Unit – 1

Complex - analysis

- **Function of Complex Variable/ Differentiation:**

If for each value of the complex variable Z= X+iY in a given region 'R' , we have one or more values of w=f(z)=u+iv, Then W is said to be a function of 'Z' , and we have w=f(z)=u+iv.

Where u and v are real and imaginary parts of f(z). z=x+iy

and

f(z)=u(x,y)+iv(x,y) is a complex function.

- **Continuity of a Function:**

Let f(z) is said to be continuous function at z=z if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- **Differentiability of a Function:**

A function f(z) is said to be differentiable at z=z if

exists. It is donated by $\lim_{\Delta z \to 0} \left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right)$ $f^{I}(z_0)$

i.e. $f^{I}(z_0) = \lim_{\Delta z \to 0} \left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right)$

- Analytical Function:

The complex function f(z) is said to be analytical function at z=a if the function f(z) has derivative at z=a and neighbourhood of z=a.

Example:

1. Let f(z) = z^2 f'(z) = 2z

At z=0, f'(z) = 2(0) = 0 (finite) f(z)

has derivative at z=0

Finally f(z) is called **analytical** function.

2. Let f(z) = $\frac{1}{z}$

f'(z) = $\frac{-1}{z^2}$ At z=0, f'(z) = $\frac{-1}{(0)^2}$ = ∞

f(z) has no derivative at z=0

Finally f(z) is called **not analytical** function.

- **Singular Point:**

Let z=a is said to be singular point if the function f(z) is not analytical at z=a.

Example:

f(z) = $\frac{1}{z}$, f'(z) = ∞ z = 0 is called singular point.

- **Cauchy – Riemann Equations in Cartesian co-ordinates:**
- If f(z) is continuous in some neighbourhood of z and differentiable at z then the first order partial derivatives satisfy the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at the point z which are called the Cauchy-Riemann equations.

proof:

Let f(z) = u+iv be an analytical function

By definition of analytical function, f(z) has derivative.

i.e. $f^{I}(z) = \lim_{\Delta z \to 0} \left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right)$ exists (finite)

1) z = x+iy f(z) = u+iv f(z) = u(x,y)+iv(x,y)

2) z = x+iy $\Delta z = \Delta x + i \Delta y$ ***3)*** $f(z + \Delta z)$ = ?

$z + \Delta z$ = x+iy+ Δx+ iΔ y

$z + \Delta z$ = (x+ Δx)+i(y+ Δ y)

$f(z + \Delta z)$ = u(x+ Δx , y+ Δ y) + iv(x+ Δx , y+ Δ y)

$$f^{I}(z) = \lim_{\Delta x + i\Delta y \to 0} \left(\frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}\right) \to ①$$

We know that Δ x+i Δy = 0+i0 Δ

x = 0, Δ y = 0

Case (1) If $\triangle y = 0$, put $\triangle y = 0$ in ①.

$$f^{1}(z) = \lim_{\triangle x \to 0}\left(\frac{[u(x+\triangle x, y)+iv(x+\triangle x, y)) - [u(x,y)+iv(x,y)]}{\triangle x}\right)$$

$$f^{1}(z) = \left(\lim_{\triangle x \to 0}\frac{[u(x+\triangle x, y)-u(x,y)]}{\triangle x} + \lim_{\triangle x \to 0}\frac{i[v(x+\triangle x, y)-u(x,y)]}{\triangle x}\right)$$

$$f^{1}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow ②$$

Case (2) If $\triangle x = 0$, put $\triangle x = 0$ in ①

$$f^{1}(z) = \lim_{\triangle y \to 0}\left(\frac{[u(x,y+\triangle y)+iv(x,y+\triangle y)) - [u(x,y)+iv(x,y)]}{i\triangle y}\right)$$

$$f^{1}(z) = -i\lim_{\triangle y \to 0}\left(\frac{[u(x,y+\triangle y)-u(x,y)]}{\triangle y} + \lim_{\triangle x \to 0}\frac{i[v(x,y+\triangle y)-u(x,y)]}{\triangle y}\right)$$

$$f^{1}(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \rightarrow ③$$

Equate ② & ③

Compare the real and imaginary parts

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\{ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \}$ (If $u_x = v_y$ and $u_y = -v_x$)

These are **Cauchy – Riemann** Equations in **Cartesian** co-ordinate System.

Cauchy – Riemann Equations in Polar co-ordinates:

Let z=x+iy

We know that x=rcosθ ,

y=rsinθ z =

rcosθ+irsinθ z =

r(cosθ+isinθ) z = $re^{i\theta}$

f(z)=u+iv f($re^{i\theta}$) = u(r, θ)+iv(r,

θ) → 1

Differentiate 1 w.r.t ‘r’,

$f^{I}(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$ → 2

Differentiate 1 w.r.t ‘θ’,

f^{I} (→ 3

Substitute 2 in 3 , We get

$$[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} re^{i\theta}) rie^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

ir

ir

Lets compare real and imaginary parts

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

These are **Cauchy – Riemann** Equations in **Polar** co-ordinate System. **Examples**

1) Show that f(z) = xy+iy is not analytical

Solution : Given , f(z) = xy+iy

f(z) = u+iv u= xy

v= y

$\frac{\partial u}{\partial x}$= y , $\frac{\partial v}{\partial x}$ = 0

$\frac{\partial u}{\partial y}$= x , $\frac{\partial v}{\partial y}$ = 1

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

It doesn't **not satisfies C-R equations** and hence its **not an analytical function**.

2) Show that f(z) = 2xy+i(x^2- y^2) is not analytical function. Solution: Given f(z) = 2xy+i(x^2- y^2)

f(z) = u+iv

u=2xy v= x^2- y^2

$\frac{\partial u}{\partial x}$= 2y , $\frac{\partial v}{\partial x}$= 2x

$\frac{\partial u}{\partial y}$= 2x , $\frac{\partial v}{\partial y}$ = - 2y

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

It doesn't **not satisfies C-R** equations and hence its **not an analytical** function.

3) Test the analyticity f(z) = e^x(cosy-isiny) and also find the f'(z) Solution: Given f(z) = e^xcosy - ie^xsiny

f(z) = u+iv u = e^xcosy

v = -e^xsiny

$\frac{\partial u}{\partial x}$ = e^x cosy , $\frac{\partial v}{\partial x}$ = - e^xsiny ,

$\frac{\partial u}{\partial y}$ = -e^x siny

$\frac{\partial v}{\partial y}$ = -e^xcosy

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

f(z) is **not analytical** function and the f'(z) does not exist.

