UNIT 1 PROBLEMS

Problem:1.1

Point charges 1mC and –2mC are located at (3, 2, -1) and (-1, -1, 4) respectively. Calculate the electric force on a 10nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

We know

$$\bar{F} = \frac{Q}{4\pi \in_0} \sum_{K=1}^{2} Q_K \frac{\bar{r} - \bar{r}_K}{\left|\bar{r} - \bar{r}_K\right|^3}$$

$$= \frac{10\times10^{-9}}{4\pi \in_0}\left[1\times10^{-3}\frac{\left(-3\bar{a}_x + \bar{a}_y + 2\bar{a}_z\right)}{\left(\sqrt{9+1+4}\right)^3} - 2\times10^{-3}\frac{\left(\bar{a}_x + 4\bar{a}_y - 3\bar{a}_z\right)}{\left(\sqrt{1+16+9}\right)^3}\right]$$

$$= 90\left[\frac{\left(-3\bar{a}_x + \bar{a}_y + 2\bar{a}_z\right)\times10^{-3}}{52.38} - 10^{-3}\frac{\left(2\bar{a}_x + 8\bar{a}_y - 6\bar{a}_z\right)}{132.57}\right]$$

$$= 90\times10^{-3}\left[\bar{a}_x\left(\frac{-3}{52.38} - \frac{2}{132.57}\right) + \bar{a}_y\left(\frac{1}{52.38} - \frac{8}{132.57}\right) + \bar{a}_z\left(\frac{2}{52.38} + \frac{6}{132.57}\right)\right]$$

$$= 90\times10^{-3}\left[-0.0723\bar{a}_x - 0.0413\bar{a}_y + 0.0834\bar{a}_z\right]$$

$$= -0.0065\,\bar{a}_x - 0.0037\,\bar{a}_y + 0.0075\,\bar{a}_z \quad \text{N.}$$

Also we know $\bar{E} = \dfrac{\bar{F}}{Q}$

$$= -\frac{0.0065}{10\times10^{-9}}\bar{a}_x - \frac{0.037}{10\times10^{-9}}\bar{a}_y + \frac{0.0075}{10\times10^{-9}}\bar{a}_z$$

$$= -650\,\bar{a}_x - 370\,\bar{a}_y + 750\,\bar{a}_z \quad \text{KV/m.}$$

Problem :1.2

Point charges 5nC and –2nC are located at $2\bar{a}_x + 4\bar{a}_z$ and $-3\bar{a}_x + 5\bar{a}_z$ respectively. (a) Determine the force on a 1nC point charge located at $\bar{a}_x - 3\bar{a}_y + 7\bar{a}_z$.(b) Find the electric field $\bar{E}$ *at* $\bar{a}_x - 3\bar{a}_y + 7\bar{a}_z$.

Solution:

(a) We know

$$\bar{F} = \frac{Q}{4\pi \in_0} \sum_{K=1}^{2} Q_K \frac{\bar{r} - \bar{r}_K}{\left|\bar{r} - \bar{r}_K\right|^3}$$

$$= 10^{-9}\times 9\times 10^{9}\times 10^{-9}\left[5\frac{\left(-\bar{a}_x-3\bar{a}_y+3\bar{a}_z\right)}{\left(\sqrt{1+9+9}\right)^3}-\frac{2\left(4\bar{a}_x-3\bar{a}_y+2\bar{a}_z\right)}{\left(\sqrt{16+9+4}\right)^3}\right]$$

$$=9\times 10^{-9}\left[\bar{a}_x\left(\frac{-5}{82.81}-\frac{8}{156.169}\right)+\bar{a}_y\left(\frac{-15}{82.81}+\frac{6}{156.169}\right)+\bar{a}_z\left(\frac{15}{82.81}-\frac{4}{156.169}\right)\right]$$

$$= 9\times 10^{-9}\left[\bar{a}_x(-0.112)+\bar{a}_y(-0.143)+\bar{a}_z(0.155)\right]$$

$$= -1.008\,\bar{a}_x-1.287\,\bar{a}_y+1.395\,\bar{a}_z \quad nN$$

(b) $\bar{E}=\frac{\bar{F}}{Q}$, here Q=1nC

$$\therefore \bar{E}=-1.008\bar{a}_x-1.287\bar{a}_y+1.395\bar{a}_z V/m$$

***Problem:1.3**

Point charges Q_1 and Q_2 are respectively located at(4,0,-3) and (2,0,1). If Q_2=4nC, Find Q_1 such that (a) The $\bar{E}$ at (5,0,6) has no Z-component. (b) The force on a test charge at (5,0,6) has no X-component.

Solution:

We have $\bar{F}=\frac{Q}{4\pi\in_0}\sum_{K=1}^{2}Q_K\frac{\bar{r}-\bar{r}_K}{\left|\bar{r}-\bar{r}_K\right|^3}$

(a) $$\bar{E}=\frac{\bar{F}}{Q}=\frac{1}{4\pi\in_0}\left[\frac{Q_1\left|(5,0,6)-(4,0,-3)\right|}{\left(\sqrt{1+81}\right)^3}+\frac{4\times 10^{-9}\left|(5,0,6)-(2,0,1)\right|}{\left(\sqrt{9+25}\right)^3}\right]$$

Given $\bar{E}$ has no Z-component $$0=\frac{1}{4\pi\in_0}\left[\frac{Q_1\times 9}{\left(\sqrt{82}\right)^3}+\frac{4\times 10^{-9}\times 5}{\left(\sqrt{34}\right)^3}\right]$$

$$\frac{Q_1\times 9}{\left(\sqrt{82}\right)^3}=-\frac{4\times 10^{-9}\times 5}{\left(\sqrt{34}\right)^3}$$

$$Q_1=-\frac{20}{9}\left(\sqrt{\frac{41}{17}}\right)^3 nC=-8.3nC$$

(b) Given the force on test charge has no X-component

$$0=\frac{Q}{4\pi\in_0}\left[\frac{Q_1}{\left(\sqrt{82}\right)^3}+\frac{4\times10^{-9}\times3}{\left(\sqrt{34}\right)^3}\right]$$

$$\frac{Q_1}{\left(\sqrt{82}\right)^3}=-\frac{4\times10^{-9}\times3}{\left(\sqrt{34}\right)^3}$$

$$Q_1=-12\left(\sqrt{\frac{41}{17}}\right)^3 nC=-44.95nC$$

Problem:1.4

Two point charges of equal mass 'm', charge 'Q' are suspended at a common point by two threads of negligible mass and length '*l*'. Show that at equilibrium the inclination angle 'α' of each thread to the vertical is given by $Q^2 = 16\,\pi\in_0 mgl^2 \sin^2\alpha$ $\tan\alpha$, (or) $\frac{\tan^3\alpha}{1+\tan^2\alpha}=\frac{Q^2}{16\pi\in_0 mgl^2}$,

if 'α' is very small

Show that $\alpha=\sqrt[3]{\frac{Q^2}{16\pi\in_0 mgl^2}}$

Solution:

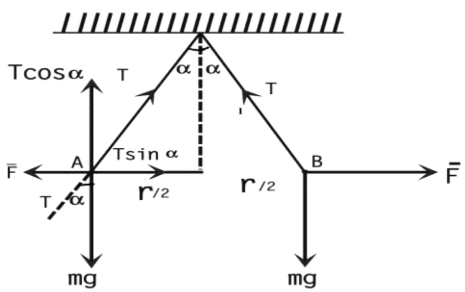


Fig: 1.3 suspended charge particles

When two charges are suspended from a common point with threads of length '*l*', we can represent graphically as sown in Fig:1.3, where T is the tension in thread 'mg' is the weight of charge towards ground due to gravitational force and $\bar{F}$ is force on charge at 'A'(B) due to charge at 'B'(A). T cos α is the vertical component of 'T' which is upwards and T sin α is the horizontal component of 'T' which is opposite to $\bar{F}$. To form equilibrium either at 'A' or 'B'

$$T\cos\alpha = mg \qquad (1.3.1)$$

$$T \sin\alpha = \overline{F} \qquad (1.3.2)$$

$$\frac{(1.3.1)}{(1.3.2)} = \frac{T\sin\alpha}{T\cos\alpha} = \frac{\overline{F}}{mg}$$

$$\Rightarrow Tan\,\alpha = \frac{\overline{F}}{mg}$$

where $\overline{F} = \dfrac{Q^2}{4\pi \in_0 r^2}$ from Fig:1.3 $\sin\alpha = \dfrac{r/2}{l}$

$\Rightarrow r = 2l\sin\alpha$

$$Tan\,\alpha = \frac{Q^2}{4mg\pi \in_0 r^2}$$

$$= \frac{Q^2}{4mg\pi \in_0 4l^2 \sin^2\alpha}$$

$$Tan\alpha = \frac{Q^2}{16mgl^2\pi \in_0 \sin^2\alpha}$$

$$Sin^2\alpha\, Tan\,\alpha = \frac{Q^2}{16mgl^2\pi \in_0} \qquad (1.3.3)$$

$$\Rightarrow Q^2 = 16\pi \in_0 mgl^2 Sin^2\alpha\, Tan\,\alpha \qquad (1.3.4)$$

From (1.3.3)

$$Cos^2\alpha \frac{Sin^2\alpha}{Cos^2\alpha} Tan\,\alpha = \frac{Q^2}{16\pi \in_0 mgl^2}$$

$$\frac{Tan^3\alpha}{Sec^2\alpha} = \frac{Q^2}{16\pi \in_0 mgl^2}$$

$$\frac{Tan^3\alpha}{1+Tan^2\alpha} = \frac{Q^2}{16\pi \in_0 mgl^2}$$

If α is very small, $\sin\alpha = Tan\,\alpha = \alpha$

From (1.3.4) $Q^2 = 16\pi\in_0 mg\, l^2\, \alpha^3$

$$\alpha^3 = \frac{Q^2}{16\pi \in_0 mg\, l^2}$$

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi \in_0 mg\, l^2}}$$

Problem:1.5

Two small identical conducting spheres have charges of $2x10^{-9}$ and $-0.5x10^{-9}C$ respectively. (a) When they are placed 4cm apart what is the force between them? (b) If they are brought into contact and then separated by 4cm. What is the force between them?

Solution:

(a) We know

$$\overline{F} = \frac{Q_1 Q_2}{4\pi \in_0 R^2}$$

$$= \frac{-2\times10^{-9}\times0.5\times10^{-9}\times9\times10^{9}}{4\times10^{-4}\times4}$$

$= -5.625\ \mu N$

(b) when they are brought into contact, charges will be added and again when they are separated charge will be distributed equally

$Q_1 = 0.758x10^{-9}C$ $\qquad$ $Q_2 = 0.75x10^{-9}C$

$$\overline{F} = 3.164\ \mu N$$

Problem:1.6

If the charges in the above problem are separated with the same distance in a kerosene ($\in_r = 2$), then find (a) and (b) as in the previous problem.

Solution:

(a)

$$\overline{F_k} = \frac{-5.625}{2}\ \mu N$$

$= -2.8125\ \mu N$

(b) $\overline{F_k} = \dfrac{3.164}{2}$ $= 1.582\ \mu N$

Problem:1.7

Three equal +Ve charges of $4x10^{-9}C$ each are located at 3 corners of a square, side 20cm. Determine the magnitude and direction of the electric field at the vacant corner point of the square.

Solution:

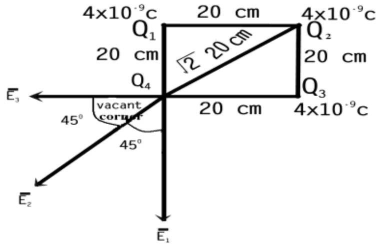


Fig:1.4

$\overline{E}_1$ = Electric field intensity at Q_4 due to Q_1

$$= \frac{Q_1}{4\pi \in_0 R^2}$$

= 900 V/m

$\overline{E}_2$ = 450 V/m

$\overline{E}_3$ = 900 V/m

The electric field intensity at vacant point is

$$\overline{E} = \overline{E}_2 + \overline{E}_1 \cos 45^0 + \overline{E}_3 \cos 45^0$$

$$= 450 + \frac{900}{\sqrt{2}} + \frac{900}{\sqrt{2}}$$

$$= 450 + 900\sqrt{2}$$

= 1722.792206 V/m

(1.14)

Problem: 1.8

A circular ring of radius ‘a’ carries a uniform charge ρ_L C/m and is placed on the XY plane with axis the same as the Z-axis.

(a) Show that $\overline{E}(0,0,h) = \dfrac{\rho_L ah}{2 \in_0 \left(h^2 + a^2\right)^{3/2}} \overline{a}_z$.

(b) What values of h gives the maximum value of $\overline{E}$

(c) If the total charge on the ring is Q. Find $\overline{E}$ as ‘a’ tends to zero.

Solution:

(a)

Here dl = a dϕ

dQ = ρ_L dl

= ρ_L adϕ

$$\therefore d\overline{E} = \frac{dQ}{4\pi \in_0 R^2}\overline{a}_r$$

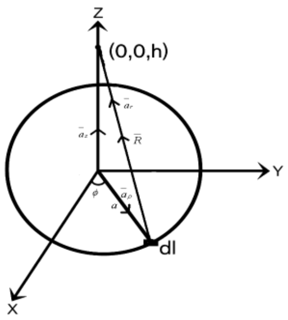


Fig:1.15

$$\overline{a}_r = \frac{\overline{R}}{|R|}\;;\;\frac{\overline{a}_r}{R^2} = \frac{\overline{R}}{R^3}$$

$$\therefore d\overline{E} = \frac{dQ}{4\pi \in_0}\frac{\left[-a\,\overline{a}_\rho + h\overline{a}_z\right]}{\left(a^2+h^2\right)^{3/2}}$$

dQ = ρ_L a dϕ

$$Q = \int \rho_L\, a\, d\phi$$

when we add up electric fields, the electric field in ρ direction gets cancelled.

$$\therefore\ \overline{E} = \frac{dQ}{4\pi \in_0}\frac{h\overline{a}_z}{\left(a^2+h^2\right)^{3/2}}$$

$$= \int\ \frac{\rho_L a\, d\phi}{4\pi \in_0}\frac{h\overline{a}_z}{\left(a^2+h^2\right)^{3/2}}$$

$$= \frac{\rho_L a}{4\pi \in_0}\frac{h\overline{a}_z}{\left(a^2+h^2\right)^{3/2}}\int_0^{2\pi} d\phi = \frac{\rho_L ah}{2\in_0\left(a^2+h^2\right)^{3/2}}\overline{a}_z$$

(b) $\frac{d\overline{E}}{dh} = 0$

$$\frac{\rho_L a}{2\in_0}\overline{a}_z\ \frac{\left(a^2+h^2\right)^{3/2}.1 - h\frac{3}{2}(a^2+h^2)^{1/2}2h}{\left(a^2+h^2\right)^3} = 0$$

$(a^2 + h^2) - 3h^2 = 0$

$a^2 - 2h^2 = 0$

$2h^2 = a^2$

$$h = \pm \frac{a}{\sqrt{2}}$$

(c) When 'a' tends to zero, it becomes a point charge 'Q' located at origin and we have to find electric field at (0,0,h) due to point charge 'Q' located at origin.

$$\therefore \ \bar{E} = \frac{Q}{4\pi \in_0 h^2} \bar{a}_z$$

***Problem: 1.9**

Derive an expression for the electric field strength due to a circular ring of radius 'a' and uniform charge density ρ_L C/m. Obtain the value of height 'h' along Z-axis at which the net electric field becomes zero. Assume the ring to be placed in X-Y plane.

Solution:

Derivation is as in Problem: 1.8.

$$\bar{E} = \frac{\rho_L ah}{2 \in_0 \left(a^2 + h^2\right)^{3/2}} \bar{a}_z$$

Which can be written as

$$\bar{E} = \frac{\rho_L a}{2 \in_0 h^2 \left(\frac{a^2}{h^2} + 1\right)^{3/2}} \bar{a}_z$$

From the above equation we can say that for h=∞, the net electric field becomes zero.

***Problem: 1.10**

A circular ring of radius 'a' carries uniform charge ρ_L C/m and is in XY-plane. Find the Electric field at point (0,0,2) along its axis.

Solution:

Replacing 'h' in problem:1.8 with '2' and solving, we get

$$\bar{E} = \frac{\rho_L a 2}{2 \in_0 \left(a^2 + 4\right)^{3/2}} \bar{a}_z$$

1.1.1 Volume Charge Distribution

Consider a sphere of radius 'a' as shown in the Fig:1.16.

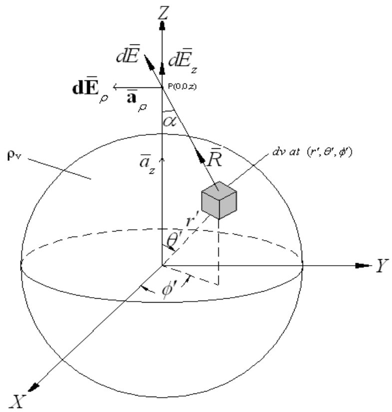


Fig:1.16 Finding $\bar{E}$ due to volume charge distribution

Assume elemental volume dv is placed at point (r', θ', ϕ'). The elemental charge dQ due to the elemental volume dv, whose volume charge density ρ_v is

$$dQ = \rho_v dv$$

$$Q = \rho_v \int_v dv$$

$$= \rho_v \frac{4}{3} \pi a^3$$

The elemental electric field $d\bar{E}$ due to elemental volume dv is

$$d\bar{E} = \frac{dQ}{4\pi \in_0 R^2} \bar{a}_R$$

$$= \frac{\rho_v dv}{4\pi \in_0 R^2} \bar{a}_R$$

where $\bar{a}_R = \cos\alpha\, \bar{a}_z + \sin\alpha\, \bar{a}_\rho$

Due to symmetry, the electric field in 'ρ' direction will be zero. Finally total electric field will be in Z-direction.

$$\bar{E}_z = \bar{E} \cdot \bar{a}_z = \int_v \frac{\rho_v dv}{4\pi \in_0 R^2} \cos\alpha$$

In spherical coordinate system

$$dv = dr' r' d\theta' r' \sin\theta' d\phi'$$

$$dv = (r')^2 \sin\theta' dr' d\theta' d\phi'$$

$$\bar{E}_z = \int_v \frac{\rho_v (r')^2 \sin\theta' dr' d\theta' d\phi' \cos\alpha}{4\pi \in_0 R^2}$$

By applying cosine rule in the Fig:1.16

$$(r')^2 = z^2 + R^2 - 2zR\cos\alpha$$

$$\cos\alpha = \frac{-(r')^2 + z^2 + R^2}{2zR}$$

Similarly

$$R^2 = z^2 + (r')^2 - 2zr'\cos\theta'$$

$$\Rightarrow \cos\theta' = \frac{z^2 + (r')^2 - R^2}{2zr'} \qquad (1.15)$$

on differentiating equation (1.15), we get

$$-\sin\theta' d\theta' = \frac{-2R}{2zr'} dR$$

$$\sin\theta' d\theta' = \frac{R}{zr'} dR$$

Here as θ' varies from 0 to π, R changes from $z - r'$ to $z + r'$ respectively

Substituting $\cos\alpha$ and $\sin\theta' d\theta'$ in $\bar{E}_z$ equation, we get

$$\bar{E}_z = \frac{\rho_v}{4\pi \in_0} \int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'^2 \frac{RdR}{zr'} dr' \frac{z^2 + R^2 - r'^2}{2zR} \frac{1}{R^2}$$

$$\bar{E}_z = \frac{\rho_v 2\pi}{8\pi \in_0 z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r' \left[1 + \frac{z^2 - r'^2}{R^2}\right] dRdr'$$

$$\bar{E}_z = \frac{\rho_v \pi}{4\pi \in_0 z^2} \int_{r'=0}^{a} r' \left[R - \frac{z^2 - r'^2}{R}\right]_{z-r'}^{z+r'} dr'$$

$$\bar{E}_z = \frac{\rho_V \pi}{4\pi \in_0 z^2} \int_{r'=0}^{a} 4r'^2 dr'$$

$$\bar{E}_z = \frac{\rho_v}{\in_0 z^2}\frac{a^3}{3} = \frac{\rho_v}{4\pi \in_0 z^2}\frac{4}{3}\pi a^3$$

$$\bar{E} = \frac{Q}{4\pi \in_0 z^2}\bar{a}_z \qquad (1.16)$$

The electric field due to a sphere of radius 'a' with volume charge density ρ_v is similar to the electric field due to a point charge which is placed at origin.

Problem: 1.11

A circular disk of radius 'a' is uniformly charged with ρ_s C/m^2. If the disk lies on the Z=0 plane with it's axis along the Z-axis

(a) Show that at point (0, 0, h), $\bar{E} = \frac{\rho_s}{2\in_0}\left[1 - \frac{h}{\left(h^2+a^2\right)^{1/2}}\right]\bar{a}_z$

(b) From this derive the $\bar{E}$ due to an infinite sheet of charge on the Z=0 plane.

(c) If a <<h, Show that $\bar{E}$ is similar to the field due to a point charge.

Solution:

(a)

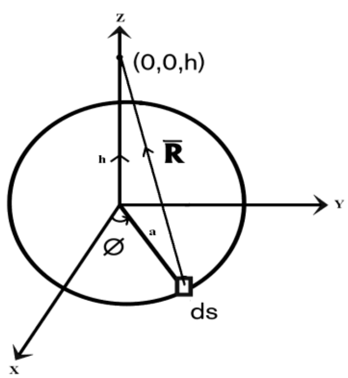


Fig:1.17

$$d\bar{E} = \frac{dQ}{4\pi \epsilon_0 R^2} \bar{a}_r$$

$dQ = \rho_s \, ds \quad ; ds = d\rho.\rho d\phi,$

$= \rho_s \, \rho d\rho \, d\phi$

$$\rho\bar{a}_\rho + \bar{R} = h\bar{a}_z$$

$$\bar{R} = h\bar{a}_z - \rho\bar{a}_\rho$$

$$\bar{E} = \int_S \frac{\rho_s \rho \, d\rho \, d\phi}{4\pi \epsilon_0} \frac{\left(h\bar{a}_z - \rho\bar{a}_\rho\right)}{\left(h^2 + \rho^2\right)^{3/2}}$$

$$\bar{E} = \frac{\rho_s}{4\pi \epsilon_0} \bar{a}_z \int_0^{2\pi} d\phi \int_0^a \frac{\rho h}{\left(h^2 + \rho^2\right)^{3/2}} d\rho$$

$$= \frac{\rho_s}{4\pi \epsilon_0} \bar{a}_z 2\pi h \int_0^a \frac{1}{2} \left(h^2 + \rho^2\right)^{-3/2} d\left(\rho^2\right)$$

$$= \frac{\rho_s h}{2 \epsilon_0} \bar{a}_z \frac{1}{2} \left[\frac{\left(h^2 + \rho^2\right)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1} \right]_0^a$$

$$= \frac{\rho_s h}{4 \epsilon_0} \bar{a}_z \left\{ -2\left[\left(h^2 + a^2\right)^{-1/2} - \left(h^2\right)^{-1/2} \right] \right\}$$

$$= \frac{-\rho_s h \bar{a}_z}{2 \epsilon_0} \left[\frac{1}{\sqrt{\left(h^2 + a^2\right)}} - \frac{1}{h} \right]$$

$$\bar{E} = \frac{\rho_s}{2 \epsilon_0} \left[1 - \frac{h}{\left(h^2 + a^2\right)^{1/2}} \right] \bar{a}_z$$

(b) $a \to \infty$; $\quad \therefore \bar{E} = \frac{\rho_s}{2 \epsilon_0} \bar{a}_z$

(c) when a<<h, the volume charge density becomes a point charge located at origin, $\therefore \bar{E} = \frac{Q}{4\pi \epsilon_0 h^2} \bar{a}_z$

Problem: 1.12

The finite sheet $0 < x < 1$, $0 < y < 1$ on the Z=0 plane has a charge density $\rho_s = xy(x^2 + y^2 + 25)^{3/2}$ nC/m^2.
Find (a) the total charge on the sheet

(b) the electric field at (0, 0, 5)

(c) the force experienced by a −1nC charge located at (0, 0, 5)

Solution:

