Unit-IV

Recurrence Relation:

Topics

1. Generating functions of Sequences

$(x+y)^n = n_{c_0}x^n + n_{c_1}x^{n-1}y + n_{c_2}x^{n-2}y^2 + \cdots + n_{c_n}x^{n-n}y^n$

Where $n \in N$ the Number of terms in the expansion are $n+1$.

Generating function:

Consider a Sequence of real numbers $a_0, a_1, a_2 \cdots a_n$. If there exist a function $f(x)$ whose expansion in a series of powers of x.

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ $n \leftarrow \in N$ +ve $-n+1$, -Ne, fractions } infinite

$= \sum_{r=0}^{\alpha} a_r x^r$

$f(x)$ is called generating function for the Sequence $a_0, a_1, a_2 \cdots$.

Ex:

1. $(1+x)^{-1}$ find the Sequence for the generating function.

Given $f(x) = (1+x)^{-1}$

$(x+y)^n = n_{c_0}x^n + n_{c_1}x^{n-1}y + n_{c_2}x^{n-2}y^2 + \cdots$

$n_{c_r} = \frac{n!}{(n-r)!\, r!}$

$n_{c_2} = \frac{n!}{(n-2)!\, 2!}$

$= \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\, 2!}$

$= \frac{n(n-1)}{2!}$

$n_{c_0} = \frac{\cancel{n!}}{\cancel{(n-0)!}\, 0!} = 1$

$n_{c_1} = \frac{n!}{(n-1)!\, 1!} = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}\, 1!} = n$

(2)

$= 1.x^n + \frac{n}{1!} x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + \ldots$

$(1+x)^{-1}$ where $x = 1$, $y = x$, $n = -1$

$(1+x)^{-1} = 1.(1)^{-1} + \frac{(-1)}{1!} 1^{-2} x + \frac{(-1)(-1-1)}{2!} 1^{-3} x^2 + \frac{(-1)(-1-1)(-1-2)}{3!} 1^{-4} x^3 + \text{----}$

$= 1 - x + \frac{\cancel{2}}{\cancel{2!}} x^2 - \frac{\cancel{6}}{\cancel{3!}} x^3 + \text{-----}$ $\quad 3! = 6$, $2! = 2$

$= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \text{---}$

Sequence is $\rightarrow$

$1, -1, 1, -1, 1, -1, 1 \text{-----}$

*

$(1-x)^{-1}$

Given $(1-x)^{-1}$

Here $x = 1$

$y = x$

$n = -1$

We know

$(x+y)^n = 1.x^n + \frac{n}{1!} x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \text{----}$

$= (1)(1)^{-1} + \frac{(-1)}{1!} (1)^{-1-1} (-x) + (-1) \frac{(-1-1)}{2!} (1)^{-1-2} (-x)^2 + \text{----}$

$= 1 + x + x^2 + \text{---}$

Sequence is

$= 1, 1, 1, \text{-----}$

* $(1-x)^{-2}$ find Sequence.

$= 1.x^n + \frac{n}{1!}x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \text{------}$

where $x=1,\ y=-x,\ n=-2$

$(1-x)^{-2} = 1(1)^{-2} + \frac{(-2)}{1!}(1)^{-2-1}(-x) + \frac{(-2)(-2-1)}{2!}(1)^{-2-2}(-x)^2 + \text{---}$

$= 1+(-2)\cdot 1^{-2-1}(-x) + \frac{(-2)(-2-1)}{2!}(1)^{-2-2}(-x)^2 +$

$\frac{(-2)(-2-1)(-2-2)}{3!}1^{-2-3}(-x)^3 + \text{---}$

$= 1+2x+\frac{\overset{3}{\cancel{6}}}{\cancel{2!}}x^2 + \frac{\overset{4}{\cancel{24}}}{\cancel{3!}}x^3 + \text{-----}$

$= 1+2x+3x^2+4x^3+5x^4+6x^5+\text{-----}$

Sequence is

$= 1, 2, 3, 4, 5, 6 \text{------}$

* $(1+x)^{-2}$

Given $(1+x)^{-2}$

Here $x=1,\ y=x,\ n=-2$

$(1+x)^{-2} = 1(1)^{-2} + \frac{(-2)}{2!}(1)^{-2-1}(x) + \frac{(-2)(-2-1)}{2!}(1)^{-2-2}(x)^2 + \text{---}$

$= 1+(-2)x+3x^2+\text{----}$

$= 1-2x+3x^2-4x^3+\text{----}$

Sequence is

$1, -2, +3, -4 \text{-----}$

* $(1+3x)^{-1/3}$ where $x=1$, $y=3x$, $n=-1/3$

$$(1+3x)^{-1/3} = 1.(1)^{-1/3} + \frac{(-1/3)}{1!}(1)^{-1/3-1}(3x) + \frac{(-1/3)(-1/3-1)}{2!}(1)^{-1/3-2}(3x)^2 + \frac{(-1/3)(-1/3-1)(-1/3-2)}{3!}(3x)^3 + \cdots$$

$$= 1 + (-1/3)(3x) + \frac{(-1/3)(-1/3-1)}{2}(3x)^2 + \frac{(-1/3)(-1/3-1)(-1/3-2)}{6}(3x)^3 + \cdots$$

$$= 1 - x + \frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{2} 9x^2 + \frac{(-1/3)(-4/3)(-7/3)}{6} 27x^3 + \cdots$$

$$= 1 - x + \frac{4x^2}{2} - \frac{28}{6}x^3 + \cdots$$

$$= 1 - x + 2x^2 - \frac{14}{3}x^3 + \cdots$$

∴ The sequence is $1, -1, 2, -14/3, \cdots$

* $(1-4x)^{-1/2}$

Given

$(1-4x)^{-1/2}$

Here

$x=1$, $y=-4x$, $n=-1/2$

$$(1-4x)^{-1/2} = 1(1)^{-1/2} + \frac{(-1/2)}{1!}(1)^{-1/2-1}(-4x) + \frac{(-1/2)(-1/2-1)}{2!}(1)^{-1/2-2}(-4x)^2 + \cdots$$

$$= 1 + 2x + 6x^2 + 20x^3 + \cdots$$

Sequence is

$= 1, 2, 6, 20, \cdots$

* $2x^2(1-x)^{-1}$

$2x^2\left[1+x+x^2+x^3+\cdots\right]$

$2x^2+2x^3+2x^4+2x^5+\cdots$

$0x^0+0x^1+2x^2+2x^3+2x^4+2x^5+\cdots$

Sequence is $0,0,2,2,2,2\cdots$

(5)

* $\frac{1}{1-x}+2x^3$

$\frac{1}{1-x}+2x^3=(1-x)^{-1}+2x^3$

$=\left[1+x+x^2+x^3+\cdots\right]+2x^3$

$=1+x+x^2+3x^3+\cdots$

Sequence is

$=1,1,1,3,\cdots$

* $(3+x)^3=3^3\left(1+\frac{x}{3}\right)^3$

$(3+x)^3=3^3\left(1+\frac{x}{3}\right)^3$

Here $x=1$, $y=x/3$, $n=3$

$\left(1+x/3\right)^3=1(1)^3+\frac{3}{1!}(1)^{3-1}\left(\frac{x}{3}\right)+\frac{3(3-1)}{2!}(1)^{3-2}\left(\frac{x}{3}\right)^2+\cdots$

$=1+x+\frac{x^2}{3}+\frac{x^3}{27}+\frac{x^4}{4!}(0)$

Now

$\Rightarrow 27\left[1+x+\frac{x^2}{3}+\frac{x^3}{27}+\frac{x^4}{4!}(0)\right]$

$=27+27x+9x^2+x^3$

Sequence is $27,27,9,1$

Find the generating function for the following Sequences

(5) (i) 1, 2, 3, 4 - - -

(ii) 1, -2, 3, -4 - - -

(iii) 0, 1, 2, 3 - - -

(iv) 0, 1, -2, 3, -4 - - -

1 Ans: 1, 2, 3, 4 - - - - -

$1x^0 + 2x^1 + 3x^2 + 4x^3 + - - - = (1-x)^{-2}$

(ii) 1, -2, 3, -4

$1x^0 - 2x + 3x^2 - 4x^3 + - - - = (1+x)^{-2}$

(iii) 0, 1, 2, 3 - - -

$0x^0 + 1x + 2x^2 + 3x^3 + - - - - -$

$\Rightarrow x + 2x^2 + 3x^3 + - - - -$

$\Rightarrow x(1 + 2x + 3x^2 + - - -)$

$\Rightarrow x(1-x)^{-2}$

(iv) 0, 1, -2, 3, -4

$0x^0 + 1x - 2x^2 + 3x^3 - 4x^4 + - - - -$

$\Rightarrow x - 2x^2 + 3x^3 - 4x^4 + - - -$

$\Rightarrow x(1 - 2x + 3x^2 - 4x^3 + - - -)$

$\Rightarrow x(1+x)^{-2}$

Find the generating functions for the following Sequences.

(i) $1x^2 + 2x^2 + 3x^2 + - - - - -$ $\quad 1^2, 2^2, 3^2 - - -$

(ii) $0x^2 + 1x^2 + 2x^2 - - - -$ $\quad 0, 1^2, 2^2 - - - -$

(iii) $1x^3, 2x^3$ $\quad 1^3, 2^3, 3^3 - - -$

(iv) $0^3, 1^3, 2^3, 3^3 - - -$

(i) $1^2, 2^2, 3^2, 4^2 - - - -$

$$0x^0 + 1 \cdot x^1 + 2x^2 + 3x^3 + - - - = \frac{x}{(1-x)^2}$$

$$0 + 1^2 \cdot x^0 + 2^2 \cdot x^1 + 3^2 \cdot x^2 + 4^2 x^3 + - - - - = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right]$$

$$= \frac{(1-x)^2 \cdot 1 - x \cdot 2(1-x)(-1)}{(1-x)^4}$$

$$\frac{u}{v} = \frac{vu' - uv'}{v^2}$$

$$= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

$$= \frac{\cancel{(1-x)}(1-x+2x)}{\cancel{(1-x)}^4(1-x)^2} = \frac{(1+x)}{(1-x)^3}$$

(ii) $0^2, 1^2, 2^2, 3^2 - - - - -$

$$0^2 \cdot x^0 + 1^2 x^1 + 2^2 \cdot x^2 + 3^2 x^3 + - - - - - = \frac{x}{(1-x)^2}$$

$$x(1^2 + 2^2 x + 3^2 x^2 + - - - - - -)$$

$$= x\frac{(1+x)}{(1-x)^3}$$

(iii) $1^3, 2^3, 3^3 - - -$

$$1^3 x^0 + 2^3 x^1 + 3^3 x^2 + - - - - - -$$

by(2) $0^2 x^2 + 1^2 x^1 + 2^2 x^2 + 3^2 x^3 + - - - = \frac{x(1+x)}{(1-x)^3}$

differentiate x

$$0 + 1^3 x^0 + 2^3 x^1 + 3^3 x^2 + - - - - - = \frac{d}{dx}\left[\frac{x^2 + x}{(1-x)^3}\right]$$

$$= \frac{(1-x)^3(2x+1) - (x^2+x)3(1-x)}{(1-x)^6}$$

$$= \frac{\cancel{(1-x)}^2[(1-x)(2x+1) + 3(x^2+x)]}{\cancel{(1-x)}^{6\,4}}$$

$$= \frac{2x+1-2x^2-x+3x^2+3x}{(1-x)^4}$$

$$= \frac{x^2+4x+1}{(1-x)^4}$$

uences.