4) Show that f(z) = z z^2 is not analytical function

Solution : Given f(z) = z z^2

$f(z) = (x+iy)|(x+iy)|^2 = (x+iy)[\sqrt{x^2+y^2}]^2$

$f(z) = x(x^2+y^2)+iy(x^2+y^2)$ f(z) =

u+iv

$u = x(x^2+y^2) = x^3+xy^2 \quad V = y(x^2+y^2) = x^2y+y^3$

$\frac{\partial u}{\partial x} = 3x^2+y^2, \quad \frac{\partial v}{\partial x} = 2xy$

$\frac{\partial u}{\partial y} = 2xy, \quad \frac{\partial v}{\partial y} = x^2+3y^2$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

f(z) is **not analytical** function

5) Show that w= logz is an analytical function and also find $\frac{dw}{dz}$

Solution : Given w = logz

put $z = re^{i\theta}$

$^{i}\theta$ = log r + log $e^{i}\theta$ w w

= log re

= log r + iθ log e

f(z) = w = log r +iθ = u+iv u

= log r v = θ

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial \theta} = 1$$

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \;\&\; \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$r(\frac{1}{r}) = 1$ & $0 = 0$ It is an **analytical** function f(z)

$= u+iv$

$$f(re^{i\theta}) = u(r, \theta)+iv(r, \theta)$$

differentiate on both sides w.r.t 'r'

$$f^{I}(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$f^{I}(z)\, e^{i\theta} = \frac{1}{r} + i(0)$$

$$f^{I}(z) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

6) Show that f(x) = sinz is an analytical function everywhere in the complex plane

Solution : Given f(x) = sinz

f(x) = sin(x+iy) f(x) = sinx

cos(iy) + sin(iy) cosx f(X) = sinx

coshy + isinhy cosx f(x) = u+iv

u = sinx coshy v= sinhy cosx

$\frac{\partial u}{\partial x}$ = cosx coshy , $\frac{\partial v}{\partial x}$ = -sinx sinhy

$\frac{\partial u}{\partial y}$ = sinx sinhy, $\frac{\partial v}{\partial y}$ = coshy cosx & It is an **analytical** function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

7) Test the analyticity of the function f(z) = e^x (cosy+isiny) and find $f^I(z)$. Solution : Given , f(z) = e^x (cosy+isiny) = u+iv

$$u = e^x \cos y \qquad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

& It is an **analytical** function $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$f(z) = u+iv$$

$$f^I(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = e^x \cos y + i\, e^x \sin y$$

$$f^I(z) = e^x(\cos y + i \sin y)$$

$$f^I(z) = e^x\, i\, e^y = e^{(x+iy)}$$

$$f^I(z) = e^z$$

8) Determine P such that the function f(z) = $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(\frac{px}{y})$ be an analytical function.

Solution :

Given , $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(\frac{px}{y})$

It is an analytical function, It satisfies the C-R equation

$u = \frac{1}{2}\log(x^2+y^2)$) $v = \tan^{-1}(\frac{px}{y})$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}2x,$ $\frac{\partial v}{\partial x} = \frac{1}{1+(\frac{px}{y})^2}\frac{p}{y}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}2y$

$\frac{\partial v}{\partial y} = \frac{1}{1+(\frac{px}{y})^2}px(\frac{-1}{y^2})$

$\frac{\partial v}{\partial y} = \frac{y^2}{y^2+(\frac{px}{y})^2}(\frac{-px}{y^2})$

similarly : $\frac{\partial v}{\partial x} = \frac{py}{p^2x^2+y^2}$ $\frac{\partial v}{\partial y} = \frac{-px}{y^2+p^2x^2},$

By given f(z) is an analytical function, f(z) satisfies C-R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{x}{x^2+y^2} = \frac{-px}{y^2+p^2x^2}$$

Comparing the equations we get:

P = -1

9) Prove that function f(z) defined by f(z) = -R equations are satisfied at the origin, yet $f^{I}(0)$ does not exist.

Solution : Given $f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$

i) To show that f(z) is continuous at z=0 let $\lim f(z) = \lim_{x\to 0} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$ (given f(0) = 0) $\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$, z≠0 and f(0) is continues and C

$z\to 0$

$y\to 0$

$$\lim_{x\to 0} \frac{x^3(1+i)}{x^2}$$

$_3$ f(z) = f(z) =

$\lim x(1+i) = 0 = f(0)$

$_{x\to 0}$ **f(z) is continuous**

ii) To show that C-R equations are satisfied at origin

$$f(z) = \frac{x^3+x^3i-y^3+iy^3}{x^2+y^2} = \frac{x^3-y^3}{x^2+y^2} + \frac{i(x^3+y^3)}{x^2+y^2} \text{ f(z)}$$

= u+iv

u $\frac{x^3-y}{x^2+y^2}$ ^{3} = , $\frac{x^3+y^3}{x^2+y^2}$

v =

$\frac{\partial u}{\partial x} = \lim_{x\to0} \frac{u(x,0)-u(0,0)}{x}$

$\frac{\partial u}{\partial x} = \lim_{x\to0} \frac{x-0}{x}$ => $\lim_{x\to0}$

$\frac{\partial u}{\partial x}$ **= 1**

= 1

$\frac{\partial u}{\partial y} = \lim_{y\to0} \frac{u(0,y)-u(0,0)}{y}$

$\frac{\partial u}{\partial y} = \lim_{y\to0} \frac{-y-0}{y}$ => $\lim_{y\to0}$ -1 = -

$\frac{\partial u}{\partial y}$ **= -1**

$\frac{\partial v}{\partial x} = \lim_{x\to0} \frac{v(x,0)-v(0,0)}{x}$

$\frac{\partial v}{\partial x} = \lim_{x\to0} \frac{x-0}{x} = \lim_{x\to0} 1 = 1$

$\frac{\partial v}{\partial x}$ **= 1**

$\frac{\partial v}{\partial y} = \lim_{y\to0} \frac{v(0,y)-v(0,0)}{y}$

$\frac{\partial v}{\partial y} = \lim_{y\to0} \frac{y-0}{y} = \lim_{x\to0} 1 = 1$

$\frac{\partial v}{\partial y}$ **= 1**

1

C – $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ **&** ∂u ∂v $\overline{\partial y}$ = - $\overline{\partial x}$

x

lim 0

R Equations are satisfied at origin **iii)** To show that f'(z) does not exist at origin

lim $\frac{f(z)-f(0)}{}$ f'(z) =

y→0 z

( )

x^3 1+i −y2 $+^3$ y(1−i) 2 –

$\frac{^3(1+i)-0}{^2+y^2}$

( )

$f^{I}(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \dfrac{x}{x + iy}$

$$\lim_{x \to 0} \frac{x}{x}$$

$f^{I}(z) =$

$x\,1+i\,^{3}\; f^{I}(z) \lim_{x \to 0} x^{3} =$

= 1+i **(Finite)**

$f^{I}(z)$ Exists

At y = mx

$f^{I}(z) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z}$ $f^{I}(z) = \lim_{\substack{x \to 0 \\ y \to mx}} \dfrac{\dfrac{x^3(1+i) - m^3x^3(1-i)}{x^2 + x^2m^2}}{x + imx}$

$f^{I}(z) = \lim_{\substack{x \to 0 \\ y \to mx}} \dfrac{x^3[(1+i) - m^3(1-i)]}{x^2(1+m^2)x(1+im)}$