(a) $dQ = \rho_s \, ds$

$$Q = \int_s \rho_s ds$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} xy\left(x^2+y^2+25\right)^{3/2} n dx dy$$

$$= n \int_{x=0}^{1} x \int_{y=0}^{1} \left(x^2+y^2+25\right)^{3/2} \frac{1}{2} d(y^2) dx$$

$$= n \int_{x=0}^{1} x \left[\left(x^2+y^2+25\right)^{5/2}\right]_0^1 \frac{2}{5}\frac{1}{2} dx$$

$$= \frac{n}{5} \int_{x=0}^{1} \left[\left(x^2+26\right)^{5/2} - \left(x^2+25\right)^{5/2}\right] \frac{1}{2} d(x^2)$$

$$= \frac{n}{5}\left[\left(x^2+26\right)^{7/2} - \left(x^2+25\right)^{7/2}\right]_0^1 \frac{1}{7}$$

$$= \frac{n}{35}\left[(27)^{7/2} - 2(26)^{7/2} + (25)^{7/2}\right]$$

$$= \frac{n}{35}\left[102275.868136 - 179240.733942 + 78125\right]$$

Q = 33.15nC

(b) Electric field at (0, 0, 5)

$$d\overline{E} = \frac{\rho_s ds}{4\pi \in_0 R^2} \overline{a}_R \qquad \text{; on Z-plane point is } (x,y,0)$$

$$\therefore \overline{R} = (0,0,5) - (x,y,0) = -x\overline{a}_x - y\overline{a}_y + 5\overline{a}_z$$

$$\frac{\bar{a}_R}{R^2} = \frac{\bar{R}}{\left|\bar{R}\right|^3} = \frac{-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z}{\left(\sqrt{x^2 + y^2 + 25}\right)^3}$$

$$\bar{E} = \int_s \frac{\rho_S ds}{4\pi \in_0} \frac{\bar{R}}{\left|\bar{R}\right|^3}$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \frac{xy\left(x^2 + y^2 + 25\right)^{3/2} \times 10^{-9}}{4\pi \in_0} \left(\frac{-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z}{\left(\sqrt{x^2 + y^2 + 25}\right)^3} \right) dx\, dy$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} \int_{y=0}^{1} -x^2 y\bar{a}_x - xy^2\bar{a}_y + 5xy\bar{a}_z dx\, dy \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} -x^2\left[\frac{y^2}{2}\right]_0^1 \bar{a}_x - x\left[\frac{y^3}{3}\right]_0^1 \bar{a}_y + 5x\left[\frac{y^2}{2}\right]_0^1 \bar{a}_z\, dx \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} -\frac{x^2}{2}\bar{a}_x - \frac{x}{3}\bar{a}_y + \frac{5}{2}x\bar{a}_z\, dx \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0}\left[\left[-\frac{x^3}{6}\right]_0^1 \bar{a}_x - \left[\frac{x^2}{6}\right]_0^1 \bar{a}_y + \frac{5}{2}\left[\frac{x^2}{2}\right]_0^1 \bar{a}_z\right] \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0}\left[-\frac{1}{6}\bar{a}_x - \frac{1}{6}\bar{a}_y + \frac{5}{4}\bar{a}_z\right] \times 10^{-9}$$

$$= 9 \times 10^9\left[-\frac{1}{6}\bar{a}_x - \frac{1}{6}\bar{a}_y + \frac{5}{4}\bar{a}_z\right] \times 10^{-9}$$

$$= -1.5\bar{a}_x - 1.5\bar{a}_y + 11.25\bar{a}_z \text{ V/m}$$

(c) $\bar{\mathrm{F}} = \mathrm{q}\bar{E}$

$$= \left(-1nC\right)\left[-1.5\bar{a}_x - 1.5\bar{a}_y + 11.25\bar{a}_z\right]$$

$$= 1.5\bar{a}_x + 1.5\bar{a}_y - 11.25\bar{a}_z \text{ nN}$$

Problem: 1.13

A square plane described by –2 < x < 2, -2 < y < 2, z = 0 carries a charge density 12|y| mC/m². Find the total charge on the plate and the electric field intensity at (0, 0, 10)

Solution:

$dQ = \rho_s\, ds$

$$Q = \int_s \rho_s ds$$

$$= \int_{x=-2}^{2} \int_{y=-2}^{2} 12\,|\,y\,|\times 10^{-3}\, dx dy$$

$$= 10^{-3} \int_{x=-2}^{2} \left[\int_{y=-2}^{0} -12 y\, dy + \int_{y=0}^{2} 12 y\, dy \right] dx$$

$$= 10^{-3} \int_{x=-2}^{2} -12\left[\frac{y^2}{2}\right]_{-2}^{0} + 12\left[\frac{y^2}{2}\right]_{0}^{2} dx$$

$$= 10^{-3} \int_{x=-2}^{2} 12(2) + 12(2) dx$$

$$= 48\times 10^{-3} \int_{x=-2}^{2} dx = 48\times 10^{-3} \times 4 = 192 mC$$

$$d\overline{E} = \frac{\rho_s ds}{4\pi \in_0 R^2}\,\overline{a}_R\,;\ \overline{R} = (0,0,10) - (x, y, 0) = -x\overline{a}_x - y\overline{a}_y + 10\overline{a}_z$$

$$d\overline{E} = \frac{\rho_s ds}{4\pi \in_0} \frac{\overline{R}}{R^3}$$

$$\overline{E} = \int_s \frac{\rho_s ds}{4\pi \in_0} \frac{\overline{R}}{R^3}$$

$$= \int_{x=-2}^{2} \int_{y=-2}^{2} \frac{12\,|\,y\,|\times 10^{-3}}{4\pi \in_0} \left(\frac{-x\overline{a}_x - y\overline{a}_y + 10\overline{a}_z}{\left(\sqrt{x^2 + y^2 + 100}\right)^3} \right) dx\, dy$$

$$= 9\times10^6\times12\int_{x=-2}^{2}\left[\int_{y=-2}^{0}\frac{xy\bar{a}_x + y^2\bar{a}_y - 10y\bar{a}_z}{\left(x^2+y^2+100\right)^{3/2}}dy + \int_{y=0}^{2}\frac{-xy\bar{a}_x - y^2\bar{a}_y + 10y\bar{a}_z}{\left(x^2+y^2+100\right)^{3/2}}dy\right]dx$$

Replacing y with –y in the first integral and simplifying

$$\bar{E} = 108\times10^6\int_{x=-2}^{2}\left[\int_{y=0}^{2}\frac{-2xy\bar{a}_x + 20y\bar{a}_z}{\left(x^2+y^2+100\right)^{3/2}}dy\right]dx$$

$$= 108\times10^6\int_{x=-2}^{2}\left[-x\int_{y=0}^{2}2y\bar{a}_x\left(x^2+y^2+100\right)^{-3/2}dy + 10\int_{y=0}^{2}2y\bar{a}_z\left(x^2+y^2+100\right)^{-3/2}dy\right]dx$$

$$= 108\times10^6\int_{x=-2}^{2}\left[-x\int_{y=0}^{2}\bar{a}_x\left(x^2+y^2+100\right)^{-3/2}d(y^2) + 10\int_{y=0}^{2}\bar{a}_z\left(x^2+y^2+100\right)^{-3/2}d(y^2)\right]dx$$

$$= 108\times10^6\int_{x=-2}^{2}\left[-x\left[\frac{\left(x^2+y^2+100\right)^{-1/2}}{-1/2}\right]_0^2\bar{a}_x + 10\left[\frac{\left(x^2+y^2+100\right)^{-1/2}}{-1/2}\right]_0^2\bar{a}_z\right]dx$$

$$= 108\times10^6\int_{x=-2}^{2}\left\{\left[2x\left(x^2+104\right)^{-1/2} - 2x\left(x^2+100\right)^{-1/2}\right]\bar{a}_x - 20\left[\left(x^2+104\right)^{-1/2} - \left(x^2+100\right)^{-1/2}\right]\bar{a}_z\right\}dx$$

$$\because x\left(x^2+104\right)^{-1/2} \;\&\; x\left(x^2+100\right)^{-1/2} \textit{ are odd functions}$$

$$\textit{and}\quad\left(x^2+104\right)^{-1/2} \;\&\; \left(x^2+100\right)^{-1/2} \textit{ are even functions}$$

$$\int_{-a}^{a} f(x)dx \quad = \quad 0 \qquad \textit{if} \quad f \quad \textit{is} \quad \textit{odd}$$

$$= \quad 2\int_{0}^{a} f(x)dx \quad \textit{if} \quad f \quad \textit{is} \quad \textit{even}$$

$$\therefore \bar{E} = -20\times108\times10^6\times2\int_{x=0}^{2}\left[\frac{1}{\sqrt{x^2+\left(\sqrt{104}\right)^2}} - \frac{1}{\sqrt{x^2+10^2}}\right]\bar{a}_z dx$$

$$= -40\times108\times10^6\left[\sinh^{-1}\left(\frac{x}{\sqrt{104}}\right) - \sinh^{-1}\left(\frac{x}{10}\right)\right]_0^2\bar{a}_z$$

$$= -40 \times 108 \times 10^{6} \left[\sinh^{-1}\left(\frac{2}{\sqrt{104}}\right) - \sinh^{-1}\left(\frac{1}{5}\right) \right] \bar{a}_z$$

$$= -40 \times 108 \times 10^{6} \left[0.19488 - 0.19869 \right] \bar{a}_z$$

$\bar{E} = 16.46\,\bar{a}_z$ MV/m.

1.6 ELECTRIC FLUX DENSITY OR DISPLACEMENT DENSITY

It is also called Electric displacement and to understand the concept one needs to know about line integral, surface integral and electric flux, which are explained as follows.

1.6.1 Line integral:

If a vector $\bar{A}$ is passing through a line as shown in the Fig:1.18. The line integral can be defined as the tangential component of vector $\bar{A}$ along the line, which can be written as

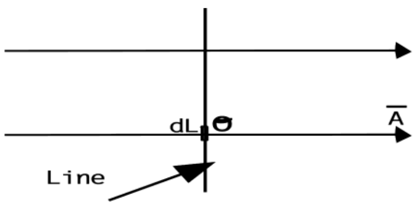


Fig: 1.18 Evaluation of Line integral

$$\int_L \left|\bar{A}\right| \cos\theta \, dL = \int_L \bar{A}.d\bar{L}$$

If a line is closed curve then the above integral can be written as $\oint_L \bar{A}.d\bar{l}$ which is called as contour line integral.

1.6.2 Surface integral:

Similarly, if a vector $\bar{A}$ is passing through a surface as shown in Fig: 1.19

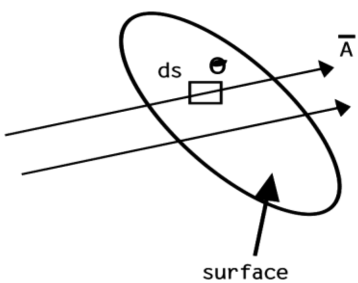


Fig: 1.19 Evaluation of Surface integral

The flux (ψ) of a vector $\bar{A}$ or surface integral can be written as

$$\psi = \int_s |\bar{A}| \cos\theta \, ds$$

$$= \int_s \bar{A}.d\bar{s} \qquad (1.17)$$

If the surface is closed surface then the above integral can be written as $\oint_s \bar{A}.d\bar{s}$ which is called as contour surface integral.

1.6.3 Electric flux:

We know that electric field intensity depends upon the medium in which it passes. Let us define a new vector $\bar{D}$ such that it is independent of medium i.e.,

$\bar{D} = \epsilon_0 \bar{E}$. Then the flux of $\bar{D}$, i.e $\psi = \oint_s \bar{D}.d\bar{s}$, where ψ is the electric flux. Which can be defined according to SI units as one line of flux originates from +1 Coloumb and terminates at –1 Coloumb. So the unit of Electric flux is also Coloumb and $\bar{D}$ is the electric flux density whose unit is columb/m^2.

The formulae for $\bar{D}$ can be obtained by multiplying the formulae of $\bar{E}$ with ϵ_0.

∴ Electric flux density due to a point charge $\bar{D}_Q = \frac{Q}{4\pi}\frac{\bar{a}_R}{R^2}$ (1.18)

and Electric flux density due to an infinite line with line chare density

$$\rho_L \text{ is } \bar{D}_L = \frac{\rho_L}{2\pi\rho}\bar{a}_\rho \qquad (1.19)$$

Problem: 1.14

Determine $\bar{D}$ at (4, 0, 3) if there is a point charge -5π mC at (4, 0, 0) and a line charge 3π mC/m along the Y-axis

Solution:

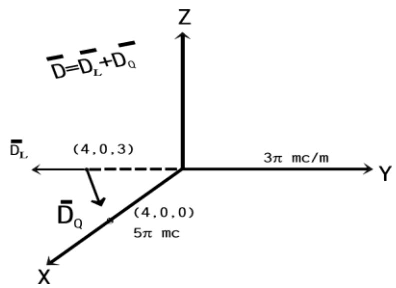


Fig: 1.20

$$\bar{D}_Q = \frac{Q}{4\pi}\frac{\bar{a}_R}{R^2} \quad \text{Where } \bar{R} = (4,0,3)-(4,0,0)=(0,0,3)$$

$$= \frac{-5\pi}{4\pi}\frac{3\bar{a}_z \times 10^{-3}}{(9)^{3/2}}$$

$$= \frac{-5}{4}\frac{3\bar{a}_z \times 10^{-3}}{27} = \frac{-5\bar{a}_z \times 10^{-3}}{36} = -0.139\bar{a}_z \times 10^{-3} \text{ C/m}^2.$$

$$\bar{a}_\rho = \frac{\bar{\rho}}{|\bar{\rho}|}$$

$$\bar{\rho} = (4,0,3)-(0,0,0) = 4\bar{a}_x + 3\bar{a}_z$$

$$\bar{D}_L = \frac{\rho_L}{2\pi\rho}\bar{a}_\rho$$

$$= \frac{3\pi}{2\pi}\times 10^{-3}\frac{4\bar{a}_x + 3\bar{a}_z}{25}$$

$$= 0.24\bar{a}_x + 0.18\bar{a}_z \text{ mC/m}^2.$$

$$\bar{D} = \bar{D}_Q + \bar{D}_L = 240\bar{a}_x + 42\bar{a}_z \quad \mu C/m^2$$

1.7 DIVERGENCE OF A VECTOR

Divergence: The divergence of a vector $\bar{A}$ at a given point is the outward flux in a volume as volume shrinks about the point. It can be represented as

$$div\,\bar{A} = \nabla\cdot\bar{A} = \lim_{\Delta v\to 0}\frac{\oiint_s \bar{A}.d\bar{s}}{\Delta v} \qquad (1.20)$$

Where ∇ is the del operator or gradient operator. ∇ can be operated on a vector or scalar. It has got different meanings when it is operating on a vector and scalar. If it is operating on a scalar V then it can be written as ∇V which is called as scalar gradient. If it is operating on a vector $\bar{A}$ with dot product then it is $\nabla.\bar{A}$ and it is called as divergence of vector $\bar{A}$ and If it is operating on a vector $\bar{A}$ with cross product then it is $\nabla\times\bar{A}$ and it is called as curl of vector $\bar{A}$.

..

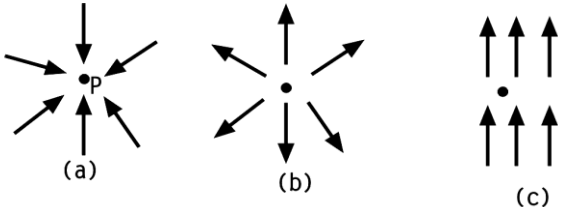


Fig:1.21 Flux lines

Physically divergence can be interpreted as the measure of how much field diverges or emanates from a point. Let us consider the Fig:1.21a in which field is reaching to the point. Divergence at that point is –Ve or it is also called as convergence. In Fig:1.21b the field is going away from the point, therefore divergence is +Ve. In Fig:1.21c some of the flux lines or field lines are reaching to the point and same number of field lines are leaving from the point hence the divergence is zero.

To determine $\nabla.\bar{A}$ let us consider the volume in Cartesian co-ordinate systems as shown in the Fig:1.22. In Cartesian co-ordinate system, the vector $\bar{A}$ with it's unit vectors and components along X, Y, Z is

$$\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$$

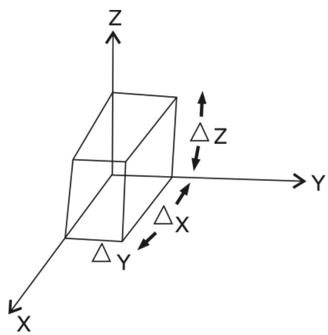


Fig:1.22 Evaluation of $\nabla.\bar{A}$

Assume the elemental volume $\Delta V = \Delta x\Delta y\Delta z$. The flux of a vector $\bar{A}$ on Y-axis that enters in to the left side of the volume is $A_y\Delta x\Delta z$. The flux which is leaving from right side of the volume on Y-axis can be written as $(A_y+\Delta A_y)\,\Delta x\Delta z$. This equation can be modified as $\left(A_y + \frac{\Delta A_y}{\Delta y}\Delta y\right)\Delta x\Delta z$. So the total flux on Y-axis is $A_y\Delta x\Delta z + \frac{\partial A_y}{\partial y}\Delta x\Delta y\Delta z$ - $A_y\Delta x\Delta z$

$$= \frac{\partial A_y}{\partial y}\,\Delta x\Delta y\Delta z$$

Similarly on X and Z-axises also.

The entire flux in all the directions is $\psi = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta x \Delta y \Delta z$. We know

$$\psi = \oiint_s \bar{A} \cdot d\bar{s}$$

$$\frac{\oiint_s \bar{A} \cdot d\bar{s}}{\Delta v} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Applying Limit on both sides

$$\frac{\oiint_s \bar{A} \cdot d\bar{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\therefore \quad \nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Conclusion:

The divergence of a vector results a scalar. The divergence of a scalar has no meaning

$$\nabla \cdot \left(\bar{A} + \bar{B}\right) = \nabla.\bar{A} + \nabla.\bar{B}$$

$$\nabla \cdot \left(V\bar{A}\right) = V\nabla.\bar{A} + \bar{A} \cdot \nabla V$$

$$\nabla \cdot \bar{A} = \left(\frac{\partial \bar{a}_x}{\partial x} + \frac{\partial \bar{a}_y}{\partial y} + \frac{\partial \bar{a}_z}{\partial z}\right).\left(A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

So from the above equation, the gradient operator is

$$\nabla = \frac{\partial \bar{a}_x}{\partial x} + \frac{\partial \bar{a}_y}{\partial y} + \frac{\partial \bar{a}_z}{\partial z} \qquad (1.21)$$

and the scalar gradient is

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

1.7.1 Divergence Theorem

Statement:

This theorem states that the outward flux flows through a closed surface is same as the volume integral of divergence of a vector.

$$\oint_s \bar{A} \cdot d\bar{s} = \int_v \nabla \cdot \bar{A}\, dv \qquad (1.22)$$

Proof:

Consider a vector $\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$.

Similarly $d\bar{s} = ds_x\bar{a}_x + ds_y\bar{a}_y + ds_z\bar{a}_z$ and we know that divergence of vector $\bar{A}$ i.e., $\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Assume dv = dx dy dz

consider the volume integral

$$\int_v \nabla \cdot \bar{A}\, dv = \iiint_v \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\, dx\, dy\, dz$$

The integral can be written as

$$\iiint_v \frac{\partial A_y}{\partial y}\, dx\, dy\, dz = \oiint_s \left[\int \frac{dA_y}{dy}\, dy\right] dx\, dz = \oiint_s A_y ds_y$$

where ds_y = The elemental surface on XZ plane.

Similarly the first and third integrals can be written as

$$\oiint_s A_x ds_x\,,\ \oiint_s A_z ds_z$$

$$\therefore \int_v \nabla \cdot \bar{A} dv = \oiint_s (A_x ds_x + A_y ds_y + A_z ds_z)$$

$$= \oiint_s \left(A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z\right)\cdot\left(ds_x\bar{a}_x + ds_y\bar{a}_y + ds_z\bar{a}_z\right) = \oiint_s \bar{A}\cdot d\bar{s}$$

Hence proved.

Formulae for Gradient:

In Cartesian co-ordinate system

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z \qquad (1.23)$$

in cylindrical co-ordinate system

$$\nabla V = \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z \qquad (1.24)$$

in spherical co-ordinate system

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi \qquad (1.25)$$

Problem:1.15

Find the gradient of the following scalar fields

(a) $V = e^{-z} \sin 2x \cos hy$
(b) $U = \rho^2 z \cos 2\phi$
(c) $W = 10r \sin^2\theta \cos\phi$

Solution:

(a) Since given V is in x and y, consider gradient in Cartesian co-ordinate system

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

$$= e^{-z}\cosh y \cos 2x\, 2\bar{a}_x + e^{-z}\sin 2x \sinh y\,\bar{a}_y + \sin 2x \cosh y\, e^{-z}(-1)\bar{a}_z$$

$$= 2\cos 2x \cosh y\, e^{-z}\bar{a}_x + \sin 2x \sinh y\, e^{-z}\,\bar{a}_y - \sin 2x \cosh y\, e^{-z}\bar{a}_z$$

(b) Since given U is in ρ, z and ϕ, consider gradient in cylindrical co-ordinate system

$$\nabla U = \frac{\partial U}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\bar{a}_\phi + \frac{\partial U}{\partial z}\bar{a}_z$$

$$= Z\cos 2\phi\, 2\rho\,\bar{a}_\rho + \rho z(-\sin 2\phi)2\,\bar{a}_\phi + \rho^2 \cos 2\phi\,\bar{a}_z$$

(c) Since given W is in r,θ and ϕ, consider gradient in spherical co-ordinate system

$$\nabla W = \frac{\partial W}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\bar{a}_\phi$$

$$=10\sin^2\theta\cos\phi\,\bar{a}_r+\left(\frac{10r}{r}\right)2\sin\theta\cos\theta\cos\phi\,\bar{a}_\theta+10r\sin^2\theta(-\sin\phi)\,\bar{a}_\phi.\frac{1}{r\sin\theta}$$

Formulae for Divergence of a Vector:

In Cartesian co-ordinate system

$$\nabla\cdot\bar{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \tag{1.26}$$

in cylindrical co-ordinate system

$$\nabla\cdot\bar{A}=\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial(A_\phi)}{\partial\phi}+\frac{\partial A_z}{\partial z} \tag{1.27}$$

in spherical co-ordinate system

$$\nabla\cdot\bar{A}=\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{1.28}$$

Problem:1.16

Determine the divergence of the following vector fields.

(a) $\bar{P}=x^2yz\,\bar{a}_x+x^3zy\,\bar{a}_y+xy^2z^3\bar{a}_z$

(b) $\bar{Q}=\rho\sin\phi\,\bar{a}_\rho+\rho^2z\,\bar{a}_\phi+z\cos\phi\,\bar{a}_z$

(c) $\bar{T}=\frac{1}{r^2}\cos\theta\;\bar{a}_r+r\sin\theta\cos\phi\,\bar{a}_\theta+\cos\theta\,\bar{a}_\phi$

(d) $\bar{N}=r^3\sin\theta\;\bar{a}_r+\sin2\theta\cos^2\phi\,\bar{a}_\theta+\cos\theta\,r^2\;\bar{a}_\phi$

Solution:

(a) Given $\bar{P}=x^2yz\,\bar{a}_x+x^3zy\,\bar{a}_y+xy^2z^3\bar{a}_z$

$$\nabla\cdot\bar{P}=\frac{\partial P_x}{\partial x}+\frac{\partial P_y}{\partial y}+\frac{\partial P_z}{\partial z}$$

$= 2xyz + x^3z + 3xy^2z^2$

(b)Given $\bar{Q}=\rho\sin\phi\,\bar{a}_\rho+\rho^2z\,\bar{a}_\phi+z\cos\phi\,\bar{a}_z$

$$\nabla\cdot\bar{Q}=\frac{1}{\rho}\frac{\partial(\rho Q_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial(Q_\phi)}{\partial\phi}+\frac{\partial Q_z}{\partial z}$$

$$=\frac{1}{\rho}2\rho\sin\phi+\frac{1}{\rho}(0)+\cos\phi$$

= 2sinϕ + cosϕ

(c) Given $\bar{T}=\frac{1}{r^2}\cos\theta\ \bar{a}_r+r\sin\theta\cos\phi\,\bar{a}_\theta+\cos\theta\,\bar{a}_\phi$

$$\nabla\cdot\bar{T}=\frac{1}{r^2}\frac{\partial(r^2T_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,T_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_\phi}{\partial\phi}$$

$$=\frac{1}{r^2}(0)+\frac{1}{r\sin\theta}r\,2\sin\theta\cos\theta\cos\phi+\frac{1}{r\sin\theta}(0)$$

= 2 cos θ cosϕ

(d) Given $\bar{N}=r^3\sin\theta\ \bar{a}_r+\sin 2\theta\cos^2\phi\,\bar{a}_\theta+\cos\theta\,r^2\,\bar{a}_\phi$

$$\nabla\cdot\bar{N}=\frac{1}{r^2}\frac{\partial(r^2N_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,N_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial N_\phi}{\partial\phi}$$

$$=\frac{1}{r^2}5r^4\sin\theta+\frac{1}{r\sin\theta}\frac{1}{2}\left(-\sin\theta+\frac{\sin 3\theta}{3}\right)\cos^2\phi+\frac{1}{r\sin\theta}(0)$$

$$=5r^2\sin\theta-\frac{1}{2r}\cos^2\phi+\frac{\sin 3\theta}{6r\sin\theta}\cos^2\phi$$

Problem:1.17

Given $\bar{D}=z\rho\cos^2\phi\bar{a}_z$ C/m². Calculate the charge density at (1, π/4, 3) and the total charge enclosed by the cylinder of radius 1m with $-2\le z\le 2$ m.

