Unit III

Torsion of Circular Shafts

The product of this turning force and the distance between the point of application of the force and the axis of the shaft is known as torque, turning moment or twisting moment. And the shaft is said to be subjected to torsion. Due to this torque, every cross-section of the shaft is subjected to some shear stress.

Assumptions for Shear Stress in a Circular Shaft Subjected to Torsion

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. Normal cross-sections of the shaft, which were plane and circular before the twist, remain plane and circular even after the twist.
4. All diameters of the normal cross-section, which were straight before the twist, remain straight with their magnitude unchanged, after the twist.

Torsional Stresses and Strain

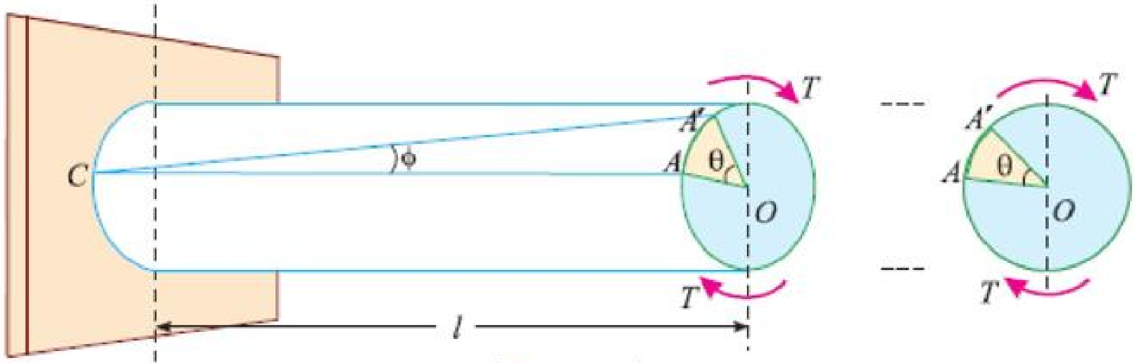


Fig. 1

Consider a circular shaft fixed at one end and subjected to a torque at the other end as shown in Fig.1

T = Torque in N-mm,

l = Length of the shaft in mm and

R = Radius of the circular shaft in mm.

As a result of this torque, every cross-section of the shaft will be subjected to shear stresses. Let the line CA on the surface of the shaft be deformed to CA′ and OA to OA′ as shown in Fig.1

$\angle ACA' = \phi$ in degrees

$\angle AOA' = \theta$ in radians

τ = Shear stress induced at the surface and

C = Modulus of rigidity, also known as torsional rigidity of the shaft material.

We know that shear strain = Deformation per unit length

$$= \frac{AA'}{l} = \tan\theta$$

$$= \phi \qquad \text{...}(\phi \text{ being very small, } \tan\phi = \phi)$$

We also know that the arc $AA' = R \cdot \theta$

$$\therefore \quad \phi = \frac{AA'}{l} = \frac{R \cdot \theta}{l} \qquad \text{...}(i)$$

If τ is the intensity of shear stress on the outermost layer and C the modulus of rigidity of the shaft, then

$$\phi = \frac{\tau}{C} \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*), we find that

$$\frac{\tau}{C} = \frac{R \cdot \theta}{l} \quad \text{or} \quad \frac{\tau}{R} = \frac{C \cdot \theta}{l}$$

If τ_X be the intensity of shear stress, on any layer at a distance x from the centre of the shaft, then

$$\frac{\tau_X}{x} = \frac{\tau}{R} = \frac{C \cdot \theta}{l} \qquad \text{...}(iii)$$

Strength of a Solid Shaft

The term, strength of a shaft means the maximum torque or power, it can transmit. As a matter of fact, we are always interested in calculating the torque, a shaft can withstand or transmit.

