression tests.

(*b*) For most brittle materials ultimate compresive stress in compression is much larger than in tension. It is because of flows and cracks present in brittle materials which weaken the material in tension but will not affect the strength in compression.

NOMINAL STRESS AND TRUE STRESS

So far our discussion on direct stress is based on the value obtained by dividing the load by original cross-sectional area. That is the reason why the value of stress started dropping after neck is formed in mild steel (or any ductile material) as seen in Fig. 10. But actually as material is stressed its cross-sectional area changes. We should divide load by the actual cross-sectional area to get true stress in the material. To distinguish between the two values we introduce the terms nominal stress and true stress and define them as given below:

$$\text{Nominal Stress} = \frac{\text{Load}}{\text{Original Cross-sectional Area}} \qquad \text{...(11}a\text{)}$$

$$\text{True Stress} = \frac{\text{Load}}{\text{Actual Cross-sectional Area}} \qquad \text{...11}b\text{)}$$

So far discussion was based on nominal stress. That is why after neck formation started (after ultimate stress), stress-strain curve started sloping down and the breaking took place at lower stress (nominal). If we consider true stress, it is increasing continuously as strain increases as shown in Fig. 13.

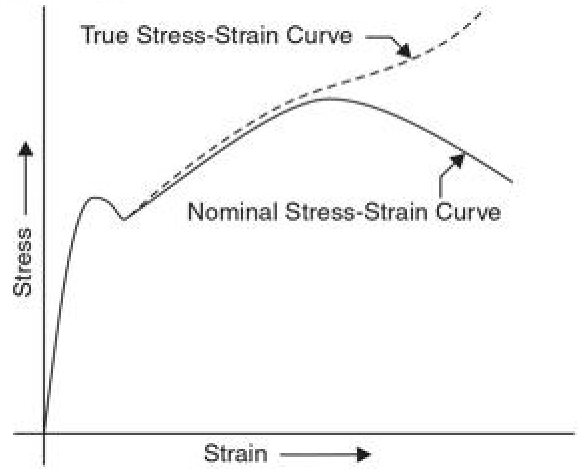


Fig. 13. Nominal Stress-Strain Curve and True Stress-Strain Curve for Mild Steel.

FACTOR OF SAFETY

In practice it is not possible to design a mechanical component or structural component permitting stressing up to ultimate stress for the following reasons:

1. Reliability of material may not be 100 per cent. There may be small spots of flaws.
2. The resulting deformation may obstruct the functional performance of the component.
3. The loads taken by designer are only estimated loads. Occasionally there can be overloading. Unexpected impact and temperature loadings may act in the lifetime of the member.
4. There are certain ideal conditions assumed in the analysis (like boundary conditions). Actually ideal conditions will not be available and, therefore, the calculated stresses will not be 100 per cent real stresses.

Hence, *the maximum stress to which any member is designed is much less than the ultimate stress, and this stress is called Working Stress. The ratio of ultimate stress to working stress is called factor of safety.* Thus

$$\text{Factor of Safety} = \frac{\text{Ultimate Stress}}{\text{Working Stress}} \qquad \text{...(8.12)}$$

In case of elastic materials, since excessive deformation create problems in the performance of the member, working stress is taken as a factor of yield stress or that of a 0.2 proof stress (if yield point do not exist).

Factor of safety for various materials depends up on their reliability. The following values are commonly taken in practice:

1. For steel – 1.85
2. For concrete – 3
3. For timber – 4 to 6

HOOKE'S LAW

Robert Hooke, an English mathematician conducted several experiments and concluded that *stress is proportional to strain up to elastic limit.* This is called Hooke's law. Thus Hooke's law is, up to elastic limit

$$p \propto e \qquad ....(13a)$$

where p is stress and e is strain

Hence, $$p = Ee \qquad ...(13b)$$

where E is the constant of proportionality of the material, known as modulus of elasticity or Young's modulus, named after the English scientist Thomas Young (1773–1829).

However, present day sophisticated experiments have shown that for mild steel the Hooke's law holds good up to the proportionality limit which is very close to the elastic limit. For other materials, Hooke's law does not hold good. However, in the range of working stresses, assuming Hooke's law to hold good, the relationship does not deviate considerably from actual behaviour. Accepting Hooke's law to hold good, simplifies the analysis and design procedure considerably. Hence Hooke's law is widely accepted. The analysis procedure accepting Hooke's law is known as Linear Analysis and the design procedure is known as the working stress method.

EXTENSION/SHORTENING OF A BAR

Consider the bars shown in Fig. 14

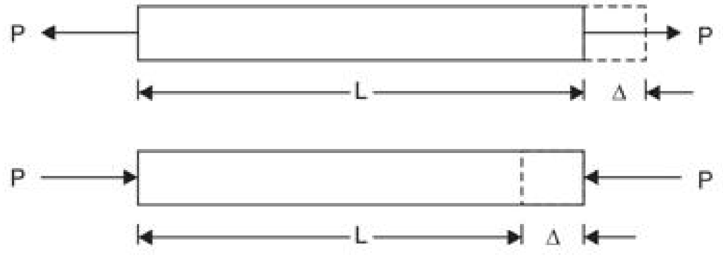


Fig. 14

From equation (8.6), Stress $p = \dfrac{P}{A}$

From equation (8.7), Strain, $e = \dfrac{\Delta}{L}$

From Hooke's Law we have,

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{p}{e} = \frac{P/A}{\Delta/L} = \frac{PL}{A\Delta}$$

or $$\Delta = \frac{PL}{AE}. \qquad ...(14)$$

***Example 1.** A circular rod of diameter 16 mm and 500 mm long is subjected to a tensile force 40 kN. The modulus of elasticity for steel may be taken as 200 kN/mm^2. Find stress, strain and elongation of the bar due to applied load.*

***Solution*:** Load $P = 40 \text{ kN} = 40 \times 1000 \text{ N}$

$$E = 200 \text{ kN/mm}^2 = 200 \times 10^3 \text{ N/mm}^2$$

$$L = 500 \text{ mm}$$

Diameter of the rod $d = 16$ mm

Therefore, sectional area $$A = \frac{\pi d^2}{4} = \frac{\pi}{4} \times 16^2$$

$$= 201.06 \text{ mm}^2$$

$$\textbf{Stress } p = \frac{P}{A} = \frac{40 \times 1000}{201.06} = \mathbf{198.94\ N/mm^2}$$

$$\textbf{Strain } e = \frac{p}{E} = \frac{198.94}{200 \times 10^3} = \mathbf{0.0009947}$$

$$\textbf{Elongation } \Delta = \frac{PL}{AE} = \frac{4.0 \times 1000 \times 500}{201.06 \times 200 \times 10^3} = \mathbf{0.497\ mm}$$

Example 2. *A Surveyor's steel tape 30 m long has a cross-section of 15 mm × 0.75 mm. With this, line AB is measure as 150 m. If the force applied during measurement is 120 N more than the force applied at the time of calibration, what is the actual length of the line?*

Take modulus of elasticity for steel as 200 kN/mm².

***Solution*:**

$$A = 15 \times 0.75 = 11.25 \text{ mm}^2$$

$$P = 120 \text{ N},\ L = 30 \text{ m} = 30 \times 1000 \text{ mm}$$

$$E = 200 \text{ kN/mm}^2 = 200 \times 10^3 \text{ N/mm}^2$$

$$\text{Elongation } \Delta = \frac{PL}{AE} = \frac{120 \times 30 \times 1000}{11.25 \times 200 \times 10^3} = 1.600 \text{ mm}$$

Hence, if measured length is 30 m.

Actual length is 30 m + 1.600 mm = 30.001600 m

$$\therefore \quad \textbf{Actual length of line } AB = \frac{150}{30} \times 30.001600 = \mathbf{150.008\ m}$$

Example 3. *A hollow steel tube is to be used to carry an axial compressive load of 160 kN. The yield stress for steel is 250 N/mm². A factor of safety of 1.75 is to be used in the design. The following three class of tubes of external diameter 101.6 mm are available.*

Class	**Thickness**
Light	3.65 mm
Medium	4.05 mm
Heavy	4.85 mm

Which section do you recommend?

***Solution*:** Yield stress = 250 N/mm²

Factor of safety = 1.75

Therefore, permissible stress

$$p = \frac{250}{1.75} = 142.857 \text{ N/mm}^2$$

$$\text{Load } P = 160 \text{ kN} = 160 \times 10^3 \text{ N}$$

but

$$p = \frac{P}{A}$$

i.e.

$$142.857 = \frac{160 \times 10^3}{A}$$

$$\therefore \quad A = \frac{160 \times 10^3}{142.857} = 1120 \text{ mm}^2$$

For hollow section of outer diameter 'D' and inner diameter 'd'

$$A = \frac{\pi}{4}(D^2 - d^2) = 1120$$

$$\frac{\pi}{4}(101.6^2 - d^2) = 1120$$

$$d^2 = 8896.53 \quad \therefore \quad d = 94.32 \text{ mm}$$

$$\therefore \qquad t = \frac{D-d}{2} = \frac{101.6 - 94.32}{2} = 3.63 \text{ mm}$$

Hence, use of light section is recommended.