Solution:

We know

$$\rho_V=\nabla\cdot\bar{D}$$

in cylindrical co-ordinate system the divergence can be written as

$$\rho_v=\frac{1}{\rho}\frac{\partial(\rho D_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial(D_\phi)}{\partial\phi}+\frac{\partial D_z}{\partial z}$$

$$\rho_v=\frac{\partial D_z}{\partial z}\quad\text{since } \bar{D} \text{ has only Z- component}$$

$$\rho_v=\rho\cos^2\phi$$

$$(\rho_v)_{\left(1,\frac{\pi}{4},3\right)} = (1)\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2} \text{ C/m}^3$$

change enclosed = Q_{enc} = $\int_v \rho_v dv$ where dv = ρdρdϕdz

$$Q_{enc} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} \int_{z=-2}^{2} \rho\cos^2\phi\rho\, d\rho\, d\phi\, dz$$

$$= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} \rho^2\cos^2\phi(4)d\rho\, d\phi$$

$$= 4\int_{\rho=0}^{1} \rho^2\left[\frac{1}{2}(2\pi)+\frac{1}{2}\sin 4\phi\right]d\rho$$

$$= 4\pi\int_{\rho=0}^{1} \rho^2 d\rho = 4\pi\left[\frac{\rho^3}{3}\right]_0^1 = \frac{4\pi}{3} \text{ C.}$$

Problem:1.18

If $\bar{D} = \left(2y^2 + z\right)\bar{a}_x + 4xy\bar{a}_y + x\bar{a}_z$ C/m². Find

(a) the volume charge density at (-1, 0, 3)

(b) the flux through the cube defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$

(c) the total charge enclosed by the cube

Solution:

According to Maxwell's I equation

$$\rho_v = \nabla \cdot \bar{D}$$

$$\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$= 0 + 4x + 0$

$= 4x$ C/m³

(a) $(\rho_v)_{(-1,0,3)} = 4(-1) = -4$ C/m²

(b) & (c) $\psi = \int_v \rho_v dv = Q_{enc}$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 4x\,dx\,dy\,dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 4x(1)\,dx\,dy$$

$$= \int_{x=0}^{1} 4x(1)\,dx$$

$$= 4\left[\frac{x^2}{2}\right]_0^1 = \frac{4}{2} = 2C$$

Problem: 1.19

Given the electric flux density $\bar{D} = 0.3r^2\bar{a}_r$ nC/m^2, in free space. Find (a) $\bar{E}$ at point (2, 25^0, 90^0)

(b) the total charge within the sphere r = 3

(c) the total electric flux leaving the sphere r = 4

Solution:

(a)

Given $\bar{D} = 0.3r^2\bar{a}_r$ nC/m^2

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{0.3r^2\bar{a}_r}{8.854\times10^{-12}}$$

$$\left(\bar{E}\right)_{(2,25^0,90^0)} = \frac{0.3(4)}{8.854\times10^{-12}}\bar{a}_r = 1.355\times10^{11}\bar{a}_r \times10^{-9} = 135.5\ \bar{a}_r \text{V/m}$$

(b) we know $\rho_v = \nabla\cdot\bar{D}$

$$= \frac{1}{r^2}\frac{\partial(r^2D_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, D_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} =$$

$$\frac{1}{r^2}0.3(4)r^3 = 1.2r$$

Also know $Q = \int_v \rho_v dv$ where dv = r sinθ dϕ r dθ dr

= r^2 sinθ dθ dϕ dr

$$\therefore\ Q = \int_{r=0}^{3} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 1.2r\ r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \int_{r=0}^{3} \int_{\theta=0}^{\pi} 1.2r^3 \sin\theta (2\pi) d\theta\, dr$$

$$= 2.4\pi \int_{r=0}^{3} r^3 [-\cos\theta]_0^{\pi}\, dr$$

$$= 2.4\pi(2)\left[\frac{r^4}{4}\right]_0^3 = 305.4\,nC$$

(c) $$Q = \int_{r=0}^{4} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 1.2\ r^3 \sin\theta\, d\theta\, d\phi\, dr$$

Upon simplifying, we get

Q= 965.09nC

Problem: 1.20

A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \rho_0 \dfrac{r}{R}, & 0 \le r \le R \\ 0, & r > R \end{cases}$$ Determine $\bar{E}$ everywhere

Solution:

Replace 'a' with 'R' in Fig:1.27, Then

Case I: Inside the sphere of radius 'R'

The charge enclosed by the sphere of radius 'r' is $Q_{enc} = \int_v \rho_v dv$

$$Q_{enc} = \int_v \rho_0 \frac{r}{R} dv$$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} \frac{r^3}{R} \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{\rho_0}{R} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=0}^{r} r^3 dr$$

$$= \frac{4\pi r^4 \rho_0}{4R}$$

$$Q_{enc} = \frac{\rho_0}{R} \pi r^4$$

The flux flowing through the spherical surface

$$\psi = \oint_s \overline{D}.d\overline{s}$$

As the flux density is normal to the surface it will have components only in 'r' direction.

$$= D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$\psi = D_r \, 4\pi \, r^2$$

According to Gauss's law charge enclosed = flux flowing through the surface i.e., $Q_{enc} = \psi$

$$\frac{\rho_0}{R} \pi r^4 = D_r 4\pi r^2$$

$$D_r = \frac{\rho_0}{4R} r^2$$

$$\overline{D} = \frac{\rho_0}{4R} r^2 \, \overline{a}_r \text{ and}$$

$$\overline{E} = \frac{\overline{D}}{\epsilon_0} = \frac{\rho_0}{4R \epsilon_0} r^2 \, \overline{a}_r$$

Case II: outside the sphere of radius 'R'

Charge enclosed by the sphere of radius 'r' is

$$Q_{enc} = \int_v \rho_v dv$$

$$Q_{enc} = \int_v \rho_0 \frac{r}{R} dv$$

$$= \frac{\rho_0}{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{R} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \rho_0 \pi R^3$$

Flux flowing through the surface

$$\psi = D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$= D_r \, 4\pi \, r^2$

$Q_{enc} = \psi$ according to Gauss's law

$\rho_0 \, \pi \, R^3 = D_r \, 4\pi \, r^2$

$$D_r = \frac{\rho_0 R^3}{4r^2}$$

$$\bar{D} = \frac{\rho_0 R^3}{4r^2} \bar{a}_r \text{ and}$$

$$\bar{E} = \frac{\rho_0 R^3}{4r^2 \in_0} \bar{a}_r$$

***Problem: 1.21**

A sphere of radius 'a' is filled with a uniform charge density of ρ_v C/m³ . Determine the electric field inside and outside the sphere.

Solution:

The answer is as derived in section "Uniformly charged sphere" case-I(inside the sphere) and case-II(outside the sphere).

Problem: 1.22

A charge distribution in free space has $\rho_v = 2r$ nC/m³ for $0 < r < 10$m and '0' otherwise. Determine $\bar{E}$ at r=2m and r=12m

Solution:

Replace 'a' with '10m' in Fig:1.27, Then

At r=2m:

The charge enclosed by the sphere of radius '2m' is $Q_{enc} = \int\limits_{v} \rho_v dv$

$$Q_{enc} = \int\limits_{v} 2rndv$$

$$= 2n \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{2} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 32\pi nC$$

The flux flowing through the spherical surface

$$\psi = \oint\limits_{s} \overline{D}.d\overline{s}$$

As the flux density is normal to the surface it will have components only in 'r' direction.

$$= D_r \int\limits_{\theta=0}^{\pi} \int\limits_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, 16\pi$$

According to Gauss's law charge enclosed = flux flowing through the surface i.e., $Q_{enc} = \psi$

$$32\pi n = D_r \, 16\pi$$

$$D_r = 2n$$

$$\overline{D} = 2n\,\overline{a}_r \text{ and}$$

$$\overline{E} = \frac{\overline{D}}{\in_0} = 226\overline{a}_r V/m$$

At r=12m:

Charge enclosed by the sphere of radius '12m' is

$$Q_{enc} = \int\limits_{v} \rho_v dv$$

$$Q_{enc} = \int\limits_{v} 2rndv$$

$$= 2n \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{10} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 20\pi \; \mu C$$

Flux flowing through the surface

$$\psi = D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$= D_r \, 4\pi \, 12^2$

$Q_{enc} = \psi$ according to Gauss's law

$20\pi \, \mu = D_r \, 4\pi \, 12^2$

$D_r = 0.0347\mu$

$\bar{D} = 0.0347\mu \, \bar{a}_r$ and

$\bar{E} = 3.92\, \bar{a}_r kV/m$

Problem: 1.23

Two point charges -4μC and 5μC are located at (2,-1,3) and (0,4,-2) respectively. Find the potential at (1, 0, 1). Assuming '0' potential at infinity.

Solution:

$$V = \frac{Q_1}{4\pi \in_0 |r - r_1|} + \frac{Q_2}{4\pi \in_0 |r - r_2|}$$

$$V = \frac{-4\times 10^{-6}}{4\pi \in_0 |(1,0,1)-(2,-1,3)|} + \frac{5\times 10^{-6}}{4\pi \in_0 |(1,0,1)-(0,4,-2)|}$$

Simplifying, we get

V= -5.872KV

Problem:1.24

A point charge 3μC is located at the origin in addition to the two charges of previous problem. Find the potential at (-1, 5, 2). Assuming V(∞) = 0.

Solution:

$r - r_1 = \sqrt{1+25+4} = 5.478$

$r - r_2 = \sqrt{9+36+1} = 6.782$

$r - r_3 = \sqrt{16+1+1} = 4.243$

$$V = \left[\frac{3\times10^3}{5.478} + \frac{-4\times10^3}{6.782} + \frac{5\times10^3}{4.243}\right]\times 9$$

$= 10.23KV$

Problem:1.25

A point charge of 5nC is located at the origin if V=2V at (0, 6, -8) find (a) the potential at A (-3, 2, 6)

(b) the potential at B (1, 5, 7)

(c) the potential difference V_{AB}

Solution:

(a) $$V_A - V = \frac{Q}{4\pi\in_0}\left(\frac{1}{r_A} - \frac{1}{r}\right)$$

r_A = (-3, 2, 6) – (0, 0, 0)= $\sqrt{3^2+2^2+6^2} = 7$

r = (0, 6, -8) – (0, 0, 0)= $\sqrt{0+6^2+8^2} = 10$

$$V_A - 2 = \frac{5\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left(\frac{1}{7} - \frac{1}{10}\right)$$

V_A=3.929 V.

(b) $$V_B - V = \frac{Q}{4\pi\in_0}\left(\frac{1}{r_B} - \frac{1}{r}\right)$$

r_B=(1,5,7)-(0,0,0,)= $\sqrt{1+5^2+7^2} = \sqrt{74}$

$$V_B - 2 = \frac{5\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left(\frac{1}{\sqrt{74}} - \frac{1}{10}\right)$$

V_B=2.696 V.

(c)V_{AB} = V_B - V_A=-1.233 V.

***Problem:1.26**

A point of 5nC is located at (-3,4,0), while line y=1,z=1 carries uniform charge 2nC/m.

(a) If V=0V at O(0,0,0), find V at A(5,0,1).

(b) If V=100V at B(1,2,1), find V at C(-2,5,3).

(c) If V=-5V at O, find V_{BC}.

Solution:

Let the potential at any point be

$V=V_Q+V_L$

Where V_Q is potential due to point charge

$$i.e.\quad V_Q=\frac{Q}{4\pi\in_0 r}$$

by neglecting constant of integration

and V_L is potential due to line charge distribution,

for infinite line, we have

$$\bar{E}=\frac{\rho_L}{2\pi\in_0\rho}\bar{a}_\rho$$

$$\therefore V_L=-\int\bar{E}.d\bar{l}=-\int\frac{\rho_L}{2\pi\in_0\rho}\bar{a}_\rho.d\rho\bar{a}_\rho$$

$$\therefore V_L=-\frac{\rho_L}{2\pi\in_0}\ln\rho$$

by neglecting constant of integration.

Here ρ is the perpendicular distance from the line y=1, z=1(which is parallel to the x-axis) to the field point.

Let the field point be (x,y,z), then

$$\rho=|(x,y,z)-(x,1,1)|=\sqrt{(y-1)^2+(z-1)^2}$$

$$\therefore V=-\frac{\rho_L}{2\pi\in_0}\ln\rho+\frac{Q}{4\pi\in_0 r}$$

by neglecting constant of integration.

(a)

$$\rho_O = |(0,0,0)-(0,1,1)| = \sqrt{2}$$

$$\rho_A = |(5,0,1)-(5,1,1)| = 1$$

$$r_O = |(0,0,0)-(-3,4,0)| = 5$$

$$r_A = |(5,0,1)-(-3,4,0)| = 9$$

$$\therefore V_O - V_A = -\frac{\rho_L}{2\pi \in_0}\ln \rho_O + \frac{\rho_L}{2\pi \in_0}\ln \rho_A + \frac{Q}{4\pi \in_0 r_O} - \frac{Q}{4\pi \in_0 r_A}$$

$$V_O - V_A = -\frac{\rho_L}{2\pi \in_0}\ln\frac{\rho_O}{\rho_A} + \frac{Q}{4\pi \in_0}\left[\frac{1}{r_O} - \frac{1}{r_A}\right]$$

$$0 - V_A = -\frac{2\times 10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}}\ln\frac{\sqrt{2}}{1} + \frac{5\times 10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left[\frac{1}{5} - \frac{1}{9}\right]$$

$$-V_A = -36\ln\sqrt{2} + 45\left[\frac{1}{5} - \frac{1}{9}\right]$$

$$V_A = 36\ln\sqrt{2} + 4 = 8.477V$$

(b)

$$\rho_B = |(1,2,1)-(1,1,1)| = 1$$

$$\rho_C = |(-2,5,3)-(-2,1,1)| = \sqrt{20}$$

$$r_B = |(1,2,1)-(-3,4,0)| = \sqrt{21}$$

$$r_C = |(-2,5,3)-(-3,4,0)| = \sqrt{11}$$

$$\therefore V_C - V_B = -\frac{\rho_L}{2\pi \in_0}\ln\frac{\rho_C}{\rho_B} + \frac{Q}{4\pi \in_0}\left[\frac{1}{r_C} - \frac{1}{r_B}\right]$$

$$V_C - 100 = -36\ln\frac{\sqrt{21}}{1} + 45\left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right]$$

$$V_C - 100 = -50.175$$

$$V_C = 49.825V$$

(c)

$$V_{BC} = V_C - V_B = 49.825 - 100 = -50.175V$$

Problem: 1.27

Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$

(a) Find the electric flux density $\overline{D}\, at \left(2, \pi/2, 0\right)$

(b) Calculate the work done in moving a 10mC charge from point A(1, 30^0, 120^0) to B(4, 90^0, 60^0)

Solution:

(a)

We have

$$\overline{E} = -\nabla V$$

Since V is given in spherical co-ordinate system, consider ∇V in spherical co-ordinate system

$$\therefore \overline{E} = -\frac{\partial v}{\partial r}\overline{a}_r + \frac{1}{r}\frac{\partial v}{\partial \theta}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}\overline{a}_\phi$$

$$= -\left(10\left(-2r^{-3}\right)\sin\theta\cos\phi\,\overline{a}_r + \frac{1}{r}\frac{10\cos\theta\cos\phi}{r^2}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{10\sin\theta\left(-\sin\phi\right)}{r^2}\overline{a}_\phi\right)$$

$$= -\left(-20r^{-3}\sin\theta\cos\phi\,\overline{a}_r + \frac{10\cos\theta\cos\phi}{r^3}\overline{a}_\theta + \frac{-10\sin\phi}{r^3}\overline{a}_\phi\right)$$

$$= \left(\frac{20\sin\theta\cos\phi}{r^3}\overline{a}_r + \frac{-10\cos\theta\cos\phi}{r^3}\overline{a}_\theta + \frac{10\sin\phi}{r^3}\overline{a}_\phi\right)$$

$$= \frac{10}{r^3}\left(2\sin\theta\cos\phi\,\overline{a}_r - \cos\theta\cos\phi\,\overline{a}_\theta + \sin\phi\,\overline{a}_\phi\right)$$

$$\overline{D} = \overline{E} \in_0$$

$$= \frac{8.825\times 10^{-11}}{r^3}\left[2\sin\theta\cos\phi\,\overline{a}_r - \cos\theta\cos\phi\,\overline{a}_\theta + \sin\phi\,\overline{a}_\phi\right]$$

$$= \frac{8.825\times10^{-11}}{r^3}\left[2.1.1\bar{a}_r - 0 + 0\right]$$

$\overline{D}_{(2,\frac{\pi}{2},0)} =$ $22.1\,\bar{a}_r$ pC/m²

(b) Work done = $-Q\int_A^B \overline{E}\cdot d\overline{L} = -Q\left(-V_{AB}\right)$

$= Q\ (V_B - V_A)$

$$V_B = \frac{10}{16}1.\frac{1}{2} = 0.3125v$$

$$V_A = \frac{10}{1}\frac{1}{2}\left(-0.5\right) = -5\times0.5 = -2.5v$$

$V_B - V_A = 2.8125V$

$W = 10^{-3} \times 10 \times (V_B - V_A) = 28.125mJ.$

Problem:1.28

Given that $\overline{E} = \left(3x^2 + y\right)\bar{a}_x + x\bar{a}_y kV/m$. Find the work done in moving a –2μC charge from (0, 5, 0) to (2, -1, 0) by taking the path

(a) (0, 5, 0) →(2, 5, 0) → (2, -1, 0)

(b) y = 5-3x

Solution:

(a)

Line equation for (0, 5, 0) to (2, 5, 0) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\frac{x - 0}{0 - 2} = \frac{y - 5}{5 - 5} = \frac{z - 0}{0 - 0}$$

y = 5 z =0

dy = 0 dz = 0

$$W_1 = -QK \int_{(0,5,0)}^{(2,5,0)} \left(\left(3x^2 + y\right)\bar{a}_x + x\bar{a}_y\right)\left(dx\,\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z\right)$$

$$= -QK \int_{(0,5,0)}^{(2,5,0)} \left(3x^2 + y\right)dx + xdy$$

$$= 2\times10^{-3} \int_{(0)}^{(2)} \left(3x^2 + 5\right)dx + 0$$

$$= 2\times10^{-3}\left(3\left(\frac{x^3}{3}\right)_0^2 + 5(2)\right)$$

= 36mJ

Line equation for (2, 5, 0) to (2, -1, 0)

Z = 0 dz = 0 $\frac{x-2}{2-2} = \frac{y-5}{5+1} = \frac{z-0}{0-0}$

x=2 dx=0

$$W_2 = -QK \int_{(2,5,0)}^{(2,-1,0)} \left(3x^2 + y\right)dx + xdy$$

$$W_2 = -QK \int_{5}^{-1} 2dy = -2Qk(-1-5) = -24mJ$$

W=W$_1$+W$_2$=12mJ.

(b)

Line equation for (0, 5, 0) to (2, 5, 0) is y=5-3x

dy=-3dx

$$W = -QK \int_{(0,5,0)}^{(2,-1,0)} \left(3x^2 + y\right)dx + xdy$$

$$W = 2\times10^{-3} \int_{0}^{2} \left(3x^2 + 5 - 3x\right)dx - 3xdx = 12\text{mJ}.$$

Problem:1.29

An electric dipole located at the origin in free space has a moment $\bar{p} = 3\bar{a}_x - 2\bar{a}_y + \bar{a}_z nCm$

(a) Find V at P_A (2,3,4)

(b) Find V at r = 2.5, $\theta = 30^0$, $\phi = 40^0$

Solution:

(a) We have

$$V = \frac{1}{4\pi \in_0 \left|r - r^1\right|^2} \bar{p}.\frac{r - r^1}{\left|r - r^1\right|}$$

$r^1 = (0, 0, 0)$

$$\left|r - r^1\right| = \sqrt{4+9+16} = \sqrt{29}$$

$$V = 9\times10^9 \frac{(3\bar{a}_x - 2\bar{a}_y + \bar{a}_z)\cdot(2\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z)}{29\sqrt{29}} \times 10^{-9}$$

$$= \frac{9\times10^9 \times (4)}{(29)^{3/2}} = 0.235V$$

(b) r = 2.5 $\theta = 30^0$ $\phi = 40^0$

x = r sin ϕ cos θ = 0.958

y = r sin ϕ sin θ = 0.8035

z = r cos θ = 2.165

upon simplifying we get

V= 1.97V

Problem:1.30

Three point charges –1nC, 4nC and 3nC are located at (0, 0, 0), (0, 0, 1) and (1, 0, 0) respectively. Find the energy in the system.

Solution:

$$W_E = W_1 + W_2 + W_3$$

$$= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32})$$

$$= Q_2.\frac{Q_1}{4\pi \in_0 \left|r_2 - r_1\right|} + \frac{Q_3}{4\pi \in_0}\left[\frac{Q_1}{\left|r_3 - r_1\right|} + \frac{Q_2}{\left|r_3 - r_2\right|}\right]$$

$$= \frac{1}{4\pi \in_0}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right)$$

$$= \frac{1}{4\pi . \frac{10^{-9}}{36\pi}} \left(-4-3+\frac{12}{\sqrt{2}} \right) .10^{-18}$$

$$= 9\left(\frac{12}{\sqrt{2}} - 7 \right) nJ = 13.37 nJ$$

Problem:1.31

Point charges Q_1 = 1nC, Q_2 = -2nC, Q_3 = 3nC and Q_4 = -4nC are positioned one at a time and in that order at (0, 0, 0), (1, 0, 0), (0, 0, -1) and (0, 0, 1) respectively. Calculate the energy in the system after each charge is positioned.

Solution:

Energy after Q_1 is positioned is W_1=0

$$W_2 = Q_2 V_{21} = Q_2 . \frac{Q_1}{4\pi \in_0 |r_2 - r_1|} = \frac{-2\times 1\times 10^{-18}}{4\pi . \frac{10^{-9}}{36\pi} |(1,0,0)-(0,0,0)|} = -18\text{nJ}$$

Energy after Q_2 is positioned W_2' = W_1+ W_2 =-18nJ

Energy after Q_3 is positioned

$$W_3' = W_2' + Q_3(V_{32}+V_{31}) = -18nJ + \frac{3\times 10^{-9}}{4\pi . \frac{10^{-9}}{36\pi}} \left[\frac{-2\times 10^{-9}}{|(0,0,-1)-(1,0,0)|} + \frac{1\times 10^{-9}}{|(0,0,-1)-(0,0,0)|} \right]$$

=-29.18nJ

Energy after Q_4 is positioned

$W_4' = W_3' + Q_4(V_{43}+V_{42}+V_{41})$=-68.27nJ.

Problem: 1.32

If $\bar{J} = \frac{1}{r^3}(2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta)$ A/m². Calculate the current passing through

(a) Hemispherical shell of radius 20cm.

(b) A spherical shell of radius 20cm.

Solution:

$$I = \int \bar{J}.d\bar{s}$$

Since it is sphere $d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$

(a) ϕ = 0 to 2π ,θ = 0 to $\pi/2$ and r=0.2m for hemispherical shell

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \frac{1}{r^3}\left(2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right)\cdot r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} 2\cos\theta \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \sin 2\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi} \left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} d\phi$$

$$= -\frac{1}{2r}(-1-1)(2\pi) = \frac{2\pi}{0.2} = 10\pi = 31.4A$$

(b) ϕ = 0 to 2π ,θ = 0 to π and r=0.2m for spherical shell

$$I = \frac{1}{r}\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin 2\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi} \left[\frac{-\cos 2\theta}{2}\right]_0^{\pi} d\phi$$

$$= -\frac{1}{2r}\int_{\varphi=0}^{2\pi} [1-1]\, d\phi = 0A$$

Problem: 1.33

For the current density $\bar{J} = 10z\sin^2\phi\,\bar{a}_\rho$ A/m². Find the current through the cylindrical surface ρ = 2, $1 \leq z \leq 5$ m .

Solution:

Since it is cylinder $d\bar{s} = \rho\, d\phi\, dz\, \bar{a}_\rho$

We have

$$I = \int \bar{J} \cdot d\bar{s}$$

$$= \int_{z=1}^{5} \int_{\phi=0}^{2\pi} 10z \sin^2 \phi\rho \, d\phi \, dz$$

$$= 10\rho \int_{z=1}^{5} z(1-\cos\phi)$$

=754A

***Problem: 1.34**

In a cylindrical conductor of radius 2mm, the current density varies with distance from the axis according to $J = 10^3 e^{-400r} A/m^2$. Find the total current I.

Solution:

Since it is cylinder $d\bar{s} = \rho \, d\phi \, dz \, \bar{a}_\rho$

Here r=ρ=0.02m, $\therefore \bar{J} = 10^3 e^{-400\rho} \bar{a}_\rho A/m^2$

We know the total current $I = \int_s \bar{J}.d\bar{s}$

$$\therefore I = \int_{\phi=0}^{2\pi} \int_{z=0}^{z} 10^3 e^{-400\rho} \rho d\phi dz$$

$$I = 2\pi z 10^3 e^{-400\rho} \rho$$

$$I = 4\pi z e^{-0.8} = z5.65A$$

Problem:1.35

If the current density $\bar{J} = \frac{1}{r^2}(\cos\theta \bar{a}_r + \sin\theta \bar{a}_\theta) A/m^2$, find the current passing through a sphere of radius 1.0m.

Solution:

We know the total current $I = \int_s \bar{J}.d\bar{s}$

Since it is spherical symmetry $d\bar{s} = r^2 \sin\theta d\theta d\phi \bar{a}_r$

$$\bar{J}.d\bar{S} = \frac{r^2}{r^2} \cos\theta \sin\theta d\phi d\theta$$

$$I = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \cos\theta \sin\theta d\phi d\theta$$

$$I = \pi \int_{0}^{\pi} \sin 2\theta d\theta$$

$$= \pi \left(\frac{-\cos 2\theta}{2} \right)_{0}^{\pi} = 0A$$

Problem:1.36

Write Laplace's equation in rectangular co-ordinates for two parallel planes of infinite extent in the X and Y directions and separated by a distance 'd' in the Z-direction. Determine the potential distribution and electric field strength in the region between the planes.