Let R = Radius of the shaft in mm and

τ = Shear stress developed in the outermost layer of the shaft in N/mm^2

Consider a shaft subjected to a torque T as shown in Fig. 2. Now let us consider an element of area da of thickness dx at a distance x from the centre of the shaft as shown in Fig. 2.

$$\therefore \quad da = 2\pi x \cdot dx \qquad \text{...}(i)$$

and shear stress at this section,

$$\therefore \quad \tau_X = \tau \times \frac{x}{R} \qquad \text{...}(ii)$$

where τ = Maximum shear stress.

$\therefore$ Turning force = Shear Stress × Area

$$= \tau_x \cdot dx$$

$$= \tau \times \frac{x}{R} \times da$$

$$= \tau \frac{x}{R} \times 2\pi x \cdot dx$$

$$= \frac{2\pi x}{R} \cdot x^2\, dx$$

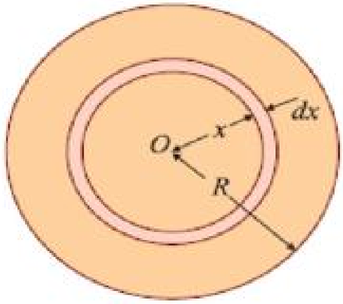


Fig. 2

We know that turning moment of this element,

dT = Turning force × Distance of element from axis of the shaft

$$= \frac{2\pi\tau}{R} x^2\, dx \cdot x = \frac{2\pi\tau}{R} x^3 \cdot dx \qquad ...(iii)$$

The total torque, which the shaft can withstand, may be found out by integrating the above equation between 0 and R *i.e.*,

$$T = \int_0^R \frac{2\pi\tau}{R} x^3 \cdot dx = \frac{2\pi\tau}{R} \int_0^R x^2 \cdot dx$$

$$= \frac{2\pi\tau}{R}\left[\frac{x^4}{4}\right]_0^R = \frac{\pi}{2}\tau \cdot R^3 = \frac{\pi}{16} \times \tau \times D^3 \quad \text{N-mm}$$

where D is the diameter of the shaft and is equal to $2R$.

EXAMPLE 1 *A solid steel shaft is to transmit a torque of 10 kN-m. If the shearing stress is not to exceed 45 MPa, find the minimum diameter of the shaft.*

SOLUTION. Given: Torque (T) = 10 kN-m = 10×10^6 N-mm and maximum shearing stress (τ) = 45 MPa = 45 N/mm^2.

Let D = Minimum diameter of the shaft in mm.

We know that torque transmitted by the shaft (T),

$$10 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 45 \times D^3 = 8.836\, D^3$$

$$\therefore \quad D^3 = \frac{10 \times 10^6}{8.836} = 1.132 \times 10^6$$

or $D = 1.04 \times 10^2 =$ **104 mm** **Ans.**

Strength of a Hollow Shaft

It means the maximum torque or power a hollow shaft can transmit from one pulley to another. Now consider a hollow circular shaft subjected to some torque.

Let R = Outer radius of the shaft in mm,

r = Inner radius of the shaft in mm, and

τ = Maximum shear stress developed in the outer most layer of the shaft material.

Now consider an elementary ring of thickness dx at a distance x from the centre as shown in Fig. 3.

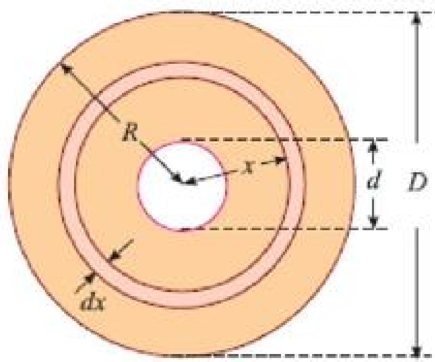


Fig. 27.3

We know that area of this ring,

$$da = 2\pi x \cdot dx \qquad ...(i)$$

and shear stress at this section,

$$\tau_X = \tau \times \frac{x}{R}$$

$\therefore$

$$\text{Turning force} = \text{Stress} \times \text{Area}$$

$$= \tau_X \cdot dx \qquad ...\left(\because \tau_X = \tau \times \frac{x}{R}\right)$$

$$= \tau \times \frac{x}{R} \times 2\pi\, xdx \qquad ...(\because da = 2\pi\, xdx)$$

$$= \frac{2\pi\tau}{R} x^2 \cdot dx \qquad ...(ii)$$

We know that turning moment of this element,

$$dT = \text{Turning force} \times \text{Distance of element from axis of the shaft}$$

$$= \frac{2\pi\tau}{R} x^2 \cdot dx \cdot x = \frac{2\pi\tau}{R} x^3 \cdot dx \qquad ...(iii)$$

The total torque, which the shaft can transmit, may be found out by integrating the above equation between r and R.