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(iv) $0^3, 1^3, 2^3, 3^3 - - - -$

$$0^3 x^0 + 1^3 \cdot x^1 + 2^3 \cdot x^2 + 3^3 \cdot x^3 + - - - - -$$

$$x \left[1^3 + 2^3 x + 3^3 x^2 + - - - - \right]$$

$$= x \left[\frac{(x^2 + 4x + 1)}{(1-x)^4} \right]$$

Find the generating function for the following Sequences

(i) $1, 1, 0, 1, 1, 1 - - - - -$

(ii) $0, 2, 6, 12, 20, 30, 42 - - - -$

(i) Given $1, 1, 0, 1, 1, 1 - - - - -$

$$1 \cdot x^0 + 1 x^1 + 0 x^2 + 1 \cdot x^3 + 1 \cdot x^4 + - - - - -$$

Add and substract x^2

$$(1x^0 + 1x^1 + 1x^2 + 1 \cdot x^3 + 1 \cdot x^4 + - - - - -) - x^2$$

$$(1-x)^{-1} - x^2$$

(ii) Given Sequence is $0, 2, 6, 12, 20, 30, 42 - - - - -$

$$(0 \cdot x^0 + 1x^1 + 2x^2 + 3x^3 + - - -) + (0^2 x^0 + 1^2 x^1 + 2^2 x^2 + 3^2 x^3 + - - -)$$

$0 = 0 + 0^2$ $\quad = x(1-x)^{-2} + \frac{x(1+x)}{(1-x)^3}$

$2 = 1 + 1^2$

$6 = 2 + 2^2$

$12 = 3 + 3^2$

$20 = 4 + 4^2$

$30 = 5 + 5^2$

$42 = 6 + 6^2$

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Calculating the coefficients of generating functions:-

$$(x+y)^n = \sum_{r=0}^{n} (^n c_r) x^r \qquad (1+x)^n = \sum_{r=0}^{n} (^n c_r) x^r$$

$$(1+x)^{-n} = \sum_{r=0}^{\alpha} \left(^{n+r-1} c_r \right) x^r$$

1. Determine the coefficient of x^{12} in $x^3(1-2x)^{10}$

Sol: Given $x^3(1-2x)^{10}$

$\because (1+x)^n = \sum_{r=0}^{n} {}^nC_r \, x^r$

$= x^3 \sum_{x=0}^{10} \binom{10}{r} (-2x)^r$

$= \sum_{r=0}^{10} {}^{10}C_r \, (-2)^r \, x^{3+r}$

To find the coefficient of x^{12} then substitute $r=9$.

${}^nC_r = \frac{n!}{(n-r)!\, r!}$

$= {}^{10}C_9 \, (-2)^9 \, x^{3+9}$

$= -10 \times 512$

$= -5120$

2. x^0 in $\left(3x^2 - \left(\frac{2}{x}\right)\right)^{15}$

$= (3x^2)^{15} \left(1 - \frac{2}{3x^3}\right)^{15}$

$= (3x^2)^{15} \sum_{r=0}^{15} \binom{15}{r} \left(\frac{-2}{3x^3}\right)^r$

$= (3x^2)^{15} \sum_{r=0}^{15} \binom{15}{r} (-2)^r \, (3x^3)^{-r}$

$= (3)^{15} \sum_{r=0}^{15} {}^{15}C_r \, (-2)^r \, (3)^r \, (x)^{30-3(r)}$

$r=0$

to get x^0 coefficient $r=10$

$= (3)^{15} \, ({}^{15}C_{10}) \, (-2)^{10} \, (3)^{-10} \, x^{30-3(10)}$

$= (3)^{15} \cdot {}^{15}C_{10} \, (-2)^{10} \, (-3)^{-10}$

Sol:-

$= 8.966909952 \times 10^{10}$

3. x^5 in $(1-2x)^{-7}$

$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$

$(1-2x)^{-7} = \sum_{r=0}^{\infty} \binom{7+r-1}{r} (2x)^r$

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Substitute $r=5$

$$= \sum_{r=5}^{\alpha} \binom{7+5-1}{5} (2)^5 \qquad n_{C_r} = \frac{n!}{(n-r)!\,r!}$$

$$= \binom{6+5}{5} (2^5) \qquad n_{C_5} = \frac{11!}{6!\,5!} = 462$$

$$= 462 \cdot 32$$

$$= 14784$$

4 x^{10} in $\left(\frac{x^3-5x}{1-x^3}\right)$

$$(x^3-5x) \cdot \frac{1}{1-x^3} = (x^3-5x)(1-x)^{-3}$$

$$= \sum_{r=0}^{1} \binom{1}{r} x^r - \sum_{r=0}^{\alpha} \binom{3+r-1}{r} x^r$$

$$= \left(\binom{1}{0} x^0\right) \sum_{r=0}^{\alpha} \binom{2+r}{r} x^r$$

$$= \sum_{r=0}^{\alpha} \binom{1}{0} \binom{2+r}{r} x^r \qquad \text{Substitute } r=10$$

$$= \binom{9}{7} - 5 \times \binom{11}{9} = -239$$

$$= \binom{12}{10}$$

5 x^{15} in $\frac{(1+x)^4}{(1-x)^4}$ $\quad (1+x)^4 (1-x)^{-4}$

$$= \sum_{r=0}^{4} \binom{4}{r} x^r \cdot \sum_{r=0}^{\alpha} \binom{4+r-1}{r} x^r$$

$$= \sum_{r=0}^{4} \binom{4}{r} x^r \cdot \sum_{r=0}^{\alpha} \binom{3+r}{r} x^r$$

$$= \left(\binom{4}{0} x^0 + \binom{4}{1} x^1 + \binom{4}{2} x^2 + \binom{4}{3} x^3 + \binom{4}{4} x^4\right) \sum_{r=0}^{\alpha} \binom{3+r}{r} x^r$$

$$= \sum_{r=0}^{\alpha} \binom{4}{0} \binom{3+r}{r} x^r + \sum_{r=0}^{\alpha} \binom{4}{1} \binom{3+r}{r} x^{r+1} + \sum_{r=0}^{\alpha} \binom{4}{2} \binom{3+r}{r} x^{r+2} +$$

$$\sum_{r=0}^{\alpha} \binom{4}{3} \binom{3+r}{r} x^{r+3} + \sum_{r=0}^{\alpha} \binom{4}{4} \binom{3+r}{r} x^{r+4}$$

Substitute $r=15$, $r=14$, $r=13$, $r=12$, $r=14$

$$= \sum_{r=0}^{\alpha} \binom{4}{0} \binom{18}{15} + \binom{4}{1} \binom{17}{14} + \binom{4}{2} \binom{16}{13} + \binom{4}{3} \binom{15}{12} + \binom{4}{4} \binom{14}{11}$$

6. x^8 in $\frac{1}{(x-3)(x-2)^2} = (x-3)^{-1}(x-2)^{2}$

$= \sum_{r=0}^{\alpha} \binom{1+r-1}{r} x^r + \sum_{r=0}^{\alpha} \binom{2+r-1}{r} x^r$

7. Find the coefficient of x^{27} in the following functions.

$(x^4+x^5+x^6+\cdots)^5$

Given $(x^4+x^5+x^6+\cdots)^5$

$=(x^4)^5(1+x+x^2+\cdots)^5$

$= x^{20}((1-x)^{-1})^5$ $\qquad (1+x)^{-n} = \sum_{r=0}^{\alpha} \binom{n+r-1}{r} x^r$

$= x^{20}(1-x)^{-5}$

$= x^{20} \sum_{r=0}^{\alpha} \binom{5+r-1}{r} x^r$

Substitute $r=7$

$= \sum_{r=0}^{\alpha} \binom{5+r-1}{r} x^{20} x^7$

$= \sum_{r=7}^{\alpha} \binom{5+7-1}{7} = \binom{5+6}{7} = \binom{11}{7} = 330$

11×10×9×8

90
×8
720
7920

8. $(x^4+2x^5+3x^6+\cdots)^5 \quad x^{27}$

Given $(x^4+2x^5+3x^6+\cdots)^5$

$= (x^4)^5(1+2x+3x^2+\cdots)^5$

$= x^{20}((1-x)^{-2})^5$

$= x^{20}(1-x)^{-10}$

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$$x^{20} \times \sum_{r=0}^{\alpha} \binom{n+r-1}{r} x^{r}$$

$$x^{20} \times \sum_{r=0}^{\alpha} \binom{10+r-1}{r} x^{r}$$

$$x^{20} \times \sum_{r=0}^{\alpha} \binom{9+r}{r} x^{r}$$

Substitute $r = 7$.