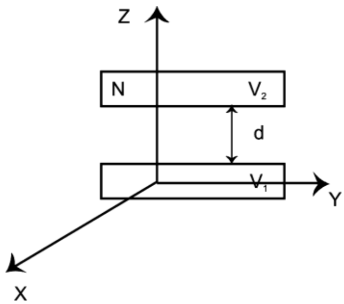


Fig: 2.15

Solution:

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ since the potential is constant in X and Y directions

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial V}{\partial z} = A$$

$$V = Az + B$$

At Z =0 $\quad V = V_1$

$V_1 = 0 + B$

At Z =d $\quad V = V_2$

V_2 = Ad +B

V_2=Ad+V_1

$$A = \frac{V_2 - V_1}{d}$$

The Potential distribution is $V = \frac{V_2 - V_1}{d} z + V_1$

The Electric field strength is

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial z}\overline{a}_z = -\frac{V_2 - V_1}{d}\overline{a}_z = \frac{V_1 - V_2}{d}\overline{a}_z$$

***Problem:1.37**

Calculate the capacitance of a parallel plate capacitor with a dielectric, mica filled between plates. $\in_r$ of mica is 6. The plates of the capacitor are square in shape with 0.254cm side. Separation between the two plates is, 0.254cm.

Solution:

We have $C = \frac{\in A}{d}$

Here $\in = \in_0 \in_r = 8.854 \times 10^{-12} \times 6$

$$C = \frac{8.854 \times 10^{-12} \times 6 \times 0.254 \times 0.254 \times 10^{-4}}{0.254 \times 10^{-2}} = 0.1349\, pF$$

***Problem:1.38**

A parallel plate capacitance has 500mm side plates of square shape separated by 10mm distance. A sulphur slab of 6mm thickness with $\in_r = 4$ is kept on the lower plate find the capacitance of the set-up. If a voltage of 100volts is applied across the capacitor, calculate the voltages at both the regions of the capacitor between the plates.

Solution:

Given Area of parallel plates, A=500mmX500mm=500X500X10^{-6} m^2.

Distance of separation d=10mm=10X10^{-3}m.

Thickness of sulphur slab d_2=6mm=6X10^{-3}m.

Relative permittivity of sulphur slab $\in_r = 4$.

Voltage applied across the capacitor V=100v.

Here the capacitor has two dielectric media,

One medium is the sulphur slab of thickness (d_2)6mm,

since the distance between the plates(d) is 10mm

The remaining distance is air $d_1 = d - d_2$ =4mm.

$\therefore$ the other dielectric medium is air with thickness(d_1) 4mm.

The capacitance of the parallel plate capacitor with two dielectric media is

$$C = \frac{\in_0 A}{\left(\frac{d_1}{\in_{r_1}} + \frac{d_2}{\in_{r_2}}\right)} \quad F$$

Here $\in_{r_1}$ (air)=1, $\in_{r_2} = \in_r = 4$

$$C = \frac{8.854\times10^{-12}\times500\times500\times10^{-6}}{\left(\frac{4\times10^{-3}}{1} + \frac{6\times10^{-3}}{4}\right)} = 0.402nF$$

The charge Q=CV=$0.402\text{X}10^{-9}\text{X}100 = 4.02\text{x}10^{-8}$ C

The value of capacitance(C_1) in delectric-1 i.e., air is

$$C_1 = \frac{\in_0 A}{d_1} = \frac{8.854\times10^{-12}\times500\times500\times10^{-6}}{4\times10^{-3}} = 0.55nF$$

Similarly, The value of capacitance(C_2) in delectric-2 i.e., sulphur is

$$C_2 = \frac{\in A}{d_2} = \frac{4\times8.854\times10^{-12}\times500\times500\times10^{-6}}{6\times10^{-3}} = 1.48nF$$

We have $V=V_1+V_2$

Where V_1 is the voltage at the region of the capacitor plate near dielectric-1 i.e., air.

And V_2 is the voltage at the region of the capacitor plate near dielectric-2 i.e., sulphur.

$$V_1 = \frac{Q_1}{C_1} = \frac{Q}{C_1} = \frac{4.02\times10^{-8}}{0.55\times10^{-9}} = 73.1V$$

$\therefore V_2$=100-73.1=26.9V.

UNIT 2 STATIC MAGNETIC FIELDS

Problem:2.1

Find $\bar{H}$ at (0, 0, 5) due to side OA and side OB of the triangular current loop shown in Fig:2.3.

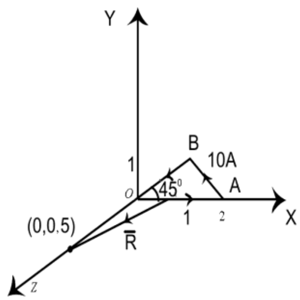


Fig: 2.3

Solution:

Magnetic field intensity due to OA

$$d\bar{H}_{OA} = \frac{Id\bar{l} \times \bar{R}}{4\pi R^3} \quad \text{where} \quad \bar{R} = (0,\ 0,\ 5) - (x,\ 0,\ 0)$$

$$= -x\bar{a}_x + 5\bar{a}_z$$

$$d\bar{l} = dx\,\bar{a}_x$$

$$d\bar{H}_{OA} = \frac{10dx\,\bar{a}_x \times \left(-x\bar{a}_x + 5\bar{a}_z\right)}{4\pi\left(x^2 + 5^2\right)^{3/2}}$$

$$d\bar{H}_{OA} = \frac{10\left(-5dx\bar{a}_y\right)}{4\pi\left(x^2 + 5^2\right)^{3/2}}$$

$$\bar{H}_{OA} = -\int_0^2 \frac{50\bar{a}_y}{4\pi\left(x^2 + 25\right)^{3/2}}\,dx$$

$$= \frac{-50}{4\pi}\bar{a}_y \left[\frac{x/25}{\left(x^2 + 25\right)^{1/2}}\right]_0^2$$

$$= \frac{-50}{4\pi}\,\frac{1}{25}\,\bar{a}_y\,\frac{2}{(29)^{1/2}}$$

$= -59.1\bar{a}_y$ mA/m

Magnetic field intensity due to OB

$\bar{R}$ = (0, 0, 5) – (1, 1, z) = $-\bar{a}_x - \bar{a}_y + (5-z)\bar{a}_z$

$d\bar{l} = dz\,\bar{a}_z$

$$d\bar{H}_{OB} = \frac{(10)dz\,\bar{a}_z \times \left(-\bar{a}_x - \bar{a}_y + (5-z)\bar{a}_z\right)}{4\pi\left[2+(5-z)^2\right]^{3/2}}$$

$$= \frac{(10)\left(-\bar{a}_y dz + \bar{a}_x dz\right)}{4\pi\left[2+(5-z)^2\right]^{3/2}}$$

$$\bar{H}_{OB} = \frac{10}{4\pi}\int_{\sqrt{2}}^{0} \frac{\left(-\bar{a}_y dz + \bar{a}_x dz\right)}{\left[2+(5-z)^2\right]^{3/2}}$$

$$= \frac{10}{4\pi}\left(-\bar{a}_y + \bar{a}_x\right)\int_{\sqrt{2}}^{0} \frac{dz}{\left(2+(5-z)^2\right)^{3/2}}$$

$$= \frac{10}{4\pi}\left(\bar{a}_y - \bar{a}_x\right)\left[\frac{(5-z)/2}{\left(2+(5-z)^2\right)^{1/2}}\right]_{\sqrt{2}}^{0}$$

$$= \frac{5}{4\pi}\left(\bar{a}_y - \bar{a}_x\right)\left[\frac{5}{\sqrt{27}} - \frac{\left(5-\sqrt{2}\right)}{\left(2+\left(5-\sqrt{2}\right)^2\right)^{1/2}}\right]$$

$$= \frac{5}{4\pi}\left(\bar{a}_y - \bar{a}_x\right)\left(0.9623 - 0.9303\right) = -12.73\bar{a}_x + 12.73\bar{a}_y\ mA/m$$

***Problem:2.2**

Show that the magnetic field due to a finite current element along Z-axis at point 'P', 'r' distance away along Y-axis is given by $\bar{H} = (I/4\pi r)(\sin\alpha_1 - \sin\alpha_2)\bar{a}_\phi$ where I is the current through the conductor, α_1 and α_2 are the angles made by the tips of the conductor at 'P'.

Solution:

Consider Fig:2.4,

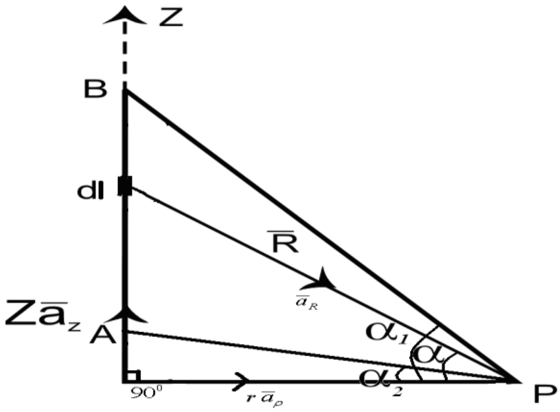


Fig: 2.4

From Fig:2.4 , $\overline{R} = r\,\overline{a}_\rho - z\,\overline{a}_z$

and $d\bar{l} = dz\,\overline{a}_z$

$$d\overline{H} = \frac{I\,dz\,\overline{a}_z \times \left(r\,\overline{a}_\rho - z\,\overline{a}_z\right)}{4\pi\left(r^2 + z^2\right)^{3/2}}$$

$$d\overline{H} = \frac{I\,dz\,r\,\overline{a}_\phi - 0}{4\pi\left(r^2 + z^2\right)^{3/2}}$$

From Fig:2.4, $Tan\alpha = \dfrac{z}{r}$

$\therefore z = rTan\alpha$

$dz = rsec^2\alpha \quad d\alpha$

$$d\overline{H} = \frac{I\,r\,\overline{a}_\phi\left(rsec^2\alpha\ d\alpha\right)}{4\pi\left(r^2 + r^2Tan^2\alpha\right)^{3/2}}$$

$$= \frac{I\,r^2\,sec^2\alpha\ \overline{a}_\phi\,d\alpha}{4\pi\,r^3 sec^3\alpha}$$

$$= \frac{I\ \overline{a}_\phi\,d\alpha}{4\pi\,rsec\alpha}$$

$$\bar{H} = \frac{I\,\bar{a}_\phi}{4\pi r}\int_{\alpha_2}^{\alpha_1} \cos\alpha\, d\alpha$$

$$= \frac{I\,\bar{a}_\phi}{4\pi r}\left[\sin\alpha\right]_{\alpha_2}^{\alpha_1}$$

$$= \frac{I\,\bar{a}_\phi}{4\pi r}\left(\sin\alpha_1 - \sin\alpha_2\right)$$

$$\bar{H} = \frac{I\,\bar{a}_\phi}{4\pi r}\left(\sin\alpha_1 - \sin\alpha_2\right)$$

***Problem:2.3**

Derive an expression for magnetic field strength H, due to a current carrying conductor of finite length placed along the Y-axis, at a point P in X-Z plane and 'r' distance from the origin. Hence deduce expressions for H due to semi-infinite length of the conductor.

Solution:

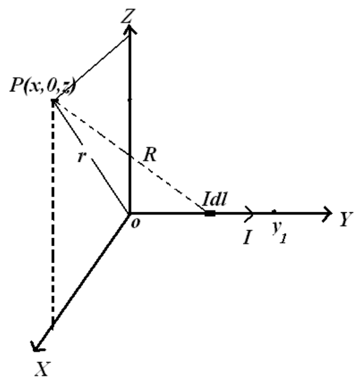


Fig: 2.5

The geometry of the given problem is shown in Fig: 2.5 with the finite length(y_1 m)current carrying conductor lying along Y-axis.

Since the point P lies in the XZ plane, for all values of X and Z the line (OP=r) makes 90^0 with Y-axis

Where $\bar{r} = x\bar{a}_x + z\bar{a}_z$

The above figure can be modified as shown in Fig:2.6

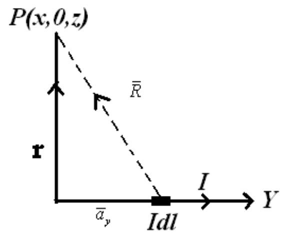


Fig: 2.6

From Fig:2.5, $\bar{R} = (x,0,z) - (0,y,0)$

$$\bar{R} = x\bar{a}_x - y\bar{a}_y + z\bar{a}_z$$

$$r^2 = x^2 + z^2$$

Consider a small differential current element Idl along Y-axis

According to Biot-Savart's law $d\bar{H} = \dfrac{Id\bar{l} \times \bar{R}}{4\pi R^3}$

Here $d\bar{l} = dy\bar{a}_y$

$$d\bar{H} = \frac{Idy\bar{a}_y \times (x\bar{a}_x + z\bar{a}_z - y\bar{a}_y)}{4\pi\left(x^2 + z^2 + y^2\right)^{3/2}}$$

$$d\bar{H} = \frac{Idy}{4\pi\left(r^2 + y^2\right)^{3/2}}\left(z\bar{a}_x - x\bar{a}_z\right)$$

Integrating w.r.t. y from y=0 to y_1, we get total magnetic field strength

$$\therefore \bar{H} = \int_{y=0}^{y_1} \frac{Idy}{4\pi\left(r^2 + y^2\right)^{3/2}}\left(z\bar{a}_x - x\bar{a}_z\right)$$

$$= \frac{I\left(z\bar{a}_x - x\bar{a}_z\right)}{4\pi}\int_{y=0}^{y_1} \frac{dy}{\left(r^2 + y^2\right)^{3/2}}$$

$$= \frac{I\left(z\bar{a}_x - x\bar{a}_z\right)}{4\pi}\left[\frac{y/r^2}{\sqrt{r^2 + y^2}}\right]_0^{y_1} \quad \because \int \frac{dx}{\left(a^2 + x^2\right)^{3/2}} = \frac{x/a^2}{\sqrt{a^2 + x^2}}$$

$$\bar{H} = \frac{I}{4\pi r^2\sqrt{\dfrac{r^2}{y_1^2} + 1}}\left(z\bar{a}_x - x\bar{a}_z\right)$$

For a semi-infinite length conductor, $y_1=\infty$

$$\therefore \bar{H} = \frac{I}{4\pi r^2\sqrt{0+1}}\left(z\bar{a}_x - x\bar{a}_z\right) = \frac{I}{4\pi r^2}\left(z\bar{a}_x - x\bar{a}_z\right)$$

Problem:2.4

Find $\bar{H}$ at (-3, 4, 0) due to the current filament shown in Fig:2.7

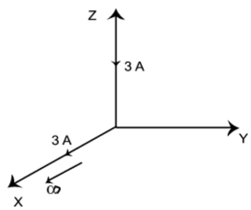


Fig: 2.7

Solution:

For the element along X-axis is

$$\bar{R} = (-3, 4, 0) - (x, 0, 0)$$

$$d\bar{l} = dx\,\bar{a}_x$$

$$\bar{H}_x = \int_0^\infty \frac{3dx\,\bar{a}_x \times \left[(-3-x)\bar{a}_x + 4\bar{a}_y\right]}{4\pi\left[(-3-x)^2+16\right]^{3/2}}$$

$$= \frac{3}{4\pi}4\bar{a}_z \int_0^\infty \frac{dx}{\left[16+(-3-x)^2\right]^{3/2}}$$

$$= \frac{3}{\pi}\bar{a}_z \int_0^\infty \frac{dx}{\left[16+(3+x)^2\right]^{3/2}}$$

$$= \frac{3\bar{a}_z}{\pi}\left[\frac{(3+x)/16}{\left[16+(3+x)^2\right]^{1/2}}\right]_0^\infty$$

$$= \frac{3\bar{a}_z}{16\pi}\left[1-\left(\frac{3}{5}\right)\right] = 23.88\bar{a}_z \quad mA/m$$

For the element along Z-axis is

$$\bar{R} = (-3, 4, 0) - (0, 0, z)$$

$$d\bar{l} = dz\,\bar{a}_z$$

$$\bar{H}_z = \int_{\infty}^{0} \frac{3\,dz\,\bar{a}_z \times \left[\left(-3\bar{a}_x + 4\bar{a}_y - Z\bar{a}_z\right)\right]}{4\pi\left[9+16+z^2\right]^{3/2}}$$

$$= \frac{3}{4\pi}\int_{0}^{\infty} \frac{\left(3\bar{a}_y + 4\bar{a}_x\right)dz}{\left(25+z^2\right)^{3/2}}$$

$$= \frac{3\left(3\bar{a}_y + 4\bar{a}_x\right)}{4\pi}\int_{0}^{\infty} \frac{dz}{\left(25+z^2\right)^{3/2}}$$

$$= \frac{3\left(3\bar{a}_y + 4\bar{a}_x\right)}{4\pi}\left[\frac{z/25}{\left(25+z^2\right)^{1/2}}\right]_0^{\infty}$$

$$= \frac{3\left(3\bar{a}_y + 4\bar{a}_x\right)}{100\pi}\left(1-0\right) = 38.2\bar{a}_x + 28.65\bar{a}_y \quad mA/m$$

$$\bar{H} = \bar{H}_x + \bar{H}_z = 38.2\bar{a}_x + 28.65\bar{a}_y + 23.88\bar{a}_z \quad mA/m$$

Problem:2.5

The +Ve Y-axis (semi infinite line w.r.t origin) carries a filamentary current of 2A in the $-\bar{a}_y$ direction. Find $\bar{H}$ at (a) A(2, 3, 0) (b) B(3, 12, -4).

Solution:

(a) $\bar{R}$ = (2, 3, 0) – (0, y, 0)

$$d\bar{l} = dy\,\bar{a}_y$$

Since 2A is along $-\bar{a}_y$, limits are ∞ to 0.

$$\bar{H}_A = \int_{\infty}^{0} \frac{2\,dy\,\bar{a}_y \times \left[\left(2\bar{a}_x + (3-y)\bar{a}_y\right)\right]}{4\pi\left[4+(3-y)^2\right]^{3/2}}$$

$$\bar{H}_A = \frac{\bar{a}_z}{\pi}\int_{0}^{\infty} \frac{dy}{\left[4+(3-y)^2\right]^{3/2}}$$

$$= -\frac{\bar{a}_z}{\pi}\left[\frac{(3-y)/4}{\left[4+(3-y)^2\right]^{1/2}}\right]_0^{\infty}$$

$$= -\frac{\bar{a}_z}{4\pi}\left(-1-\frac{3}{\sqrt{13}}\right) = 145.8\bar{a}_z mA/m$$

(b) $\overline{R}$ = (3, 12, -4) – (0, y, 0)

$$\therefore\ \bar{H}_B = \int_{\infty}^{0}\frac{2dy\,\bar{a}_y \times \left[3\bar{a}_x + (12-y)\bar{a}_y - 4\bar{a}_z\right]}{4\pi\left[25+(12-y)^2\right]^{3/2}}$$

$$\bar{H}_B = \frac{6\bar{a}_z + 8\bar{a}_x}{4\pi}\int_0^{\infty}\frac{dy}{\left[25+(12-y)^2\right]^{3/2}}$$

$$= -\frac{6\bar{a}_z + 8\bar{a}_x}{4\pi}\left[\frac{(12-y)/25}{\left[25+(12-y)^2\right]^{1/2}}\right]_0^{\infty}$$

$$= -\frac{6\bar{a}_z + 8\bar{a}_x}{100\pi}\left(-1-\frac{12}{13}\right) = 48.97\bar{a}_x + 36.73\bar{a}_z \quad mA/m$$

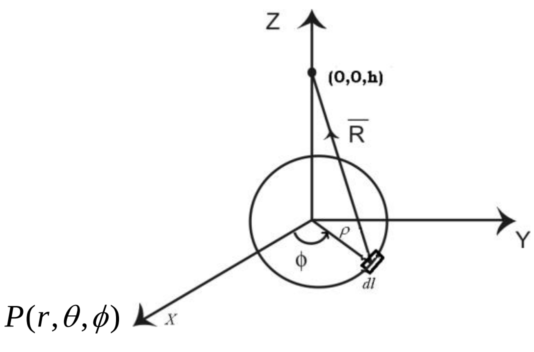
Problem:2.6

Show that the magnetic field intensity $\bar{H}$ at (0, 0, h) due to a circle which lies on XY plane with radius 'ρ' carries a current I as

$$\bar{H} = \frac{I\rho^2\,\bar{a}_z}{2\left[\rho^2+h^2\right]^{3/2}}$$

Solution:

Consider the circular loop shown in Fig:2.8



From Fig:2.8 $\bar{R} = -\rho\bar{a}_\rho + h\bar{a}_z$

and $d\bar{l} = \rho d\phi \bar{a}_\phi$

We have $d\bar{H} = \dfrac{Id\bar{l} \times \bar{R}}{4\pi R^3}$

$$d\bar{H} = \frac{I\rho d\phi \bar{a}_\phi \times (-\rho\bar{a}_\rho + h\bar{a}_z)}{4\pi(\rho^2 + h^2)^{3/2}}$$

$$d\bar{H} = \frac{I\rho d\phi(\rho\bar{a}_z + h\bar{a}_\rho)}{4\pi(\rho^2 + h^2)^{3/2}}$$

due to symmetry of circle the components in ρ direction will get cancelled.

$$d\bar{H} = \frac{I\rho^2 d\phi \bar{a}_z}{4\pi(\rho^2 + h^2)^{3/2}}$$

Integrating

$$\bar{H} = \frac{I}{4\pi}\int_0^{2\pi}\left[\frac{\rho^2\bar{a}_z d\phi}{\left(\rho^2 + h^2\right)^{3/2}}\right]$$

$$\bar{H} = \frac{I\rho^2\bar{a}_z}{2\left[\rho^2 + h^2\right]^{3/2}}$$

Problem:2.7

A circular loop located on $x^2+y^2=9$, z=0 carries a direct current of 10A along $\bar{a}_\phi$ direction. Determine $\bar{H}$ at (0, 0, 4) and (0, 0, -4).

Solution:

Here $\rho = 3$, h=4 and I=10A

$$\therefore \bar{H} = \frac{10}{4\pi}\frac{3^2}{\left(3^2 + 4^2\right)^{3/2}}\bar{a}_z\int_0^{2\pi} d\phi$$

$$= 0.36\bar{a}_z \text{ A/m}$$

Similarly $\bar{H}_{(0,0,-4)} = \bar{H}_{(0,0,4)} = 0.36\bar{a}_z A/m$

***Problem:2.8**

Find the field at the centre of a circular loop of radius 'a', carrying current I along ϕ direction in Z=0 plane.

Solution:

We have $\bar{H} = \dfrac{I\rho^2 \bar{a}_z}{2\left[\rho^2 + h^2\right]^{3/2}}$

Here ρ=a, and h=0

$$\bar{H} = \int_{\phi=0}^{2\pi} \frac{I a^2 d\phi \bar{a}_z}{4\pi\left[a^2 + 0\right]^{3/2}} = \frac{I\,\bar{a}_z}{2a} A/m$$

Problem:2.9

A thin ring of radius 5cm is placed on plane z=1cm so that its center is at (0, 0, 1 cm). If the ring carries 50mA along $\bar{a}_\phi$. Find $\bar{H}$ at

(a) (0, 0, -1cm) (b) (0, 0, 10cm)

Solution:

(a) Here ρ=5cm, h=2cm and I=50mA

We have $\bar{H} = \dfrac{I\rho^2 \bar{a}_z}{2\left[\rho^2 + h^2\right]^{3/2}}$

$$\bar{H} = \frac{50\times10^{-3}(5\times10^{-2})^2\,\bar{a}_z}{2\left[(5\times10^{-2})^2 + (2\times10^{-2})^2\right]^{3/2}}$$

$$\bar{H} = \frac{125\,\bar{a}_z}{2[29]^{3/2}}$$

$$\bar{H} = 400\bar{a}_z mA/m$$

(b) Here ρ=5cm, h=9cm and I=50mA

We have $\bar{H} = \dfrac{I\rho^2 \bar{a}_z}{2\left[\rho^2 + h^2\right]^{3/2}}$

$$\bar{H} = \frac{50\times10^{-3}(5\times10^{-2})^2\,\bar{a}_z}{2\left[(5\times10^{-2})^2+(9\times10^{-2})^2\right]^{3/2}}$$

$$\bar{H} = \frac{125\,\bar{a}_z}{2[106]^{3/2}}$$

$$\bar{H} = 57.3\bar{a}_z mA/m$$

Problem:2.10

A square conducting loop 2a m on each side carries a current of I amp. Calculate the magnetic field intensity at the center of the loop.

Solution:

Consider a square loop with each side 2a m as shown in the Fig:2.9.