$\therefore$

$$T = \int_r^R \frac{2\pi\tau}{R} x^3 \cdot dx = \frac{2\pi\tau}{R} \int_r^R x^3 \cdot dx$$

$$= \frac{2\pi\tau}{R}\left[\frac{x^4}{4}\right]_r^R = \frac{2\pi\tau}{R}\left(\frac{R^4 - r^4}{4}\right) = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d^4}{D}\right) \text{ N-mm}$$

where D is the external diameter of the shaft and is equal to $2R$ and $2d$ is the internal diameter of the shaft and is equal to $2r$.

Power Transmitted by a Shaft

We have already discussed that the main purpose of a shaft is to transmit power from one shaft to another in factories and workshops. Now consider a rotating shaft, which transmits power from one of its ends to another.

Let N = No. of revolutions per minute and

T = Average torque in kN-m.

$$\text{Work done per minute} = \text{Force} \times \text{Distance} = T \times 2\pi N = 2\pi NT$$

$$\text{Work done per second} = \frac{2\pi NT}{60} \text{ kN-m}$$

$$\text{Power transmitted} = \text{Work done in kN-m per second}$$

$$= \frac{2\pi NT}{60} \text{ kW}$$

Example 2: A hollow shaft is to transmit 200 kW at 80 r.p.m. If the shear stress is not to exceed 60 MPa and internal diameter is 0.6 of the external diameter, find the diameters of the shaft.

SOLUTION. Given : Power (P) = 200 kW ; Speed of shaft (N) = 80 r.p.m. ; Maximum shear stress (τ) = 60 MPa = 60 N/mm2 and internal diameter of the shaft (d) = 0.6D (where D is the external diameter in mm).

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 60 \times \left[\frac{D^4 - (0.6D)^4}{D}\right] \text{ N-mm}$$

$$= 10.3\, D^3 \text{ N-mm} = 10.3 \times 10^{-6} D^3 \text{ kN-m} \qquad \text{...}(i)$$

We also know that power transmitted by the shaft (P),

$$200 = \frac{2\pi NT}{60} = \frac{2\pi \times 80 \times (10.3 \times 10^{-6} D^3)}{60} = 86.3 \times 10^{-6} D^3$$

$$\therefore \quad D^3 = \frac{200}{(86.3 \times 10^{-6})} = 2.32 \times 10^6 \text{ mm}^3$$

or $D = 1.32 \times 10^2 = \mathbf{132\ mm}$ **Ans.**

and $d = 0.6\,D = 0.6 \times 132 = \mathbf{79.2\ mm}$ **Ans.**

Polar Moment of Inertia

The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure, is called polar moment of inertia with respect to the point, where the axis intersects the plane. In a circular plane, this point is always the centre of the circle. We know that

$$\frac{\tau}{R} = \frac{C \cdot \theta}{l} \qquad \text{...}(i) \qquad \text{... (from Art. 27.3)}$$

and

$$T = \frac{\pi}{16} \times \tau \times D^3 \qquad \text{...}(ii) \qquad \text{... (from Art. 27.3)}$$

or

$$\tau = \frac{16T}{\pi D^3}$$

Substituting the value of τ in equation (i),

$$\frac{16T}{\pi D^3 \times R} = \frac{C \cdot \theta}{l}$$

or

$$\frac{T}{\frac{\pi}{16} \times D^3 \times R} = \frac{C \cdot \theta}{l}$$

$$\frac{T}{\frac{\pi}{32} \times D^4} = \frac{C \cdot \theta}{l} \qquad \text{...}\left(\text{Radius, } R = \frac{D}{2}\right)$$

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \qquad \text{...}(iii)$$

where $J = \frac{\pi}{32} \times D^4$. It is known as polar moment of inertia.