$$x^{27} = \binom{9+7}{7} = \binom{16}{7} = 11,440$$

9. coefficient of x^{18} in $(x+x^2+x^3+x^4+x^5)(x^2+x^3+x^4+\cdots)^5$

$(x+x^2+x^3+x^4+x^5)(x^2+x^3+x^4+\cdots)^5$

$(x+x^2+x^3+x^4+x^5)(1+x+x^2+\cdots)^5$

$\Rightarrow x^{11}(1+x+x^2+x^3+x^4)((1-x)^{-1})^5$

$\Rightarrow x^{11}(1+x+x^2+x^3+x^4)(1-x)^{-5}$

$\Rightarrow x^{11}(1+x+x^2+x^3+x^4) \sum_{r=0}^{\alpha} \binom{5+r-1}{r} x^{r}$

$\Rightarrow x^{11}(1+x+x^2+x^3+x^4)$

$\Rightarrow \sum_{r=0}^{\alpha} \binom{4+r}{r} x^{11+r} + \sum_{r=0}^{\alpha} \binom{4+r}{r} x^{12+r} + \sum_{r=0}^{\alpha} \binom{4+r}{r} x^{13+r} +$

$\sum_{r=0}^{\alpha} \binom{4+r}{r} x^{14+r} + \sum_{r=0}^{\alpha} \binom{4+r}{r} x^{15+r}$

Substitute $r = 7, r = 6, r = 5, r = 4, r = 3$

$\Rightarrow \binom{11}{7} + \binom{10}{6} + \binom{9}{5} + \binom{8}{4} + \binom{7}{3}$

10. $(x+x^3+x^5+x^7+x^9)(x^3+2x^4+3x^5+\cdots)^3$

$\Rightarrow x^{10}(1+x^2+x^4+x^6+x^8)(1+2x+3x^2+\cdots)^3$

$\Rightarrow x^{10}(1+x^2+x^4+x^6+x^8)((1-x)^{-2})^3$

$\Rightarrow x^{10}(1+x^2+x^4+x^6+x^8)(1-x)^{-6}$

$\Rightarrow x^{10}(1+x^2+x^4+x^6+x^8) \sum_{r=0}^{\alpha} \binom{6+r-1}{r} x^{r}$

$\Rightarrow x^{10}(1+x^2+x^4+x^6+x^8) \sum_{r=0}^{\alpha} \binom{5+r}{r} x^{r}$

$\Rightarrow \sum_{r=0}^{\alpha} \binom{5+r}{r} x^{10+r} + \sum_{r=0}^{\alpha} \binom{5+r}{r} x^{12+r} + \sum_{r=0}^{\alpha} \binom{5+r}{r} x^{14+r} + \sum_{r=0}^{\alpha} \binom{5+r}{r} x^{16+r} + \sum_{r=0}^{\alpha} \binom{5+r}{r} x^{18+r}$

Substitute $r = 8, r = 6, r = 4, r = 2, r = 0$

$\Rightarrow \binom{13}{8} + \binom{11}{6} + \binom{9}{4} + \binom{7}{2} + \binom{5}{0}$

Counting Technique:

11. $x_1+x_2+x_3+\cdots\cdots x_n=r$

$x_1+x_2+\cdots\cdots x_n=r$ Under the Constraints.,

x_1 can take integer values. $P_{11}, P_{12}, P_{13}\cdots$

x_2 can take $P_{21}, P_{22}, P_{23}\cdots$

⋮

x_n can take $P_{n1}, P_{n2}, P_{n3}\cdots$

To solve this problem we define the functions

$f_1(x)$ or $f(x_1)$

$f_1(x)=x^{P_{11}}+x^{P_{12}}+x^{P_{13}}\cdots$

$f_2(x)=x^{P_{21}}+x^{P_{22}}+x^{P_{23}}\cdots$

⋮

$f_n(x)=x^{P_{n1}}+x^{P_{n2}}+x^{P_{n3}}+\cdots$

$f(x)=f_1(x).f_2(x).f_3(x)\cdots\cdots f_n(x)$.

To determine the number of integer solutions we need to fined

⇒ Find the coefficient of x^r.

1. Find the generating function that determines the number of non-negative integer solution of that equation

* $x_1+x_2+x_3+x_4+x_5=20$ under the Constraints $x_1\leq 3$, $x_2\leq 4$, $2\leq x_3\leq 6$, $2\leq x_4\leq 5$, x_5 is odd with $x_5\leq 9$.

Sol: Given $x_1+x_2+x_3+x_4+x_5=20$

(i) $x_1\leq 3$ 0,1,2,3

$f_1(x)=x^0+x^1+x^2+x^3$

(ii) $x_2\leq 4$ 0,1,2,3,4

$f_2(x)=x^0+x^1+x^2+x^3+x^4$

(iii) $2\leq x_3\leq 6$ 2,3,4,5,6

$f_3(x)=x^2+x^3+x^4+x^5+x^6$

(iv) $2 \le x_4 \le 5$ $\quad 2, 3, 4, 5$

$f_4(x) = x^2 + x^3 + x^4 + x^5$

(v) $x_5 \le 9$ odd

1 3 5 7 9

$f_5(x) = x^1 + x^3 + x^5 + x^7 + x^9$

2 find the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ under the constraints $x_i \ge 0$ for $i = 1, 2, 3, 4, 5$ and x_2 is even, x_3 is odd.

Ans: Given $x_1 + x_2 + x_3 + x_4 + x_5 = 30$

$x_1 \ge 0 \quad 0, 1, 2, 3, 4, 5 - - - -$

$f_1(x) = x^0 + x^1 + x^2 + x^3 + - - - -$

$= 1 + x + x^2 + x^3 + - - - -$

$f_1(x) = (1-x)^{-1}$

(ii) $0, 2, 4, 6, 8$

$f_2(x) = x^0 + x^2 + x^4 + x^6 + x^8$

$= (1-x^2)^{-1}$

(iii) $x_3 \ge 0$ and is odd

$1, 3, 5, 7, 9$

$f_3(x) = x^1 + x^3 + x^5 + x^7 + - - -$

$= x(1 + x^2 + x^4 + x^6 + - - - -)$

$= x(1-x^2)^{-1}$

(iv) $x_4 \ge 0 \quad 0, 1, 2, 3, 4 - - - -$

$f_4(x) = x^0 + x^1 + x^2 + - - -$

$= (1-x)^{-1}$

(v) $x_5 \ge 0 \quad 0, 1, 2, 3 - - -$

$f_5(x) = x^0 + x^1 + x^2 + - - - = (1-x)^{-1}$

$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x) \cdot f_5(x)$

$= (1-x)^{-1} \cdot (1-x^2)^{-1} \cdot x(1-x^2)^{-1} \cdot (1-x)^{-1} \cdot (1-x)^{-1}$

$f(x) = x(1-x)^{-3}(1-x^2)^{-2}$

coefficient of x^{30}

$$= x \sum_{r=0}^{\alpha} \binom{3+r-1}{r} x^r \cdot \sum_{r=0}^{\alpha} \binom{2+r-1}{r} x^r$$

$$= \sum_{r=0}^{\alpha} \binom{3+r-1}{r} x^{r+1} \sum_{r=0}^{\alpha} \binom{2+r-1}{r} x^{2r}$$

Substitute $r = 1, 14 \mid r = 3, 13$.

$$= \binom{3}{1}\binom{15}{14} + \binom{5}{3}\binom{14}{13} + \binom{7}{5}\binom{13}{12} + \cdots$$

take $r = 29, 0$

$$= \binom{31}{29}\binom{1}{0}$$

3. Find the generating functions for the no. of Solutions for the equation $c_1 + c_2 + c_3 + c_4 = 20$ $-3 \leq c_1$, $-3 \leq c_2$, $-5 \leq c_3 \leq 5$, $0 \leq c_4$ find the number of solutions.

Ans: Let $x_1 = c_1 + 3$

$x_2 = c_2 + 3$

$x_3 = c_3 + 5$

$x_4 = c_4$

$x_1 = c_1 + 3$	$x_2 = c_2 + 3$	$x_3 = c_3 + 5$	$x_4 = c_4$
$c_1 = x_1 - 3$	$c_2 = x_2 - 3$	$c_3 = x_3 - 5$	$x_4 \geq 0$
$-\cancel{3} \leq x_1 - \cancel{3}$	$-\cancel{3} \leq x_2 - \cancel{3}$	$-5 \leq x_3 - 5 \leq 5$	
$0 \leq x_1$	$0 \leq x_2$	$0 \leq x_3 \leq 10$	
$x_1 \geq 0$	$x_2 \geq 0$		

convert the given equation also in x terms

$c_1 + c_2 + c_3 + c_4 = 20$

$x_1 - 3 + x_2 - 3 + x_3 - 5 + x_4 = 20$

$x_1 + x_2 + x_3 + x_4 - 31 = 0$

$x_1 + x_2 + x_3 + x_4 = 31$

(16)

(i) $x_1 \geq 0$ $\quad 0,1,2,3 \ldots$

$$f_1(x) = x^0 + x^1 + x^2 + x^3 + \ldots$$
$$= 1 + x + x^2 + x^3 + \ldots$$
$$= (1-x)^{-1}$$

(ii) $x_2 \geq 0$ $\quad 0,1,2,3 \ldots$

$$f_2(x) = x^0 + x^1 + x^2 + x^3 + \ldots$$
$$= 1 + x + x^2 + \ldots$$
$$= (1-x)^{-1}$$

(iii) $0 \leq x_3 \leq 10$ $\quad 0,1,2,3,4,5,6,7,8,9,10$

$$f_3(x) = x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}$$

(iv) $x_4 \geq 0$ $\quad 0,1,2 \ldots$

$$f_4(x) = x^0 + x^1 + x^2 + \ldots$$
$$= 1 + x + x^2 \ldots$$
$$= (1-x)^{-1}.$$

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$$
$$= (1-x)^{-1} (1-x)^{-1} (x^0 + x^1 + \ldots x^{10}) \cdot (1-x)^{-1}$$

coefficient of x^{31}

$$= (1 + x + x^2 + x^3 + \ldots + x^{10})(1-x)^{-3}$$
$$= (1 + x + x^2 + x^3 + \ldots + x^{10}) \times \sum_{r=0}^{\alpha} \binom{3+r-1}{r} x^r$$
$$= (1 + x + x^2 + \ldots + x^{10}) \times \sum_{r=0}^{\alpha} \binom{2+r}{r} x^r$$

Substitute $r = 31, 30, 29 \ldots 21.$

$$C_{31} = \binom{33}{31} + \binom{32}{30} + \binom{31}{29} + \ldots + \binom{23}{21}$$

4. In how many ways can 12 oranges be distributed among 3 children a,b,c so that a gets atleast four b and c gets atleast 2 but c gets no more than 5.