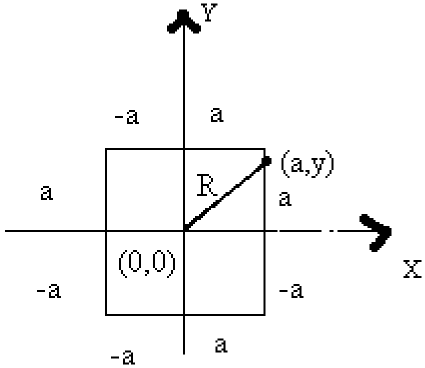


Fig:2.9 A square loop

We need to find magnetic field intensity at (0,0) due to elemental current flowing in one side of square loop at (a,y)

According to Biot Savart's law, we have

$$d\bar{H} = \frac{Id\bar{l}\times\bar{R}}{4\pi R^3}$$

From Fig.2.9, $d\bar{l} = dy\bar{a}_y$

$$\bar{R} = (0,0)-(a,y) = -a\bar{a}_x - y\bar{a}_y$$

$$\therefore d\bar{H}_{oneside} = \frac{Idy\bar{a}_y\times(-a\bar{a}_x - y\bar{a}_y)}{4\pi(a^2+y^2)^{3/2}}$$

$$\therefore d\bar{H}_{oneside} = \frac{Iady\bar{a}_z}{4\pi(a^2+y^2)^{3/2}}$$

$$\bar{H}_{oneside} = \int_{-a}^{a} \frac{Iady\bar{a}_z}{4\pi(a^2+y^2)^{3/2}}$$

$$\bar{H}_{oneside} = \frac{Ia}{4\pi}\bar{a}_z \int_{-a}^{a} \frac{dy}{(a^2+y^2)^{3/2}}$$

$$\bar{H}_{oneside} = \frac{Ia}{4\pi}\bar{a}_z \left[\frac{y/a^2}{(a^2+y^2)^{1/2}}\right]_{-a}^{a}$$

$$\bar{H}_{oneside} = \frac{Ia}{4\pi a^2}\bar{a}_z \left[\frac{a}{(a^2+a^2)^{1/2}} + \frac{a}{(a^2+a^2)^{1/2}}\right]$$

$$\bar{H}_{oneside} = \frac{I}{4\pi a}\bar{a}_z \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right]$$

$$\bar{H}_{oneside} = \frac{I\sqrt{2}}{4\pi a}\bar{a}_z$$

Magnetic field intensity at (0,0) due to four sides is

$$\bar{H} = 4\bar{H}_{oneside} = \frac{4I\sqrt{2}}{4\pi a}\bar{a}_z$$

$$\bar{H} = \frac{I\sqrt{2}}{\pi a}\bar{a}_z$$

Problem:2.11

A square conducting loop 3cm on each side carries a current of 10A. Calculate the magnetic field intensity at the center of the loop.

Solution:

We have $\bar{H} = \frac{I\sqrt{2}}{\pi a}\bar{a}_z$

Here a=1.5x10^{-2}m and I=10A

$$\therefore \bar{H} = \frac{10\sqrt{2}}{\pi \times 1.5 \times 10^{-2}}\bar{a}_z = 300.105\bar{a}_z A/m$$

Problem:2.12

Planes z=0 and z=4 carry current $\overline{K} = -10\overline{a}_x$ A/m and $\overline{K} = 10\overline{a}_x$ A/m respectively. Determine $\overline{H}$ at (a) (1, 1, 1) (b) (0, -3, 10)

Solution:

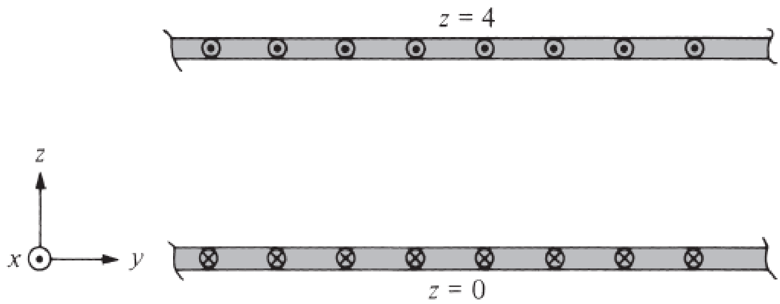


Fig: 2.13

(a)

We have $\overline{H} = \frac{1}{2}\overline{K} \times \overline{a}_n$

here $\overline{K} = -10\overline{a}_x A/m$ on Z=0 plane and Since point (1,1,1) is lying above the plane Z=0, $\overline{a}_n = \overline{a}_z$

$$\therefore \overline{H}_0 = \frac{1}{2}(-10)\overline{a}_x \times \overline{a}_z$$

$$= -5(-\overline{a}_y)$$

$$= 5\overline{a}_y$$

here $\overline{K} = 10\overline{a}_x A/m$ on Z=4 plane and Since point (1,1,1) is lying below the plane Z=4, $\overline{a}_n = -\overline{a}_z$

$$\therefore \overline{H}_4 = \frac{1}{2}10\overline{a}_x \times \left(-\overline{a}_z\right)$$

$$= -5(-\overline{a}_y)$$

$$= 5\overline{a}_y$$

$$\overline{H} = \overline{H}_0 + \overline{H}_4 = 10\overline{a}_y\ A/m$$

(b)

We have $\bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_n$

here $\bar{K} = -10\bar{a}_x A/m$ on Z=0 plane and Since point (0,-3,10) is lying above the plane Z=0, $\bar{a}_n = \bar{a}_z$

$$\therefore \bar{H}_0 = \frac{1}{2}(-10\bar{a}_x) \times \bar{a}_z$$

$$= -5(-\bar{a}_y)$$

$$= 5\bar{a}_y$$

here $\bar{K} = 10\bar{a}_x A/m$ on Z=4 plane and Since point (0,-3,10) is lying above the plane Z=4, $\bar{a}_n = \bar{a}_z$

$$\therefore \bar{H}_4 = \frac{1}{2}10\bar{a}_x \times \bar{a}_z$$

$$= 5(-\bar{a}_y)$$

$$= -5\bar{a}_y$$

$$\bar{H} = \bar{H}_0 + \bar{H}_4 = 5\bar{a}_y - 5\bar{a}_y = 0$$

Problem:2.13

Plane Y = 1 carries current $\bar{K} = 50\bar{a}_z$ mA/m. Find $\bar{H}$ at (a) (0, 0, 0) (b) (1, 5, -3)

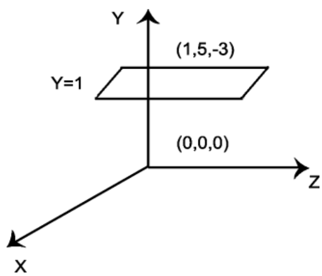


Fig:2.14

Solution:

(a)

here $\bar{K} = 50\bar{a}_z mA/m$ on Y=1 plane and Since point (0,0,0) is lying below the plane Y=1, $\bar{a}_n = -\bar{a}_y$

$$\therefore \bar{H} = \frac{1}{2}50\bar{a}_z \times (-\bar{a}_y)$$

$$= -25\left(-\bar{a}_x\right)$$

$$= 25\bar{a}_x \, \text{mA/m}$$

(b)

here $\bar{K} = 50\bar{a}_z mA/m$ on Y=1 plane and Since point (1,5,-3) is lying above the plane Y=1, $\bar{a}_n = \bar{a}_y$

$$\bar{H} = \frac{1}{2}50\bar{a}_z \times \bar{a}_y$$

$$= 25\left(-\bar{a}_x\right)$$

$$= -25\bar{a}_x \, \text{mA/m}$$

***Problem:2.14**

A long coaxial cable has an inner conductor carrying a current of 1mA along +ve Z direction, its axis coinciding with Z-axis. Its inner conductor diameter is 6mm. If its outer conductor has an inside diameter of 12mm and thickness of 2mm, determine $\bar{H}$ at (0,0,0), (0,4.5mm,0) and (0,1cm,0). (No derivations).

Solution:

As per the derivations in the section “Infinitely long co-axial transmission line”
Given I=1mA, a=3mm,b=6mm and t=2mm.

Let the given points be P_1(0,0,0), P_2(0,4.5mm,0) and P_3(0,1cm,0).

Given points are in rectangular coordinate system $\therefore \rho = \sqrt{x^2 + y^2}$

For P_1, ρ=0, i.e.ρ<a, Hence case(i) formula from the section “**Infinitely long co-axial transmission line”**

$$\bar{H} = \frac{I\rho}{2\pi a^2}\bar{a}_\phi = 0A/m$$

For P_2, ρ=4.5mm, i.e a<ρ<b, Hence case(ii) formula from the section “**Infinitely long co-axial transmission line”**

$$\bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi = \frac{1\times10^{-3}}{2\pi\times4.5\times10^{-3}}\bar{a}_\phi = 0.03537\bar{a}_\phi A/m$$

For P_3, ρ=1cm=10mm, i.e. ρ>b+t, Hence case(iv) formula from the section **"Infinitely long co-axial transmission line"**

$$\bar{H} = 0A/m$$

Problem:2.15

A solenoid of length 'l' and radius 'a' consists of 'N' turns of wire carrying current 'I'. Show that at point p along it's axis $\bar{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\bar{a}_z$ where $n = \frac{N}{l}$, θ_1, θ_2 are the angles subtended at 'p' by the end turns as illustrated in Fig:2.15. Also Show that if l >>a at the center of the solenoid $\bar{H} = nI\,\bar{a}_z$

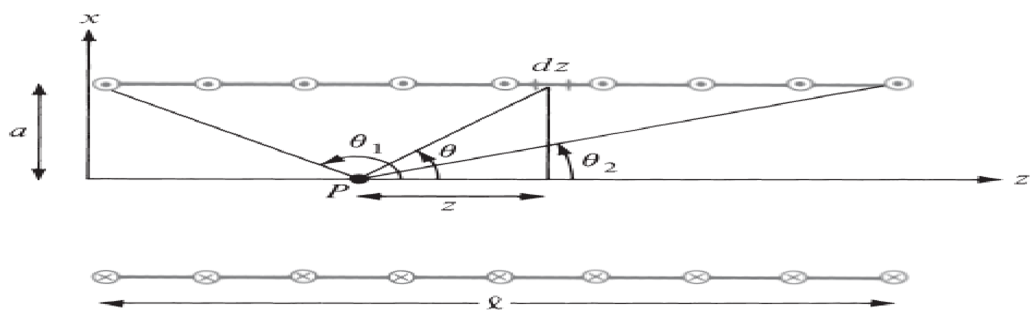


Fig: 2.15 cross section of a solenoid

Solution:

We know the elemental magnetic field intensity dH_z due to one turn (circle) at point p is $dH_z = \frac{I\,a^2 d\phi}{4\pi(a^2+z^2)^{3/2}}\bar{a}_z$

$\therefore H_z = \frac{I\,a^2}{2(a^2+z^2)^{3/2}}\bar{a}_z$ which is the magnetic field intensity at point 'p' due to one turn. As the solenoid contains 'N' number of turns by considering the elemental length dl, the elemental magnetic field intensity due to a solenoid of length 'L' and having 'N' number of turns at point 'P' is

$$dH_z = \frac{I\,a^2 dl\,\bar{a}_z}{2\,(a^2+z^2)^{3/2}}$$

where dl = ndz = $\frac{N}{l}dz$

from Fig:2.15 $Tan\theta = \frac{a}{z}$

z = a cot θ

dz = -a $\text{cosec}^2\theta$ dθ

$$dH_z = \frac{I\,a^2 n\left(-a\cos ec^2\theta\, d\theta\right)}{2\left(a^2 + a^2\cot^2\theta\right)^{3/2}}\bar{a}_z$$

$$= \frac{I\,a^2 n\left(-a\cos ec^2\theta\, d\theta\right)}{2\,a^3\cos ec^3\theta}\bar{a}_z$$

$$= -\frac{nI}{2}\sin\theta\, d\theta\,\bar{a}_z$$

$$H_z = -\frac{nI}{2}\int_{\theta_2}^{\theta_1}\sin\theta\, d\theta\,\bar{a}_z$$

$$= -\frac{nI}{2}\left(\cos\theta_1 - \cos\theta_2\right)\bar{a}_z$$

$$= \frac{nI}{2}\left(\cos\theta_2 - \cos\theta_1\right)\bar{a}_z$$

At the center of the solenoid we can write

$$\cos\theta_2 = \frac{l/2}{\sqrt{a^2 + (l/2)^2}} = -\cos\theta_1$$

as l >>a

$$H_z = \frac{nI}{2}2\cos\theta_2\,\bar{a}_z$$

$$= nI\frac{l/2}{\sqrt{a^2 + (l/2)^2}}\bar{a}_z$$

$$= nI\frac{l/2}{l/2}\bar{a}_z = nI\,\bar{a}_z$$

Problem:2.16

A Toroid whose dimensions are shown in Fig:2.16 has 'N' turns and carries current I. Determine $\bar{H}$ inside and outside the Toroid.

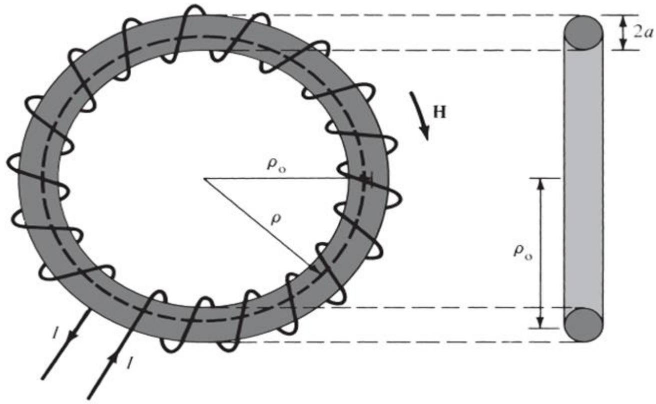


Fig: 2.16 a toroid with a circular cross section

Solution:

Inside the toroid consider the closed path L_1, According to Ampere's circuit law

$$\oint_{L_1} \bar{H}\cdot d\bar{l} = I_{enc}$$

$$\Rightarrow H_\phi 2\pi\,\rho = nI$$

$$H_\phi = \frac{nI}{2\pi\rho}$$

$$\bar{H} = \frac{nI}{2\pi\rho}\bar{a}_\phi$$

outside the Toroid $\oint_L \bar{H}\cdot d\bar{l} = I_{enc}$

$= nI - nI$

$\Rightarrow H_\phi = 0 \Rightarrow \bar{H} = 0$

NOTE: By bending a solenoid in to a form of circle we get a toroid.

***Problem:2.17**

A long straight conductor with radius 'a' has a magnetic field strength $\bar{H} = \frac{Ir}{2\pi a^2}\bar{a}_\phi$ within the conductor (r<a) and $\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi$ outside the conductor (r>a). Find the current density $\bar{J}$ in both the regions(r<a and r>a).

Solution:

we have $\bar{J} = \nabla \times \bar{H}$

Given $\bar{H} = \dfrac{Ir}{2\pi a^2}\bar{a}_\phi$ within the conductor (r<a)

Which has cylindrical symmetry, here ρ=r,

$$\bar{J} = \nabla \times \bar{H} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$= \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & r\dfrac{Ir}{2\pi a^2} & 0 \end{vmatrix} = \frac{I}{\pi a^2}\bar{a}_z A/m^2$$

And also given $\bar{H} = \dfrac{I}{2\pi r}\bar{a}_\phi$ outside the conductor (r>a)

$$\therefore \bar{J} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & r\dfrac{I}{2\pi r} & 0 \end{vmatrix} = 0A/m^2$$

***Problem:2.18**

A conducting plane at y=1 carries a surface current of $10\,\bar{a}_z$ mA/m. Find H and B at(a) (0,0,0) and (b) (2,2,2).

Solution:

(a)

We have $\bar{H} = \dfrac{1}{2}\bar{K} \times \bar{a}_n$

here $\bar{K} = 10\bar{a}_z mA/m$.

Since the point (0,0,0) is lying below the plane Y=1, $\bar{a}_n = -\bar{a}_y$

$$\therefore \bar{H} = \frac{1}{2} 10\bar{a}_z \times (-\bar{a}_y)$$

$$= -5(-\bar{a}_x)$$

$$= 5\bar{a}_x \text{ mA/m}$$

$$\bar{B} = \mu_0 \bar{H} = 4\pi \times 10^{-7} \times 5\bar{a}_x = 62.83 \times 10^{-10} \bar{a}_x T$$

(b)

We have $\bar{H} = \frac{1}{2} \bar{K} \times \bar{a}_n$

here $\bar{K} = 10\bar{a}_z mA/m$.

Since the point (2,2,2) is lying above the plane Y=1, $\bar{a}_n = \bar{a}_y$

$$\therefore \bar{H} = \frac{1}{2} 10\bar{a}_z \times \bar{a}_y$$

$$= 5(-\bar{a}_x)$$

$$= -5\bar{a}_x \text{ mA/m}$$

$$\bar{B} = \mu_0 \bar{H} = 4\pi \times 10^{-7} \times -5\bar{a}_x = -62.83 \times 10^{-10} \bar{a}_x T$$

***Problem:2.19**

An infinitely long straight conducting rod of radius 'a' carries a current of I in +ve Z-direction. Using Ampere's Circuital Law, find $\bar{H}$ in all regions and sketch the variation of H as a function of radial distance. If I=3mA and a=2cm, find $\bar{H}$ and $\bar{B}$ at (0,1cm,0) and (0,4cm,0).

Solution:

Consider cylindrical co-ordinate system

Case(i): inside the conductor(ρ<a)

According to Ampere's circuit law

$$\oint_L \bar{H} \cdot d\bar{l} = I_{enc} = \int_s \bar{J} \cdot d\bar{s}$$

$$\oint_L \bar{H}\cdot d\bar{l} = \int_0^{2\pi} H_\phi \bar{a}_\phi \cdot \rho d\phi \bar{a}_\phi = H_\phi 2\pi\rho$$

$\bar{J}$ of the internal conductor is $\frac{I}{\pi a^2}\bar{a}_z$ and $d\bar{s} = \rho d\phi d\rho \bar{a}_z$

$$\therefore I_{enc} = \int_s \bar{J}\cdot d\bar{s}$$

$$= \iint \frac{I}{\pi a^2}\bar{a}_z.\rho\, d\phi\, d\rho\, \bar{a}_z$$

$$= \int_0^{2\pi}\int_0^{\rho} \frac{I}{\pi a^2}\rho\, d\phi\, d\rho$$

$$= \frac{I}{\pi a^2}\int_0^{\rho} \rho(2\pi)\, d\rho = \frac{2\pi I}{\pi a^2}\left(\frac{\rho^2}{2}\right) = \frac{I\rho^2}{a^2}$$

$$\therefore 2\pi H_\phi \rho = \frac{I\rho^2}{a^2}$$

$$H_\phi = \frac{I\rho}{a^2 2\pi} \Rightarrow \bar{H} = \frac{I\rho}{2\pi a^2}\bar{a}_\phi$$

Case(ii): outside the conductor($\rho>a$)

According to Ampere's circuit law

$$\oint_L \bar{H}\cdot d\bar{l} = I_{enc}$$

$$\Rightarrow \int_0^{2\pi} H_\phi \bar{a}_\phi.\rho\, d\phi\, \bar{a}_\phi = I_{enc}$$

$$\Rightarrow 2\pi H_\phi \rho = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

$$\bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi$$

Given points let be $P_1(0,1cm,0)$ and $P_2(0,4cm,0)$

For P_1 radial distance $\rho = \sqrt{0^2+1^2+0^2} = 1cm$

Also given a=2cm and I=3mA i.e. ρ<a (inside the conductor)

$$\therefore \bar{H} = \frac{I\rho}{2\pi a^2}\bar{a}_\phi = \frac{3\times10^{-3}\times1\times10^{-2}}{2\pi(2\times10^{-2})^2}\bar{a}_\phi = 0.0119\bar{a}_\phi A/m \text{ and } \bar{B} = \mu_0\bar{H}$$

$$\therefore \bar{B} = 4\pi\times10^{-7}\times0.0119\bar{a}_\phi = 0.15\times10^{-7}\bar{a}_\phi T$$

For P_2 radial distance $\rho = \sqrt{0^2+4^2+0^2} = 4cm$

Here ρ>a (outside the conductor)

$$\therefore \bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi = \frac{3\times10^{-3}}{2\pi\times4\times10^{-2}}\bar{a}_\phi = 0.0119\bar{a}_\phi A/m \text{ and } \bar{B} = \mu_0\bar{H}$$

$$\therefore \bar{B} = 4\pi\times10^{-7}\times0.0119\bar{a}_\phi = 0.15\times10^{-7}\bar{a}_\phi T$$

Sketch of H_ϕ:

We have $H_\phi = \dfrac{I\rho}{2\pi a^2}$ $for\,\rho < a$ i.e. $H_\phi \propto \rho$

And $H_\phi = \dfrac{I}{2\pi\rho}$ $for\,\rho > a$ i.e. $H_\phi \propto 1/\rho$

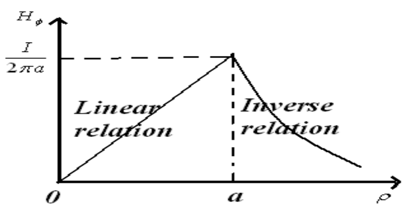


Fig: 2.18

***Problem:2.20**

Determine the magnetic flux, for the surface described by

(a) ρ=1m., $0 \le \phi \le \pi/2, 0 \le z \le 2m$

(b) a sphere of radius 2m.

If the magnetic field is of the form $\bar{H} = \dfrac{1}{\rho}\cos\phi\bar{a}_\rho A/m$

Solution:

We have magnetic flux $\psi = \int_s \bar{B}.d\bar{s} = \mu_0\int_s \bar{H}.d\bar{s}$

(a) here $d\bar{s} = \rho d\phi dz\bar{a}_\rho$ (cylindrical symmetry)

$$\psi = \mu_0 \int_s \frac{1}{\rho}\cos\phi \bar{a}_\rho.\rho d\phi dz \bar{a}_\rho = \mu_0 \int_{z=0}^{2} dz \int_{\phi=0}^{\pi/2} \cos\phi d\phi = 2\mu_0 = 25.13\times10^{-7} Wb$$

(b) here $d\bar{s} = r^2 \sin\theta d\theta d\phi \bar{a}_r$ (spherical symmetry)

$$\psi = \mu_0 \int_s \frac{1}{r}\cos\phi \bar{a}_r.r^2 \sin\theta d\theta d\phi \bar{a}_r = \mu_0 r \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} \cos\phi d\phi = 0Wb$$

***Problem:2.21**

In a conducting medium $\bar{H} = y^2 z\bar{a}_x + 2(x+1)yz\bar{a}_y - (x+1)z^2\bar{a}_z A/m$. Find the current density at (1,0,-3) and calculate the current passing through Y=1 plane, $0 \le x \le 1, 0 \le z \le 1$.

Solution:

We have current density $\bar{J} = \nabla\times\bar{H}$

$$\bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & 2(x+1)yz & -(x+1)z^2 \end{vmatrix}$$

$$\bar{J} = -2(x+1)y\bar{a}_x + (z^2+y^2)\bar{a}_y A/m^2$$

$$\bar{J}_{(1,0,-3)} = 9\bar{a}_y A/m^2$$

Current is $I = \int_s \bar{J}.d\bar{s}$, here $d\bar{s} = dxdz\bar{a}_y$

$$\therefore I = \int_s \left(-2(x+1)y\bar{a}_x + (z^2+y^2)\bar{a}_y\right).dxdz\bar{a}_y = \int_{x=0}^{1} dx \int_{z=0}^{1} (z^2+y^2)dz = 1.33A$$

Problem:2.22

Find the flux density at the center of a square loop of 10 turns carrying a current of 10A. The loop is in air and has a side of 2m.

Solution:

We have $\bar{H} = \frac{I\sqrt{2}}{\pi a}\bar{a}_z$

$\therefore$ magnetic flux density is $\bar{B} = \frac{\mu_0 I\sqrt{2}}{\pi a}\bar{a}_z$

Here no. of turns N=10

Total current I=N X current in each turn

I=10 X 10=100A

And a=1m

$$\therefore \bar{B} = \frac{\mu_0 100\sqrt{2}}{\pi \times 1}\bar{a}_z$$

$$\therefore \bar{B} = \frac{4\pi \times 10^{-7} \times 100\sqrt{2}}{\pi \times 1}\bar{a}_z = 56.569\mu\bar{a}_z Tesla$$

Problem:2.22

Given the magnetic vector potential $\bar{A} = \frac{-\rho^2}{4}\bar{a}_z$ Wb/m. Calculate the total magnetic flux crossing the surface $\phi = \frac{\pi}{2}$, $1 \le \rho \le 2$m, $0 \le z \le 5$m.

Solution:

We have

$$\nabla \times \bar{A} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \bar{A} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{\rho^2}{4} \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\bar{a}_\rho(0) - \rho\bar{a}_\phi \frac{\partial}{\partial\rho}\left(\frac{-\rho^2}{4}\right) + 0\right]$$

$$= \frac{1}{\rho}\left[-\rho\bar{a}_\phi\left(-\frac{1}{4}2\rho\right)\right] = \frac{\rho}{2}\bar{a}_\phi$$

$\because$ ϕ is constant and , ρ and z are varying, $d\bar{s} = d\rho\, dz\bar{a}_\phi$

The magnetic flux crossing the surface is $\psi = \int_S (\nabla \times \bar{A}) \cdot d\bar{s}$

$$= \int_{\rho=1}^{2}\int_{z=0}^{5} \frac{\rho}{2} d\rho dz$$

= 3.75 Wb

Problem:2.23

A current distribution gives rise to the vector magnetic potential $\bar{A} = x^2 y \bar{a}_x + y^2 x \bar{a}_y - 4xyz \bar{a}_z$ Wb/m. Calculate (a) $\bar{B}$ $at\left(-1,2,5\right)$ (b) The flux through the surface defined by Z=1, $0 \leq x \leq 1$, $-1 \leq y \leq 4$.