Example 4. *A specimen of steel 20 mm diameter with a gauge length of 200 mm is tested to destruction. It has an extension of 0.25 mm under a load of 80 kN and the load at elastic limit is 102 kN. The maximum load is 130 kN.*

The total extension at fracture is 56 mm and diameter at neck is 15 mm. Find

(*i*) *The stress at elastic limit.*

(*ii*) *Young's modulus.*

(*iii*) *Percentage elongation.*

(*iv*) *Percentage reduction in area.*

(*v*) *Ultimate tensile stress.*

***Solution*:** Diameter $d = 20$ mm

$$\text{Area } A = \frac{\pi d^2}{4} = 314.16 \text{ mm}^2$$

(*i*) **Stress at elastic limit** $= \dfrac{\text{Load at elastic limit}}{\text{Area}}$

$$= \frac{102 \times 10^3}{314.16} = \mathbf{324.675 \ N/mm^2}$$

(*ii*) **Young's modulus** $E = \dfrac{\text{Stress}}{\text{Strain}}$ within elastic limit

$$= \frac{P/A}{\Delta/L} = \frac{80 \times 10^3/314.16}{0.25/200}$$

$$= \mathbf{203718 \ N/mm^2}$$

(*iii*) **Percentage elongation** $= \dfrac{\text{Final extension}}{\text{Original length}}$

$$= \frac{56}{200} \times 100 = \mathbf{28}$$

(*iv*) **Percentage reduction in area**

$$= \frac{\text{Initial area} - \text{Final area}}{\text{Initial area}} \times 100$$

$$= \frac{\frac{\pi}{4} \times 20^2 - \frac{\pi}{4} \times 15^2}{\frac{\pi}{4} \times 20^2} \times 100 = \mathbf{43.75}$$

(*v*) **Ultimate Tensile Stress** $= \dfrac{\text{Ultimate Load}}{\text{Area}}$

$$= \frac{130 \times 10^3}{314.16} = \mathbf{413.80 \ N/mm^2}.$$

BARS WITH CROSS-SECTIONS VARYING IN STEPS

A typical bar with cross-sections varying in steps and subjected to axial load is as shown in Fig. 15(*a*). Let the length of three portions be L_1, L_2 and L_3 and the respective cross-sectional areas of the portion be A_1, A_2, A_3 and E be the Young's modulus of the material and P be the applied axial load.

Figure 15(*b*) shows the forces acting on the cross-sections of the three portions. It is obvious that to maintain equilibrium the load acting on each portion is P only. Hence stress, strain and extension of each of these portions are as listed below:

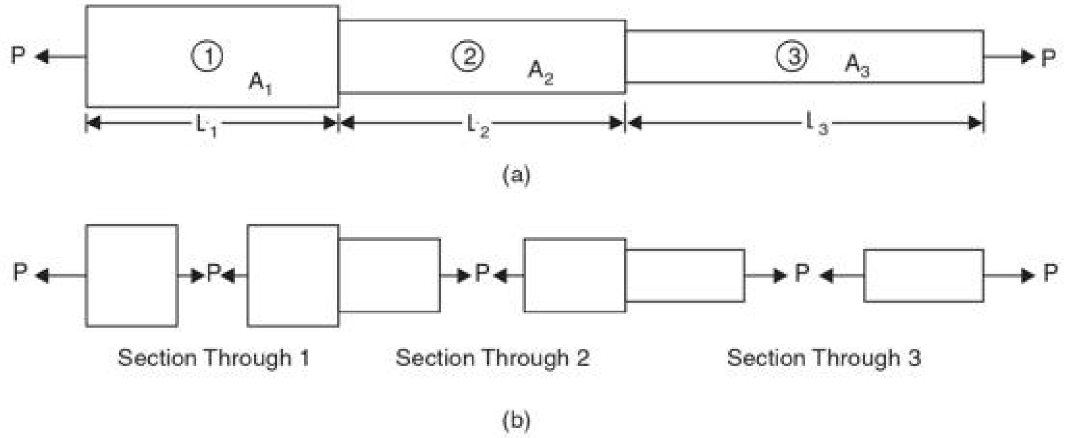


Fig.15. Typical Bar with Cross-section Varying in Step

Portion	*Stress*	*Strain*	*Extension*
1	$p_1 = \frac{P}{A_1}$	$e_1 = \frac{p_1}{E} = \frac{P}{A_1E}$	$\Delta_1 = \frac{PL_1}{A_1E}$
2	$p_2 = \frac{P}{A_2}$	$e_2 = \frac{p_2}{E} = \frac{P}{A_2E}$	$\Delta_2 = \frac{PL_2}{A_2E}$
3	$p_3 = \frac{P}{A_3}$	$e_3 = \frac{p_3}{E} = \frac{P}{A_3E}$	$\Delta_3 = \frac{PL_3}{A_3E}$

Hence total change in length of the bar

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = \frac{PL_1}{A_1E} + \frac{PL_2}{A_2E} + \frac{PL_3}{A_3E} \qquad ...(15)$$

Example 5. *The bar shown in Fig. 16 is tested in universal testing machine. It is observed that at a load of 40 kN the total extension of the bar is 0.280 mm. Determine the Young's modulus of the material.*

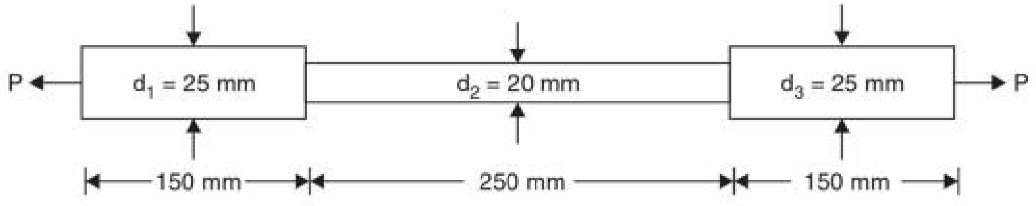


Fig. 16

Solution: Extension of portion 1,
$$\frac{PL_1}{A_1E} = \frac{40 \times 10^3 \times 150}{\frac{\pi}{4} \times 25^2 E}$$

Extension of portion 2,
$$\frac{PL_2}{A_2E} = \frac{40 \times 10^3 \times 250}{\frac{\pi}{4} \times 20^2 E}$$

Extension of portion 3, $\dfrac{PL_3}{A_3E} = \dfrac{40 \times 10^3 \times 150}{\dfrac{\pi}{4} \times 25^2\, E}$

$$\text{Total extension} = \frac{40 \times 10^3}{E} \times \frac{4}{\pi} \left\{ \frac{150}{625} + \frac{250}{400} + \frac{150}{625} \right\}$$

$$0.280 = \frac{40 \times 10^3}{E} \times \frac{4}{\pi} \times \frac{1.112}{E}$$

$$\mathbf{E = 200990\ N/mm^2}$$

BARS WITH CONTINUOUSLY VARYING CROSS-SECTIONS

When the cross-section varies continuously, an elemental length of the bar should be considered and general expression for elongation of the elemental length derived. Then the general expression should be integrated over entire length to get total extension.

Example 8. *A bar of uniform thickness 't' tapers uniformly from a width of b_1 at one end to b_2 at other end in a length 'L' as shown in Fig. 18. Find the expression for the change in length of the bar when subjected to an axial force P.*

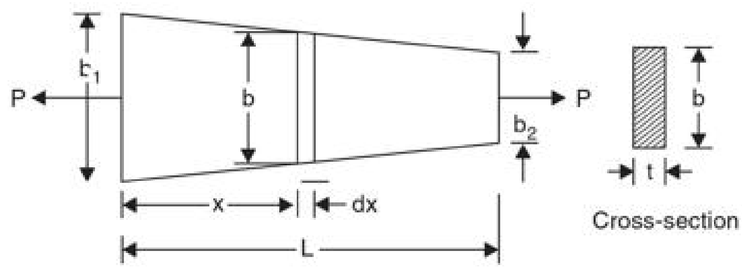


Fig. 19

Solution**:** Consider an elemental length dx at a distance x from larger end. Rate of change of breadth is $\dfrac{b_1 - b_2}{L}$.