Ans: Given that $x_1+x_2+x_3=12$

$$\left.\begin{array}{r} x_1 \geq 4 \\ x_2 \geq 2 \\ 2 \leq x_3 \leq 5 \end{array}\right\} \text{constraints}$$

(i) $f_1(x) = x^4+x^5+x^6 \cdots$ 4,5,6 ---

(ii) $x_2 \geq 2$ 2,3,4 ---

$f_2(x) = x^2+x^3+x^4+\cdots$

(iii) $2 \leq x_3 \leq 5$ 2,3,4,5

$f_3(x) = x^2+x^3+x^4+x^5$

$f(x) = f_1(x).f_2(x).f_3(x)$

$= (x^4+x^5+x^6+\cdots)(x^2+x^3+x^4\cdots)(x^2+x^3+x^4+x^5)$

$= x^4(1+x+x^2+\cdots)x^2(1+x+x^2+\cdots)x^2(1+x+x^2+x^3)$

$= x^8(1-x)^{-1}(1-x)^{-1}(1+x+x^2+x^3)$

$= x^8(1+x+x^2+x^3)(1-x)^{-2}$

$= x^8(1+x+x^2+x^3) \times \sum_{r=0}^{\infty} \binom{2+r-1}{r} x^r$

$= x^8(1+x+x^2+x^3) \times \sum_{r=0}^{\infty} \binom{1+r}{r} x^r$

coefficient of x^{12}

Substitute $r = 4,3,2,1$.

$= \binom{4+1}{4} + \binom{3+1}{3} + \binom{2+1}{2} + \binom{1+1}{1}$

$= {}^5C_4 + {}^4C_3 + {}^3C_2 + {}^2C_1$

$= 5+4+3+2 = 14$

5. IN how many ways can we distribute 24 pencils to 4 children so that each child gets atleast 3 pencils but not more than 8.

Ans: $x_1+x_2+x_3+x_4 = 24$

Constraints $3 \le x_1 \le 8$

$3 \le x_2 \le 8$ $3 \le x_4 \le 8$

$3 \le x_3 \le 8$

(i) $3 \le x_1 \le 8$ 3, 4, 5, 6, 7, 8

$f_1(x) = x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

(ii) $f_2(x) = x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

(iii) $f_3(x) = x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

(iv) $x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$

$= (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$

$= (x^3)^4 (1 + x + x^2 + x^3 + x^4 + x^5)^4$

$= x^{12} \left(\frac{1(1-x^6)}{1-x} \right)^4$ Sum of n terms.

GP $= \frac{a(1-r^n)}{1-r}$

$= x^{12} (1-x^6)^4 (1-x)^{-4}$

AP $= a+(n-1)d$

$= x^{12} \sum_{r=0}^{4} \binom{4}{r} (-x^6)^r \sum_{r=0}^{\alpha} \binom{4+r-1}{r} x^r$

$= \sum_{r=0}^{4} \binom{4}{r} (-1)^r x^{6r+12} \sum_{r=0}^{\alpha} \binom{3+r}{r} x^r$

Substitute $r = 0, 12 / 1, 6 / 2, 0$

Coefficient of $x^{24} = \binom{4}{0}(-1)^0 \binom{15}{12} + \binom{4}{1}(-1)^1 \binom{9}{6} + \binom{4}{2}(-1)^2 \binom{3}{0}$

6. A bag contains a large number of red, green, black White Marbles with (atleast) 24 of each colour. In How many ways can I can select 24 of these Marbles so that there are even number of white

Marbles and atleast 6 black Marbles. (19)

Ans: $x_1+x_2+x_3+x_4=24$

Red: $x_1 \geq 0$ $\quad 0,1,2---$

Green: $x_2 \geq 0$ $\quad 0,1,2---$

White : even numbers $0,2,4,6----=x_3$ $\quad 0,2,4-$

black : $x_4 \geq 6$. $\quad 6,7,8----$

(i) $f_1(x) = x^0+x^1+x^2+----$

(ii) $f_2(x) = x^0+x^1+x^2+---$

(iii) $f_3(x) = x^0+x^2+x^4+----$

(iv) $f_4(x) = x^6+x^7+x^8+---$

$f_8(x) = f_1(x).f_2(x).f_3(x).f_4(x)$

$= (1+x+x^2+--)(1+x+x^2)(1+x^2+x^4--)(x^6+x^7+x^8---)$

$=(1+x)^{-1}(1-x)^{-1}(1-x^2)^{-1}x^6(1+x+x^2+---)$

$=(1+x)^{-1})(1-x)^{-1}(1-x^2)^{-1}x^6(1-x)^{-1}$

$= x^6(1-x)^{-3}(1-x^2)^{-1}$

$= x^6 \sum_{r=0}^{\alpha}\binom{3+r-1}{r}x^r \cdot \sum_{r=0}^{\alpha}\binom{1+r-1}{r}x^r$

$= x^6 \sum_{r=0}^{\alpha}\binom{2+r}{r}x^r \sum_{r=0}^{\alpha}(x^2)^r$

coefficient of x^{24}

Substitute $r = 18,16,14,12,10,8,6,4,2,0$

$=\binom{20}{18}+\binom{18}{16}+\binom{16}{14}+\binom{14}{12}+\binom{12}{10}+\binom{10}{8}+\binom{8}{6}+\binom{6}{4}+\binom{4}{2}+\binom{2}{0}$

$= 190+153+120+91+66+45+28+15+6+1$

$= 715$

Recurrense Relations:

The relation a and it it is expressed in terms of a and -1 a_{n-1} a and -2 a_{n-2} ... Such a relation is called recurrense relation for the Sequence.

process of determining a_n from a recurrence relation is called solving of the relation.

General Solution: A value a_n that satisfies a recurrence relation is called general solution.

First Order Recurrence Relation:

$a_n = c.a_{n-1} + f(n)$

c is constant

f(n) is known function.

If f(n)=0 the relation is called homogeneous. Otherwise it is non-homogeneous

$$a_n = c.a_{n-1} + f(n)$$

replace n with n+1

$$a_{n+1} = c.a_n + f(n+1)$$

Substitute n = 0, 1, ...

If $n=0 \Rightarrow a_1 = ca_0 + f(1)$

If $n=1 \Rightarrow a_2 = ca_1 + f(2)$

$$= c[ca_0 + f(1)] + f(2)$$

$$a_2 = c^2a_0 + cf(1) + f(2)$$

if $n=2 \Rightarrow a_3 = ca_2 + f(3)$

$$= c[c^2a_0 + cf(1) + f(2)] + f(3)$$

$$= c^3a_0 + c^2f(1) + cf(2) + f(3)$$

$$a_n = c^na_0 + c^{n-1}f(1) + c^{n-2}f(2) + \cdots + c^{n-n}f(n)$$

$$\boxed{a_n = c^na_0 + \sum_{K=1}^{n} c^{n-K} f(K)}$$

if f(n)=0 then the Solution is $a_n = c^na_0$.

* if $f(n)=0$ then the solution is $a_n = c^n a_0$.

1. Solve the recurrence relation $a_{n+1} = 4a_n$ for $n \geq 0$. Given $a_0 = 3$.

Ans: It is first order homogeneous recurrence relation

Given $a_{n+1} = 4a_n$

$a_n = c^n a_0$

$a_n = 4^n a_0 \rightarrow$ General Solution

given $n=3 \rightarrow$ particular solution

Substitution

$\therefore a_n = 4^n \cdot 3$.

2. Solve the recurrence relation $a_n = 7a_{n-1}$ when $n \geq 0$ given $a_2 = 98$.

A: Given $a_n = 7a_{n-1}$

$n \rightarrow n+1$

$a_{n+1} = 7a_n$ | $a_n = 7^n a_0 \rightarrow$ General solution.

Given $a_2 = 98$

Substitute $n=2$

$a_2 = 7^2 a_0$

$a_2 = 49 \cdot a_0$

$98 = 49 a_0$

$a_0 = 98/49$

$a_0 = 2$.

3. If a_n is a solution of recurrence relation $a_{n+1} = Ka_n$ and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$ what is K.

A: Given $a_{n+1} = Ka_n$.

Given a_n is the solution

$a_n = K^n a_0$

$a_3 = 153/49$

$n=3$,

$a_3 = K^3 a_0$

$\frac{153}{49} = K^3 a_0 \rightarrow (1)$

$a_5 = \frac{1377}{2401}$

$a_5 = K^5 a_0$

$\frac{1377}{2401} = K^5 a_0 \longrightarrow (2)$

$$\frac{\frac{1377}{2401}}{\frac{153}{49}} = \frac{K^{\cancel{5}^{2}} \cancel{a_0}}{K^{\cancel{3}} \cancel{a_0}}$$

$$\frac{1377}{2401} \times \frac{49}{153} = K^2$$

$$K^2 = \frac{9}{49}$$

$$K = \pm \frac{3}{7}$$

4. $4a_n - 5a_{n-1} = 0 \quad n \geq 1 \; , a_0 = 1 .$

5. $2a_n - 3a_{n-1} = 0 \quad n \geq 1 \; , a_4 = 81$

6. $a_{n+1} - a_n = 3n^2 - n \quad n \geq 0 \; , a_0 = 3.$

Given equation is non-homogeneous.

Second Order homogeneous Recurrence Relation:

$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0.$
where C_n, C_{n-1}, C_{n-2} are Constants.
Assume $a_n = CK^n$ is a solution

$C_n C K^n + C_{n-1} C K^{n-1} + C_{n-2} C K^{n-2} = 0 \longrightarrow (1)$

$C [C_n K^n + C_{n-1} K^{n-1} + C_{n-2} K^{n-2}] = 0$

$C K^{n-2} [C_n K^2 + C_{n-1} K + C_{n-2}] = 0$

$C_n K^2 + C_{n-1} K + C_{n-2} = 0$
It is a Quadratic equation
It is also called auxilary equation (or) characteristic equation.