The above equation (iii) may also be written as :

$$\frac{\tau}{R} = \frac{T}{J} = \frac{C \cdot \theta}{l} \qquad \text{...}\left(\because \frac{\tau}{R} = \frac{C \cdot \theta}{l}\right)$$

EXAMPLE 3. *Find the angle of twist per metre length of a hollow shaft of 100 mm external and 60 mm internal diameter, if the shear stress is not to exceed 35 MPa. Take C = 85 GPa.*

SOLUTION. Given: Length of the shaft (l) = 1 m = 1×10^3 mm ; External diameter (D) = 100 mm; Internal diameter (d) = 60 mm ; Maximum shear stress (τ) = 35 MPa = 35 N/mm^2 and modulus of rigidity (C) = 85 GPa = 85×10^3 N/mm^2.

Let θ = Angle of twist in the shaft.

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 35 \times \left[\frac{(100)^4 - (60)^4}{100}\right] \text{ N-mm}$$

$$= 5.98 \times 10^6 \text{ N-mm}$$

We also know that polar moment of inertia of a hollow circular shaft,

$$J = \frac{\pi}{32}[D^4 - d^4] = \frac{\pi}{32}[(100)^4 - (60)^4] = 8.55 \times 10^6 \text{ mm}^4$$

and relation for the angle of twist,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{5.98 \times 10^6}{8.55 \times 10^6} = \frac{(85 \times 10^3)\,\theta}{1 \times 10^3} = 85.\theta$$

$$\therefore \quad \theta = \frac{5.98 \times 10^6}{(8.55 \times 10^6) \times 85} = 0.008 \text{ rad} = \mathbf{0.5^\circ} \quad \textbf{Ans.}$$

EXAMPLE 4. *A solid shaft is subjected to a torque of 1.6 kN-m. Find the necessary diameter of the shaft, if the allowable shear stress is 60 MPa. The allowable twist is 1° for every 20 diameters length of the shaft. Take C = 80 GPa.*

SOLUTION. Given: Torque $(T) = 1.6\,\text{kN-m} = 1.6 \times 10^6$ N-mm; Allowable shear stress $(\tau) = 60$ MPa = 60 N/mm^2; Angle of twist $(\theta) = 1° = \dfrac{\pi}{180}$ rad ; Length of shaft $(l) = 20\,D$ and modulus of rigidity (C) = 80 GPa = 80×10^3 N/mm^2.

First of all, let us find out the value of diameter of the shaft for its strength and stiffness.

1. *Diameter for strength*

We know that torque transmitted by the shaft (T),

$$1.6 \times 10^6 = \frac{\pi}{16} \times \tau \times D_1^3 = \frac{\pi}{16} \times 60 \times D_1^3 = 11.78\,D_1^3$$

$$\therefore \qquad D_1^3 = \frac{1.6 \times 10^6}{11.78} = 0.136 \times 10^6\ \text{mm}^3$$

or $$D_1 = 0.514 \times 10^2 = 51.4\ \text{mm} \qquad \ldots(i)$$

2. *Diameter for stiffness*

We know that polar moment of inertia of a solid circular shaft,

$$J = \frac{\pi}{32} \times (D_2)^4 = 0.098\,D_2^4$$

and relation for the diameter,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{1.6 \times 10^6}{0.098\,D_2^4} = \frac{(80 \times 10^3) \times (\pi/180)}{20 D_2}$$

$$\therefore \qquad D_2^3 = \frac{(1.6 \times 10^6) \times 20}{0.098 \times (80 \times 10^3) \times (\pi/180)} = 234 \times 10^3\ \text{mm}^3$$

or $$D_2 = 6.16 \times 10^1 = 61.6\ \text{mm} \qquad \ldots(ii)$$

We shall provide a shaft of diameter of 61.6 mm (*i.e.*, greater of the two values). **Ans.**

EXAMPLE 5. *A solid shaft of 200 mm diameter has the same cross-sectional area as a hollow shaft of the same material with inside diameter of 150 mm. Find the ratio of*

(*a*) *powers transmitted by both the shafts at the same angular velocity.*

(*b*) *angles of twist in equal lengths of these shafts, when stressed to the same intensity.*

SOLUTION. Given: Diameter of solid shaft $(D_1) = 200$ mm and inside diameter of hollow shaft (d) = 150 mm.