$f^{I}(z) = \lim_{x \to 0} \dfrac{[(1+i) - m^3(1-i)]}{(1+m^2)(1+im)}$

$= \dfrac{[(1+i) - m^3(1-i)]}{(1+m^2)(1+im)}$

$f^{I}(z) =$ **(Infinite) $f^{I}(z)$ depends upon the 'm' value, so that the $f^{I}(z)$ does not exist at origin**

Part – B

Laplace Equations

the equation of the form $\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0$ **or** $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$

Harmonic Function

The function u and v are said to be harmonic, if it satisfies Laplace Equations

i.e

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Milne – Thomson Method

When u is given find f(z) :

1) To find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

2) To find $f^{I}(z) = u+iv$

Differentiate w.r.t ‘x’ we get

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f^{I}(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (From C-R equation)

$f^{I}(z) =$

$\frac{\partial u}{\partial x} = \emptyset_1(z_1$

,0)

$\frac{\partial u}{\partial y} = \emptyset_2(z_2$

,0) $f^{I}(z)$ =

$\emptyset_1(z_1,0) - i\,\emptyset_2(z_2,0)$

Integrate w.r.t 'z' **f(z) = $_1\emptyset$ (z₁,0) dz - i $_2\emptyset$ (z₂,0)**

dz + c When v is given find f(z):

1) To find $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}$

2) To find f(z) = u+iv

Differentiate w.r.t 'x' , we get

$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f^{I}(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = \emptyset_1(z_1$,0)

$\frac{\partial v}{\partial x} = \emptyset_2(z_2$,0) (From C-R equation)

$f'(z) = \emptyset_1(z_1,0) + i\,\emptyset_2(z_2,0)$

Integrate w.r.t 'z' **f(z) =** $\int$**[**$\emptyset_1$**(**z_1**,0) + i** $\emptyset_2$**(**z_2**,0)**

]dz + c

1) Construct an analytical function f(z) when $u = x^3 - 3x\,y^2 + 3x + 1$ is given

$$\frac{\partial u}{\partial x} = 3x^2 - 3\,y^2 + 3$$

$$\frac{\partial u}{\partial y} = -6xy$$

Solution:

By Milne Thomson Method

f(z) =u+iv

$$\frac{\partial u}{\partial x} = \emptyset_1(z,0) = 3\,z^2 + 3$$

$$\frac{\partial u}{\partial y} = \emptyset_2(z,0) = -6(z)(0) = 0$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \emptyset_1(z,0) - i\,\emptyset_2(z,0)$$

Integrate w.r.t 'z' f(z) =

$\int[\emptyset_1(z,0) + i\,\emptyset_2(z,0)]\,dz + c$

f(z) $= \int(3z^2 + 3 - 0)\,dz + c$

f(z) $= \frac{3z^3}{3} + 3z + c$

f(z) = $z^3 + 3z + c$

2) Construct an analytical function f(z) when u = sinx coshy is given

Solution: = cosx sinhy $\frac{\partial u}{\partial x}$

$\frac{\partial u}{\partial y}$ = sinx sinhy

By Milne Thomson Method

f(z) =u+iv

$\frac{\partial u}{\partial x} = \emptyset_1(z,0)$ = cosz(1) = cosz

$\frac{\partial u}{\partial y} = \emptyset_2(z,0)$ = sinz(0) = 0

$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$f^{I}(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \emptyset_1(z,0) - i\,\emptyset_2(z,0)$

Integrate w.r.t 'z' f(z) = ∫[$\emptyset_1(z,0) - i\,\emptyset_2(z,0)$] dz + c f(z) = ∫cosz dz + c **f(z) = sinz + c**

3) Find the analytical function f(z) = u+iv if u+v = $\frac{\sin 2x}{(\cosh 2y - \cos 2x)}$

Solution: u+v = $\frac{\sin 2x}{(\cosh 2y - \cos 2x)}$

f(z) = u+iv

if(z) = ui-v

(1 + i)f(z) = (u-v)+i(u+v)

f(z) = u+iv

$$\frac{\partial V}{\partial x} = \frac{[coh2y - cos2x]2cos2x - sin2x[0+2sin2x]}{[cosh2y - cos2x]^2}$$

$$\frac{\partial V}{\partial x} = \frac{2cos2x\ coshy - 2cos^2 2x - 2sin^2 2x}{[cosh2y - cos2x]^2}$$

$$\frac{\partial V}{\partial x} = \frac{2cos2x\ coshy - 2}{[cosh2y - cos2x]^2}$$

$$\frac{\partial V}{\partial x} = \emptyset_2(z_{,0})$$

$$\frac{\partial V}{\partial x} = \frac{2cos2z\ cosh0 - 2}{[cosh0 - cos2z]^2} = \frac{2[cos2z - 1]}{[1 - cos2z]^2} = \frac{-2[1 - cos2z]}{[1 - cos2z]^2}$$

$$\frac{\partial V}{\partial x} = \frac{-2}{2sin^2 z}$$

Where F(z) = (1+i)f(z)

u+v = V

$$\overline{\partial x} = \emptyset_2(z_{\partial V,0}) = -\mathbf{cosec2z}$$

$$\frac{\partial v}{\partial y} = \emptyset_1(z_{,0}) = \frac{[coh2y - cos2x]\ 0 - sin2x[sinh2y(2)]}{[cosh2y - cos2x]^2}$$

$$\emptyset_1(z,0) = \frac{\partial v}{\partial y} = \frac{-2sin2xsinhy}{[cosh2y - cos2x]^2}$$

$$\frac{\partial v}{\partial y} = \frac{-0\ sin2z}{[cosh2y - cos2z]^2} = 0$$

f(z) = u+iv

$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ $f^{I}(z) =$

f(z) = ⍰[$\emptyset_1$(z,0) + i $\emptyset_2$(z,0)] dz + c

f(z) = ⍰ $-\mathrm{cosec}^2 z$ (i) dz+ c

f(z) = -i(-cotz) + c = i cotz + c

f(z) = i cot z + c

(1+i) f(z) $\frac{}{1+i}$ + = $\frac{c}{1+i}$ =icotz + c

i

f(z) = $\frac{i(1-i)}{2}$

cotz f(z) =

cotz + c_1

i+1

f(z) = $\frac{}{2}$ cotz + c_1

$\frac{\partial u}{\partial x} = e^x x^2$ cosy + 2x e^x cosy - e y

$\emptyset_1(z,0) = \frac{\partial u}{\partial x} = e^z z^2$

4) Find the analytical function , whose real part is u = e^x[(x^2- y^2)(cosy − 2xysiny)]

$\boldsymbol{\emptyset_1(z,0) = \frac{\partial u}{\partial x} = e^z z^2 + 2z\, e^z}$