(i) The two roots K_1 and K_2 are real and distinct then
$a_n = A_1 K_1^n + B_1 K_2^n$ $\quad a_n = A_1 K_1^n + B_1 K_2^n.$

(ii) K_1 and K_2 are equal $a_n = (A + Bn) K^n$

(iii) K_1 and K_2 are Complex.
$a_n = r^n (A \cos n\theta + B \sin n\theta)$

$K_1 = p + iq$
$K_2 = p - iq$ then
$a_n = r^n (A \cos n\theta + B \sin n\theta)$
$r = \sqrt{B^2 + A^2}$
$\theta = \tan^{-1} \frac{q}{p}$

* Solve the recurrence relation $A_n + a_{n-1} - 6a_{n-2}$
$a_n + a_{n-1} - 6a_{n-2} = 0$
$a_0 = -1 \quad a_1 = 8$

$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0$
$C_n = 1 \quad a_{n-1} = 1 \quad C_{n-2} = -6$
$a_n = CK^n$
$C_n C K^n + C_{n-1} C K^{n-1} + C_{n-2} C K^{n-2} = 0$
$1 \cdot C K^n + 1 \cdot C K^{n-1} - 6 \cdot C K^{n-2} = 0$

$c K^{n-2} [K^2 + K - 6] = 0$

$K^2 + K - 6$

$K^2 + 3K - 2K - 6 = 0$

$K(K+3) - 2(K+3) = 0$

$(K+3)(K-2) = 0$

∴ The roots are real and ~~(equal)~~ distint

$a_n = A(-3)^n + B(-2)^n \rightarrow (1)$

Sub $n=0$ & $n=1$

$a_0 = A(-3)^0 + B(2)^0$

$-1 = A + B \rightarrow (2)$

Sub $n=1$ in 1

$a_1 = A(-3)^1 + B(2)^1$

$8 = -3A + 2B \rightarrow (3)$

$3A - 2B + 8 = 0$

Solving (1) & (2)

$+3A + 2B = 8$

$+2A + 2B = -2$

$-5A = 10$

$A = -2$

$A + B = -1$

$-2 + B = -1$

$B = 1$

* $a_n = 3a_{n-1} - 2a_{n-2}$ $a_1 = 5$ $a_2 = 3$.

Given $a_n = 3a_{n-1} - 2a_{n-2}$ $a_1 = 5$, $a_2 = 3$.

$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0$ $a_n = cK^n$

$C_n = 1, C_{n-1} = -3, C_{n-2} = 2$

$c \cdot K^n - 3c \cdot K^{n-1} + 2 c K^{n-2} = 0$

$c K^{n-2} [K^2 - 3K + 2] = 0$

$K^2 - 3K + 2 = 0$

$K^2 - 2K - K + 2 = 0$

$K(K-2) - 1(K-2) = 0$

$(K-1)(K-2) = 0$

∴ The roots are real and distinct.

$a_n = A \times 2^n + B \times 1^n \rightarrow (1)$ $a_2 = Ax$

$a_1 = A \times 2^n + B \times 1^n$

$5 = 2A + B \rightarrow (2)$

$3 = 4A + B \rightarrow (3)$

$2A + \cancel{B} = 5$

$-\; 4A + \cancel{B} = 3$

$-2A = 8$

$A = \frac{8}{2} 4$

$\boxed{A = -4}$

$A = -4 \Rightarrow 5 = 2(-4) + B$

$5 = -8 + B$

$B = 5 + 8$

$\boxed{B = 12}$

* $a_n - 6a_{n-1} + 9a_{n-2} = 0 \qquad a_0 = 5 \quad a_1 = 12$

(25) $C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0.$

$C_n = 1, \quad a_{n-1} = -6, \quad a_{n-2} = 9$

$a_n = cK^n$

$C_n \cdot cK^n + C_{n-1} cK^{n-1} + C_{n-2} \, cK^{n-2} = 0$

$cK^{n-2} [K^2 - 6K + 9]$

$1 \cdot cK^n - 6 \, cK^{n-1} + 9 \, c \cdot K^{n-2} = 0$

$cK^{n-2} [K^2 - 6K + 9] = 0.$

$K^2 - 6K + 9 = 0.$

$K^2 - 3K - 3K + 9 = 0.$

$K(K-3) - 3K(K-3) = 0.$

$(K-3)(K-3) = 0.$

$K = 3, 3.$

$\therefore$ The roots are real and equal

$a_n = (A + Bn) K^n.$

$a_n = (A + Bn) 3^n \rightarrow (1).$

Sub $n = 0$ in (1) $\Rightarrow a_0 = (A) 3^0$

$a_0 = 5$

$5 = A$

Sub $n = 1$ in (1) $\Rightarrow a_1 = (B(3)) 3^1$

$12 = B.$

* $a_n - 2(a_{n-1} - a_{n-2}) \qquad a_0 = 1 \quad a_1 = 2.$

$a_n - 2a_{n-1} + 2a_{n-2} = 0.$

Let $a_n = c \cdot k^n$. Is a solution.

$cK^n - 2c \, K^{n-1} + 2 \, cK^{n-2} = 0$

$cK^{n-2} [K^2 - 2K + 2] = 0.$

$K^2 - 2K + 2 = 0.$

*

$$K = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$K = \frac{2 \pm \sqrt{4-4\times1\times2}}{2}$$

$$K = \frac{2 \pm 2i}{2} = 1 \pm i$$

$a_n = r^n (A\cos n\theta + B\sin n\theta)$

$r = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}(q/p)$

$= \tan^{-1}(1/1)$

$\theta = \pi/4$

$a_n = (\sqrt{2})^n \left(A\cos n\frac{\pi}{4} + B\sin n\pi/4\right)$

Substitute $n=0$.

$a_0 = 1.(A+B)$

$a_0 = 1$

$\boxed{1 = A}$

Sub $n=1$

$a_1 = \sqrt{2}\,(A\cos \pi/4 + B\sin \pi/4)$

$2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + B.\frac{1}{\sqrt{2}}\right)$

$2 = \cancel{\sqrt{2}}\left(\frac{1+B}{\cancel{\sqrt{2}}}\right)$

$2 = 1+B$

$\boxed{B=1}$

$a_n = (\sqrt{2})^n (1\cos n\pi/4 + 1\sin n\pi/4)$

* $f_{n+2} = f_{n+1} + f(n)$. $f_0 = 0$, $f_1 = 1$

replace n with n-2

$f_{n-2+2} = f_{n-2+1} + f_{n-2}$

$f_n = f_{n-1} + f_{n-2}$.

Let $f_n = cK^n$ be the solution

$cK^n -$ $f_n - f_{n-1} - f_{n-2} = 0$

(27) $f_n = c \cdot K^n$

$c \cdot K^n - c \cdot K^{n-1} - cK^{n-2} = 0$

$(K^2 - K - 1) = 0$

roots are $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$. $a_n = (A+Bn)\left(\frac{1-\sqrt{5}}{2}\right)^n$

$a_n = (A+Bn)K^n$

$a_n = (A+Bn)\left(\frac{1+\sqrt{5}}{2}\right)^n$. $K = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2}(1 \pm \sqrt{5})$

Sub $n=0$ $F_n = A\left[\frac{1+\sqrt{5}}{2}\right]^n + B\left[\frac{1-\sqrt{5}}{2}\right]^n$

$0 = A\left[\frac{1+\sqrt{5}}{2}\right]^0 + B\left[\frac{1-\sqrt{5}}{2}\right] = A+B$

$1 = A\left[\frac{1+\sqrt{5}}{2}\right] + B\left[\frac{1-\sqrt{5}}{2}\right]$ $A = -B = \frac{1}{\sqrt{5}}$

$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$

- $a_{n+2} = a_{n+1} \cdot a_n$. where $a_0 = 1$ $a_1 = 2$

replace n with $n-2$

$a_{n-2+2} = a_{n-2+1} \cdot a_{n-2}$

$a_n = a_{n-1} \cdot a_{n-2}$

Apply log on both sides

$\log a_n = \log(a_{n-1} \cdot a_{n-2})$

$\log a_n = \log a_{n-1} + \log a_{n-2}$

$\underline{\log a_n = c_n}$.

$c_n = c_{n-1} + c_{n-2}$.

$0 = F_0 = a_0 = \log a_0$

$1 = F_1 = a_1 = \log a_1$

$a_0 = 1, a_1 = 2$,

$a_n = F_n$ $a_n = \log_2 a_n$

$\therefore a_n = 2^{a_n} = 2^{F_n}$.

$\cdot\ a^2_{n+2} - 5a^2_{n+1} + 4a_n^2 = 0.$ $a_0 = 4$

replace n with n-2 $a_1 = 13.$ — 10 example

$a^2_{n+2-2} - 5a^2_{n-2+1} + 4a(n-2)^2 = 0.$

$a^2_n - 5a^2_{n-1} + 4a(n-2)^2 = 0.$

$a_n^2 = C_K^n$

$CK^n - 5C^{n-1} + 4C^{n-2} = 0.$

$C_K^{n-2}(K^2 - 5K + 4) = 0$

$K^2 - 5K + 4 = 0$

$K^2 - 4K - K + 4 = 0.$

$K(K-4) - 1(K-4) = 0$

$(K-1)(K-4) = 0.$

$K = 1, 4$

$a_n = A \times 4^n + B \times 1^n$

$a_0 = 4 \quad a_1 = 13.$

$16 = A + B \qquad 169 = 4A + B.$

$A = 51$

$B = \cdot 35$

$(1) \Rightarrow a_n = 51 \times 4^n - 35$

$a_n = \pm\sqrt{(51 \times 4^n - 35)}$

Third and higher order linear homogeneous Recurrence relations:

(29)

* solve the recurrence relation to

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \rightarrow ①$$

$$a_0 = 0, a_1 = 1, a_2 = 2$$

Substitute $n = n-3$ in (1)

$$2a_n = a_{n-1} + 2a_{n-2} - a_n$$

$$2a_n - a_{n-1} - 2a_{n-2} + a_n = 0$$

$$a_n = CK^n \Rightarrow$$

$$2K^3 - K^2 - 2K + 1 = 0$$

$$(K-1)(2K^2 + K - 1) = 0$$

$$\begin{array}{r|rrrr} 1 & 2 & -1 & -2 & 1 \\ & 0 & 2 & 1 & -1 \\ \hline & 2 & 1 & -1 & 0 \end{array}$$

$$2K^2 + K - 1 = 0$$

$$2K^2 + 2K - K - 1 = 0$$

$$2K(K+1) - 1(K+1) = 0$$

$$(2K-1)(K+1) = 0$$

$$K = 1/2 \quad K = -1, 1$$

∴ The roots are real and distinct.

$$a_n = A.1^n + B(-1)^n + C.(1/2)^n \rightarrow (2).$$

Substitute $n = 0$.

$$a_0 = A(1)^0 + B(-1)^0 + C(1/2)^0$$

$$\boxed{0 = A + B + C}$$

Sub $n = 1$

$$a_1 = A.1^1 + B(-1)^1 + C(1/2)^1$$

$$\boxed{1 = A - B + C/2}$$

Sub $n = 2$

$$a_2 = A(1)^2 + B(-1)^2 + C(1/2)^2$$

$$\boxed{2 = A + B + \frac{C}{4}}$$

$0 = A + \cancel{B} + C$
$1 = A - \cancel{B} + C/2$

$2A + \frac{3}{2}C = 1$

$A - \cancel{B} + C/2 = 1$
$A + \cancel{B} + C/4 = 2$

$2A + \frac{3C}{4} = 3$

solve

$\cancel{2A} + \frac{3}{2}C = 1$
$\cancel{2A} + \frac{3}{4}C = 3$
$- \quad - \quad -$

$C\left(\frac{3}{2} - \frac{3}{4}\right) = -2.$

$C\left[\frac{6-3}{4}\right] = -2$

$\frac{3C}{4} = -2$

$\boxed{C = -8/3}$

Sub $C = -8/3$ in above eqn.