(*a*) ***Ratio of powers transmitted by both the shafts***

We know that cross-sectional area of the solid shaft,

$$A_1 = \frac{\pi}{4} \times D_1^2 = \frac{\pi}{4} \times (200)^2 = 10\,000\,\pi\ \text{mm}^2$$

and cross-sectional area of hollow shaft,

$$A_2 = \frac{\pi}{4} \times (D^2 - d^2) = \frac{\pi}{4} \times [D^2 - (150)^2] = \frac{\pi}{4}(D^2 - 22\,500)$$

Since the cross-sectional areas of both the shafts are same, therefore equating A_1 and A_2,

$$\frac{\pi}{4}(200)^2 = \frac{\pi}{4}(D^2 - 22\,500)$$

$$\therefore \quad 40\,000 = D^2 - 22\,500$$

$$D^2 = 40\,000 + 22\,500 = 62\,500 \text{ mm}^2$$

or

$$D = 250 \text{ mm}$$

We also know that torque transmitted by the solid shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times D_1^3 = \frac{\pi}{16} \times \tau \times (200)^3 = 500 \times 10^3 \, \pi \, \tau \text{ N-mm} \qquad \text{...}(i)$$

Similarly, torque transmitted by the hollow shaft,

$$T_2 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{(250)^4 - (150)^4}{250}\right] \text{N-mm}$$

$$= 850 \times 10^3 \, \pi \, \tau \text{ N-mm}$$

$$\therefore \quad \frac{\text{Power transmitted by hollow shaft}}{\text{Power transmitted by solid shaft}}$$

$$= \frac{T_2}{T_1} = \frac{50 \times 10^3 \, \pi \, \tau}{500 \times 10^3 \, \pi \, \tau} = 1.7 \qquad \textbf{Ans.}$$

(b) Ratio of angles of twist in both the shafts

We know that relation for angle of twist for a shaft,

$$\frac{\tau}{R} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \theta = \frac{\tau l}{RC}$$

$\therefore$ Angle of twist for the solid shaft,

$$\theta_1 = \frac{\tau l}{RC} = \frac{\tau l}{100C} \qquad \text{...}\left(\text{where } R = \frac{D_1}{2} = \frac{200}{2} = 100 \text{ mm}\right)$$

Similarly angle of twist for the hollow shaft,

$$\theta_2 = \frac{\tau l}{RC} = \frac{\tau l}{125C} \qquad \text{...}\left(\text{where } R = \frac{D_1}{2} = \frac{250}{2} = 125 \text{ mm}\right)$$

$$\therefore \quad \frac{\text{Angle of twist of hollow shaft}}{\text{Angle of twist of solid shaft}} = \frac{\theta_2}{\theta_1} = \frac{\dfrac{\tau l}{125C}}{\dfrac{\tau l}{100C}} = \frac{100}{125} = 0.8 \qquad \textbf{Ans.}$$

EXAMPLE 6. *A shaft ABC of 500 mm length and 40 mm external diameter is bored, for a part of its length AB to a 20 mm diameter and for the remaining length BC to a 30 mm diameter bore as shown in Fig. . If the shear stress is not to exceed 80 MPa, find the maximum power, the shaft can transmit at a speed of 200 r.p.m.*

If the angle of twist in the length of 20 mm diameter bore is equal to that in the 30 mm diameter bore, find the length of the shaft that has been bored to 20 mm and 30 mm diameter.