Solution: u = $e^x x^2$ cosy - $e^x y^2$ cosy – 2xy e^x siny

$\frac{\partial u}{\partial y} = - e^x x^2$ x 2

x 2 cosy – 2y e^x siny – 2xy e^x siny

$\boldsymbol{\emptyset_2(z,0) = \frac{\partial u}{\partial y}}$ **= 0 + 0 - 0 – 0 = 0** cos(0) + 2z e^z cos(0) – 0 – 0 – 0

siny + e siny y - 2y e^x cosy – 2x e^x siny – 2xy e^x cosy

f(z) = u+iv

$f^{I}(z) \quad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \quad \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y}$

$f^{I}(z) =$

f(z) = ,0)] dz + c z(Ø2i -,0) z([Ø1

f(z) = (e^z z^2 + 2z e^z - 0) dz + c f(z)

= e^z (z^2 +2z)dz + c f(z) = e^z z^2

dz + 2 ze^z dz

u = z^2 dv = e^z dz du = 2z dz v= e^z f(z) = **e^z**

z^2 - 2 z dz e^z dz + 2 z e^z dz + c **f(z) = e^z**

z^2 + c

5) The analytical function whose imaginary part is v(x,y) = 2xy **Solution:**

$$v = 2xy$$

$$\frac{\partial v}{\partial x} = 2y = \emptyset_2(z,0) = 2(0) = 0$$

$$\frac{\partial v}{\partial y} = 2x = \emptyset_1(z,0) = 2(z) = 2z \; f(z)$$

= ∫ [∅₁(z ,0) + i ∅₂(z ,0)] dz + c

= ∫ 2z dz + c

$$f(z) = 2\,\frac{z^2}{2} + c$$

$$\mathbf{f(z) = z^2 + c}$$

6) Find harmonic conjugate at $u = e^{x^2-y^2}\cos 2xy$ and also find f(z)

Solution : $u = e^{x^2-y^2}\cos 2xy$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2}\cos 2xy\,(2x) - e^{x^2-y^2}\sin 2xy\,(2y)$$

$$\emptyset_1(z,0) = e^{z^2-0}\cos 0\,(2z) - e^{x^2-y^2}(0)$$

$$\boldsymbol{\emptyset_1(z,0) = e^{z^2}\,2z} \quad \frac{\partial u}{\partial y} = e^{x^2-y^2}\cos 2xy\,(-2y) -$$

$$e^{x^2-y^2}\sin 2xy\,(2x)$$

$\emptyset_2(z,0) = 0 - 0$

$\emptyset_2$**(z,0) = 0** f(z)

= u+iv $f^I(z)$ =

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ $f^I(z)$ =

$f^I(z) = \emptyset_1(z,0) - i\,\emptyset_2(z,0)$

f(z) = ⍰[$\emptyset_1(z,0) - i\,\emptyset_2(z,0)$] dz + c f(z) = ⍰ $e^{z^2}2z$

dz + c **(put $z^2 = t$ => 2z dz = dt)** f(z) = ⍰ e^t dt +

c = e^t + c

f(z) = e^{z^2} + c f(z) = $e^{(x+iy)^2}$ f(z) =

$e^{x^2-y^2+2xyi}$ + c f(z) = $e^{x^2-y^2}e^{2xyi}$ + c u+iv =

$e^{x^2-y^2}$ [cos2xy+isin2xy] + c u+iv = $e^{x^2-y^2}$

cos2xy + i e $e^{x^2-y^2}$ (sin2xy) + c

v = $e^{x^2-y^2}$ sin2xy + c

7) Find the analytical function f(z) such that Re[$f^I(z)$] = 3 x^2 - 4y -3 y^2 and f(1+i) = 0.

Solution : $Re[f^{I}(z)] = 3x^2 - 4y - 3y^2$

$f(z) = u+iv$

$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$Re[f^{I}(z)] = \frac{\partial u}{\partial x}$

$\frac{\partial u}{\partial x} = 3x^2 - 4y - 3y^2$

Integrate w.r.t 'x' we get **&** $u = \frac{3x^3}{3} - 4xy - 3y^2 x + f(y)$

$u = x^3 - 4xy - 3y^2x + f(y)$

Differentiate w.r.t 'y' we get

$\frac{\partial u}{\partial y} = -4x - 6xy + f^{I}(y)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = 3x^2 - 4y - 3y^2$

Integrate w.r.t 'y' we get

$v = 3 x^2y - \frac{4y^2}{2} - \frac{3y^3}{3} + f(x)$

$v = 3x^2y - y^3 - 2y^2 + f(x)$

Differentiate w.r.t 'x' we get

$\frac{\partial v}{\partial x} = 6xy + f^{I}(x)$

From C-R equations

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-4x - 6xy + f^{I}(y) = -6xy - f^{I}(x)$$

$$-4x + f^{I}(y) = -f^{I}(x)$$

Compare equation on both sides

i.e $f^{I}(x) = 4x$, $f^{I}(y) = 0$

$f(x) = 4 \int x\, dx$ $\quad$ $f(y) = c$ $f(x)$

$= \frac{4x^2}{2} + c$

$f(x) = 2x^2 + c$ **$f(y) = c$**

$f(z) = u+iv$ $f(z) = [x^3 - 4xy - 3y^2x] + i[3x^2y - y^3 - 2y^2] +$

$2x^2 + c$

given $f(1+i) = 0$ $f(z) = u+iv$

$z = x+iy = (1+i)$

put $x = 1, y = 1$ $f(z) = [1-4-3] + i[3-2-1]$

$+2 + c$ $f(1+i) = 0 = -6 + 2i + c$ **c**

$= 6 - 2i$

$f(z) = [x^3 - 4xy - 3y^2x] + i[3x^2y - y^3 - 2y^2] + 2x^2 + 6 - 2i$

8) Find the analytic function $f(z) = u+iv$ if $u-v = e^x(\cos y - \sin y)$ **Solution:**

$f(z) = u+iv$ $i f(z) = iu-v$

$(1+i) f(z) = (u-v) + i(u+v)$

$f(z) = u+iv$ **$u = u-v = e^x$**

$(\cos y - \sin y)$

$F(z) = (1+i) f(z) \cos y - e^x \sin y =$

$\frac{\partial u}{\partial x} = e^x$

$\emptyset_1(z,0) = e^z - 0 = e^z \quad \sin y - e^x$

$\frac{\partial u}{\partial y} = - e^x$

$\cos y = \emptyset_2(z,0) = 0 - e^z = - e^z$

$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y}$

$f(z) = ? [\emptyset_1(z,0) - i\, \emptyset_2(z,0)]\, dz + c$

$f(z) = ? (e^z + i\, e^z)\, dz + c$

$f(z) = (e^z + i\, e^z) + c$ $(1+i)$

$f(z) = e^z + i\, e^z + c$

$f(z) = \frac{e^z(1+i)}{(1+i)} + \frac{c}{1+i}$ **$f(z)$**

$= e^z + c$

Harmonic Conjugate

1) Show that function u= 2xy+3y is harmonic and find harmonic conjugate.