Hence, width at section x is $b = b_1 - \dfrac{b_1 - b_2}{L}\, x = b_1 - kx$

where $k = \dfrac{b_1 - b_2}{L}$

$\therefore$ Cross-section area of the element $= A = t(b_1 - kx)$

Since force acting at all sections is P only,

Extension of element $= \dfrac{Pdx}{AE}$ [where length $= dx$]

$$= \frac{Pdx}{(b_1 - kx)tE}$$

Total extension of the bar $= \displaystyle\int_0^L \frac{Pdx}{(b_1 - kx)tE} = \frac{P}{tE}\int_0^L \frac{dx}{(b_1 - kx)}$

$$= \frac{P}{tE}\left(\frac{1}{-k}\right)\Big[\log(b_1 - kx)\Big]_0^L$$

$$= \frac{P}{tEk}\left[-\log\left(b_1 - \frac{b_1 - b_2}{L}\, x\right)\right]_0^L$$

$$= \frac{P}{tEk}[-\log b_2 + \log b_1] = \frac{P}{tEk}\log\frac{b_1}{b_2}$$

$$= \frac{\mathbf{PL}}{\mathbf{tE(b_1 - b_2)}}\log\frac{\mathbf{b_1}}{\mathbf{b_2}}. \qquad ...(16)$$

A tapering rod has diameter d_1 at one end and it tapers uniformly to a diameter d_2 at the other end in a length L as shown in Fig. 20. If modulus of elasticity of the material is E, find its change in length when subjected to an axial force P.

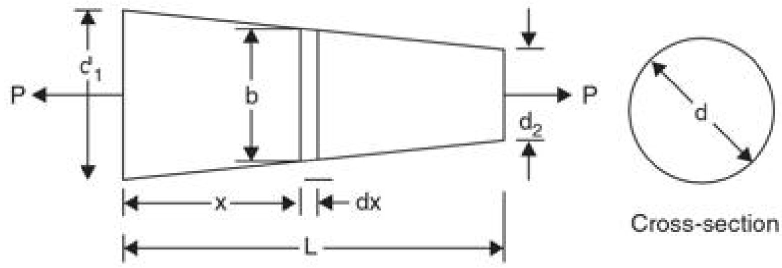


Fig. 20

***Solution*:** Change in diameter in length L is $d_1 - d_2$

$\therefore$ Rate of change of diameter, $k = \dfrac{d_1 - d_2}{L}$

Consider an elemental length of bar dx at a distance x from larger end. The diameter of the bar at this section is

$$d = d_1 - kx.$$

Cross-sectional area $\qquad A = \dfrac{\pi d^2}{4} = \dfrac{\pi}{4}(d_1 - kx)^2$

$\therefore$ Extension of the element $\qquad = \dfrac{P\,dx}{\dfrac{\pi}{4}(d_1 - kx)^2 E}$

Extension of the entire bar $\qquad \Delta = \displaystyle\int_0^L \frac{P\,dx}{\frac{\pi}{4}(d_1 - kx)^2 E}$

$$= \frac{4P}{\pi E}\int_0^L \frac{dx}{(d_1 - kx)^2}$$

$$= \frac{4P}{\pi Ek}\left(\frac{1}{d_1 - kx}\right)_0^L$$

$$= \frac{4P}{\frac{\pi E(d_1 - d_2)}{L}}\left(\frac{1}{d_2} - \frac{1}{d_1}\right), \text{ since } d_1 - kL = d_2$$

$$\Delta = \frac{4PL}{\pi E(d_1 - d_2)} \times \frac{(d_1 - d_2)}{d_1 d_2} = \frac{\mathbf{4PL}}{\mathbf{\pi E d_1 d_2}}. \qquad ...(17)$$

Example 6. *A steel flat of thickness 10 mm tapers uniformly from 60 mm at one end to 40 mm at other end in a length of 600 mm. If the bar is subjected to a load of 80 kN, find its extension. Take E = 2 × 10^5 MPa. What is the percentage error if average area is used for calculating extension?*

***Solution*:** Now, $t = 10$ mm $\quad b_1 = 60$ mm $\quad b_2 = 40$ mm

$L = 600$ mm $\quad P = 80$ kN $= 80000$ N

Now, $1 \text{ MPa} = 1 \text{ N/mm}^2$

Hence $E = 2 \times 10^5 \text{ N/mm}^2$

Extension of the tapering bar of rectangular section

$$\Delta = \frac{PL}{tE(b_1 - b_2)} \log \frac{b_1}{b_2}$$

$$= \frac{80000 \times 600}{10 \times 2 \times 10^5 (60 - 40)} \log \frac{60}{40}$$

$$= \mathbf{0.4865 \text{ mm}}$$

If averages cross-section is considered instead of tapering cross-section, extension is given by

$$\Delta = \frac{PL}{A_{av} E}$$

Now $$A_{av} = \frac{60 \times 10 + 40 \times 10}{2} = 500 \text{ mm}^2$$

$$\Delta = \frac{80000 \times 600}{500 \times 2 \times 10^5} = 0.480 \text{ mm}$$

∴ **Percentage error** $$= \frac{0.4865 - 0.48}{0.4865} \times 100$$

$$= \mathbf{1.348}$$

SHEAR STRESS

Figure 22 shows a bar subject to direct shearing force *i.e.*, the force parallel to the cross-section of bar. The section of a rivet/bolt subject to direct shear is shown in Fig. 23. Let Q be the shearing force and q the shearing stress acting on the section. Then, with usual assumptions that stresses are uniform we get,

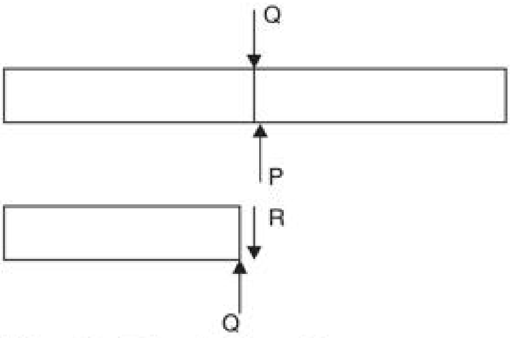


Fig. 22. Direct Shear Force on a Section

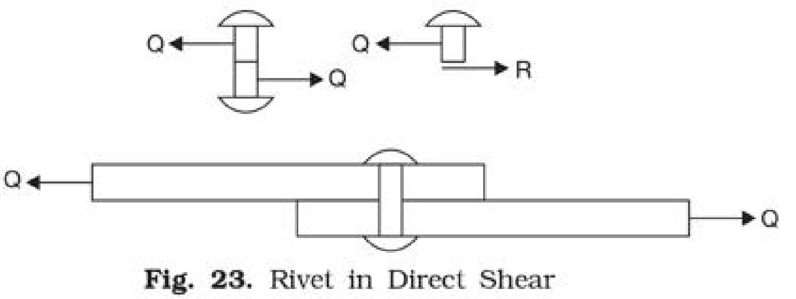


Fig. 23. Rivet in Direct Shear

$$R = \int q \, dA = q \int dA = qA$$

For equilibrium $$Q = R = qA$$

i.e., $$q = \frac{Q}{A} \qquad ...(18)$$

Thus, the direct stress is equal to shearing force per unit area.

POISSON'S RATIO

When a material undergoes changes in length, it undergoes changes of opposite nature in lateral directions. For example, if a bar is subjected to direct tension in its axial direction it elongates and at the same time its sides contract (Fig. 27).

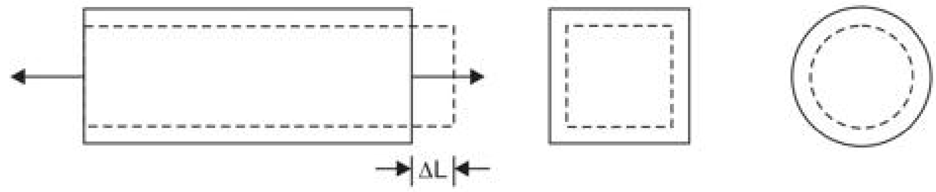


Fig. 27. Changes in Axial and Lateral Directions

If we define the ratio of change in axial direction to original length as linear strain and change in lateral direction to the original lateral dimension as lateral strain, it is found that *within elastic limit there is a constant ratio between lateral strain and linear strain. This constant ratio is called Poisson's ratio.* Thus,

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Linear strain}} \qquad ...(19)$$

It is denoted by $\frac{1}{m}$, or μ. For most of metals its value is between 0.25 to 0.33. Its value for steel is 0.3 and for concrete 0.15.

VOLUMETRIC STRAIN

When a member is subjected to stresses, it undergoes deformation in all directions. Hence, there will be change in volume. The *ratio of the change in volume to original volume is called volumetric strain.*

Thus $$e_v = \frac{\delta V}{V} \qquad ...(20)$$

where e_V = Volumetric strain

δ_V = Change in volume

V = Original volume

It can be shown that volumetric strain is sum of strains in three mutually perpendicular directions.

i.e., $$e_v = e_x + e_y + e_z$$

For example consider a bar of length L, breadth b and depth d as shown in Fig. 28.

Fig. 28

Now, $V = Lbd$

Since volume is function of L, b and d.

$$\delta V = \delta L\ bd + L\ \delta b\ d + Lb\ \delta d$$

$$\frac{\delta V}{V} = \frac{\delta v}{Lbd}$$

$$e_V = \frac{\delta L}{L} + \frac{\delta b}{b} + \frac{\delta d}{d}$$

$$e_V = e_x + e_y + e_z$$

Now, consider a circular rod of length L and diameter 'd' as shown in Fig. 29.