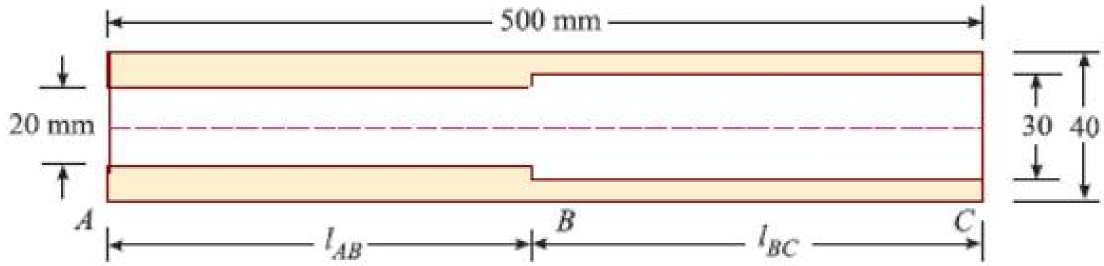


Fig.

SOLUTION. Given: Total length of the shaft (l) = 500 mm; External diameter of the shaft (D) = 40 mm ; Internal diameter of shaft AB (d_{AB}) = 20 mm ; Internal diameter of shaft BC (d_{BC}) = 30 mm ; Maximum shear stress (τ) = 80 MPa = 80 N/mm^2 and speed of the shaft (N) = 200 r.p.m.

Maximum power the shaft can transmit

We know that torque transmitted by the shaft AB,

$$T_{AB} = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d_{AB}^4}{D}\right) = \frac{\pi}{16} \times 80 \times \left[\frac{(40)^4 - (20)^4}{40}\right] \text{ N-mm}$$

$$= 942.5 \times 10^3 \text{ N-mm} \qquad ...(i)$$

Similarly,

$$T_{BC} = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d_{BC}^4}{D}\right) = \frac{\pi}{16} \times 80 \times \left[\frac{(40)^4 - (30)^4}{40}\right] \text{ N-mm}$$

$$= 687.3 \times 10^3 \text{ N-mm} \qquad ...(ii)$$

From the above two values, we see that the safe torque transmitted by the shaft is minimum of the two, *i.e.*, 687.3×10^3 N-mm = 687.3 N-m. Therefore maximum power the shaft can transmit,

$$P = \frac{2\pi NT}{60} = \frac{2 \times \pi \times 200 \times (687.3)}{60} = 14\ 394 \text{ W}$$

$$= \mathbf{14.39\ kW} \qquad \textbf{Ans.}$$

Length of the shaft, that has been bored to 20 mm diameter

Let l_{AB} = Length of the shaft AB (*i.e.*, 20 mm diameter bore) and

l_{BC} = Length of the shaft BC (*i.e.*, 30 mm diameter bore) equal to $(500 - l_{AB})$ mm.

We know that polar moment of inertia for the shaft AB,

$$J_{AB} = \frac{\pi}{32} \times (D^4 - d_{AB}^4) = \frac{\pi}{32} \times [(40)^4 - (20)^4] \text{ mm}^4$$

Similarly,
$$J_{BC} = \frac{\pi}{32} \times (D^4 - d_{BC}^4) = \frac{\pi}{32} \times [(40)^4 - (30)^4] \text{ mm}^4$$

We know that relation for the angle of twist:

$$\frac{T}{J} = \frac{C\,\theta}{l} \quad \text{or} \quad \theta = \frac{T \cdot l}{JC}$$

$$\therefore \quad \theta_{AB} = \frac{T \cdot l_{AB}}{J_{AB} \cdot C} \quad \text{and} \quad \theta_{BC} = \frac{T \cdot l_{BC}}{J_{BC} \cdot C}$$