Solution: $u = 2xy+3y$

$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2x+3$

$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0$

2 2

$\frac{\partial u}{\partial x^2}+\frac{\partial u}{\partial y^2}= 0$ **u satisfies laplace equation**

'u' is a **Harmonic** function

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

dv = -(2x+3) dx + 2y dy v

= ? -(2x+3) dx + 2y dy

v = -+ c $(\frac{2x}{2}+3x) + \frac{2y}{2}$ 2 2

v = - x^2+ y^2 - 3x + c

2) Show that u = 2log (x^2+ y^2) is harmonic and find its harmonic conjugate.

Solution: u = 2log (x^2+ y^2)

$\frac{\partial u}{\partial x} = 2\frac{1}{x^2+y^2}2x$ $\qquad$ $\frac{\partial u}{\partial y} = 2\frac{1}{x^2+y^2}2y$

$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(4)-4x(2x)}{(x^2+y^2)^2}$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(4)-4y(2y)}{(x^2+y^2)^2}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4x^2+4y^2-8x^2+4x^2+4y^2-8y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$dv = \frac{-4y}{x^2+y^2}dx + \frac{4x}{x^2+y^2}dy$$

$$dv = \frac{-4}{x^2+y^2}(y\,dx - x\,dy)$$

$$v = -4\int\left[\frac{x\,dy - y\,dx}{x^2+y^2}\right]v$$

$$= -4\int d\tan^{-1}\left(\frac{y}{x}\right)$$

$$\mathbf{v = -4\tan^{-1}\left(\frac{y}{x}\right) + c}$$

$$d\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{x\,dy - y\,dx}{x^2}\right]_x$$

3) Find f(z) if the imaginary part is $r^2\cos 2\theta + r\sin\theta$ **Solution:**

$$V = r^2\cos 2\theta + r\sin\theta$$

Integrate w.r.t r, we get

$\frac{\partial v}{\partial r} = 2r\cos 2\theta + \sin\theta$

$u = -r^2 \sin 2\theta + r\cos\theta + f(\theta)$

Differentiate w.r.t θ

$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta - r\sin\theta + f^{I}(\theta) \rightarrow ③$

Compare the equations ② & ③

$\frac{\partial v}{\partial r} = -2r\sin 2\theta + \cos\theta \rightarrow ①$

$f^{I}(\theta) = 0$

$f(\theta) = c$

$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$

$u = -r^2 \sin 2\theta + r\cos\theta + c$

$f(z) = (-r^2 \sin 2\theta + r\cos\theta + c) + i(r^2 \cos 2\theta + r\sin\theta)$

$\frac{\partial u}{\partial \theta} = -r[2r\cos 2\theta + \sin\theta]$

$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta - r\sin\theta \rightarrow ②$

4) Show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right][\text{real } f(z)]^2 = 2|f^{I}(z)|^2$

Solution: $f(z) = u+iv$

real $f(z) = u$

$[\text{real } f(z)]^2 = u^2$

$$\frac{\partial (u^2)}{\partial x} = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 (u^2)}{\partial x^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} \rightarrow ①$$

Similarly,

$\rightarrow ②$

Add equation ② $\frac{\partial^2 (u^2)}{\partial y^2} = 2u\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$ &

$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]u^2 = 2[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2$ ① $] + 2u[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}]$

$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]u^2 = 2[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]$

$\{ f(z) = u+iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$|f'(z)|^2 = (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \}$

$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]u^2$ **[real f(z)]^2 = 2 |f'(z)|^2**

5) If f(z) is analytical function with constant modulus ,then show that f(z) is constant.

Solution:

let f(z) is constant modulus

f(z) = u+iv

$|f(z)| = \sqrt{u^2 + v^2}$ = constant

$\sqrt{u^2 + v^2} = c$

$u^2 + v^2 = c^2 = c_1$

Differentiate w.r.t **'x'**

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \rightarrow$ ①

$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$ Differentiate w.r.t 'y'

$= 0 \rightarrow$ ② By C-R

equations $2u\frac{\partial v}{\partial y} - 2v\frac{\partial u}{\partial y}$

① ❾ $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$ $= 0 \rightarrow$ ③

$= 0 \rightarrow$ ④ $\frac{\partial v}{\partial y} - v^2\frac{\partial u}{\partial y}$

Multiply ③ * v ❾ uv $u^2\frac{\partial u}{\partial y} + uv\frac{\partial v}{\partial y}$ $= 0$

④ * u ❾ $= 0$

Subtract then uv

$u = \frac{\partial v}{\partial y} - v^2 \frac{\partial u}{\partial y} - u^2 \frac{\partial u}{\partial y} - uv \frac{\partial v}{\partial y} = 0$

u = c

$- \frac{\partial u}{\partial y}(u^2 + v^2) = 0$

Similarly

$u^2 + v^2 \neq 0$

$\frac{\partial u}{\partial y} = 0$

$\int \frac{\partial u}{\partial y} = c$

v = c f(z) is

constant

Conformal Mapping :

A transformation w = f (z) is said to be conformal if it preserves angel between oriented curves in magnitude as well as in orientation.

Bilinear Transformation :

The transformation $w = f(z) = \frac{az+b}{cz+d}$ is called the bilinear transformation or mobius transformation. Where a,b,c,d are complex constants.

The method to find the bilinear transformation if three points and their images are given as follows:

We know that we need four equations to find 4 unknowns. To find a bilinear transformation we need three points and their images.

in cross ration, three are four points $(w,w_1, w_2,w_3,) = (z,z_1, z_2, z_3,)$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Since we have to get $w = \frac{az+b}{cz+d}$, we take one point as 'z' and its image as 'w'

Problems about bilinear transformation:

1) Find the bilinear transformation on which maps the points (-1, 0, 1) into the points (0,i,3i) in w-plane

Solution : In z-plane, $z_1 = -1$, $z_2 = 0$, $z_3 = 1$

In w-plane, $w_1 = 0$, $w_2 = i$, $w_3 = 3i$

In cross ration,

$(w,0,i,3i) = (z,-1,0,1)$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} \quad \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{(w)(-2i)}{(-i)(3i-w)} = \frac{-(z+1)}{-(1-z)} \quad =$$

$-2wi(1-z) = (z+1) [- [i(3i-w)]]$

$-2wi + 2wiz = -[-3-wi](z+1)$

$-2wi + 2wiz = 3z + wiz + 3 +wi$

$$\frac{(w-0)(0-1)}{(0-i)(1-0)} = \frac{(0-1)(i-0)}{(1-0)(0-z)}$$

$$\frac{-w}{-i} = \frac{-i}{-z} \qquad w = \frac{i^2}{z} = \frac{-1}{z} \qquad \mathbf{w = \frac{-1}{z}}$$

-3wi + wiz = (3z + 3)

w[i(3-z)] = z(z+1) **w** $= \frac{-3(z+1)}{-i(3-z)}$

$z_1 = \alpha = \frac{1}{z_1'} = \frac{1}{0}$ $[z_1' = 0]$, $z_2 = i$, $z_3 = 0$

2) Find the bilinear transformation which maps the points $(\alpha,i,0)$ in the z-plane into $(0,i,\alpha)$ in the w-plane.