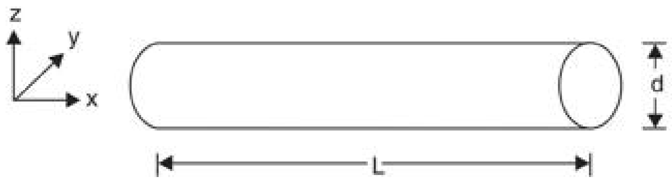


Fig. 29

Volume of the bar $\qquad V = \frac{\pi}{4}\, d^2 L$

$\therefore \qquad \delta V = \frac{\pi}{4}\, 2d\delta d\; L + \frac{\pi}{4}\, d^2\, \delta L \qquad$ (since v is function of d and L).

$$\therefore \qquad \frac{\delta V}{\frac{\pi}{4} d^2 L} = 2\frac{\delta d}{d} + \frac{\delta L}{L}$$

$$e_V = e_x + e_y + e_z;\ \text{since } e_y = e_z = \frac{\delta d}{d}$$

In general for any shape *volumetric strain may be taken as sum of strains in three mutually perpendicular directions*.

ELASTIC CONSTANTS

Modulus of elasticity, modulus of rigidity and bulk modulus are the three elastic constants. Modulus of elasticity (Young's Modulus) 'E' has been already defined as the ratio of linear stress to linear strain within elastic limit. Rigidity modulus and Bulk modulus are defined in this article.

Modulus of Rigidity: It is defined as the *ratio of shearing stress to shearing strain within elastic limit and is usually denoted by letter G or N.* Thus

$$G = \frac{q}{\phi} \qquad ...(21)$$

where $\quad G$ = Modulus of rigidity

$\qquad q$ = Shearing stress

and $\quad \phi$ = Shearing strain

Bulk Modulus: When a body is subjected to identical stresses p in three mutually perpendicular directions, (Fig. 30), the body undergoes uniform changes in three directions without undergoing distortion of shape. The ratio of change in volume to original volume has been defined as volumetric strain (e_v). Then the bulk modulus, K is defined as

$$K = \frac{p}{e_v}$$

where $\quad p$ = identical pressure in three mutually perpendicular directions

$\qquad e_v = \frac{\Delta_v}{v}$, Volumetric strain

$\qquad \Delta_v$ = Change in volume

$\qquad v$ = Original volume

Thus bulk modulus may be defined as the ratio of identical pressure 'p' acting in three mutually perpendicular directions to corresponding volumetric strain.

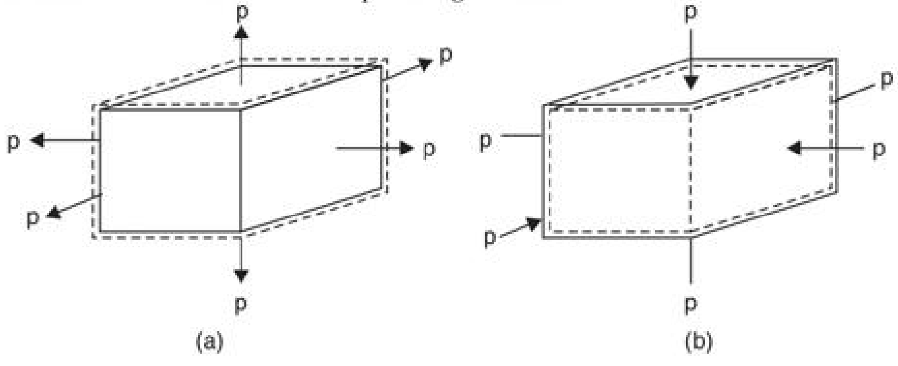


Fig. 30

Figure 30 shows a body subjected to identical compressive pressure 'p' in three mutually perpendicular directions. Since hydrostatic pressure, the pressure exerted by a liquid on a body within it, has this nature of stress, such a pressure 'p' is called as hydrostatic pressure.

RELATIONSHIP BETWEEN MODULUS OF ELASTICITY AND MODULUS OF RIGIDITY

Consider a square element $ABCD$ of sides 'a' subjected to pure shear 'q' as shown in Fig. 8.31. $AEC'D$ shown is the deformed shape due to shear q. Drop perpendicular BF to diagonal DE. Let ϕ be the shear strain and G modulus of rigidity.

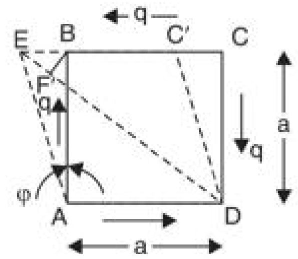


Fig. 31

Now, strain in diagonal $$BD = \frac{DE - DF}{DF} = \frac{EF}{DB} = \frac{EF}{AB\sqrt{2}}$$

Since angle of deformation is very small we can assume $\angle BEF = 45°$, hence $EF = BE \cos 45°$

$\therefore$ Strain in diagonal
$$BD = \frac{EF}{BD} = \frac{BE \cos 45°}{AB\sqrt{2}}$$
$$= \frac{a \tan\phi \cos 45°}{a\sqrt{2}}$$
$$= \frac{1}{2}\tan\phi = \frac{1}{2}\phi \qquad \text{(Since } \phi \text{ is very small)}$$
$$= \frac{1}{2} \times \frac{q}{G}, \text{ since } \phi = \frac{q}{G} \qquad ...(1)$$

Now, we know that the above pure shear gives rise to axial tensile stress q in the diagonal direction of DB and axial compression q at right angles to it. These two stresses cause tensile strain along the diagonal DB.

Tensile strain along the diagonal $DB = \frac{q}{E} + \mu\frac{q}{E} = \frac{q}{E}(1+\mu)$ ...(2)

From equations (1) and (2), we get

$$\frac{1}{2} \times \frac{q}{G} = \frac{q}{E}(1+\mu)$$
$$E = 2G(1+\mu) \qquad ...(22)$$

RELATIONSHIP BETWEEN MODULUS OF ELASTICITY AND BULK MODULUS

Consider a cubic element subjected to stresses p in the three mutually perpendicular direction x, y, z as shown in Fig. 32.

Now the stress p in x direction causes tensile strain $\frac{p}{E}$ in x direction while the stress p in y and z direction cause compressive strains $\mu\ \frac{p}{E}$ in x direction.

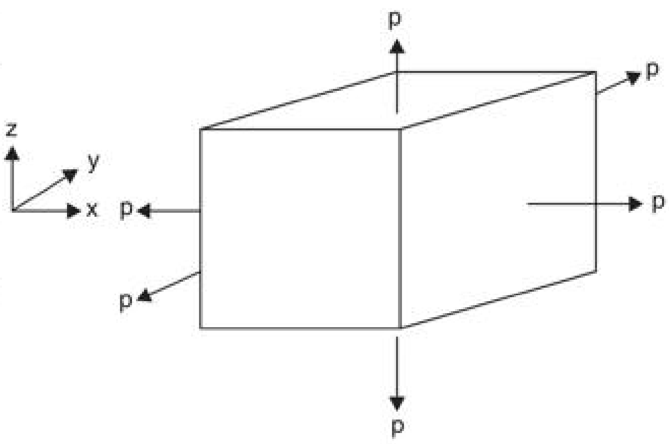


Fig. 32

Hence, $$e_x = \frac{p}{E} - \mu\frac{p}{E} - \mu\frac{p}{E}$$
$$= \frac{p}{E}(1-2\mu)$$

Similarly $e_y = \frac{p}{E}(1-2\mu)$

$$e_z = \frac{p}{E}(1 - 2\mu) \quad \text{...(1)}$$

$\therefore$ Volumetric strain $\quad e_v = e_x + e_y + e_z = \dfrac{3p}{E}(1 - 2\mu)$

From definition, bulk modulus K is given by

$$K = \frac{p}{e_v} = \frac{p}{\dfrac{3p(1-2\mu)}{E}}$$

or $\quad E = 3K(1 - \mu) \quad \text{...(2)}$

Relationship between EGK:

We know $\quad E = 2G(1 + \mu) \quad \text{...}(a)$

and $\quad E = 3K(1 - 2\mu) \quad \text{...}(b)$

By eliminating μ between the above two equations we can get the relationship between E, G, K, free from the term μ.