$2A + \frac{\cancel{3}}{\cancel{2}}\left(\frac{-\overset{4}{\cancel{8}}}{\cancel{3}}\right) = 1$

$2A - 4 = 1$

$2A = 5$

$A = 5/2$

$A + B + C = 0$

$\frac{5}{2} + B - \frac{8}{3} = 0$

$B = \left[\frac{8}{3} - \frac{5}{2}\right] = \left[\frac{16-15}{6}\right] = 1/6$

$\therefore A = 5/2, \; B = 1/6, \; C = -8/3.$

Sub in (2)

$a_n = \frac{5}{2} \cdot 1^n + 1/6(-1)^n + \left(-\frac{8}{3}\right)(+1/2)^n.$

* $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0.$

$a_0 = 1, \; a_1 = 5, \; a_2 = 1$

Given $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0.$

$a_n = CK^n.$

$CK^{n-3}\left[K^3 + K^2 - 8K - 12\right] = 0.$

$K^3 + K^2 - 8K - 12 = 0.$

$(K-1)(K^2+4K+4)=0$

$(K-1)(K^2+2K+2K+4)=0$

$(K-1)(K(K+2)+2(K+2))=0$

$(K-1)(K+2)(K+2)=0$

$K=3,-2,-2.$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -8 & -12 \\ & 0 & 1 & 2 & -6 \\ \hline & 1 & 2 & -6 & -28 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -8 & -12 \\ & 0 & 2 & 6 & 4 \\ \hline & 1 & 3 & -2 & -10 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 1 & 1 & -8 & -12 \\ & 0 & 3 & 12 & 12 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$a_n=(A+Bn)(-2)^n+C\times 3^n$

Substitute $n=0$

$a_0=(A+B0)(-2)^0+C\times 3^0$

$1=A+C.$

Substitute $n=1$

$a_1=(A+B(1))(-2)^1+C\times 3^1$

$5=-2(A+B)+3C.$

Substitute $n=2$

$a_2=(A+B(2))(-2)^2+C\times 3^2$

$1=4(A+2B)+9C.$

Solving $A=0, B=-1, C=1$

① $\Rightarrow a_n=(-n)\times(-2)^n+3^n.$

* solve $a_n + a_{n-3} = 0$

Given $a_n + a_{n-3} = 0$

$a_n = c \cdot n^k$

$k^3 + 1 = 0$

$(k+1)(k^2 - k + 1) = 0$

$k = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2}$

$k = \frac{1 \pm \sqrt{3}i}{2}$

$k = -1, \frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}$

$$\begin{array}{r|rrrr} & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 2 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & +1 & -1 \\ \hline & 1 & -1 & +1 & 0 \end{array}$$

$a_n = A(-1)^n + r^n(A\cos n\theta + B\sin n\theta)$

$r = \sqrt{p^2 + q^2} = \sqrt{1/4 + 3/4} = 1$

$\theta = \tan^{-1}(p/q) = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \tan^{-1}\sqrt{3}$
$= \pi/3$

$\therefore a_n = A(-1)^n + 1^n\left(B\cos\frac{n\pi}{3} + C\sin\frac{n\pi}{3}\right)$

Non-Homogeneous Recurrence Relations:

$a_n = a_n^h + a_n^p$

a_n^h is the General Solution of the homogeneous part.

a_n^p is particular Solution.

$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + \cdots C_{n-k} a_{n-k} = f(n)$

$f(n) \neq 0$.

If $f(n)$ is a polynomial of degree 3 and 1 is not a root of characteristic equation then

$a_n^p = a_0 + a_1 n + a_2 n^2 + \cdots + a_3 n^3$.

$= A_0 + A_1 n + A_2 n^2 + \cdots + A_3 n^3$.

- $f(n)$ is a polynomial of degree Q and 1 is a root of multiplicity m of the characteristic equation then $a_n^P = n^m[A_0 + A_1 n + A_2 n^2 + \cdots + A_3^3 n^3]$.

$f(n) = \alpha B^n$ where α is a constant, B is not a root of characteristic equation then $a_n^P = A_0 b^n$.

$f(n) = \alpha \beta^n$ where α is a constant, and B is a root of multiplicity m then $a_n^P = A_0 n^m b^n$.

- Solve the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 8$. for $n \geq 2$ and $a_0 = 1$, $a_1 = 2$.

Given $a_n + 4a_{n-1} + 4a_{n-2} = 8 \rightarrow (1)$

Solve for homogeneous

$a_n + 4a_{n-1} + 4a_{n-2} = 0$

$K^2 + 4K + 4 = 0$

$K^2 + 2K + 2K + 4 = 0$

$(K+2)(K+2) = 0$

$k = -2, -2$

$\therefore$ Roots are real and equal.

$a_n = a_n^h + a_n^P$

$a_n^h = (A + Bn)(-2)^n$

$f(n)$ is polynomial of degree 0

$f(n) = 8$

and 1 is not a root of characteristic equation

$a_n^P = A_0 + A_1 n + A_2 n^2 + \cdots A_2 n^q$

$a_n^P = A_0$

Substitute a_n^P in eq(1)

$A_0 + 4A_0 + 4A_0 = 8$

$9A_0 = 8$

$A_0 = 8/9$

$a_n = (A+Bn)(-2)^n + \frac{8}{9}$

given $a_0 = 1$, $a_1 = 2$

Substitute $n=0$, $n=1$

$a_0 = A + B(0)\,(-2)^0 + \frac{8}{9}$

$1 = A + \frac{8}{9}$

$A = 1/9$

$a_1 = (\frac{1}{9} + B(1))\,(-2)^1 + \frac{8}{9}$

$= \frac{1}{9}(-2)B + \frac{8}{9}$

$= 2 + \frac{2}{9} - \frac{8}{9}$.

$B = -2/3$

$a^2_{n+2} - 5a^2_{n+1} + 6a_n^2 = 7n$ given $a_0 = a_1 = 1$

Substitute n with $n-2 \rightarrow$

$a^2_n - 5a^2_n + 6a^2_{n-2} = 7(n-2)$

Substitute $b_n = a_n^2 \rightarrow (3)$

$b_n - 5b_n + 6b_{n-2} = 7(n-2) \rightarrow (1)$

Solve homogeneous

$b_n - 5b_{n-1} + 6b_{n-2} = 0$

$K^2 - 5K + 6 = 0$

$K^2 - 3K - 2K + 6 = 0$

$K(K-3) - 2(K-3) = 0$

$(K-3)(K-2) = 0$

$K = 2, 3$

Roots are real and distinct

$b_n^n = A\,2^n + B \cdot 2^n$

now solve non-homogeneous

$f(n)$ is a polynomial of degree 1

'1' is not a root of characteristic equation

$b_n^0 = A_0 + A_1 n \rightarrow (2)$

Substitute in eqn -

$(A_0 + A_1 n) - 5(A_0) + A_1(n-1) + 6(A_0) + (A_1(n-2)) = 7n - 14$

$A_0 - A_1 n - 5A_0 - 5A_1 n + 5A_1 + 6A_0 + 6A_1 n - 12A_1 = 7n - 14$

$n(A_1 - 5A_1 + 6A_1) + A_0 - 5A_0 + 5A_1 + 6A_0 + 2A_1 = 7n - 14$

$\Rightarrow n(2A_1) + 2A_0 - 7A_1 = 7n - 14$

Compare 'n'

$2A_1 = 7$

$A_1 = 7/2$

$2A_0 - 7A_1 = -4$

$2A_0 - 7 \cdot \frac{7}{2} = -14$

Substitute in eq^n

$A_0 = 21/4$

(4b) $b_n^P = \frac{21}{4} + \frac{7}{2} n$

$b_n = A 2^n + B 3^n + \frac{21}{4} + \frac{7}{2} n \longrightarrow (4)$

Substitute $A_0 = 1$

$b_n = a_n^2$ this from eq(3)

$b_0 = a_0^2$

$b_0 = 1^2$

$b_0 = 1$

$b_0 = A \cdot 2^0 + B\ 3^0 + \frac{21}{4} + \frac{7}{2} \cdot 0$

$1 = A + B + \frac{21}{4}$

$A + B = 1 - \frac{21}{4}$

$A + B = -17/4 \longrightarrow (6)$

$A_1 = 1$

$b_1 = a_1^2$

$b_1 = 1^2$

$b_1 = 1$

$b_1 = A\, 2^1 + B\, 3^1 + \frac{21}{4} + \frac{7}{2} \cdot 1$

$1 = 2A + 3B + \frac{35}{4}$

$2A + 3B = \frac{1 - 35}{4} = \frac{-31}{4} \longrightarrow (7)$

$6 \times 2 = \cancel{2A} + 2B = -\frac{34}{4}$

$7 = \cancel{2A} + 3B = -\frac{31}{4}$

$\quad - \quad - \quad +$

$-B = -\frac{34}{4} + \frac{31}{4} \Rightarrow \cancel{-}B = \frac{\cancel{-}3}{4} = 3/4$

Sub in eq(5)