$w_1 = 0$, $w_2 = i$, $w_3 = \frac{1}{0} = \alpha = \frac{1}{w_3'}$ $[w_3' = 0]$

Solution: In z-plane, z

In w-plane,

$$\frac{(w-w_1)(w_3'w_2-1)}{(w_2-\frac{1}{w_3'})(1-w_1 w_3')} = \frac{(zz_1'-1)(z_2-z_3)}{(z-\frac{1}{z_1'})(z_2-z_3)(1-z_3 z_1')}$$

(w−w₁) (w −w)(−w) (−z)(z z)₁ ₂ w₃' =

z₁' ₂ ₃– ((ww−w₁−w₂) (ww₃) (−z₁z₂) (z₃−z) w₃' z₁'

3) Find the bilinear transformation that maps the points $(0,i,\alpha)$ respectively into $(0,1,\alpha)$.

Solution: In z-plane, $z_1 = 0$, $z_2 = i$, $z_3 = \frac{1}{0} = \frac{1}{\alpha} = \frac{1}{z_3'}$ $[z_3' = 0]$

$$w_1 = 0,\ w_2 = 1,\ w_3 = \frac{1}{w_3'} = \frac{1}{0} = \alpha\ [w_3' = 0]$$

In w-plane,

$$\frac{(w-w_1)\left(w_2-\frac{1}{w_3'}\right)}{(w_1-w_2)\left(\frac{1}{w_3'}-w\right)} = \frac{(z-z_1)\left(z_2-\frac{1}{z_3'}\right)}{(z_1-z_2)\left(\frac{1}{z_3'}-z\right)}$$

$$\frac{(w-w_1)(w_2w_3'-1)}{(w_1-w_2)(1-ww_3')} = \frac{(z-z_1)(z_2z_3'-1)}{(z_1-z_2)(1-zz_3')}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-0)(i(0)-0)}{(0-i)(1-0)}$$

$$\frac{-w}{-1} = \frac{-z}{-i}$$

$$w = -iz$$

Fixed point :

The transformation $w = \frac{az+b}{cz+d}$

The roots of this transformation are called fixed points or invariant points.

$z = \frac{az+b}{cz+d}$ (we know that w = f(z)) $z(cz+d) = az+b$ $cz^2 + dz = az+b$ $\mathbf{cz^2+(d-a)z - b = 0}$

Problems:

1) Find the fixed points of the transformation w =

Solution: The roots of above transformation are called fixed points

$\frac{z-1+i}{z+2}$ put w $\frac{z-1}{z+1}$

$= \mathbf{z}\ z = \frac{z-1}{z+1} z(z+1)$

$= z-1\ z^2 + z - z + 1$

$= 0\ z^2$

$+1 = 0\ z^2 = -1\ \mathbf{z} = \pm$

i fixed points $\pm$ i

2) The fixed points of the transformation w =

Solution: **put w = z**

$$\mathbf{z} = \frac{z-1+i}{z+2}$$

z(z+2) = (z-i+1) (a =1, b =1, c =1-i)

$z^2 + 2z = z - i + l$

$z^2 + z + i - l = 0$

$$\mathbf{z} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 + 4(1-i)}}{2}$$

$$= \frac{-1 \pm \sqrt{1 + 4 - 4i}}{2} = \frac{-1 \pm \sqrt{3-4i}}{2}$$

$$\frac{-1 + \sqrt{3-4i}}{2} \quad \& \quad \frac{-1 - \sqrt{3-4i}}{2}$$

3) Determine the bilinear transformation whose fixed points are 1,-1 **Solution:**

Given fixed points are z = 1,-1

The roots of the transformation is $w = \frac{az+b}{cz+d}$ are called fixed points **put w = z**

$$z = \frac{az+b}{cz+d}$$

$$cz^2+(d-a)z - b = 0 \quad (z+1)(z-1) = 0$$

$$z^2 - 1 = 0 \qquad (c = 1, d = 0, a = 0, b = 1)$$

$$\mathbf{w} = \frac{0z+1}{1z+0} = \frac{1}{z}$$

Problems on images:

1) Write the image of the triangle with vertices (i,1+i,1) in the z-plane under the transformation w = 3z+4-2i

Solution:

y

(x,y) = (1,0)

In w-plane:

y

in z-plane Transformation $z = i$ ❾ $x+iy = 0+i$ $w = 3z+4-2i$ $(x,y) = (0,1)$ $w = 3(x+iy)+4-2i$ $z = 1+i$ ❾ $x+iy = 1+i$ $u+iv = w$

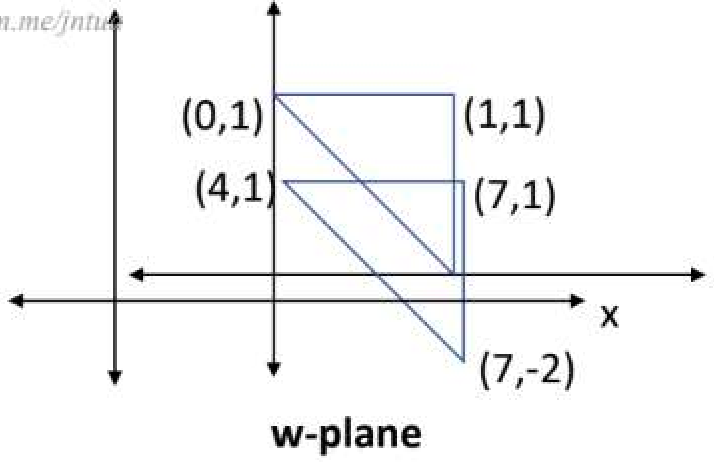
$(x,y) = (1,1)$ $u = 3x+4,\ v = 3y-2$

x **z- plane**

(1,0)

$z = 1$ ❾ $x+iy = 1$

i) (x,y) = (0,1) ❾ (u,v) = (4,1) ii) (x,y) = (1,1) ❾ (u,v) = (7,1) iii) (x,y) = (1,0) ❾ (u,v) = (7,-2)



Conclusion:

The image of the triangle whose vertices (i,1+i,1) is mapped as triangle whose vertices (4,1) ,(7,1), (7,-2) in w-plane under the transformation **w=3z+4-2i**

2) Find the image of the infinite strip $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution: **In z –plane**

the infinite strip between the lines y =0, y = .