From equation (a) $\mu = \dfrac{E}{2G} - 1$

Substituting it in equation (b), we get

$$E = 3K\left[1 - 2\left(\frac{E}{2G} - 1\right)\right]$$

$$= 3K\left(1 - \frac{E}{G} + 2\right) = 3K\left(3 - \frac{E}{G}\right)$$

$$= 9K - \frac{3KE}{G}$$

$\therefore \quad E\left(1 + \dfrac{3K}{G}\right) = 9K$

or $\quad E\left(\dfrac{G + 3K}{G}\right) = 9K \quad \text{...}(c)$

or $\quad E = \dfrac{9KG}{G + 3K} \quad \text{...}(23a)$

Equation (c) may be expressed as

$$\frac{9}{E} = \frac{G + 3K}{KG}$$

Example 7. *A circular rod of 25 mm diameter and 500 mm long is subjected to a tensile force of 60 kN. Determine modulus of rigidity, bulk modulus and change in volume if Poisson's ratio = 0.3 and Young's modulus E = 2 × 10⁵ N/mm².*

Solution: From the relationship

$$E = 2G(1 + \mu) = 3k(1 - 2\mu)$$

We get, $$\boldsymbol{G} = \frac{E}{2(1+\mu)} = \frac{2\times10^5}{2(1+0.3)} = \mathbf{0.7692 \times 10^5\ N/mm^2}$$

and $$\boldsymbol{K} = \frac{E}{3(1+2\mu)} = \frac{2\times10^5}{3(1-2\times0.3)} = \mathbf{1.667 \times 10^5\ N/mm^2}$$

$$\text{Longitudinal stress} = \frac{P}{A} = \frac{60\times10^3}{\frac{\pi}{4}\times25^2} = 122.23\ \text{N/mm}^2$$

$$\text{Linear strain} = \frac{\text{Stress}}{E} = \frac{122.23}{2\times10^5} = 61.115\times10^{-5}$$

$$\text{Lateral strain} = e_y = -\mu e_x \quad \text{and} \quad e_z = -\mu e_x$$

$$\text{Volumetric strain} \quad e_v = e_x + e_y + e_z$$

$$= e_x(1 - 2\mu)$$

$$= e_v \quad = 61.115 \times 10^{-5}\ (1 - 2 \times 0.3)$$

$$= 24.446 \times 10^{-5}$$

but $$\frac{\text{Change in volume}}{v}$$

$\therefore$ **Change in volume** $= e_v \times v$

$$= 24.446\times10^{-5}\times\frac{\pi}{4}\times(25^2)\times500$$

$$= \mathbf{60\ mm^3}$$

Example 8. *A 400 mm long bar has rectangular cross-section 10 mm × 30 mm. This bar is subjected to*

(i) 15 kN tensile force on 10 mm × 30 mm faces,

(ii) 80 kN compressive force on 10 mm × 400 mm faces, and

(iii) 180 kN tensile force on 30 mm × 400 mm faces.

Find the change in volume if E = 2 × 10⁵ N/mm² and μ = 0.3.

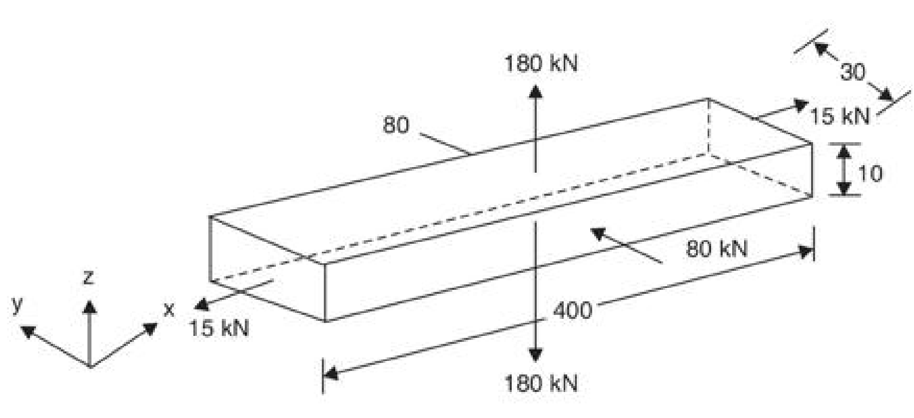


Fig 33

Example 9. *In a laboratory, tensile test is conducted and Young's modulus of the material is found to be 2.1×10^5 N/mm^2. On the same material torsion test is conducted and modulus of rigidity is found to be 0.78×10^5 N/mm^2. Determine Poisson's Ratio and bulk modulus of the material.*

[**Note:** This is usual way of finding material properties in the laboratory].

***Solution*:**

$$E = 2.1 \times 10^5 \text{ N/mm}^2$$

$$G = 0.78 \times 10^5 \text{ N/mm}^2$$

Using relation $E = 2G(1 + \mu)$

we get $2.1 \times 10^5 = 2 \times 0.78 \times 10^5 (1 + \mu)$

$$1.346 = 1 + \mu$$

or $\boldsymbol{\mu = 0.346}$

From relation $E = 3K(1 - 2\mu)$

we get $2.1 \times 10^5 = 3 \times K(1 - 2 \times 0.346)$

$$\mathbf{K = 2.275 \times 10^5 \text{ N/mm}^2}$$

Example 10. *A material has modulus of rigidity equal to 0.4×10^5 N/mm^2 and bulk modulus equal to 0.8×10^5 N/mm^2. Find its Young's Modulus and Poisson's Ratio.*

***Solution*:**

$$G = 0.4 \times 10^5 \text{ N/mm}^2$$

$$K = 0.8 \times 10^5 \text{ N/mm}^2$$

Using the relation $E = \dfrac{9GK}{3K + G}$

$$E = \frac{9 \times 0.4 \times 10^5 \times 0.8 \times 10^5}{3 \times 0.8 \times 10^5 + 0.4 \times 10^5}$$

$$\mathbf{E = 1.0286 \times 10^5 \text{ N}}$$

From the relation $E = 2G(1 + \mu)$

we get $1.0286 \times 10^5 = 2 \times 0.4 \times 10^5(1 + \mu)$

$$1.2857 = 1 + \mu$$

or $\boldsymbol{\mu = 0.2857}$

COMPOSITE/COMPOUND BARS

Bars made up of two or more materials are called composite/compound bars. They may have same length or different lengths as shown in Fig. 34. The ends of different materials of the bar are held together under loaded conditions.

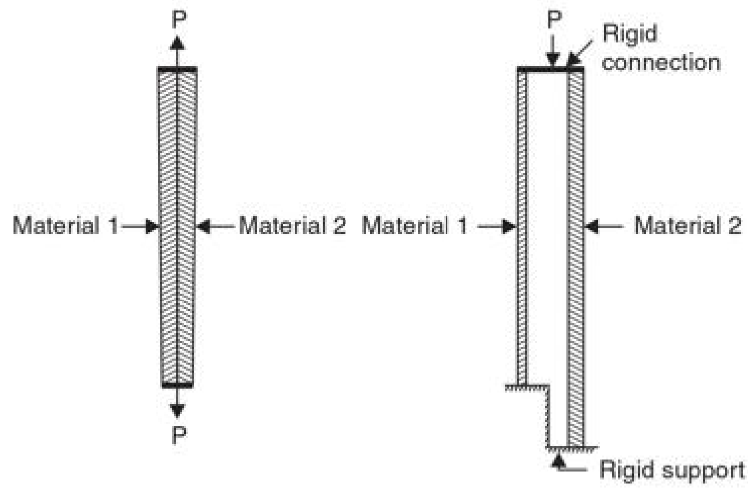


Fig. 34

Consider a member with two materials. Let the load shared by material 1 be P_1 and that by material 2 be P_2. Then

(*i*) From equation of equilibrium of the forces, we get

$$P = P_1 + P_2 \qquad \text{...24a)}$$

(*ii*) Since the ends are held securely, we get

$$\Delta l_1 = \Delta l_2$$

where Δl_1 and Δl_2 are the extension of the bars of material 1 and 2 respectively

i.e.
$$\frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L_2}{A_2 E_2} \qquad \text{...24b)}$$

Using equations 8.24(*a*) and (*b*), P_1 and P_2 can be found uniquely. Then extension of the system can be found using the relation $\Delta l = \frac{P_1 L_1}{A_1 E_1}$ or $\Delta l = \frac{P_2 L_2}{A_2 E_2}$ since $\Delta l = \Delta l_1 = \Delta l_2$.

The procedure of the analysis of compound bars is illustrated with the examples below:

Example 11. *A compound bar of length 600 mm consists of a strip of aluminium 40 mm wide and 20 mm thick and a strip of steel 60 mm wide × 15 mm thick rigidly joined at the ends. If elastic modulus of aluminium and steel are 1 × 10⁵ N/mm² and 2 × 10⁵ N/mm², determine the stresses developed in each material and the extension of the compound bar when axial tensile force of 60 kN acts.*

Solution: The compound bar is shown in the figure 8.36.