Solution:

(a)

$$\bar{B} = \nabla \times \bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & y^2 x & -4xyz \end{vmatrix}$$

$$\bar{B} = \bar{a}_x \left(-4xz - 0\right) - \bar{a}_y \left(-4yz - 0\right) + \bar{a}_z \left(y^2 - x^2\right)$$

$$\bar{B} = -4xz\bar{a}_x + 4yz\bar{a}_y + \left(y^2 - x^2\right)\bar{a}_z$$

$$\bar{B}_{(-1,2,5)} = 20\bar{a}_x + 40\bar{a}_y + 3\bar{a}_z \text{ Wb/m}^2$$

(b)

Here $d\bar{s} = dx\,dy\,\bar{a}_z$

$$\psi = \int_S \nabla \times \bar{A} \cdot d\bar{s}$$

$$\psi = \int_{x=0}^{1} \int_{y=-1}^{4} \left(-4xz\bar{a}_x + 4yz\bar{a}_y + \left(y^2 - x^2\right)\bar{a}_z\right) \cdot dxdy\bar{a}_z$$

$$\psi = \int_{x=0}^{1} \int_{y=-1}^{4} \left(y^2 - x^2\right) dxdy$$

$$\psi = \int_{y=-1}^{4} \left[y^2 x - \frac{x^3}{3}\right]_0^1 dy = \int_{y=-1}^{4} \left[y^2 - \frac{1}{3}\right] dy$$

$$\psi = \frac{1}{3}\left[y^3 - y\right]_{-1}^{4} = \frac{1}{3}\left[64 - 4 + 1 - 1\right] = 20Wb$$

Problem:2.24

A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown in Fig.2.21. Show that the force experienced by the loop is given by

$$\bar{F} = -\frac{\mu_0 I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_0} - \frac{1}{\rho_0 + a}\right]\bar{a}_\rho N$$

Solution:

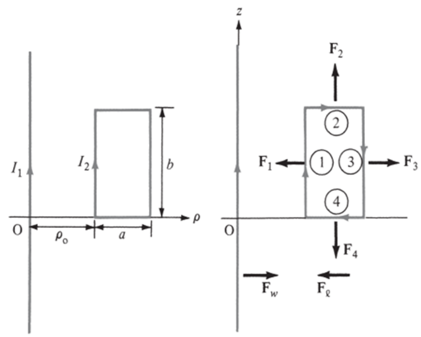


Fig:2.21

Let $\bar{F}_\ell$ be the force on the loop

$$\bar{F}_\ell = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= I_2 \oint d\bar{l}_2 \times \bar{B}_1$$

Where $\bar{F}_1, \bar{F}_2, \bar{F}_3$ *and* $\bar{F}_4$ are the forces exerted on sides of the loop.

We know for infinitely long wire $\bar{B} = \frac{\mu_0 I}{2\pi\rho}\bar{a}_\phi$

To evaluate $\bar{F}_1$, $d\bar{l}_2 = dz\bar{a}_z$, z ranges from 0 to b, $I = I_1$ and $\rho = \rho_0$

$$\bar{F}_1 = I_2 \int d\bar{l}_2 \times \bar{B}_1 = I_2 \int_{z=0}^{b} dz\bar{a}_z \times \frac{\mu_0 I_1}{2\pi\rho_0}\bar{a}_\phi$$

$$\therefore \bar{F}_1 = -\frac{\mu_0 I_1 I_2 b}{2\pi\rho_0}\bar{a}_\rho \qquad (attractive)$$

To evaluate $\bar{F}_3$, $d\bar{l}_2 = dz\bar{a}_z$, z ranges from b to 0, $I = I_1$ and $\rho = \rho_0 + a$

$$\bar{F}_3 = I_2 \int d\bar{l}_2 \times \bar{B}_1 = I_2 \int_{z=b}^{0} dz\bar{a}_z \times \frac{\mu_0 I_1}{2\pi(\rho_0 + a)}\bar{a}_\phi$$

$$\therefore \bar{F}_3 = \frac{\mu_0 I_1 I_2 b}{2\pi(\rho_0 + a)}\bar{a}_\rho \qquad (repulsive)$$

To evaluate $\bar{F}_2$, $d\bar{l}_2 = d\rho\bar{a}_\rho$, ρ ranges from ρ_0 to $\rho_0 + a$, I=I$_1$ and $\rho = \rho$

$$\bar{F}_2 = I_2 \int_{\rho=\rho_0}^{\rho_0+a} d\rho\bar{a}_\rho \times \frac{\mu_0 I_1}{2\pi\rho}\bar{a}_\phi$$

$$\therefore \bar{F}_2 = \frac{\mu_0 I_1 I_2}{2\pi}\ln\left(\frac{\rho_0 + a}{\rho_0}\right)\bar{a}_z \qquad (parallel)$$

To evaluate $\bar{F}_4$, $d\bar{l}_2 = d\rho\bar{a}_\rho$, ρ ranges from $\rho_0 + a$ to ρ_0, I=I$_1$ and $\rho = \rho$

$$\bar{F}_4 = I_2 \int_{\rho=\rho_0+a}^{\rho_0} d\rho\bar{a}_\rho \times \frac{\mu_0 I_1}{2\pi\rho}\bar{a}_\phi$$

$$\therefore \bar{F}_4 = -\frac{\mu_0 I_1 I_2}{2\pi}\ln\left(\frac{\rho_0 + a}{\rho_0}\right)\bar{a}_z \qquad (parallel)$$

Then the total force $\bar{F}_\ell$ on the loop is

$$\bar{F}_\ell = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$\bar{F}_\ell = -\frac{\mu_0 I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_0} - \frac{1}{\rho_0 + a}\right]\bar{a}_\rho N$$

and

$$\bar{F}_w = -\bar{F}_\ell$$

Problem:2.25

A charged particle of mass 2kg and 1C starts at origin with velocity $3\bar{a}_y$ m/s and travels in a region of uniform magnetic field $\bar{B} = 10\bar{a}_z$ Wb/m^2 at t=4sec. Calculate (a) velocity and acceleration of particle (b) the magnetic force on it.

Solution:

(a)

We have

$$\bar{F} = m\frac{d\bar{u}}{dt} = Q\bar{u} \times \bar{B}$$

Acceleration is

$$\bar{a} = \frac{d\bar{u}}{dt} = \frac{Q}{m}\bar{u} \times \bar{B}$$

Hence $$\frac{d}{dt}(u_x\bar{a}_x + u_y\bar{a}_y + u_z\bar{a}_z) = \frac{1}{2}\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ u_x & u_y & u_z \\ 0 & 0 & 10 \end{vmatrix} = 5(u_y\bar{a}_x - u_x\bar{a}_y)$$

By equating components, we get

$$\frac{du_x}{dt} = 5u_y, \quad \frac{du_y}{dt} = -5u_x, \quad \frac{du_z}{dt} = 0 \Rightarrow u_z = c_0$$

u_x or u_y an be eliminated in the above equations by taking second derivative of one equation and utilizing the other. Thus

$$\frac{d^2u_x}{dt^2} = 5\frac{du_y}{dt} = -25u_x$$

or

$$\frac{d^2u_x}{dt^2} + 25u_x = 0$$

Which is a linear differential equation whose solution is

$$u_x = c_1 \cos 5t + c_2 \sin 5t$$

$$\frac{du_x}{dt} = 5u_y = -5c_1 \sin 5t + 5c_2 \cos 5t$$

or

$$u_y = -c_1 \sin 5t + c_2 \cos 5t$$

Let us determine c_0,c_1 and c_2 using initial conditions. At t=0, $\bar{u} = 3\bar{a}_y$.

Hence,

$$u_x = 0 \Rightarrow 0 = c_1.1 + c_2.0 \Rightarrow c_1 = 0$$
$$u_y = 3 \Rightarrow 3 = -c_1.0 + c_2.1 \Rightarrow c_2 = 3$$
$$u_z = 0 \Rightarrow 0 = c_0$$

Substituting the values of c_0,c_1 and c_2, gives velocity as

$$\bar{u} = (u_x, u_y, u_z) = (3\sin 5t, 3\cos 5t, 0)$$

Hence, velocity at t=4sec is

$$\bar{u} = (3\sin 20, 3\cos 20, 0)$$
$$= 2.739\bar{a}_x + 1.224\bar{a}_y \quad m/s$$

Acceleration is $\bar{a} = \dfrac{d\bar{u}}{dt} = (15\cos 5t, -15\sin 5t, 0)$

Hence, acceleration at t=4sec is

$$\bar{a} = 6.101\bar{a}_x - 13.703\bar{a}_y \quad m/s^2$$

(b) $\bar{F} = m\dfrac{d\bar{u}}{dt} = m\bar{a} = 12.2\bar{a}_x - 27.4\bar{a}_y \quad N$.

Problem:2.26

A flux density of $0.05\bar{a}_y$ tesla in a material having magnetic susceptibility 2.5, find magnetic field current density and magnetization.

Solution:

Given $\bar{B} = 0.05\bar{a}_y$ and $\chi_m = 2.5$

Relative permiability $\mu_r = 1 + \chi_m = 1 + 2.5 = 3.5$

Permiability of material $\mu = \mu_0\mu_r = 4\pi \times 10^{-7} \times 3.5 = 4.398 \times 10^{-6}$ H/m

Magnetic field intensity $\bar{H} = \dfrac{\bar{B}}{\mu} = \dfrac{0.05\bar{a}_y}{4.398 \times 10^{-6}} = 11368.8\bar{a}_y$ A/m

Magnetic field current density $\bar{J} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 11368.8 & 0 \end{vmatrix} = 0$ A/m²

Magnetization $\bar{M} = \chi_m \bar{H} = 2.5 \times 11368.8\bar{a}_y = 28422\bar{a}_y$ A/m

Problem:2.27

In a certain material, $\chi_m = 4.2$ and $\bar{H} = 0.2x\bar{a}_y$ A/m. Determine: (a) μ_r, (b) μ, (c) $\bar{M}$,(d) $\bar{B}$,(e) $\bar{J}$, (f) $\bar{J}_b$.

Solution:

(a) $\mu_r = 1 + \chi_m = 1 + 4.2 = 5.2$

(b) $\mu = \mu_0 \mu_r = 4\pi \times 10^{-7} \times 5.2 = 6.534 \times 10^{-6}$ H/m

(c) $\bar{M} = \chi_m \bar{H} = 4.2(0.2x\bar{a}_y) = 0.84x\bar{a}_y$ A/m

(d) $\bar{B} = \mu\bar{H} = 6.534 \times 10^{-6} \times 0.2x\bar{a}_y = 1.307x\bar{a}_y$ μWb/m^2

(e) $$\bar{J} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0.2x & 0 \end{vmatrix} = 0.2\bar{a}_z \text{ A/m}^2$$

(f) $\bar{J}_b = \chi_m \bar{J} = 4.2 \times 0.2\bar{a}_z = 0.84\bar{a}_z$ A/m^2

Problem:2.28

Determine the self inductance of a co-axial cable of inner radius ‘a’ and outer radius ‘b’ and length of the co-axial cable is ‘l’.

Solution:

Assume inner conductor carries a current I in Z-direction and outer conductor carries a current I in opposite direction as shown in Fig:2.26.

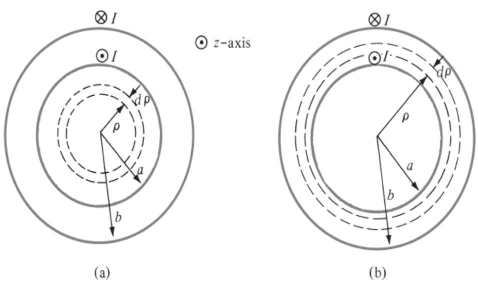


Fig: 2.26 (a) For finding L_{in} (b) For finding L_{out}

Inner Conductor:

We know for inner conductor $\bar{H} = \dfrac{I\rho}{2\pi a^2}\bar{a}_\phi$

Since it is a cylinder, we can use the cylindrical coordinate system to solve the problem

We have

$$W_m = \frac{1}{2}\int_v \mu H^2 dv$$

$$W_m = \frac{1}{2}\int_v \mu \frac{I^2 \rho^2}{4\pi^2 a^4} \rho\, d\rho\, d\phi\, dz$$

$$W_m = \frac{1}{2}\int_{z=0}^{l}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} \mu \frac{I^2 \rho^2}{4\pi^2 a^4} \rho d\rho\, d\phi\, dz$$

$$L_{in} = \frac{2W_m}{I^2} = \frac{\mu}{4\pi^2}\int_{z=0}^{l}\int_{\varphi=0}^{2\pi} \frac{1}{a^4}\frac{a^4}{4} d\phi\, dz$$

$$L_{in} = \frac{\mu}{4\pi^2} 2\pi\, l = \frac{\mu\, l}{8\pi}$$

Outer conductor:

We know for outer conductor $\bar{H} = \dfrac{I}{2\pi\rho}\bar{a}_\phi$

$$L_{out} = \frac{2W_m}{I^2} = \frac{2}{I^2}\frac{1}{2}\int_{z=0}^{l}\int_{\rho=a}^{b}\int_{\phi=0}^{2\pi} \mu\left(\frac{I}{2\pi\rho}\right)^2 \rho\, d\rho\, d\phi\, dz$$

$$L_{out} = \frac{\mu}{4\pi^2}\times 2\pi \times l \ln\frac{b}{a} = \frac{\mu l}{2\pi}\ln\frac{b}{a}$$

The self inductance of co-axial cable is

$$L = L_{in} + L_{out} = \frac{\mu\, l}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right)$$

UNIT 3 MAXWELL'S EQUATIONS FOR TIME VARYING FIELDS

Problem:3.1

A conducting bar can slide freely over two conducting rails as shown in Fig:3.4. Calculate the induced voltage in bar, if the bar is stationed at y=8cm and $\overline{B} = 4\cos\left(10^6 t\right)\overline{a}_z$ mWb/m^2.

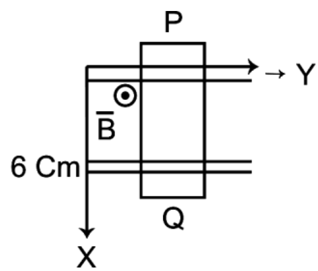


Fig: 3.4

Solution:

$$\frac{\partial \overline{B}}{\partial t} = -4\sin\left(10^6 t\right)10^6\,\overline{a}_z\, mwb/m^2 \quad \text{here} \quad d\overline{s} = dx\,dy\,\overline{a}_z$$

$$V_{emf} = -\int \frac{\partial \overline{B}}{\partial t}\cdot d\overline{s}$$

$$= -\int_{x=0}^{0.06}\int_{y=0}^{0.08} -4\times10^6 \sin\left(10^6 t\right) dx\,dy$$

= 19.2 Sin 10^6 t V

***Problem:3.2**

In figure let $\overline{B} = 0.2\cos 120\pi t \quad T$, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by I(t) is negligible. Find (a) $V_{ab}(t)$ (b) I(t).

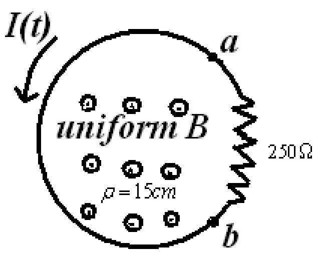


Fig: 3.5

Solution:

(a) We have $V_{emf} = -\frac{d\psi}{dt}$

Here $V_{ab}(t) = V_{emf} = -\frac{d\psi}{dt}$

$\psi = Ba$, where 'a' is the area of cross-section of the loop= $\pi\rho^2$

$\therefore \psi = 0.2\cos 120\pi t \times \pi \times (15\times10^{-2})^2 = 0.0141\cos 120\pi t$

Hence $V_{ab}(t) = -\frac{d\psi}{dt} = -\frac{d(0.0141\cos 120\pi t)}{dt} = 5.326\sin 120\pi t V$

(b) from Fig:3.5, $I(t) = \frac{V_{ba}(t)}{R} = \frac{-V_{ab}(t)}{R} = \frac{-5.326\sin 120\pi t}{250} = -0.0213\sin 120\pi t A$

***Problem:3.3**

A circular loop conductor of radius 0.1m lies in the Z=0 plane and has a resistance of 5Ω given $\overline{B} = 0.20\sin 10^3 t\overline{a}_z \quad T$. Determine the current.

Solution:

We have $\psi = \int_s \overline{B}.\overline{ds}$, here $\overline{ds} = \rho d\phi d\rho \overline{a}_z$

$$\psi = \int_s 0.20\sin 10^3 t\overline{a}_z.\rho d\phi d\rho \overline{a}_z = 0.20\sin 10^3 t \int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{0.1} \rho d\rho = \frac{2\pi\sin 10^3 t}{10^3}$$

We also have $V_{emf} = -\frac{d\psi}{dt}$

$$V_{emf} = -\frac{d\left(\frac{2\pi\sin 10^3 t}{10^3}\right)}{dt} = -2\pi\cos 10^3 t \quad V$$

The current in the loop $I = \frac{V_{emf}}{R} = \frac{2\pi\cos 10^3 t}{5} = 1.26\cos 10^3 t \quad A$

***Problem:3.4**

The electric field intensity in the region $0 < x < 5, 0 < y < \pi/12, 0 < z < 0.06m$ in free space is given by $\overline{E} = c\sin 12y \sin az \cos 2\times10t\overline{a}_x V/m$. Beginning with the $\nabla\times\overline{E}$ relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than '0'.

Solution:

Given $\overline{E} = c\sin 12y \sin az \cos 2\times10t\overline{a}_x V/m$ for a>0

Consider $\nabla\times\bar{E}=\begin{vmatrix}\bar{a}_x & \bar{a}_y & \bar{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=\begin{vmatrix}\bar{a}_x & \bar{a}_y & \bar{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ c\sin 12y\sin az\cos 2\times 10t & 0 & 0\end{vmatrix}$

$$\nabla\times\bar{E}=ac\sin 12y\cos az\cos 2\times 10t\bar{a}_y-12c\cos 12y\sin az\cos 2\times 10t\bar{a}_z$$

We have $\nabla\times\bar{E}=\frac{-\partial\bar{B}}{\partial t}$

$$\frac{-\partial\bar{B}}{\partial t}=ac\sin 12y\cos az\cos 2\times 10t\bar{a}_y-12c\cos 12y\sin az\cos 2\times 10t\bar{a}_z$$

Integrating

$$\bar{B}=\left(\frac{-ac\sin 12y\cos az\sin 2\times 10t\bar{a}_y}{20}+\frac{12c\cos 12y\sin az\sin 2\times 10t\bar{a}_z}{20}\right)$$

But $\bar{B}=\mu_0\bar{H}$

$$\bar{H}=\left(\frac{-ac\sin 12y\cos az\sin 2\times 10t\bar{a}_y}{20\mu_0}+\frac{12c\cos 12y\sin az\sin 2\times 10t\bar{a}_z}{20\mu_0}\right)$$

$$\nabla\times\bar{H}=\begin{vmatrix}\bar{a}_x & \bar{a}_y & \bar{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ H_x & H_y & H_z\end{vmatrix}=\begin{vmatrix}\bar{a}_x & \bar{a}_y & \bar{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & \frac{-ac\sin 12y\cos az\sin 2\times 10t}{20\mu_0} & \frac{12c\cos 12y\sin az\sin 2\times 10t}{20\mu_0}\end{vmatrix}$$

$$\nabla\times\bar{H}=-\left(\frac{12^2+a^2}{20\mu_0}\right)c\sin 12y\sin az\sin 2\times 10t\bar{a}_x$$

But for free space($\rho_v=0$), we have $\nabla\times\bar{H}=\frac{\partial\bar{D}}{\partial t}=\in_0\frac{\partial\bar{E}}{\partial t}$

$$-\left(\frac{12^2+a^2}{20\mu_0}\right)c\sin 12y\sin az\sin 2\times 10t\bar{a}_x=\in_0\frac{\partial(c\sin 12y\sin az\cos 2\times 10t)\bar{a}_x}{\partial t}$$

$$-\left(\frac{12^2+a^2}{20\mu_0}\right)c\sin 12y\sin az\sin 2\times 10t\bar{a}_x = -20\in_0 c\sin 12y\sin az\sin 2\times 10t\bar{a}_x$$

$$\left(\frac{12^2+a^2}{20\mu_0}\right) = 20\in_0 \Rightarrow 12^2+a^2 = 20^2\in_0\mu_0$$

$$a^2 = 20^2\times 8.854\times 10^{-12}\times 4\pi\times 10^{-7} - 12^2 \Rightarrow a = j12$$

***Problem:3.5**

A certain material has $\sigma = 0 and \in_R = 1$ if $\bar{H} = 4\sin(10^6 t - 0.01z)\bar{a}_y A/m$ make use of Maxwell's equations to find μ_r .

Solution:

$$\nabla\times\bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 4\sin(10^6 t - 0.01z) & 0 \end{vmatrix}$$

$$\nabla\times\bar{H} = -\frac{\partial[4\sin(10^6 t - 0.01z)\bar{a}_x]}{\partial z} = 0.04\cos(10^6 t - 0.01z)\bar{a}_x$$

Since $\sigma = 0$, we have $\nabla\times\bar{H} = \frac{\partial\bar{D}}{\partial t} = \in_0\in_r\frac{\partial\bar{E}}{\partial t} = \in_0\frac{\partial\bar{E}}{\partial t}$

$$\bar{E} = \frac{1}{\in_0}\int 0.04\cos(10^6 t - 0.01z)\bar{a}_x dt$$

$$\bar{E} = \frac{0.04\sin(10^6 t - 0.01z)\bar{a}_x}{10^6\in_0}$$

$$\nabla\times\bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{0.04\sin(10^6 t - 0.01z)}{10^6\in_0} & 0 & 0 \end{vmatrix}$$

$$\nabla\times\bar{E} = -\frac{4\times 10^{-4}\cos(10^6 t - 0.01z)}{10^6\times 8.854\times 10^{-12}}\bar{a}_y$$

$$\nabla\times\bar{E}=-45.2\cos(10^6 t-0.01z)\bar{a}_y$$

We have $\nabla\times\bar{E}=\dfrac{-\partial\bar{B}}{\partial t}=-\mu_0\mu_r\dfrac{\partial\bar{H}}{\partial t}$

$$-45.2\cos(10^6 t-0.01z)\bar{a}_y=-\mu_0\mu_r\frac{\partial\bar{H}}{\partial t}$$

$$45.2\cos(10^6 t-0.01z)\bar{a}_y=4\times10^6\,\mu_0\mu_r\cos(10^6 t-0.01z)\bar{a}_y$$

$$\mu_r=\frac{45.2}{4\times10^6\,\mu_0}=\frac{45.2}{4\times10^6\times4\pi\times\ 10^{-7}}=8.99$$

Problems:3.6

Show that the displacement current through the capacitor is equal to the conduction current.

Solution:

We know the conduction current $I=C\dfrac{dv}{dt}$. Assume that the capacitor plates are having area 'A' and are separated by a distance 'd' as shown in Fig:3.6,

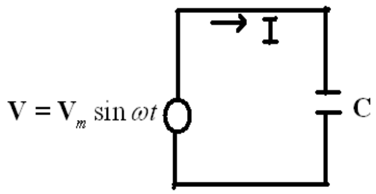


Fig: 3.6

Then $C=\dfrac{\in A}{d}$

$$I=\frac{\in A}{d}\frac{d}{dt}\left(V_m\sin\omega t\right)$$

$$=\frac{A\in}{d}V_m\omega\cos\omega t$$

We have

$$J_D=\frac{\partial D}{\partial t}$$

But $J_D=\dfrac{I_D}{A}$

$$\frac{I_D}{A}=\frac{\partial D}{\partial t}\qquad\text{where } D=\in E$$

$$\frac{I_D}{A}=\in\frac{\partial E}{\partial t}\qquad\text{where } E=\frac{V}{d}$$

$$\frac{I_D}{A}=\frac{\in}{d}\frac{\partial V}{\partial t}=\frac{\in}{d}\frac{\partial}{\partial t}\left(V_m\sin\omega t\right)=\frac{\in}{d}V_m\omega\cos\omega t$$

$$I_D=\frac{A\in}{d}V_m\omega\cos\omega t$$

$\therefore$ conduction current = displacement current

Problem:3.7

A parallel plate capacitor with plate area of $5cm^2$ and plate separation of 3mm has a voltage 50 sin (10^3t)V applied to it's plates. Calculate the displacement current assuming $\in = 2\in_0$.

Solution:

$$I_D=\frac{A\in}{d}V_m\omega\cos\omega t$$

$$=\frac{5\times10^{-4}\times2\in_0}{3\times10^{-3}}50\times10^3\cos10^3t$$

$= 10^3 x16.65 \in_0 \cos 10^3t$

$= 16.65\in_0 \cos 10^3t$ KA

$= 147.4 \cos 10^3t$ nA

Problem:3.8

A 'Cu' wire carries a conduction current of 1A at 60Hz. What is the displacement current in the wire. For 'Cu' wire $\in = \in_0$, $\sigma=5.8x10^7$ and $\mu = \mu_0$.

Solution:

$$I_D=I_C\left(\frac{\omega\in}{\sigma}\right)=1\left(\frac{2\pi\times60\times8.854\times10^{-12}}{5.8\times10^7}\right)$$

$= 575.2x10^{-19}$A

Problem:3.9

In free space $\overline{E}=20\cos(\omega t-50x)\overline{a}_y$ V/m. Calculate $\overline{J}_D$.