Since $\theta_{AB} = \theta_{AC}$ and T as well as C is equal in both these cases, therefore

$$\frac{l_{AB}}{J_{AB}} = \frac{l_{BC}}{J_{BC}} \quad \text{or} \quad \frac{l_{AB}}{\frac{\pi}{32} \times [(40)^4 - (20)^4]} = \frac{l_{BC}}{\frac{\pi}{32} \times [(40)^4 - (30)^4]}$$

or
$$\frac{l_{AB}}{l_{BC}} = \frac{(40)^4 - (20)^4}{(40)^4 - (30)^4} = \frac{2400000}{1750000} = 1.37$$

$$\therefore \quad l_{AB} = 1.37\, l_{BC}$$

$$1.37\, l_{BC} + l_{BC} = 500 \qquad \text{...}(\because l_{AB} + l_{BC} = 500)$$

$$\therefore \quad l_{BC} = \frac{500}{2.37} = 211 \text{ mm} \qquad \textbf{Ans.}$$

and
$$l_{AB} = 500 - 211 = 289 \text{ mm} \qquad \textbf{Ans.}$$

Thin Cylinders

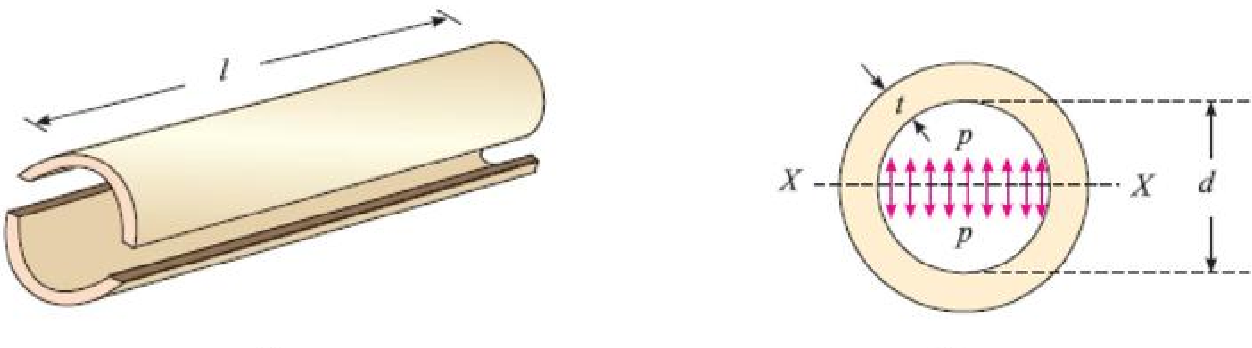
In general, if the thickness of the wall of a shell is less than 1/10th to 1/15th of its diameter, it is known as a thin shell.

Stresses in a Thin Cylindrical Shell

The walls of the cylindrical shell will be subjected to the following two types of tensile stresses:

1. Circumferential stress
2. Longitudinal stress.

Circumferential Stress



(a) (b)

Fig.

Consider a thin cylindrical shell subjected to an internal pressure as shown in Fig.(a) and (b). We know that as a result of the internal pressure, the cylinder has a tendency to split up into two troughs as shown in the figure.

Let l= Length of the shell

d = Diameter of the shell,

t = Thickness of the shell and

p = Intensity of internal pressure.

Total pressure along the diameter (say *X-X* axis) of the shell,

P = Intensity of internal pressure × Area = $p \times d \times l$

and circumferential stress in the shell,

$$\sigma_c = \frac{\text{Total pressure}}{\text{Resisting section}} = \frac{pdl}{2tl} = \frac{pd}{2t} \qquad \text{...(}\because\text{ of two sections)}$$