$B = 3/4$

$A + B = -17/4$

$A + \frac{3}{4} = -\frac{17}{4}$

$A = \frac{-17}{4} - \frac{3}{4} = \frac{-20}{4}$

$a_{n+2} + 3a_{n+1} + 2a_n = 3^n \qquad a_0 = 0 \quad a_1 = 1$

Sub n with $n-2$

$a_{n+2-2} + 3a_{n-2+1} + 2a_{n-2} = 3^{n-2}$

$a_n + 3a_{n-1} + 2a_{n-2} = 3^{n-2} \longrightarrow (1)$

$a_n + 3a_{n-1} + 2a_{n-2} = 0$

$K^2 + 3K + 2 = 0$

$K^2 + 2K + K + 2 = 0$

$K(K+2) + 1(K+2) = 0$

$(K+2)(K+1) = 0$

$K = -1, -2$

roots are real and distinct

$a_n^n = A(-1)^n + B(-2)^n$

It is in a form of $\alpha(\beta^n)$

$\alpha = 1/2 \quad \beta = 3$

3 is not root of characteristic eqn

$a_n^P = A_0 + A_1^n$

$a_n^P = A_0 \beta^n$

Substitute in eqn (1)

$A_0 3^n + 3A_0 3^{n-1} + 2A_0 3^{n-2} = 3^{n-2}$

$A_0 3^{n-2}(9+9+2) = 3^{n-2}$

$A_0 20 = 1$

$A_0 = 1/20$

$a_n^P = \frac{1}{20} 3^n$

$a_n = A(-1)^n + B(-2)^n + \frac{1}{20} 3^n \quad \text{---} (2)$

$a_0 = 0, \; a_1 = 1$

Sub $n = 0$ in eq (2)

$a_0 = A((-1)^0 + B(-2)^0 + \frac{1}{20} 3^0$

$0 = A + B + \frac{1}{20}$

$A + B = -1/20$

$a_1 = A(-1) + B(-2)^{-1} - 1 + \frac{1}{20}(3)^1$

$1 = -A - 2B + \frac{3}{20}$

$-A - 2B = \frac{+17}{20}$

$A + B = -1/20$

$-A - 2B = 17/20$

$-B = \cancel{16}^{4}/\cancel{20}_{5}$

$B = -4/5$

$a_n = \frac{3}{4}(-1)^n + -\frac{4}{5}(-2)^n + \frac{1}{20}3^n.$

Method of generating function

First Order recurrence relation

$a_n = c\,a_{n-1} + f(n)$

replace n with n-1

$a_{n+1} = c\,a_n + \phi(n) \quad f(x)$

$\phi n = f(n+1)$

$f(x) = \frac{a_0 + x\,g(x)}{1 - cx}$

$g(x) = \sum_{n=0}^{\alpha} \phi(n).x^n.$

find a generating function for the recurrence relation

$a_{n+1} - a_n = 3^n$ $a_0 = 1$ and hence solve the relation

$a_0 = 1 \quad c = -1 \quad \phi(n) = 3^n$

$f(x) = \frac{a_0 + x\,g(x)}{1 - cx}$

$g(x) = \sum_{n=0}^{\alpha} \phi(n)\,x^n$

$$g(x) = \sum_{n=0}^{\alpha} 3^n x^n$$

$$g(x) = \sum_{n=0}^{\alpha} (3x)^n = [1+3x+(3x)^2+(3x)^3 - - -]$$
$$= (1-3x)^{-1}$$

$$g(x) = \frac{1}{1-3x}$$

$$f(x) = \frac{1 + x \cdot \frac{1}{3} x}{1-x}$$

$$= \frac{1-3x+x}{(1-x)(1-3x)}$$

$$f(x) = \frac{1-2x}{(1-x)(1-3x)}$$

$$\frac{1-2x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$1-2x = A(1-3x) + B(1-3x)$$

put $x = 1$

$$1-2(1) = A(1-3(1) + B(1-1)$$
$$1-2 = A(1-3)$$
$$-1 = A(-2)$$
$$A = 1/2$$

$$1-2 \cdot 1/3 = A \cdot 0 + B(1-1/3)$$
$$-1/3 = -2/3 A$$
$$B = 1/2$$

$$f(x) = \frac{1/2}{1-x} + \frac{1/2}{1-3x}$$

$$f(x) = 1/2 \left[(1-x)^{-1} + (1-3x)^{-1} \right]$$

$$1/2 \left[\sum_{r=0}^{\alpha} \binom{1+r-1}{r} x^r + \sum_{r=0}^{\alpha} \binom{1+r-1}{r} (3x)^r \right.$$

$$1/2 \left[\sum x^r + \sum (3x)^r \right]$$

coefficient of x^n

$a_n = \frac{1}{2}[1+3^n]$

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Find the generating function for the recurrense relation for $a_{n+1}-a_n=n^2$ $a_0=1$ and hence solve it.

$a_0=1 \quad c=1 \quad \phi(n)=n^2$

$$f(x)=\frac{a_0+x\,g(x)}{1-cx}$$

$$g(x)=\sum_{r=0}^{\alpha} x^n \phi(n)$$

$$g(x)=\sum_{n=0}^{\alpha} x^n n^2$$

$$=[0+1^2x+2^2x^2+3^2x^3+\cdots]$$

$$g(x)=\frac{x(1+x)}{(1-x)^3}$$

$$f(x)=\frac{1+x^2(1+x)}{(1-x)3}$$

$$=\frac{\frac{1+x^2(1+x)}{(1-x)^3}}{1-x.}$$

$$=\frac{(1-x)^3+x^2+x^3}{(1-x)4}$$

$$(1+x)^4={}^nc_0+{}^nc_1x+{}^nc_2x^2+\cdots{}^nc_nx^n$$

$$={}^3c_0+{}^3c_1(x)+{}^3c_2(x)^2+{}^3c_3(x)^3$$

$$=1-3x+3x^2-x^3$$

$$f(x)=(1-3x+4x^2)(1-x)^{-4}$$

$$=(1-3x+4x^2)\left(\sum_{r=0}^{\alpha}\binom{4+r-1}{r}x^r\right)$$

Coefficient of x^n

$$a_n = \binom{4+n-1}{n} - 3\binom{4+n-1-1}{n-1} + 4\binom{4+n-2-1}{n-2}$$

$$a_n = \binom{3+n}{n} - 3\binom{2+n}{n-1} + \binom{n+1}{n-2}$$

* $a_n - 3(a_{n-1}) = n$ given $a_0 = 1$

Given $a_n - 3a_{n-1} = 0$.

$C = -3$, $a_0 = 1$

Substitute $n \to n+1$

$a_{n+1} - 3(a_{n+1-1})$

$a_{n+1} - 3a_n = n+1$

$$g(n) = \sum_{r=0}^{\alpha} \phi(n) x^n$$

$$g(n) = \sum_{r=0}^{\alpha} (n+1) x^n$$

$$\sum_{r=0}^{\alpha} (n+1)x^n = [x^0 + 2x^1 + 3x^2 + 4x^3 \ldots]$$

$g(x) = (1-x)^{-2}$

$f(x) = (1-x)^{-2}$

$= \frac{1}{(1-x)^2}$

$$f(x) = \frac{x^2 - x + 1}{(1-x)^2(1-3x)}$$

$$\frac{x^2 - x + 1}{(1-x)^2(1-3x)} = \frac{A}{1-3x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^2}$$

$x^2 - x + 1 = A(1-x^2) + B(1-x)(1-3x) + C(1-3x)$

$x = 1$

$$f(x) = \frac{1 + x\left(\frac{1}{(1+x)^2}\right)}{1-3x}$$

$$= \frac{1 + \frac{x}{(1-x)^2}}{1-3x}$$

$$= \frac{(1-x)^2 + x}{(1-x)^2(1-3x)}$$

$$= \frac{1-2x+x^2+x}{(1-x)^2(1-3x)}$$

$x=1$

$$\Rightarrow \cancel{1}-\cancel{1}+1 = A(0)+B(0)+C(1-3)$$

$$1 = -2C$$

$$C = -1/2$$

$x = 1/3$ $\quad 1/9 - 1/3 + 1 = A(1-1/3)^2 + B\cdot 0 + C\cdot 0$

$$7/9 = A(4/9)$$

$$A = 7/4$$

If $x=0$ $\quad 1 = A+B+C$

$$1 = 7/4 + B - 1/2$$

$$1 - 5/4 = B$$

$$B = -1/4$$

$$\frac{x^2-x+1}{(1-x)^2(1-3x)} = \frac{7}{4}(1-3x)^{-1} - \frac{1}{4}(1-x)^{-1} - 1/2(1-x)^{-2}$$

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Methods of generating function for second order Recurrense Relations:-

1. $a_n + A \cdot a_{n-1} + B \cdot a_{n-2} = f(n)$.

Given $a_n + A \cdot a_{n-1} + B \cdot a_{n-2} = f(n)$

replace n with $n+2$.

$$a_{n+2} + A a_{n+1} + B a_n = \phi(n)$$

$$\phi(n) = f(n+2)$$

Generating function $f(x) = \frac{a_0 + (a_1 + a_0 A) + x^2 g(x)}{1 + Ax + Bx^2}$

$$g(x) = \sum_{r=0}^{\alpha} \phi(n) x^n$$

* Find the generating function for the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$, $a_0 = -1$, $a_1 = 8$.