Solution:

$$\overline{D}=\in_0\overline{E}=20\in_0\cos\left(\omega t-50x\right)\overline{a}_y$$

$$\overline{J}_D=\frac{\partial\overline{D}}{\partial t}=-20\omega\in_0\sin\left(\omega t-50x\right)\overline{a}_y \quad A/m^2$$

Problem:3.10

A two dimensional Electric field is given by $\overline{E}=x^2\overline{a}_x+x\overline{a}_y$ v/m. Show that this electric field can not arise from a static distribution of charge.

Solution:

For static Electric field, we have $\nabla \times \bar{E} = 0$

$$\nabla \times \bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & x & 0 \end{vmatrix} = \bar{a}_x(0-0) - \bar{a}_y(0-0) + \bar{a}_z(1) = \bar{a}_z \neq 0$$

$\therefore$ The given $\bar{E}$ is not due to static distribution of charge

Problem:3.11

A conductor carries a steady current of 'I' amp. The components of current density vector $\bar{J}$ are $J_x = 2ax$ and $J_y = 2ay$. Find the third component J_z.

Solution:

Given steady current, that indicates static fields. For static fields, we have

$$\nabla \cdot \bar{J} = \frac{\partial \rho_V}{\partial t} = 0$$

$$\Rightarrow \frac{\partial}{\partial x} 2ax + \frac{\partial}{\partial y} 2ay + \frac{\partial}{\partial z} J_z = 0$$

$$\Rightarrow 2a + 2a + \frac{\partial}{\partial z} J_z = 0$$

$$\frac{\partial}{\partial z} J_z = -4a$$

Integrating, we get third component as

$J_z = -4az + c$, where c is constant of integration.

Problem:3.12

Do the fields $\bar{E} = E_m \sin x \sin t \, \bar{a}_y$

$\bar{H} = \frac{E_m}{\mu_0} \cos x \cos t \, \bar{a}_z$ satisfy Maxwell's equations.

Solution:

We have

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t} \bar{B}$$

$$\nabla \times \bar{E} = -\mu_0 \frac{\partial}{\partial t} \bar{H}$$

Assume the electromagnetic wave is traveling along X-direction, then the components of $\overline{E}\ and\ \overline{H}$ along X-direction will be zero. Variations along y and z directions are zero i.e., $\frac{\partial}{\partial y}\ and\ \frac{\partial}{\partial z}$ are zero.

$$\nabla\times\overline{E}=\begin{vmatrix}\overline{a}_x & \overline{a}_y & \overline{a}_z\\ \frac{\partial}{\partial x} & 0 & 0\\ 0 & E_m\sin x\sin t & 0\end{vmatrix}=\overline{a}_z E_m\cos x\sin t$$

$$-\mu_0\frac{\partial}{\partial t}\overline{H}=-E_m\cos x\sin t\,\overline{a}_z$$

$$\therefore\nabla\times\overline{E}=-\frac{\partial}{\partial t}\overline{B}$$

$\therefore \overline{E}\ and\ \overline{H}$ satisfy the Maxwell's equations.

Problem:3.13

Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma=2\times10^{-4}$ □ $/m$ and $\in_r = 81$.

Solution:

We know

$$\frac{J_D}{J_C}=\frac{\omega\in}{\sigma}$$

$$\Rightarrow 1=\frac{2\pi f\times8.854\times10^{-12}\times81}{2\times10^{-4}}$$

f= 44.384KHz

Problem:3.14

Calculate the ratio J_D/J_C for 'Al' at frequencies of 50Hz and 50MHz ,given $\sigma=10^5$ □ $/m$ and $\in_r = 1$.

Solution:

For f = 50Hz

$$\frac{J_D}{J_C}=\frac{2\pi 50\times8.854\times10^{-12}\times1}{10^5}$$

$= 2.782 \times 10^{-14}$.

And for $f = 50$MHz

$\frac{J_D}{J_C} = 2.782 \times 10^{-8}$.

Problem:3.15

Two extensive homogeneous isotropic dielectrics meet on plane z=0. For $z \geq 0$, $\in_{r_1} = 4$ and for $z \leq 0$, $\in_{r_2} = 3$. A uniform electric field $\overline{E}_1 = 5\overline{a}_x - 2\overline{a}_y + 3\overline{a}_z$ kv/m exists for $z \geq 0$. Find (a) $\overline{E}_2$ for $z \leq 0$ (b) the angles $\overline{E}_1$ and $\overline{E}_2$ make with the interface (c) the energy densities in J/m^3 in both dielectrics (d) the energy with in a cube of side 2m centered at (3, 4, -5).

Solution:

Consider the Fig:3.13.

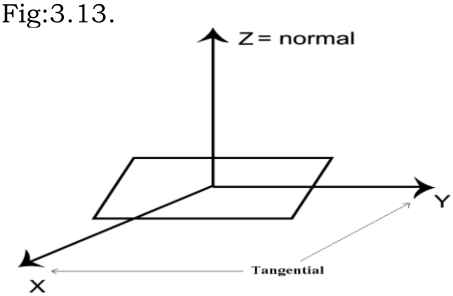


Fig:3.13

Given $\overline{E}_1 = 5\overline{a}_x - 2\overline{a}_y + 3\overline{a}_z$ KV/m

We have

$\overline{E}_1 = \overline{E}_{1t} + \overline{E}_{1n}$, comparing the above two equations

The normal component is 3 $\overline{a}_z$

$\therefore \overline{E}_{1n} = 3\overline{a}_z$

$\overline{E}_{1t} = \overline{E}_1 - \overline{E}_{1n} = 5\overline{a}_x - 2\overline{a}_y$

We know $\overline{E}_{1t} = \overline{E}_{2t}$

$\therefore \overline{E}_{2t} = 5\overline{a}_x - 2\overline{a}_y$

Also known $\in_1 \overline{E}_{1n} = \in_2 \overline{E}_{2n}$

$\Rightarrow \overline{E}_{2n} = 4\overline{a}_z$

$\therefore \overline{E}_2 = \overline{E}_{2t} + \overline{E}_{2n} = 5\overline{a}_x - 2\overline{a}_y + 4\overline{a}_z$

(b) Consider the Fig:3.14.

Here $\alpha_1 = 90-\theta_1$ and $\alpha_2 = 90-\theta_2$

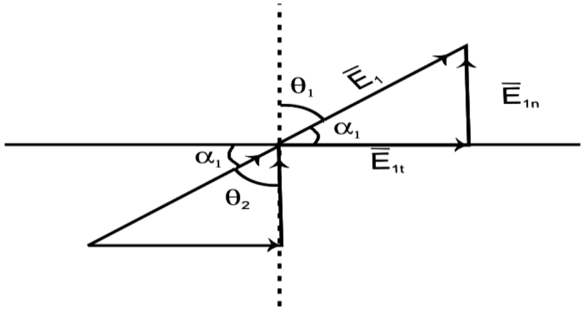


Fig:3.14

$$\bar{E}_{1t} = \bar{E}_1 \sin\theta_1$$

$$\sin\theta_1 = \frac{\bar{E}_{1t}}{\bar{E}_1} = \frac{|5\bar{a}_x - 2\bar{a}_y|}{|5\bar{a}_x - 2\bar{a}_y + 3\bar{a}_z|} = \frac{\sqrt{29}}{\sqrt{38}} \Rightarrow \theta_1 = 60.88^0$$

$\alpha_1 = 90-\theta_1 = 29.12^0$

And $$\sin\theta_2 = \frac{|\bar{E}_{2t}|}{|\bar{E}_2|} = \frac{\sqrt{29}}{\sqrt{45}} \Rightarrow \theta_2 = 53.4^0$$

$\therefore \alpha_2 = 36.6^0$

(c) Energy density in first medium is

$$W_1 = \frac{1}{2}\in_1 |E_1|^2$$

$$= \frac{1}{2}\in_0\in_{r_1}(38)$$

$= 672.9\ \mu J/m^3$

And Energy density in second medium is

$$W_2 = \frac{1}{2}\in_0\in_{r_2}(45)$$

$= 597.6\ \mu J/m^3$

(d) Given cube of side 2m centered at (3, 4, -5)

i.e., $z = -5 < 0$,We have to consider the second medium

Limits are $2 \leq x \leq 4$

$3 \leq y \leq 5$

$-6 \leq z \leq -4$

The energy in a given cube is

$$w_2 = \int W_2 dv \quad \text{where} \quad dv = dx\ dy\ dz$$

$$= \int_V 597.6\ \mu J / m^3\ dx\, dy\, dz$$

= 4.776 mJ

***Problem:3.16**

X-Z plane is a boundary between two dielectrics. Region y<0contains dielectric material $\in_{r_1} = 2.5$ while region y>0 has dielectric with $\in_{r_2} = 4.0$. If $\bar{E} = -30\bar{a}_x + 50\bar{a}_y + 70\bar{a}_z\ V/m$, find normal and tangential components of the E field on both sides of the boundary.

Solution:

Here y<0 is medium-1 and y>0 is medium-2
Assume given $\bar{E}$ belongs to medium-1

$$\bar{E} = -30\bar{a}_x + 50\bar{a}_y + 70\bar{a}_z\ V/m$$

x-z components are tangential and y component is normal
$\bar{E}_{1t} = -30\bar{a}_x + 70\bar{a}_z$ and $\bar{E}_{1n} = 50\bar{a}_y$
The boundary condition on tangential component of $\bar{E}$ is $\bar{E}_{1t} = \bar{E}_{2t}$

$$\therefore \bar{E}_{2t} = -30\bar{a}_x + 70\bar{a}_z \text{ V/m}$$

The boundary condition on normal component of $\bar{E}$ is $\bar{E}_{2n} = \frac{\in_{r_1}}{\in_{r_2}} \bar{E}_{1n}$

$$\therefore \bar{E}_{2n} = \frac{2.5}{4} 50\bar{a}_y = 31.25\bar{a}_y$$

***Problem:3.17**

Region 1 ,for which $\mu_{r_1} = 3$ is defined by x<0 and region 2, x>0 has $\mu_{r_2} = 5$ given $\bar{H}_1 = 4\bar{a}_x + 3\bar{a}_y + 6\bar{a}_z A/m$. Determine $\bar{H}_2$ for x>0 and the angles that $\bar{H}_1$ and $\bar{H}_2$ make with the interface.

Solution:

Consider the Fig:3.17.

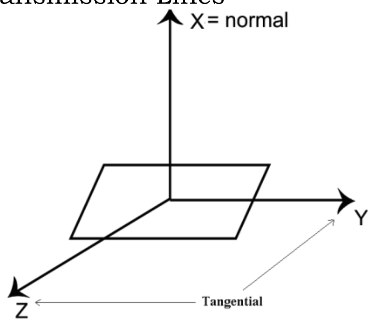


Fig:3.17

Given $\bar{H}_1 = 4\bar{a}_x + 3\bar{a}_y + 6\bar{a}_z A/m$

We have

$\bar{H}_1 = \bar{H}_{1t} + \bar{H}_{1n}$, comparing the above two equations

The normal component is 4 $\bar{a}_x$

$\therefore \bar{H}_{1n} = 4\bar{a}_x$

$\bar{H}_{1t} = \bar{H}_1 - \bar{H}_{1n} = 3\bar{a}_y + 6\bar{a}_z$

We know $\bar{H}_{1t} = \bar{H}_{2t}$

$\therefore \bar{H}_{2t} = 3\bar{a}_y + 6\bar{a}_z$

Also known $\mu_1 \bar{H}_{1n} = \mu_2 \bar{H}_{2n}$

$$\Rightarrow \bar{H}_{2n} = \frac{\mu_1}{\mu_2} 4\bar{a}_x = \frac{\mu_{r_1}\mu_0}{\mu_{r_2}\mu_0} 4\bar{a}_x = \frac{3}{5} 4\bar{a}_x = 2.4\bar{a}_x$$

$\therefore \bar{H}_2 = \bar{H}_{2t} + \bar{H}_{2n} = 2.4\bar{a}_x + 3\bar{a}_y + 6\bar{a}_z$

Consider the Fig:3.18.

Here $\alpha_1 = 90 - \theta_1$ and $\alpha_2 = 90 - \theta_2$

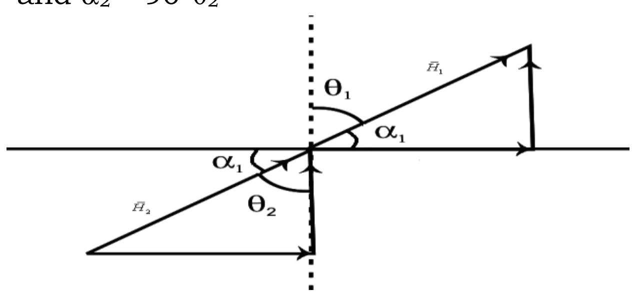


Fig:3.18

From Fig:3.18, $\bar{H}_{1t} = \bar{H}_1 \sin\theta_1$

$$\sin\theta_1 = \frac{\bar{H}_{1t}}{\bar{H}_1} = \frac{\left|3\bar{a}_y + 6\bar{a}_z\right|}{\left|4\bar{a}_x + 3\bar{a}_y + 6\bar{a}_z\right|} = \frac{\sqrt{45}}{\sqrt{61}} \Rightarrow \theta_1 = 59.19^0$$

$\alpha_1 = 90 - \theta_1 = 30.8^0$

And $$\sin\theta_2 = \frac{\left|\bar{H}_{2t}\right|}{\left|\bar{H}_2\right|} = \frac{\sqrt{45}}{\sqrt{50.76}} \Rightarrow \theta_2 = 70.313^0$$

$\therefore \alpha_2 = 19.686^0$

UNIT 4 EM WAVE CHARACTERISTICS

Problem:4.1

An uniform plane wave with an intensity of electric field = 1v/m is traveling in free space. Find the magnitude of associated magnetic field.

Solution:

Electric field intensity = 1v/m

The magnitude of the magnetic field is found by $\frac{|\overline{E}|}{|\overline{H}|} = \sqrt{\frac{\mu_0}{\epsilon_0}}$

$|\overline{H}|$ = 2.6 mA/m

Problem:4.2

Show that electric and magnetic energy densities in a traveling plane wave are equal.

Solution:

$$\frac{E}{H} = \sqrt{\frac{\mu}{\in}}$$

$$\Rightarrow \frac{E^2}{H^2} = \frac{\mu}{\in}$$

$$\Rightarrow \in E^2 = \mu H^2$$

$$\Rightarrow \frac{1}{2} \in E^2 = \frac{1}{2} \mu H^2$$

Energy density for electric field = energy density for magnetic field

Problem:4.3

For a uniform plane wave traveling in X-direction in free space E_y=10 sin (2π 10^8 t – βx). Find the phase constant, phase velocity and equation for H_z if $E_z = H_{y=0.}$

Solution:

Given E_y=10 sin (2π 10^8 t – βx)

$\omega = 2\pi\ 10^8$; $\beta^2=\omega^2\mu_0\epsilon_0=(2\pi 10^8)^2 \times 8.854\times10^{-12} \times 4\pi\times10^{-7}$

The Phase constant β=2.096.

The Phase Velocity $v = \frac{\omega}{\beta} = \frac{2\pi \times 10^8}{2.096} = 2.99 \times 10^8$ m/s

We have $\frac{E_y}{H_z} = 377$

$$\Rightarrow H_z = \frac{10}{377} \sin(2\pi 10^8 t - \beta x)$$

Problem:4.4

A plane wave traveling in +Ve X-direction in a loss less unbounded medium having permeability 4.5 times that of free space and a permittivity twice that of free space. (a) Find the phase velocity of the wave (b) If the electric field $\bar{E}$ has only a 'y' component with an amplitude of 20V/m. Find the amplitude and direction of magnetic field intensity.

Solution:

(a) Phase velocity $v = \frac{1}{\sqrt{\mu \in}} = \frac{1}{\sqrt{4.5\mu_0 \, 2 \in_0}}$

$= 1 \times 10^8$ m/s

(b) $H_z = \sqrt{\frac{\in}{\mu}} \, E_y = 0.0354 \; A/m$, it's direction is along 'z'.

***Problem:4.5**

A wave propagating in a lossless dielectric has the components $\bar{E} = 500\cos\left(10^7 t - \beta z\right)\bar{a}_x \, V/m$ and $\bar{H} = 1.1\cos\left(10^7 t - \beta z\right)\bar{a}_y \, A/m$.

If the wave is traveling at v=0.5c. Where 'c' is the velocity in free space. Find (a) μ_r (b) $\in_r$ (c) β (d) λ (e) Z .

Solution:

The wave is propagating along Z-direction, $\bar{E}$ is along x and $\bar{H}$ is along 'y' directions

(e) Z = η = intrinsic impedance = $\frac{|E|}{|H|} = \frac{500}{1.1} = 454.5\Omega$

(a) v = 0.5c

$$V = 0.5\frac{1}{\sqrt{\mu_0 \in_0}}$$

$$\frac{1}{\sqrt{\mu_0\mu_r \in_0 \in_r}} = 0.5\frac{1}{\sqrt{\mu_0 \in_0}}$$

$$\frac{1}{\sqrt{\mu_r \in_r}} = 0.5$$

$$\sqrt{\mu_r \in_r} = \frac{1}{0.5} \Rightarrow \mu_r \in_r = 4$$

$$z = 454.5 = \sqrt{\frac{\mu}{\in}} = \sqrt{\frac{\mu_r\, \mu_0}{\in_0 \in_r}} = (3.767)\times 10^{-10}\sqrt{\frac{\mu_r}{\in_r}}$$

$$\Rightarrow \frac{\mu_r}{\in_r} = \left(\frac{454.5}{3.767\times 10^{-10}}\right)^2 = 1.456\times 10^{24}$$

$$\Rightarrow \mu_r = \sqrt{4\times 1.456\times 10^{24}} = 2.41\times 10^{12}$$

(b) $\in_r = \sqrt{4/1.456\times 10^{24}} = 1.66\times 10^{-12}$

(c) given $\omega = 10^7$, as $\frac{\sigma}{\omega \in} \ll 1$ for low loss dielectric,

$$\beta = \omega\sqrt{\mu \in} = 10^7\sqrt{8.854\times 10^{-12}\times 1.66\times 10^{-12}\times 4\times \pi\times 10^{-7}\times 2.41\times 10^{12}} \quad = 0.0667$$

(d) Wavelength $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.0667} = 94.2m$

***Problem:4.6**

Dry ground has a conductivity of 5x10^{-4} mhos/m and relative dielectric constant of 10 at a frequency of 500 MHz. Compute (i) the intrinsic impedance (ii) the propagation constant (iii) the phase velocity.

Solution:

Given $\sigma = 5\times 10^{-4} mhos/m$, $\varepsilon_r = 10$ and $f = 500MHz$

Loss tangent $\frac{\sigma}{\omega\varepsilon} = \frac{5\times 10^{-4}}{2\pi\times 50\times 10^6\times 10\times \frac{10^{-9}}{36\pi}} = 0.0018$

i.e. $\frac{\sigma}{\omega\varepsilon} \to 0$, it is perfect dielectric medium

(i) intrinsic impedance $\eta = \sqrt{\dfrac{\mu}{\in}}$

$$\eta = \sqrt{\frac{\mu_0 \mu_r}{\in_0 \in_r}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{10^{-9}}{36\pi} \times 10}} = 119.22\Omega$$

(ii) the propagation constant $\gamma = j\omega\sqrt{\mu \in}$

$$\gamma = j2\pi \times 500 \times 10^6 \sqrt{4\pi \times 10^{-7} \times \frac{10^{-9}}{36\pi} \times 10} = j33.11$$

(iii) the phase velocity $v_p = \dfrac{1}{\sqrt{\mu \in}}$

$$v_p = \frac{1}{\sqrt{4\pi \times 10^{-7} \times \frac{10^{-9}}{36\pi} \times 10}} = 94.868 \times 10^6 \, m/s$$

***Problem:4.7**

Find α, β, γ *and* η for ferrite at 10GHz. $\varepsilon_r = 9, \mu_r = 4, \sigma = 10 mhos/m$

Solution:

Loss tangent $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{10}{2\pi \times 10 \times 10^9 \times 9 \times \frac{10^{-9}}{36\pi}} = 2$

i.e. $\dfrac{\sigma}{\omega\varepsilon} > 1$, it is a good conductor

$$\alpha = \sqrt{\frac{\omega\sigma\mu}{2}}$$

$$\alpha = \sqrt{\frac{2\pi \times 10 \times 10^9 \times 10 \times 4\pi \times 10^{-7} \times 4}{2}} = 12.566 Np/m$$

$$\beta = \alpha = \sqrt{\frac{\omega\sigma\mu}{2}} 12.566 \; rad/m$$

$$\gamma = \alpha + j\beta$$

$$\gamma = 12.566 + j12.566 \, m^{-1}$$

$$\eta = \sqrt{\frac{\mu\omega}{\sigma}}\angle 45^0 = \sqrt{\frac{2\pi\times 10\times 10^9\times 4\pi\times 10^{-7}\times 4}{10}}\angle 45^0 = 177.71\angle 45^0\Omega$$

***Problem:4.8**

A non magnetic medium has an intrinsic impedance of $240\angle 30^0\Omega$. Find its (i) Loss tangent. (ii) Dielectric constant. (iii) Complex permittivity. (iv) Attenuation constant at 1MHz.

Solution:

Given $\eta = 240\angle 30^0\Omega$

i.e. $|\eta| = 240 \, and \quad \theta_\eta = \angle 30^0$

(i) we have

$$\text{loss tangent} = \frac{\sigma}{\omega\in} = \tan 2\theta_\eta = \tan(2\times 30^0) = 1.732$$

(ii) we have

$$|\eta| = \frac{\sqrt{\frac{\mu}{\in}}}{\left(1+\left(\frac{\sigma}{\omega\in}\right)^2\right)^{1/4}} = 240$$

$$\frac{\sqrt{\frac{\mu_0\mu_r}{\in_0\in_r}}}{\left(1+(1.732)^2\right)^{1/4}} = 240$$

$$\frac{\sqrt{\frac{1\times 4\pi\times 10^{-7}}{\in_r\times\frac{10^{-9}}{36\pi}}}}{\left(1+(1.732)^2\right)^{1/4}} = 240$$

$$\frac{84.853\pi}{\sqrt{\in_r}} = 240$$

Dielectric constant $\in_r = 1.2337$

(iii) we have Complex permittivity

$$\in_c = \in\left[1 - \frac{j\sigma}{\omega\in}\right]$$

$$\in_c = \frac{10^{-9}}{36\pi}\times 1.2337[1 - j1.732] = (1.09 - j1.8893)\times 10^{-11} F/m$$

***Problem:4.9**

A lossy dielectric has an intrinsic impedance of $200\angle 30^0\Omega$ at a particular radian frequency ω. If at that frequency the plane wave propagating through the dielectric has the magnetic field component $\bar{H} = 10e^{-\alpha x}\cos(\omega t - 0.5x)\bar{a}_y A/m$. Find $\bar{E}$ *and* α. Determine the skin depth and wave polarization.

Solution:

From the given $\bar{H}$**,** we can say that wave travels along x-axis

We have $\bar{a}_d = \bar{a}_E \times \bar{a}_H$

here $\bar{a}_d = \bar{a}_x and\ \bar{a}_H = \bar{a}_y$

$\therefore \bar{a}_E = -\bar{a}_z$

Also we have

$$\eta = \frac{|\bar{E}|}{|\bar{H}|}$$

$$\therefore |\bar{E}| = |\bar{H}|\eta = 10\times 200\angle 30^0 = 2000e^{j\pi/6}$$

$\bar{E}$ *and* $\bar{H}$ will have the same form except magnitude and phase

$$\therefore \bar{E} = -2000e^{-\alpha x}\cos(\omega t - 0.5x + \pi/6)\bar{a}_z V/m$$

From the above expression β=0.5, α can be determined as

$$\alpha = \omega\sqrt{\mu\in}\left[\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\in}\right)^2} - 1\right)\right]^{1/2}$$

$$\beta = \omega\sqrt{\mu\in}\left[\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\in}\right)^2} + 1\right)\right]^{1/2}$$

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1+\left(\frac{\sigma}{\omega \in}\right)^2} - 1}{\sqrt{1+\left(\frac{\sigma}{\omega \in}\right)^2} + 1} \right]^{1/2}$$

But

$$\frac{\sigma}{\omega \in} = \tan 2\theta_\eta = \tan 60^0 = \sqrt{3}$$

$$\frac{\alpha}{\beta} = \left[\frac{2-1}{2+1} \right]^{1/2}$$

$$\Rightarrow \alpha = \frac{0.5}{\sqrt{3}} = 0.2887 Np/m$$

skin depth $\delta = \frac{1}{\alpha} = 2\sqrt{3} = 3.464m$

Since Electric field points along z-axis, the polarization of the wave is z-direction.

Problem:4.10

A plane wave of 16GHz frequency and E = 10V/m propagates through the body of salt water having constants $\in_r$=100, μ_r=1 and σ=100 /m. Determine attenuation constant, phase shift constant, phase velocity and intrinsic impedance of medium.

Solution:

$\frac{\sigma}{\omega \in}$ = 1.12 > 1, it is a good conductor.

Attenuation constant $\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$ = 2513.27

Phase shift constant $\beta = \alpha$ = 2513.27 .

Propagation constant γ = $\alpha + j\beta$ = 2513.27 + j 2513.27

= 3554.3 $\angle 45^o$

Phase velocity $V = \frac{\omega}{\beta} = 40 Mm/s$

intrinsic impedance $\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = 35.543\sqrt{j}$

= 35.543$\angle 45^0$

Problem:4.11

Determine the propagation constant at 500KHz for a medium in which $\mu_r = 1$, $\epsilon_r = 15$, $\sigma = 0$, at what velocity will an EM wave travel in this medium.