Transformation:

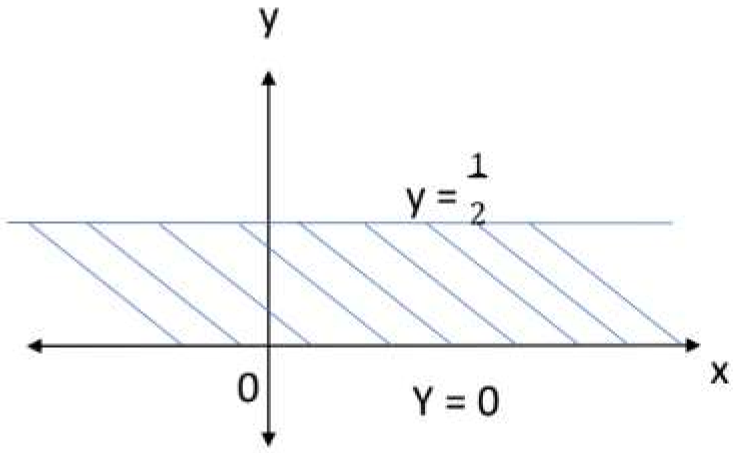
$w = \frac{1}{z}$

$z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv}\,\frac{u-iv}{u-iv}$

$x+iy = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$



In w –plane **z - plane** $\frac{1}{2}$

i) $y = \rightarrow 0 = \frac{-v}{u^2+v^2}$ **0** **ii)** $y = \frac{1}{2} \rightarrow \frac{1}{2} = \frac{-v}{u^2+v^2}$

$0 = -v$ $u^2 + v^2 = -2v$ **v = 0 Conclusion:** 1

The image of infinite strip $0 < y <$ is transferred as straight line (v=0) or circle under the transformation w = $\frac{}{2}$

π z

3) Find the image of the region in the z- $\frac{1}{2}$ plane between the lines y = 0 and y = under the transformation w = e 2

Solution: **In z –plane**

The lines are $y = 0$, $y = \frac{\pi}{2}$

Transformation

$w = e^z$

$u+iv = e^{x+iy} = e^x e^{iy}$ $y = 0$ $u+iv = e^x$

$[\cos y + i\sin y]$ $u = e^x \cos y$ $v = e^x \sin y$

In w-plane

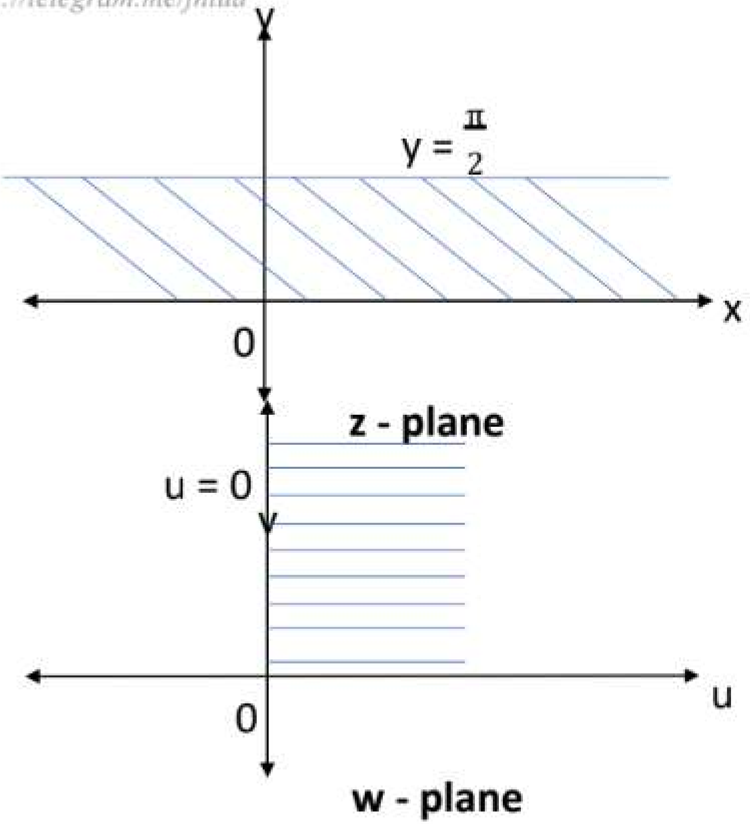
i) $y = 0$ ➒ $u = e^x$, $v = 0$

π x

ii) $y =$ ➒ $u = 0$, $v = e$

2

v = 0



Conclusion:

The image of the region lines $y = 0$ & $y = \frac{\pi}{2}$ are transferred as first quadrant in the w-plane under the transformation $w = e^z$

4) Show that transformation $w = z + \frac{1}{z}$ maps the circle $|z| = c$ into the eclipse $u = (c + \frac{1}{c})\cos\theta$, $v = (c - \frac{1}{c})\sin\theta$. Also discuss the case when $c = 1$ in detail.

Solution: **Z –plane** **Transformation**

1

The | | circle z = c

| |

w

$\sqrt{}$ = z

+

z

$x + iy = c$ $w = r\, e^{i\theta} + \frac{1}{re^{i\theta}}$

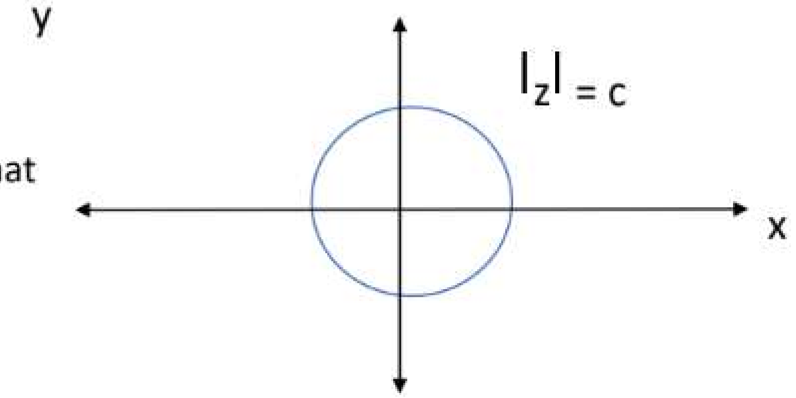
$x^2 + y^2 = c$ $u+iv = r(r\cos\theta + i\sin\theta) + \frac{1}{r}(r\cos\theta - i\sin\theta)$ $\mathbf{x^2 + y^2 = c^2}$ $u+iv = (r+\frac{1}{r})\cos\theta + i(r-\frac{1}{r})\sin\theta$ $\mathbf{u = (r+\frac{1}{r})\cos\theta}$ $\mathbf{v = (r-\frac{1}{r})\sin\theta}$

w –plane

$|z| = c$

$|z| = r$ (r =c)

we know that $\mathbf{\cos^2\theta + \sin^2\theta = 1}$ $\frac{u^2}{(c+\frac{1}{c})^2} + \frac{v^2}{(c-\frac{1}{c})^2} = 1$



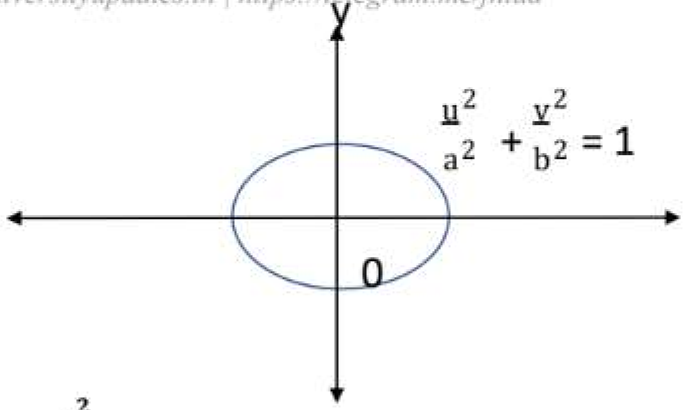
Case: **When c = 1**

$|z| = 1$, $\mathbf{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1}$ r = 1