Data available is

$$L = 600 \text{ mm}$$
$$P = 60 \text{ kN} = 60 \times 1000 \text{ N}$$
$$A_a = 40 \times 20 = 800 \text{ mm}^2$$
$$A_s = 60 \times 15 = 900 \text{ mm}^2$$
$$E_a = 1 \times 10^5 \text{ N/mm}^2,\ E_s = 2 \times 10^5 \text{ N/mm}^2.$$

Let the load shared by aluminium strip be P_a and that shared by steel be P_s. Then from equilibrium condition

$$P_a + P_s = 60 \times 1000 \qquad \text{...(1)}$$

From compatibility condition, we have

$$\Delta_a = \Delta_s$$

$$\frac{P_a L}{A_a E_a} = \frac{P_s L}{A_s E_s}$$

i.e.

$$\frac{P_a \times 600}{800 \times 1 \times 10^5} = \frac{P_s \times 600}{900 \times 2 \times 10^5}$$

$$P_s = 2.25\ P_a \qquad \text{...(2)}$$

Substituting it in eqn. (1), we get

$$P_a + 2.25\ P_a = 60 \times 1000$$

i.e.

$$P_a = 18462 \text{ N.}$$

$\therefore$

$$P_s = 2.25 \times 18462 = 41538 \text{ N.}$$

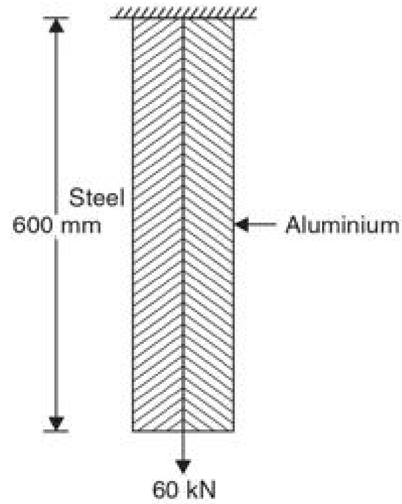


Fig. 35

$\therefore$ **Stress in aluminium strip** $= \dfrac{P_a}{A_a} = \dfrac{18462}{800}$

$$= \mathbf{23.08\ N/mm^2}$$

Stress in steel strip $= \dfrac{P_s}{A_s} = \dfrac{41538}{900} = \mathbf{46.15\ N/mm^2}$

Extension of the compound bar $= \dfrac{P_a L}{A_a E_a} = \dfrac{18462 \times 600}{800 \times 1 \times 10^5}$

$$\boldsymbol{\Delta l = 0.138 \text{ mm.}}$$

Example 12. *A compound bar consists of a circular rod of steel of 25 mm diameter rigidly fixed into a copper tube of internal diameter 25 mm and external diameter 40 mm as shown in Fig. 36. If the compound bar is subjected to a load of 120 kN, find the stresses developed in the two materials.*

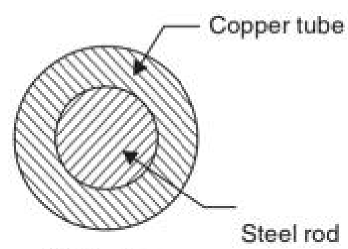


Fig. 36

Take $E_s = 2 \times 10^5$ N/mm^2
and $E_c = 1.2 \times 10^5$ N/mm^2.

Solution: Area of steel rod $A_s = \dfrac{\pi}{4} \times 25^2 = 490.87$ mm^2

Area of copper tube $A_c = \dfrac{\pi}{4}(40^2 - 25^2) = 765.76$ mm^2

From equation of equilibrium,

$$P_s + P_c = 120 \times 1000 \qquad \text{...(1)}$$

where P_s is the load shared by steel rod and P_c is the load shared by the copper tube.

From compatibility condition, we have

$$\Delta_s = \Delta_c$$

$$\frac{P_s L}{A_s E_s} = \frac{P_c L}{A_c E_c}$$

$$\frac{P_s}{490.87 \times 2 \times 10^5} = \frac{P_c}{765.76 \times 1.2 \times 10^5}$$

$\therefore$

$$P_s = 1.068\ P_c \qquad \text{...(2)}$$

From eqns. (1) and (2), we get

$$1.068\ P_c + P_c = 120 \times 1000$$

$$\therefore \quad P_c = \frac{120 \times 1000}{2.068} = 58027 \text{ N}$$

$$\therefore \quad P_s = 1.068\ P_c = 61973 \text{ N}$$

$\therefore$ **Stress in copper** $= \dfrac{58027}{9765.76} =$ **75.78 N/mm²**

Stress in steel $= \dfrac{61973}{490.87} =$ **126.25 N/mm²**

Example 13. *Three pillars, two of aluminium and one of steel support a rigid platform of 250 kN as shown in Fig. 38. If area of each aluminium pillar is 1200 mm² and that of steel pillar is 1000 mm², find the stresses developed in each pillar.*

Take $E_s = 2 \times 10^5$ N/mm² *and* $E_a = 1 \times 10^6$ N/mm².

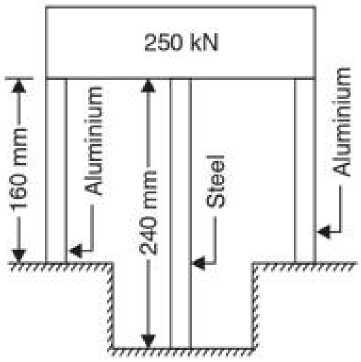


Fig. 38

Solution: Let force shared by each aluminium pillar be P_a and that shared by steel pillar be P_s.

$\therefore$ The forces in vertical direction = 0 →

$$P_a + P_s + P_a = 250$$

$$2P_a + P_s = 250 \qquad \text{...(1)}$$

From compatibility condition, we get

$$\Delta_s = \Delta_a$$

$$\frac{P_s L_s}{A_s E_s} = \frac{P_a L_a}{A_a E_a}$$

$$\frac{P_s \times 240}{1000 \times 2 \times 10^5} = \frac{P_a \times 160}{1200 \times 1 \times 10^5}$$

$$\therefore \quad P_s = 1.111\ P_a \qquad \text{...(2)}$$

From eqns. (1) and (2), we get

$$P_a\ (2 + 1.111) = 250$$

$$\therefore \quad P_a = 80.36 \text{ kN}$$

Hence from eqn. (1),

$$P_s = 250 - 2 \times 80.36 = 89.28 \text{ kN}$$

$\therefore$ Stresses developed are

$$\sigma_s = \frac{P_s}{A_s} = \frac{89.28 \times 1000}{1000} = \mathbf{89.28\ N/mm^2}$$

$$\sigma_a = \frac{80.36 \times 1000}{1200} = \mathbf{66.97\ N/mm^2}$$

Example 14. *A steel bolt of 20 mm diameter passes centrally through a copper tube of internal diameter 28 mm and external diameter 40 mm. The length of whole assembly is 600 mm. After tight fitting of the assembly, the nut is over tightened by quarter of a turn. What are the stresses introduced in the bolt and tube, if pitch of nut is 2 mm? Take* $E_s = 2 \times 10^5$ N/mm² *and* $E_c = 1.2 \times 10^5$ N/mm².

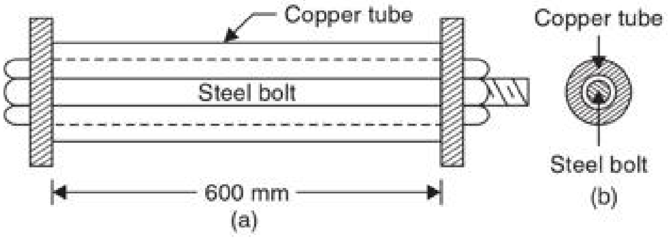


Fig. 39

Solution: Figure 8.40 shows the assembly. Let the force shared by bolt be P_s and that by tube be P_c. Since there is no external force, static equilibrium condition gives

$$P_s + P_c = 0 \quad \text{or} \quad P_s = -P_c$$

i.e., the two forces are equal in magnitude but opposite in nature. Obviously bolt is in tension and tube is in compression.

Let the magnitude of force be P. Due to quarter turn of the nut, the nut advances by $\frac{1}{4} \times$ pitch $= \frac{1}{4} \times 2 = 0.5$ mm.

[**Note.** Pitch means advancement of nut in one full turn]

During this process bolt is extended and copper tube is shortened due to force P developed. Let Δ_s be extension of bolt and Δ_c shortening of copper tube. Final position of assembly be Δ, then

$$\Delta_s + \Delta_c = \Delta$$

i.e.

$$\frac{P_s L_s}{A_s E_s} + \frac{P_c L_c}{A_c E_c} = 0.5$$

$$\frac{P \times 600}{(\pi/4) \times 20^2 \times 2 \times 10^5} + \frac{P \times 600}{(\pi/4)(40^2 - 28^2) \times 1.2 \times 10^5} = 0.5$$

$$\frac{P \times 600}{(\pi/4) \times 10^5}\left[\frac{1}{20^2 \times 2} + \frac{1}{(40^2 - 28^2) \times 1.2}\right] = 0.5$$

$$\therefore \quad P = 28816.8 \text{ N}$$

$$\therefore \quad p_s = \frac{P_s}{A_s} = \frac{28816.8}{(\pi/4) \times 20^2} = \mathbf{91.72\ N/mm^2}$$

$$p_c = \frac{P_c}{A_c} = \frac{28816.8}{(\pi/4)(40^2 - 28^2)} = \mathbf{44.96\ N/mm^2}$$

THERMAL STRESSES

Every material expands when temperature rises and contracts when temperature falls. It is established experimentally that the change in length Δ is directly proportional to the length of the member L and change in temperature t. Thus

$$\Delta \propto tL$$

$$= \alpha \, tL \qquad \text{...(8.25)}$$

The constant of proportionality α is called coefficient of thermal expansion and is defined as change in unit length of material due to unit change in temperature. Table 8.1 shows coefficient of thermal expansion for some of the commonly used engineering materials:

Table 1

Material	*Coefficient of thermal expansion*
Steel	$12 \times 10^{-6}/°C$
Copper	$17.5 \times 10^{-6}/°C$
Stainless steel	$18 \times 10^{-6}/°C$
Brass, Bronze	$19 \times 10^{-6}/°C$
Aluminium	$23 \times 10^{-6}/°C$

If the expansion of the member is freely permitted, as shown in Fig. 8.41, no temperature stresses are induced in the material.