Solution:

$$\frac{\sigma}{\omega \epsilon} = 0 = \text{Perfect dielectric}$$

$\alpha = 0$

$\beta = \omega\sqrt{\mu \epsilon} = 0.041.$

$\gamma = \alpha + j\beta = 0.041 \angle 90^\circ$

Wave travels with velocity $v = \dfrac{\omega}{\beta}$ = 76.624 M m/s

Problem:4.12

For silver the conductivity is $\sigma = 3\text{x}10^6$ □ /m, at what frequency will the wave travels, if the depth of penetration is 1mm.

Solution: $\delta = \sqrt{\dfrac{2}{\mu\omega\sigma}}$

$$1\times10^{-3} = \sqrt{\frac{2}{2\pi f \times 4\pi\times10^{-7}\times3\times10^{6}}}$$

$$f = \frac{4}{10^{-16}\times4\pi\times10^{-7}\times3\times10^{6}} = 84.4 \text{ K Hz}$$

Problem:4.13

In a medium $\bar{E} = 16e^{-x/20}\sin\left(2\times10^{8}t - 2x\right)\bar{a}_z$ V/m. Find the direction of propagation, propagation constant, wave length, speed of wave and skin depth.

Solution:

Direction of propagation is +Ve X-direction

$\alpha = 1/20, \qquad \beta = 2$

$\omega = 2 \text{ x } 10^8$

$$\gamma = \alpha + j\beta = 2\angle 88.57^0$$

$$v = \frac{1}{\sqrt{\mu \epsilon}} = 3\times10^{8} m/s$$

Skin depth $\delta = \dfrac{1}{\alpha} = 20$

Problem:4.14

An EM wave propagated through a material μ_r = 5, ϵ_r = 10. Determine (a) Velocity of propagation (b) Intrinsic impedance in free space and in material (c) wavelength in free space and in material, when f = 1 G Hz.

Solution:

(a) $$v = \frac{1}{\sqrt{\mu_0 \mu_r \in_0 \in_r}} = 42.4 \text{ M m/s.}$$

(b) Intrinsic impedance

in free space $$\eta = \sqrt{\frac{\mu_0}{\in_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}} = 377\Omega$$

in material $$\eta = \sqrt{\frac{\mu_0 \mu_r}{\in_0 \in_r}} = \sqrt{\frac{4\pi \times 10^{-7} \times 5}{8.854 \times 10^{-12} \times 10}} = 266.39\Omega$$

(c) Wavelength

in free space $$\lambda = \frac{v_0}{f} = \frac{3\times 10^8}{1\times 10^8} = 3m$$

in material $$\lambda = \frac{v}{f} = \frac{42.4\times 10^6}{1\times 10^8} = 0.424m$$

Problem:4.15

A 'Cu' wire carries a conduction current of 1A. Determine the displacement current in wire at 100M Hz. For 'Cu' σ = 5.8x10⁷ /m and permittivity is same as that of free space.

Solution:

$$\frac{I_D}{I_c} = \frac{\omega \in}{\sigma}$$

$$\Rightarrow I_D = 1A \times \frac{\omega \in}{\sigma} = \frac{1\times 2\times \pi \times 100\times 10^6 \times 8.854\times 10^{-12}}{5.8\times 10^7}$$

$$= 9.59\times 10^{-11} A$$

***Problem:4.16**

A traveling wave has two linearly polarized components $E_x = 2\cos \omega t$ and E_y = 3 cos $\left(\omega t + \frac{\pi}{2}\right)$

(a) What is the axial ratio

(b) What is the tilt angle of the major axis of the polarization ellipse.

(c) What is the sense of rotation.

Solution:

(a) Ratio of major axis to minor axis = 3/2 = 1.5

(b) Tilt angle is the phase difference between E_x and E_y

$$\therefore \theta = 90^0$$

(c) $$\frac{E_x}{2} = \cos\omega t, \frac{E_y}{3} = \cos\left(\omega t + \frac{\pi}{2}\right) = -\sin\omega t$$

$\left(\frac{E_x}{2}\right)^2 + \left(\frac{E_y}{3}\right)^2 = \cos^2\omega t + \sin^2\omega t = 1$, which is an equation for ellipse. Hence sense of rotation is an ellipse.

Problem:4.17

An EM wave traveling in free space incidents on a dielectric medium with relative dielectric constant = 2 at an angle of 45°. Find the angle by which E tilts as the wave crosses the boundary.

Solution:

$\in_1 = \in_0 = 8.854 \times 10^{-12}$ Wb/m

$\in_2 = \in_0 \in_r = 2 \times 8.854 \times 10^{-12}$ Wb/m

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\in_2}{\in_1}}$$

$$\sin\theta_2 = \sin\theta_1\sqrt{\frac{\in_1}{\in_2}} = 0.5$$

$$\theta_2 = 30^0$$

Problem:4.18

Determine the critical angle for EM wave passing from glass $\in_r = 9$ to air.

Solution:

$$\theta_c = \sin^{-1}\sqrt{\frac{\in_2}{\in_1}} = \sin^{-1}\sqrt{\frac{1}{9}} = 19.47^0$$

Problem:4.19

The dielectric constant (relative permittivity) of pure water is 80.(a) Determine the Brewster angle for parallel polarization and the corresponding angle of transmission.(b) If a plane wave of perpendicular polarization impinges at this angle, find the reflection and transmission coefficients.

Solution:

(a) θ_1= Brewster angle = $Tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = Tan^{-1}\sqrt{80}$ = 83.621^0

$$\sin\theta_2 = \sin\theta_1\sqrt{\frac{\epsilon_1}{\epsilon_2}} = \sin\theta_1\sqrt{\frac{1}{80}} \Rightarrow 6.379^0$$

(b)

$$\frac{E_r}{E_i} = \frac{\cos\theta_1 - \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2\theta_1}} = -0.978$$

$$\frac{E_t}{E_i} = 1 + \frac{E_r}{E_i} = 1 - 0.978 = 0.02143$$

Problem:4.20

Find the critical angle for the (a)glass (ϵ_r =4), (b) Polythene (ϵ_r = 2.25) and (c)polystyrene (ϵ_r = 2.52) to air surface.

Solution:

(a) Glass to air surface

$$\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = 30^0$$

(b) $\theta_c = \sin^{-1}\sqrt{\frac{1}{2.25}} = 41.8^0$

(c) $\theta_c = \sin^{-1}\sqrt{\frac{1}{2.52}} = 39.046^0$

UNIT 5 TRANSMISSION LINES

***Problem:5.1**

A coaxial line with an outer diameter of 5mm has 50ohm characteristic impedance. If the dielectric constant is 1.60. Calculate the inner diameter.

Solution:

Given Z_0=50 ohm, D=5mm, ε_r=1.6
We have

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\in}}\log_e(D/d)$$

$$50 = \frac{1}{2\pi}\sqrt{\frac{\mu_0\mu_r}{\in_0\in_r}}\log_e(D/d) = \frac{1}{2\pi}\sqrt{\frac{4\pi\times10^{-7}\times1}{8.854\times10^{-12}\times1.6}}\log_e(8mm/d)$$

$$1.0548 = \log_e(8mm/d)$$

$$e^{1.0548} = (8mm/d)$$

inner diameter d=2.788mm.

Problem:5.2

The characteristic impedance of a certain line is 510 $\angle$-16^o and the frequency is 1kHz. At this frequency the attenuation is 0.01 Nepere/km and the phase function is 0.035 radians/km. Calculate the resistance, conductance, inductance and the capacitance per km and velocity of propagation.

Solution:

Given Z_o = 510 $\angle$-16^o ; f = 1KHz
α = 0.01 Nepere /Km β = 0.035 radians/km

$\therefore \gamma = \alpha + j\beta$
$= 0.01 + j\,0.035$
$=0.0364\angle 74.055^o$

We have

$$\frac{\gamma}{Z_o} = G + j\omega C$$

$$\frac{0.0364\angle 74.055^o}{510\angle -16^o} = G + j\omega C$$

$\Rightarrow 7.137\text{x}10^{-5} \angle 90.055^o$ = G+jωC

$-6.851x10^{-8} + j7.137x10^{-5} = G+j\omega C$

Equating real part Conductance G = $-6.851x10^{-8}$ /km

Equating imaginary part $\omega C = 7.137x10^{-5}$

Capacitance $C = \dfrac{7.137\times10^{-5}}{2\pi\times10^{3}} = 1.1359\times10^{-8}$ F/km

We have $Z_o \gamma = R+j\omega L$,

$$510\angle-16^o \times 0.0364\angle 74.055^o = R + j\omega L$$

$$18.564\angle 58.055^o = R + j\omega L$$

$$9.822 + j15.753 = R + j\omega L$$

Equating real part resistance R=9.822 Ω/km

Equating imaginary part $\omega L=15.753$

$$L = \frac{15.753}{2\pi\times10^{3}} = 2.507\times10^{-3} \text{ H/km}$$

and velocity o propagation is

$V_p = \dfrac{\omega}{\beta}$ = 179519.58 Km/h

Problem:5.3

An open wire Telephone line has R=10 Ω/km, L=0.0035H/km, C=0.0053x 10^{-6} F/km and G=0.4x10^{-6} /km. Determine Z_o, α and β at 1000Hz.

Solution:

We have series impedance $Z = R + j\omega L$

$$Z = 10 + j2\pi\times1000\times0.0035 = 10 + j21.99$$

$$= 24.157\angle 65.54^o$$

and shunt admittance $Y = G + j\omega C$

$$Y = 0.4\times10^{-6} + j2\pi\times1000\times0.0053\times10^{-6} = (0.4 + j33.3)\times10^{-6}$$

$$= 3.33\times10^{-5}\angle 89.31^o$$

Propagation constant $\gamma = \sqrt{ZY}$

$$\gamma = \sqrt{8.044\times10^{-4}\angle 154.85^o} = \sqrt{8.044\times10^{-4}}\left(e^{j154.85^o}\right)^{\frac{1}{2}} = 0.02836e^{j77.425^o}$$

$$= 0.02836\angle 77.425^o$$

$$\gamma = 6.174\times10^{-3} + j0.0277 = \alpha + j\beta$$

$\therefore \alpha = 6.174\times10^{-3}$ Nepere/Km and β=0.0277 radians/Km

The characteristic impedance $Z_o = \sqrt{\dfrac{Z}{Y}}$

$$Z_o = \sqrt{\frac{24.157\angle 65.54^o}{3.33\times10^{-5}\angle 89.31^o}} = \sqrt{725435.43\angle -23.77^o}$$

$$Z_o = \sqrt{725435.43}\left(e^{-j23.77^o}\right)^{\frac{1}{2}} = 851.725e^{-j11.89^o} = 851.725\angle -11.89^o$$

$$Z_o = 833.45 - j175.48\ \Omega/\text{Km}$$

***Problem:5.4**

At 5MHz the characteristic impedance of transmission line is (40-j2) ohm and the propagation constant is (0.01+j0.15) per meter. Find the primary constants.

Solution:

Given f=5MHz, Z_0=40-j2 ohm and γ =0.01+j0.15 per meter

$$Z_o = 40.05\angle -2.86^o \quad \text{and} \quad \gamma = 0.15\angle 86.19^o$$

We have $Z_o\,\gamma$ =R+jωL and $\dfrac{\gamma}{Z_O} = G + j\omega C$

$$R + j\omega L = Z_o\gamma = 40.05\angle -2.86^o \times 0.15\angle 86.19^o = 6\angle 83.33^o$$

$$R + j\omega L = 0.697 + j5.96$$

Equating real and imaginary parts, the primary constants are

R=0.697 ohm/m and ωL=5.96 $\Rightarrow L = \dfrac{5.96}{2\pi\times5\times10^6} = 0.1897\,\mu H\ /\ m$

$$G + j\omega C = \frac{\gamma}{Z_o} = \frac{0.15\angle 86.19^o}{40.05\angle -2.86^o} = 3.745\times10^{-3}\angle 89.05^o$$

$$G + j\omega C = 6.21\times10^{-5} + j3.75\times10^{-3}$$

Equating real and imaginary parts, the primary constants are

$G = 6.21\times10^{-5}$ □ /m and

$\omega C = 3.75\times10^{-3} \Rightarrow C = \dfrac{3.75\times10^{-3}}{2\pi\times5\times10^6} = 119.37\,pF\ /\ m$

***Problem:5.5**

A telephone wire 20m long has the following constants per loop km resistance 90 ohm, capacitance 0.062 μF, inductance 0.001H and leakage=1.5x10^{-6} mhos. The line is terminated in its characteristic impedance and a potential difference of 2.1 V having a frequency of 1000Hz is applied at the sending end. Calculate:

(a) The characteristic impedance (b) wavelength and (c) The velocity of propagation

Solution:

Given R=90 ohm/km, C=0.062 μF/km, L=0.001H/km, G=1.5x10^{-6} mhos/km, V=2.1V and f=1000Hz.

Series impedance Z=R+jωL=90+j2πx1000x0.001=90+j6.253=$90.219\angle 3.99^0\,\Omega$

And shunt admittance Y=G+jωC=1.5x10^{-6}+j2πx1000x0.062

=(1.5+j359.555) x10^{-6}=$389.56\times 10^{-6}\angle 89.78^0$

(a) The characteristic impedance

$$Z_0=\sqrt{\frac{Z}{Y}}=\sqrt{\frac{90.219\angle 3.99^0}{389.56\times 10^{-6}\angle 89.78^0}}$$

$$Z_0=\sqrt{231592.0526\angle -85.79^0}=\sqrt{231592.0526}\left(e^{-j85.79^0}\right)^{1/2}$$

$$Z_0=481.24e^{(-j85.79^0/2)}=481.24\angle -42.895^0\,\Omega$$

(b) Wavelength,

We know propagation constant $\gamma=\alpha+j\beta=\sqrt{ZY}$

$$\alpha+j\beta=\sqrt{ZY}=\sqrt{(90.219\angle 3.99^0)(389.56\times 10^{-6}\angle 89.78^0)}$$

$$\alpha+j\beta=\sqrt{35145.71\angle 93.77^0}=\sqrt{35145.71}\left(e^{j93.77^0}\right)^{1/2}=187.47e^{j(93.77^0/2)}$$

$$\alpha+j\beta=187.47\angle 46.885^0=128.128+j136.849$$

$$\therefore \beta=136.849 radians$$

$$i.e. \beta=0.137 radians/km$$

Hence the wavelength $\lambda=\frac{2\pi}{\beta}=\frac{2\pi}{0.137}=45.86km$

(c) Velocity of propagation

$$v_p=\frac{\omega}{\beta}=\frac{2\pi\times 1000}{0.137}=45.86\times 10^3 km/\sec$$

Problem:5.6

O.C. and S.C. impedances of a transmission line at 1.6KHz are $900\angle -30^o\ \Omega$ and $400\angle -10^o\ \Omega$ respectively. Calculate it's Z_o.

Solution:

$Z_{OC}\, Z_{SC} = Z_o^2$

$Z_o^2 = 300\text{x}10^3 \angle\text{-}40^o$

$Z_o = 600 \angle\text{-}20^o\,\Omega$

***Problem:5.7**

A low transmission line of 100 ohm characteristic impedance is connected to a load of 400 ohm. Calculate the reflection coefficient and standing wave ratio. Derive the Relationships used.

Solution:

Given Z_0=100 ohm and Z_R=400 ohm

We have Reflection coefficient $k = \dfrac{Z_R - Z_o}{Z_R + Z_o}$

$$k = \frac{400-100}{400+100} = 0.6$$

Also we have voltage standing wave ratio $VSWR = \dfrac{1+|k|}{1-|k|}$

$$VSWR = \frac{1+0.6}{1-0.6} = 4$$

Derivations are as in section 5.14.

***Problem:5.8**

Explain the significance of V_{max} and V_{min} positions along the transmission line, for a complex load Z_R. Hence calculate the impedances at these positions.

Solution:

Explanation is as in the section "Standing Wave Ratio".

At a voltage maximum or current minimum,

$$Z_{in} = Z_{\max} = \frac{V_{\max}}{I_{\min}} = \frac{V_{\max}}{V_{\min}}\frac{V_{\min}}{I_{\min}} = VSWR \times Z_0$$

At a voltage minimum or current maximum,

$$Z_{in} = Z_{\min} = \frac{V_{\min}}{I_{\max}} = \frac{V_{\min}}{V_{\max}}\frac{V_{\max}}{I_{\max}} = Z_0 / VSWR$$

Problem:5.9

Transmission line 10km long is terminated properly at the far end at a frequency 1000Hz. The attenuating and phase constants of the line are respectively 0.03 Nepere/km and 0.03 radians/km. If the far end voltage at 1000Hz is $4\angle 0^o$V. Calculate sending end voltage of the line.

Solution:

If transmission line is terminated with Z_R then

$V = V_R \cosh \gamma (l\text{-}x) + I_R Z_o \sinh \gamma (l\text{-}x)$

At $x = 0$, $V = V_s$

$V_s = V_R \cosh \gamma l + I_R Z_o \sinh \gamma l$

Terminated properly $\therefore Z_R = Z_o$

$$I_R = \frac{V_R}{Z_R} = \frac{V_R}{Z_o}$$

$V_s = V_R \cosh \gamma l + V_R \sinh \gamma l$

$= V_R (\cosh \gamma l + \sinh \gamma l)$

$V_s = V_R\, e^{\gamma l}$

$V_R = 4\angle 0^o$

$l = 10$km

$\gamma l = (\alpha + j\beta) l = (0.03 + j\,0.03)10\text{km} = 424.26\angle 45^o$

$V_s = 4\, e^{424.26\angle 45^o}$ V

Problem:5.10

A 30m long lossless transmission line with $Z_o = 50\Omega$ operating at 2MHz is terminated with a load $Z_R = 60+j40\Omega$. If $v_p = 0.6v_0$ on the line. Find

(a) The reflection coefficient 'K'
(b) The SWR 'S'
(c) The input impedance

Solution:

Method:1(without smith chart)

(a) Reflection coefficient $K = \dfrac{Z_R - Z_o}{Z_R + Z_o}$

$= 0.3523\ \angle 56^o$

(b) SWR $= S = \dfrac{1+|K|}{1-|K|} = \dfrac{1+0.3523}{1-0.3523} = 2.08$

(c) Z_{in} for lossless transmission line is

$$Z_{in} = Z_0 \frac{Z_R + jZ_o \, Tan\,\beta l}{Z_o + jZ_R Tan\beta l}$$

βl = Electrical length of transmission line

β = ?

$$v_p = \frac{\omega}{\beta}$$

$$\beta = \frac{\omega}{v_p} = \frac{2\pi f}{v_p}$$

$$\beta l = \frac{2\pi fl}{v_p} = \frac{2\pi \times 2 \times 10^6 \times 30}{0.6 \times 3 \times 10^8} = 2.0944 \times \frac{180^o}{\pi} = 120^o$$

$$Z_{in} = 50 \frac{60 + j40 + j50(-1.732)}{50 + j(60 + j40)(-1.732)}$$

$$= 50 \frac{73.14\angle 34.9^o}{48.59\angle 26^o} = 75.26\angle 8.9^o$$

$= 23.95 + j1.35\Omega$

Method:2 (with smith chart)

(a) Calculate the normalized load impedance $Z_r = \frac{Z_R}{Z_o} = 1.2 + j0.8$

Locate Z_r on the Smith chart of Fig:5.20 at point A where the R=1.2 circle and the X=0.8 circle meet. To get K at Z_r, extend OA to meet the R=0 circle at B and measure OA and OB. Since OB corresponds to |K|=1, then at A,

$|K| = \frac{OA}{OB} = \frac{3.5cm}{10cm} = 0.350$, here OA and OB may vary depending on the size of the Smith chart used, but the ratio remains same.

Angle θ_K is read directly on the chart as the angle between OC and OA ;

i.e. $\theta_K = 56^0$

$\therefore K = 0.350\angle 56^0$

(b) To obtain the standing wave ratio S, draw a circle with radius OA and center at O. This is the constant S or |K| circle. Locate point C where the S-circle meets the K_r axis. The value of R on the Smith chart is S, i.e. S=R=2.1

(c) To obtain Z_{in}, first express l in terms of λ or in degrees.

$$\lambda = \frac{V_p}{f} = \frac{0.6 \times 3 \times 10^8}{2 \times 10^6} = 90m$$

$$l = 30m = \frac{30}{90}\lambda = \frac{\lambda}{3}$$

Since $\lambda \to 720^0$; $\quad \frac{\lambda}{3} \to \frac{720^0}{3} = 240^0$

To get input impedance, we need to move towards the generator i.e. move in the clockwise direction by 240^0 on the S circle from point A to point D. At D, we get

normalized $\quad z_{in} = 0.47 + j0.035$

$\therefore$ actual $Z_{in} = Z_0 z_{in} = 50(0.47 + j0.035) = 23.5 + j1.75\Omega$

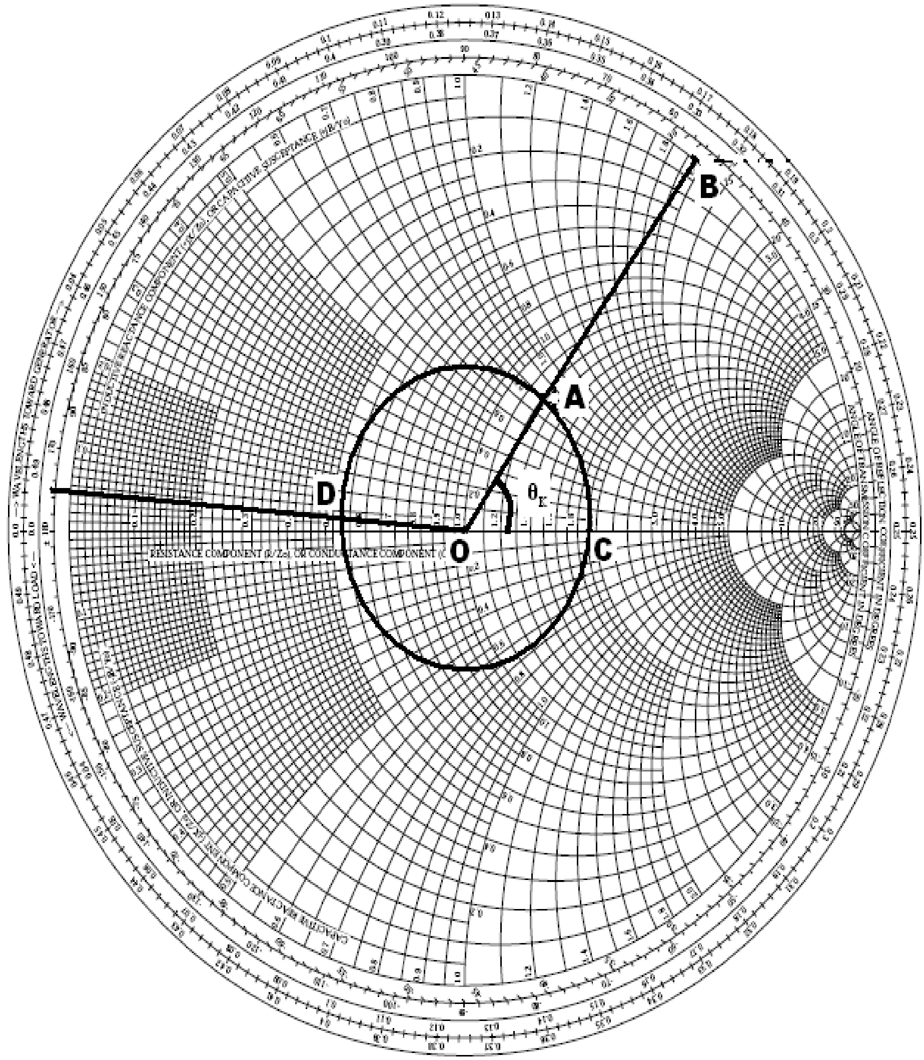


Fig:5.20

***Problem:5.11**

An aerial of(200-j300) ohm is to be matched with 500 ohm lines. The matching is to be done by means of low loss 600 ohm stub line. Find the position and length of the stub line used if the operating wave length is 2 meters.

Solution:

Given Z_R=200-j300 ohm=360.55∠-56.3^0, Z_0=500 ohm and λ=20 m

$$k=\frac{Z_R-Z_o}{Z_R+Z_o}=\frac{200-j300-500}{200-j300+500}=-0.2068-j0.5172=0.557\angle-111.8^0\ \therefore$$

|k|=0.555 and θ=-111.5⁰=-0.6211π

Position of the stub $l_2=\dfrac{\theta+\pi-\cos^{-1}(|k|)}{2\beta}=\lambda\dfrac{\theta+\pi-\cos^{-1}(|k|)}{4\pi}$

$$l_2=20\frac{-0.6211\pi+\pi-\cos^{-1}(0.557)}{4\pi}=0.33465m$$

Length of the stub $l_1=\beta\tan^{-1}\left(\dfrac{\sqrt{1-|k|^2}}{2|k|}\right)$

$$l_1=\frac{2\pi}{\lambda}\tan^{-1}\left(\frac{\sqrt{1-|k|^2}}{2|k|}\right)=\frac{2\pi}{20}\tan^{-1}\left(\frac{\sqrt{1-(0.557)^2}}{2(0.557)}\right)=0.201m$$