$u = 2\cos\theta$, $v = 0$

u+iv = 2 cosθ + i (0) 2 sinθx **u = 2 v = 0**

Conclusion:



2

| |

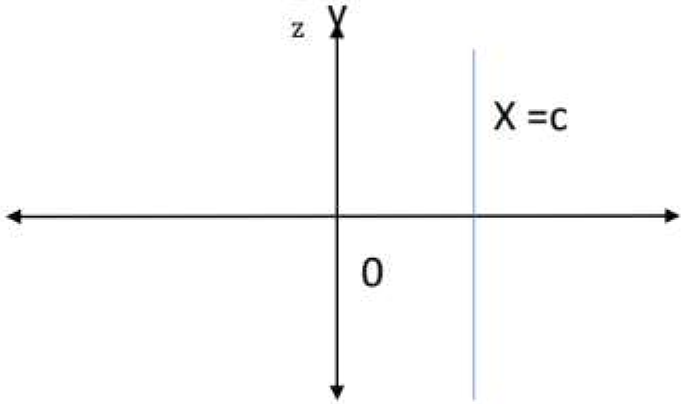
The image of circle z = c is transferred as eclipse **= 1** | | in w – plane and also the image of circle z = 1 when c = 1 is transferred as straight lines u =2 & v = 0 in w – plane under the transformation w = z + **.**

u $\frac{v^2}{2}$

$\frac{1}{z}$

5) Discuss the transformation of w = sinz using example. a²

\+ b



Solution: Transformation w = sinz

w = sin (x+iy) w = sinxcosiy + cosxsiniy

u+iv = sinxcoshy + icosxsinhyx

u = sinxcoshy v = cosxsinhy

Example: In z –plane **In w –plane**

| |

u v z = 1 ,

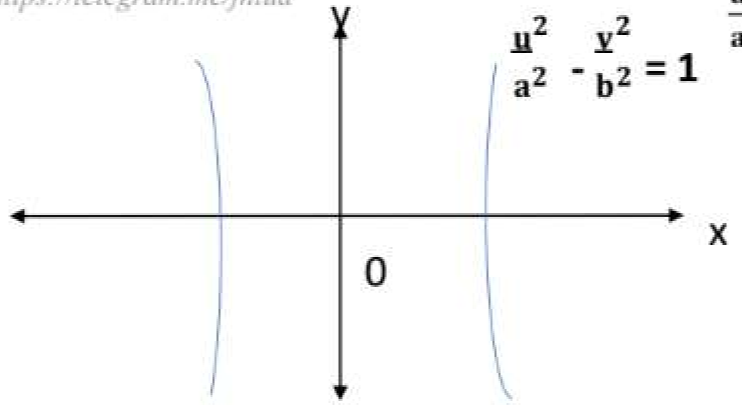
x = c coshy = ____ , sinhy = ____

sinx cosx

$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$

put x = c $\frac{u^2}{sin^2x} - \frac{v^2}{cos^2x}$

Conclusion: $\frac{u^2}{sin^2c} - \frac{v^2}{cos^2c} = 1$ $\cos^2hy - \sinh^2 y = 1$



$\frac{\mathbf{u^2}}{\mathbf{a^2}} - \frac{\mathbf{v^2}}{\mathbf{b^2}} = \mathbf{1}$

The image line x

= c is transferred as hyperbola $= 1$

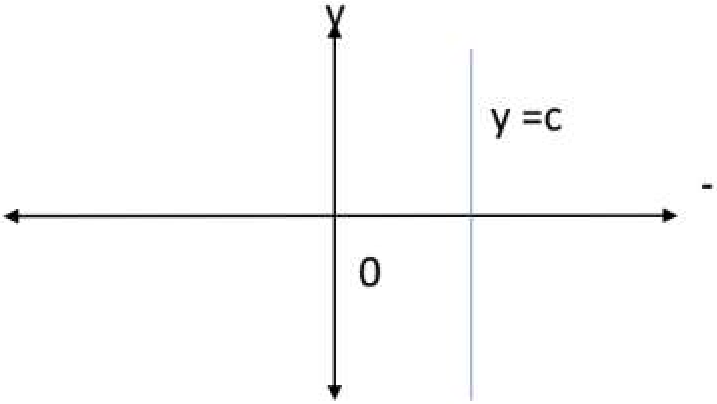
$\frac{\mathbf{u^2}}{\mathbf{a^2}} - \frac{\mathbf{v^2}}{\mathbf{b^2}} = \mathbf{1}$ in w – plane under

the transformation w = sinz.

6) Discuss the transformation of w = cosz

Solution: Transformation on w = cosz

w = cos(x+iy) w = cosxcosiy – sinxsiniy u+iv = $\frac{\mathbf{u}}{\mathbf{coshy}}$, **v =**



cosxcoshy – isinxsinhyx **u = cosxcoshy**

sinxsinhy In

z- $\frac{\mathbf{v}}{\mathbf{sinhy}}$ **plane In w-plane y = c cosx = sinx = -**

$\cos^2x + \sin^2x = 1$

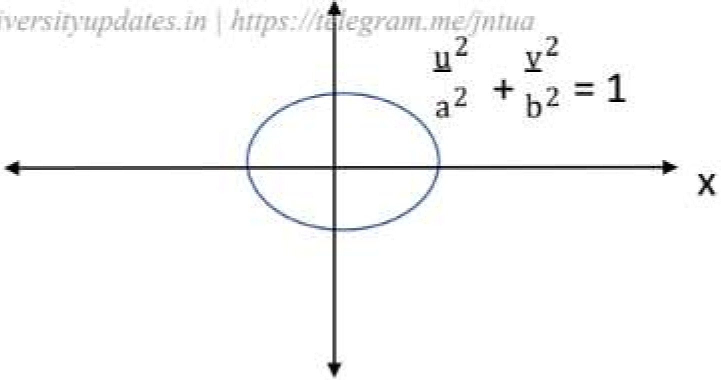
$\frac{u^2}{cosh^2y} + \frac{v^2}{sinh^2y} = 1$

put y = c

y

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

$$\frac{\mathbf{u^2}}{\mathbf{a^2}} + \frac{\mathbf{v^2}}{\mathbf{b^2}} = \mathbf{1}$$



Conclusion:

The image of line y = c is transferred as ellipse $\frac{\mathbf{u^2}}{\mathbf{a^2}} + \frac{\mathbf{v^2}}{\mathbf{b^2}} = \mathbf{1}$ under the transformation w = cosz.