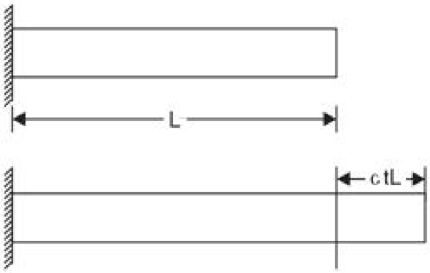


Fig. 40 Free Expansion Permitted

If the free expansion is prevented fully or partially the stresses are induced in the bar, by the support forces. Referring to Fig. 41,

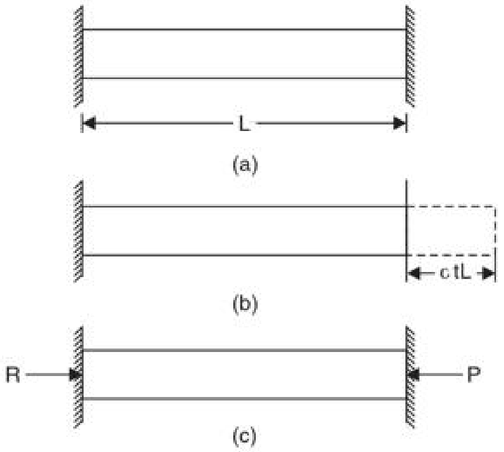


Fig. 41

If free expansion is permitted the bar would have expanded by

$$\Delta = \alpha \, tL$$

Since support is not permitting it, the support force P develops to keep it at the original position. Magnitude of this force is such that contraction is equal to free expansion, *i.e.*

$$\frac{PL}{AE} = \alpha \, tL$$

or

$$p = E \, \alpha \, t. \qquad \text{...(26)}$$

s. It is compressive in nature in this case.

Consider the case shown in Fig. 8.43 in which free expansion is prevented partially.

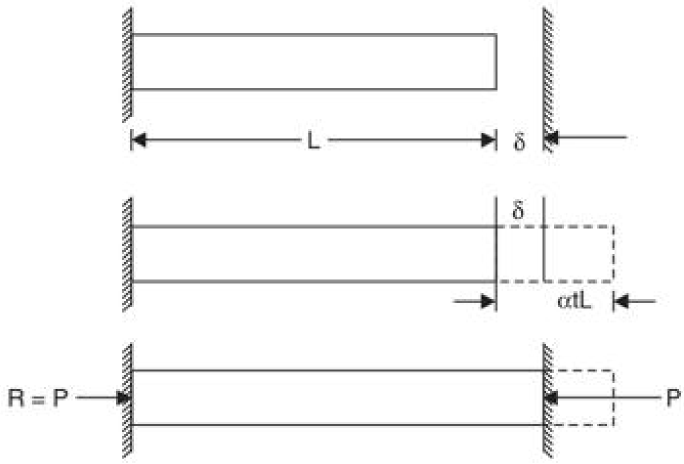


Fig. 42

In this case free expansion = $\alpha\, tL$

Expansion prevented $\Delta = \alpha\, tL - \delta$

The expansion is prevented by developing compressive force P at supports

$$\therefore \quad \frac{PL}{AE} = \Delta = \alpha\, tL - \delta. \qquad ...(27)$$

Example 15. *A steel rail is 12 m long and is laid at a temperature of 18°C. The maximum temperature expected is 40°C.*

(*i*) *Estimate the minimum gap between two rails to be left so that the temperature stresses do not develop.*

(*ii*) *Calculate the temperature stresses developed in the rails, if:*

(*a*) *No expansion joint is provided.*

(*b*) *If a 1.5 mm gap is provided for expansion.*

(*iii*) *If the stress developed is 20 N/mm², what is the gap provided between the rails?*

Take $E = 2 \times 10^5$ N/mm² *and* $\alpha = 12 \times 10^{-6}$/°C.

Solution:

(*i*) The free expansion of the rails

$$= \alpha\, tL = 12 \times 10^{-6} \times (40 - 18) \times 12.0 \times 1000$$

$$= 3.168 \text{ mm}$$

∴ **Provide a minimum gap of 3.168 mm between the rails, so that temperature stresses do not develop.**

(*ii*) (*a*) If no expansion joint is provided, free expansion prevented is equal to 3.168 mm.

i.e. $\Delta = 3.168$ mm

$$\therefore \quad \frac{PL}{AE} = 3.168$$

$$\therefore \quad p = \frac{P}{A} = \frac{3.168 \times 2 \times 10^5}{12 \times 1000} = \mathbf{52.8\ N/mm^2}$$

(*b*) If a gap of 1.5 mm is provided, free expansion prevented $\Delta = \alpha\, tL - \delta = 3.168 - 1.5 = 1.668$ mm.

∴ The compressive force developed is given by $\frac{PL}{AE} = 1.668$

$$\text{or} \quad p = \frac{P}{A} = \frac{1.668 \times 2 \times 10^5}{12 \times 1000} = \mathbf{27.8\ N/mm^2}$$

(*iii*) If the stress developed is 20 N/mm^2, then $p = \frac{P}{A} = 20$

If δ is the gap, $\quad \Delta = \alpha\ tL - \delta$

$$\therefore \qquad \frac{PL}{AE} = 3.168 - \delta$$

$$\text{i.e.} \qquad 20 \times \frac{12 \times 1000}{2 \times 10^5} = 3.168 - \delta$$

$$\therefore \qquad \boldsymbol{\delta} = 3.168 - 1.20 = \mathbf{1.968\ mm}$$

Example 16. *The composite bar shown in Fig. 43 is rigidly fixed at the ends A and B. Determine the reaction developed at ends when the temperature is raised by 18°C. Given*

$$E_a = 70\ kN/mm^2$$
$$E_s = 200\ kN/mm^2$$
$$\alpha_a = 11 \times 10^{-6}/°C$$
$$\alpha_s = 12 \times 10^{-6}/°C$$

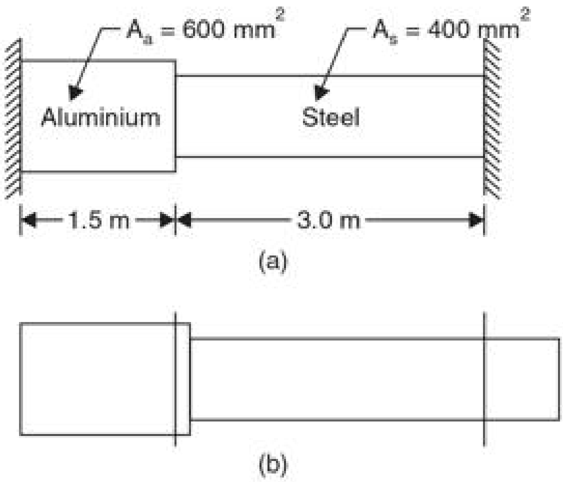


Fig.43

Solution: Free expansion $= \alpha_a\ tL_a + \alpha_s tL_s$

$$= 11 \times 10^{-6} \times 18 \times 1500 + 12 \times 10^{-6} \times 18 \times 3000$$
$$= 0.945\ \text{mm}$$

Since this is prevented

$$\Delta = 0.945\ \text{mm.}$$
$$E_a = 70\ \text{kN/mm}^2 = 70000\ \text{N/mm}^2\ ;$$
$$E_s = 200\ \text{kN/mm}^2 = 200 \times 1000\ \text{N/mm}^2$$

If P is the support reaction,

$$\Delta = \frac{PL_a}{A_a E_a} + \frac{PL_s}{A_s E_s}$$

$$\text{i.e.} \qquad 0.945 = P\left[\frac{1500}{600 \times 70000} + \frac{3000}{400 \times 200 \times 1000}\right]$$

$$0.945 = 73.214 \times 10^{-6}\ P$$

$$\text{or} \qquad \boldsymbol{P} = \mathbf{12907.3\ N}$$

THERMAL STRESSES IN COMPOUND BARS

When temperature rises the two materials of the compound bar experience different free expansion. Since they are prevented from seperating, the two bars will have common position. This is possible only by extension of the bar which has less free expansion and contraction of the bar which has more free expansion. Thus one bar develops tensile force and another develops the compressive force. In this article we are interested to find such stresses.