Given $a_n + a_{n-1} - 6a_{n-2} = 0$

The given eq^n is homogeneous.

replace n with $n+2$.

$$a_{n+2} + a_{n+1} - 6a_n = 0.$$

$$\phi(n) = 0$$

$$g(x) = 0$$

$$f(x) = a_0 + \frac{(a_1 + a_0 A) + x^2 g(x)}{1 + Ax + Bx^2}$$

$$= \frac{-1 + (8 + -1 \cdot A)}{1 + Ax + Bx^2}$$

$A = 1$, $B = -6$ comparing with equation

Substituting we get

$$= \frac{-1 + 8 + (-1) \cdot 1}{1 + x - 6x^2}$$

$$= \frac{6}{1 + x - 6x^2}$$

Find the G.F for the recurrence relation

$a_{n+2} - 3a_{n+1} + 2a_n = 0. \quad a_0 = 1, \; a_1 = 6.$

$\phi(n) = 0$

$g(n) = 0 \qquad A = -3, \; B = 2$

$$f(x) = \frac{a_0 + (a_1 + a_0 A) + x^2 g(x)}{1 + Ax + Bx^2}$$

$$= \frac{1 + (6 + (1)A) + 0}{1 + Ax + Bx^2} = \frac{1 + (6 + (1)(-3))}{1 - 3x + 2x^2} = \frac{4}{1 - 3x + 2x^2}$$

$1 - 3x + 2x^2 = (x - \alpha)(x - \beta)$

$$f(x) = \frac{4}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} \Rightarrow 4 = A(x-\beta) + B(x-\alpha)$$

$x = \alpha \Rightarrow 4 = A(\alpha - \beta) \qquad x = \beta \Rightarrow 4 = B(\beta - \alpha)$

$$A = \frac{4}{\alpha - \beta} = \frac{4}{1 - 1/2} = 4/\tfrac{1}{2} \qquad B = \frac{4}{\beta - \alpha} = \frac{4}{1/2 - 1} = 4/-\tfrac{1}{2}$$

$A = 8 \qquad B = -8.$

$$\frac{4}{(x-\alpha)(x-\beta)} = \frac{8}{x-\alpha} + \frac{-8}{x-\beta}$$

$$= 2\left[\frac{1}{\alpha}\sum_{r=0}^{\alpha}(x/\alpha)^r + \frac{1}{\beta}\sum_{r=0}^{\alpha}(x/\beta)^r\right]$$

$$\frac{4}{(x-\alpha)(x-\beta)} = 8\left[\frac{1}{x-\alpha} + \frac{-1}{x-\beta}\right]$$

coef of x^n

$$a_n = 2\left[-1/\alpha^{n+1} + \frac{1}{\beta^{n+1}}\right]$$

$$= 2\left[-1/(1)^{n+1} + 1/(1/2)^{n+1}\right]$$

$$\frac{1}{(x-\alpha)(x-\beta)} = 2\left[\frac{1}{-\alpha(1 - x/\alpha)} + \frac{1}{\beta(1 - x/\beta)}\right] = 2\left[\frac{-1}{\alpha}(1 - x/\alpha) + \frac{1}{\beta}(1 - x/\beta)^{-1}\right]$$

$$= 2\left[-1/\alpha \sum_{r=0}^{\alpha}\binom{1+r-1}{r}(x/\alpha)^r + 1/\beta \sum_{r=0}\binom{1+r-1}{r}\left(\frac{x}{\beta}\right)^r\right.$$

* Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$. $a_0 = 1$, $a_1 = 2$ by the method of generating function.

The given eq^n is non-homogeneous

$A = -2$

$\phi(n) = 2^n \qquad B = 1.$

$$g(x) = \sum_{n=0}^{\alpha} \phi(n) x^n = \sum_{n=0}^{\alpha} 2^n x^n = \sum_{n=0}^{\alpha}(2x)^n$$

$$= (1 + 2x + (2x)^2 + \cdots)$$

$$g(x) = (1 - 2x)^{-1}$$

$$f(x) = \frac{1 + (2 + 1A) + x^2(1-2x)^{-1}}{1 + Ax + Bx^2} = \frac{1 + 2 + 1\cdot(-2) + \frac{x^2}{1-2x}}{1 - 2x + x^2}$$

$$= \frac{1 + \frac{x^2}{1-2x}}{1 - 2x + x^2} = \frac{1 - 2x + x^2}{(x-1)^2(1-2x)} = \frac{(x-1)^2}{(x-1)^2(1-2x)} = \frac{1}{1-2x}$$

$f(x) = \frac{1}{1-2x} = (1-2x)^{-1}$

$= \sum_{r=0}^{\alpha} \binom{1+r-1}{r} (2x)^r$

Coefficient of x^n $\binom{n}{n} 2^n$

$a_n = 2^n$

* $a_{n+2} - 5a_{n+1} + 6a_n = 2$ $a_0 = 3$ $a_1 = 7$ and Hence Solv

Sol: $\phi(n) = 2$, $g(x) = \sum_{n=0}^{\alpha} \phi(n) x^n = \sum_{n=0}^{\alpha} 2x^n = 2 + 2x + 2x^2 + \cdots$
$= 2(1+x+x^2+\cdots) = 2(1-x)$

$f(x) = \frac{3 + (7 + (3)(-5)) + x^2 \cdot 2(1-x)^{-1}}{1-5x+6x^2}$

$6x^2 - 5x + 1$

$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-5 + \frac{2x^2}{1-x}}{(x-\alpha)(x-\beta)} = \frac{-5+5x+2x^2}{(x-1/2)(x-1/3)(1-x)}$

$= \frac{-5 \pm \sqrt{25 - 4(6)(1)}}{2(6)}$

$= \frac{5 \pm 1}{12}$, $6/12 = 1/2 = \alpha$

$= 4/12 = 1/3 = \beta$

$\frac{2x^2+5x-5}{(x-1/2)(x-1/3)(1-x)} = \frac{A}{x-1/2} + \frac{B}{x-1/3} + \frac{C}{1-x}$

$2x^2+5x-5 = A(x-1/3)(1-x) + B(x-1/2)(1-x) + C(x-1/2)(x-1/3)$

$x = 1/2 \Rightarrow 2(1/4) - 5(1/2) + 5 = A(1/2 - 1/3)(1 - 1/2)$

$1/2 + 5/2 - 5 = A(1/6)(1/2) \Rightarrow \frac{1+5-10}{2} = A(1/12)$

$\frac{-4^2}{2} = A(1/12)$

$A = -24$

$x = 1/3 \Rightarrow 2(1/9) + 5(1/3) - 5 = B(1/3 - 1/2)(1 - 1/3)$

$\frac{2}{9} + \frac{5}{3} - 5 = B\left(\frac{-1/6}{3}\right)\left(\frac{2}{3}\right)$

$\frac{2+15-45}{9} = B(-1/9) \Rightarrow -28 = -B \Rightarrow B = 28$

$x = 1 \Rightarrow 2(1)^2 + 5(1) - 5 = C(1-1/2)(1-1/3)$

$2+5-5 = C(1/2)(2/3) \Rightarrow 2 = C(1/3) \Rightarrow C = 6$

$\frac{2x^2+5x-5}{(x-1/2)(x-1/3)(1-x)} = \frac{-24}{x-1/2} + \frac{28}{x-1/3} + \frac{6}{1-x} = \frac{-24}{\frac{1}{2}[1-2x]} + \frac{28}{-1/3(1-3x)} + \frac{6}{1-x}$

$= \frac{+48}{1-2x} - \frac{84}{1-3x} + \frac{6}{1-x} = 48(1-2x)^{-1} - 84(1-3x)^{-1} + 6(1-x)^{-1}$

$= 48 \sum_{r=0}^{\alpha} \binom{1+r-1}{r} (2x)^r - 84 \sum_{r=0}^{\alpha} \binom{1+r-1}{r} (3x)^r + 6 \sum_{r=0}^{\alpha} \binom{1+r-1}{r} x^r$

$= 48 \sum_{r=0}^{\alpha} 2^r x^r - 84 \sum_{r=0}^{\alpha} 3^r x^r + 6 \sum_{r=0}^{\alpha} 1$

Coefficient of x^n $a_n = (48) 2^n - (84) 3^n + 6$

* Find the G.F for the fibonacci series and hence obtain an expression for $f(n)$.

Sol: $f_{n+2} = f_{n+1} + f_n$

$f_{n+2} - f_{n+1} - f_n = 0$

$\phi(n) = 0 \quad f_0 = 0, \; f_1 = 1$

$g(x) = 0 \qquad A = -1, \; B = -1$

$$f(x) = \frac{0 + (1 + 0\cdot(-1))}{1 - x - x^2} = \frac{0+1+0}{1-x-x^2} = \frac{1}{1-x-x^2}$$

$$1 - x - x^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1+4(1)\cdot 1}}{2(-1)} = \frac{1 \pm \sqrt{5}}{-2}$$

Assume

$$\alpha = \frac{1+\sqrt{5}}{-2}$$

$$\beta = \frac{1-\sqrt{5}}{-2}$$

$-x^2 - x + 1 = (x-\alpha)(x-\beta)$

$$f(x) = \frac{1}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} \Rightarrow \frac{1/\alpha-\beta}{x-\alpha} + \frac{1/\beta-\alpha}{x-\beta}$$

$x = \alpha$

$1 = A(x-\beta) + B(x-\alpha)$

$1 = A(\alpha - \beta)$

$$A = \frac{1}{\alpha - \beta}$$

$x = \beta$

$1 = B(\beta - \alpha)$

$$B = \frac{1}{\beta - \alpha}$$

$$\Rightarrow \frac{1}{\alpha-\beta}\left[\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right]$$

Take α as common

$$\frac{1}{\alpha-\beta}\left[\frac{1}{-\alpha(1-\frac{x}{\alpha})} + \frac{1}{\beta(1-\frac{x}{\beta})}\right]$$

$$= \frac{1}{\alpha-\beta}\left[\frac{-1}{\alpha}\left(1-\frac{x}{\alpha}\right)^{-1} + \frac{1}{\beta}\left(1-\frac{x}{\beta}\right)^{-1}\right]$$

$$\Rightarrow \frac{1}{\alpha-\beta}\left[\frac{-1}{\alpha}\sum_{r=0}^{\alpha}\binom{1+r-1}{r}\left(\frac{x}{\alpha}\right)^r + \frac{1}{\beta}\binom{1+r-1}{r}\left(\frac{x}{\beta}\right)^r\right]$$

$$= \frac{1}{\alpha-\beta}\left[\frac{-1}{\alpha}\sum_{r=0}^{\alpha}\left(\frac{x}{\alpha}\right)^r + \frac{1}{\beta}\sum_{r=\alpha}^{\alpha}\left(\frac{x}{\beta}\right)^r\right]$$

Coefficient of x^n

$$a_n = \frac{1}{\alpha-\beta}\left[\frac{-1}{\alpha}\cdot\frac{1}{\alpha^n} + \frac{1}{\beta}\frac{1}{(\beta)^n}\right]$$

$$= \frac{1}{\alpha-\beta}\left[\frac{-1}{\alpha^{n+1}} + \frac{1}{(\beta)^{n+1}}\right]$$

$$= \frac{1}{\left(\frac{1+\sqrt{5}}{-2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \left[\frac{-1}{\left(\frac{1+\sqrt{5}}{-2}\right)^{n+1}} + \frac{1}{\left(\frac{1-\sqrt{5}}{-2}\right)^{n+1}} \right]$$