Consider the compound bar shown in Fig. 45(*a*). Let α_1, α_2 be coefficient of thermal expansion and E_1, E_2 be moduli of elasticity of the two materials respectively. If rise in temperature is 't',

$$\text{Free expansion of bar } 1 = \alpha_1\, tL$$

$$\text{Free expansion of bar } 2 = \alpha_2\, tL$$

Let $\alpha_1 > \alpha_2$. Hence the position of the two bars, if the free expansions are permitted are at AA and BB as shown in Fig.

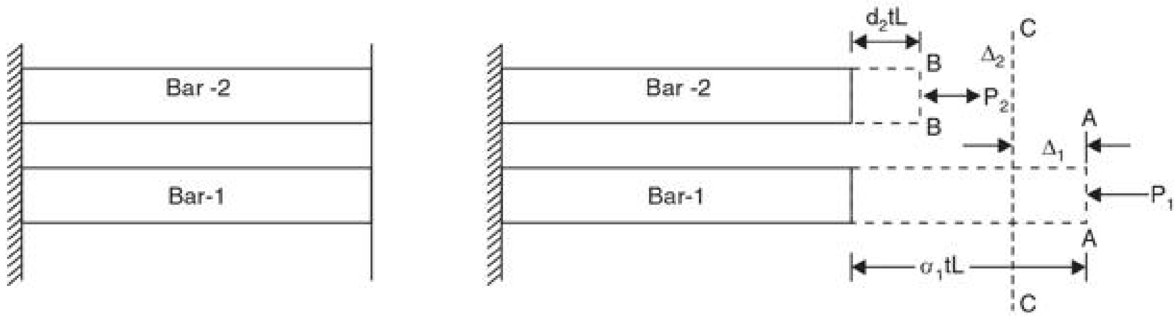


Fig. 45

Since the two bars are rigidly connected at the ends, the final position of the end will be somewhere between AA and BB, say at CC. It means Bar–1 will experience compressive force P_1 which contracts it by Δ_1 and Bar–2 experience tensile force P_2 which will expand it by Δ_2.

The equilibrium of horizontal forces gives,

$$P_1 = P_c, \text{ say } P$$

From the Fig. 8.46 (*b*), it is clear,

$$\alpha_1\, tL - \Delta_1 = \alpha_2\, tL + \Delta_2$$

$$\therefore \qquad \Delta_1 + \Delta_2 = \alpha_1\, tL - \alpha_2\, tL = (\alpha_1 - \alpha_2)\, tL.$$

If the cross-sectional areas of the bars are A_1 and A_2, we get

$$\frac{PL}{A_1 E_1} + \frac{PL}{A_2 E_2} = (\alpha_1 - \alpha_2)\, t\, L \qquad \text{...(8.28)}$$

From the above equation force P can be found and hence the stresses P_1 and P_2 can be determined.

Example 17. *A bar of brass 20 mm is enclosed in a steel tube of 40 mm external diameter and 20 mm internal diameter. The bar and the tubes are initially 1.2 m long and are rigidly fastened at both ends using 20 mm diameter pins. If the temperature is raised by 60°C, find the stresses induced in the bar, tube and pins.*

Given:

$$E_s = 2 \times 10^5 \text{ N/mm}^2$$
$$E_b = 1 \times 10^5 \text{ N/mm}^2$$
$$\alpha_s = 11.6 \times 10^{-6}/°C$$
$$\alpha_b = 18.7 \times 10^{-6}/°C.$$

Solution:

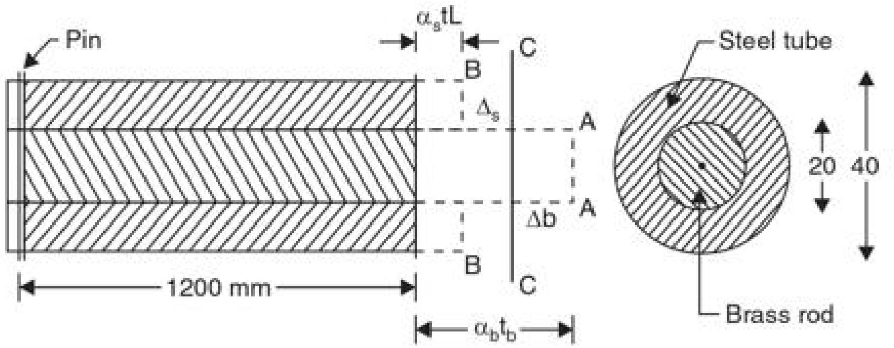


Fig. 46

$$t = 60° \quad E_s = 2 \times 10^5 \text{ N/mm}^2 \quad E_b = 1 \times 10^5 \text{ N/mm}^2$$
$$\alpha_s = 11.6 \times 10^{-6}/°C \quad \alpha_b = 18.7 \times 10^{-6}/°C$$
$$A_s = \frac{\pi}{4}(40^2 - 20^2) \quad A_b = \frac{\pi}{4} \times 20^2$$
$$= 942.48 \text{ mm}^2 \quad = 314.16 \text{ mm}^2$$

Since free expansion of brass ($\alpha_b\ tL$) is more than free expansion of steel ($\alpha_s\ tL$), compressive force P_b develops in brass and tensile force P_s develops in steel to keep the final position at CC (Ref: Fig. 46).

Horizontal equilibrium condition gives $P_b = P_s$, say P. From the figure, it is clear that

$$\Delta_s + \Delta_b = \alpha_b\ tL - \alpha_s tL = (\alpha_b - \alpha_s)tL.$$

where Δ_s and Δ_b are the changes in length of steels and brass bars.

$$\therefore \quad \frac{PL}{A_s E_s} + \frac{PL}{A_b E_b} = (18.7 - 11.6) \times 10^{-6} \times 60 \times 1200.$$

$$P \times 1200 \left[\frac{1}{942.48 \times 2 \times 10^5} + \frac{1}{314.16 \times 1 \times 10^5}\right] = 7.1 \times 10^{-6} \times 60 \times 1200$$

$$\therefore \quad P = 11471.3 \text{ N}$$

$$\therefore \quad \textbf{Stress in steel} = \frac{P}{A_s} = \frac{11471.3}{942.48} = \mathbf{12.17\ N/mm^2}$$

and $$\textbf{Stress in brass} = \frac{P}{A_b} = \frac{11471.3}{314.16} = \mathbf{36.51\ N/mm^2}$$

The pin resist the force P at the two cross-sections at junction of two bars.

$$\therefore \quad \textbf{Shear stress in pin} = \frac{P}{2 \times \text{Area of pin}}$$

$$= \frac{11471.3}{2 \times \pi/4 \times 20^2} = \mathbf{18.26\ N/mm^2}$$

IMPORTANT FORMULAE

1. If stress is uniform

$$p = \frac{P}{A}$$

2. (*i*) Linear strain $= \frac{\text{Change in length}}{\text{Original length}}$

(*ii*) Lateral strain $= \frac{\text{Change in lateral dimension}}{\text{Original lateral dimension}}$

3. Poisson's ratio $= \frac{\text{Lateral strain}}{\text{Linear strain}}$, within elastic limit.

4. Percentage elongation $= \frac{L' - L}{L} \times 100.$

5. Percentage reduction in area $= \frac{A - A'}{A} \times 100.$

6. Nominal stress $= \frac{\text{Load}}{\text{Original cross-sectional area}}.$

7. True stress $= \frac{\text{Load}}{\text{Actual cross-sectional area}}.$

8. Factor of safety $= \frac{\text{Ultimate stress}}{\text{Working stress}}$

However in case of steel, $= \frac{\text{Yield stress}}{\text{Working stress}}.$

9. Hooke's Law, $p = Ee$.

10. Extension/shortening of bar $= \frac{PL}{AE}.$

11. Extension of flat bar with linearly varying width and constant thickness $= \frac{PL}{tE(b_1 - b_2)} \log \frac{b_1}{b_2}.$

12. Extension of linearly tapering rod $= \frac{4PL}{\pi E\, d_1 d_2} = \frac{PL}{(\pi/4\; d_1 d_2)\, E}.$

13. Direct shear stress $= \frac{Q}{A}.$

14. Volumetric strain $\quad e_v = \frac{\delta V}{V} = e_x + e_y + e_z.$

15. $E = 2G\,(1 + \mu) = 3K\,(1 - 2\mu)$

or
$$\frac{9}{E} = \frac{3}{G} + \frac{1}{K}.$$

16. Extension due to rise in temperature:

$$\Delta = \alpha\; tL$$

17. Thermal force, P is given by

$$\frac{PL}{AE} = \text{extension prevented.}$$