This is a tensile stress across the *X-X*. It is also known as **hoop stress.**

Note. If η is the efficiency of the riveted joints of the shell, then stress,

$$\sigma_c = \frac{pd}{2t\eta}$$

Longitudinal Stress

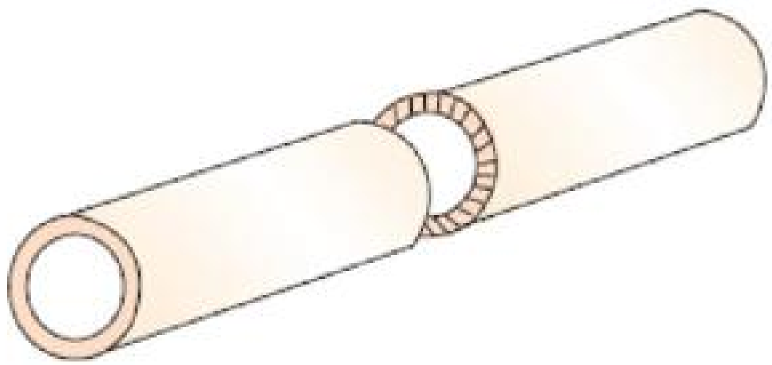
Consider the same cylindrical shell, subjected to the same internal pressure as shown in Fig. (a) and (b). We know that as a result of the internal pressure, the cylinder also has a tendency to split into two pieces as shown in the figure.

Let p = Intensity of internal pressure,

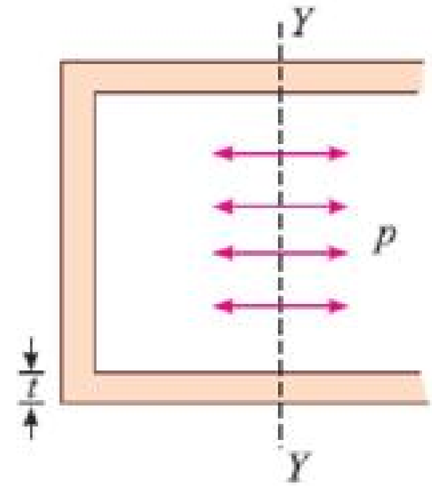
l = Length of the shell,

d = Diameter of the shell and

t = Thickness of the shell.



(a)



(b)

Fig. . Longitudinal stress.

Total pressure along its length (say Y-Y axis) of the shell

$$P = \text{Intensity of internal pressure} \times \text{Area}$$

$$= p \times \frac{\pi}{4}(d)^2$$

and longitudinal stress in the shell,

$$\sigma_l = \frac{\text{Total pressure}}{\text{Resisting section}} = \frac{p \times \frac{\pi}{4}(d)^2}{\pi d t} = \frac{pd}{4t}$$

This is also a tensile stress across the section Y-Y. It may be noted that the longitudinal stress is half of the circumferential or hoop stress.

Note. If η is the efficiency of the riveted joints of the shell, then the stress,

$$\sigma_l = \frac{pd}{4t\eta}$$

EXAMPLE 1. *A stream boiler of 800 mm diameter is made up of 10 mm thick plates. If the boiler is subjected to an internal pressure of 2.5 MPa, find the circumferential and longitudinal stresses induced in the boiler plates.*

SOLUTION. Given : Diameter of boiler (d) = 800 mm ; Thickness of plates (t) = 10 mm and internal pressure (p) = 2.5 MPa = 2.5 N/mm^2.

Circumferential stress induced in the boiler plates

We know that circumferential stress induced in the boiler plates,

$$\sigma_c = \frac{pd}{2t} = \frac{2.5 \times 800}{2 \times 10} = 100 \text{ N/mm}^2 = \mathbf{100 \text{ MPa}} \qquad \textbf{Ans.}$$

Longitudinal stress induced in the boiler plates

We also know that longitudinal stress induced in the boiler plates,

$$\sigma_l = \frac{pd}{4t} = \frac{2.5 \times 800}{4 \times 10} = 50 \text{ N/mm}^2 = \mathbf{50 \text{ MPa}} \qquad \textbf{Ans.}$$

EXAMPLE 2. *A cylindrical shell of 1.3 m diameter is made up of 18 mm thick plates. Find the circumferential and longitudinal stress in the plates, if the boiler is subjected to an internal pressure of 2.4 MPa. Take efficiency of the joints as 70%.*

SOLUTION. Given: Diameter of shell (d) = 1.3 m = 1.3 × 10^3 mm ; Thickness of plates (t) = 18 mm; Internal pressure (p) = 2.4 MPa = 2.4 N/mm^2 and efficiency (η) = 70% = 0.7.

Circumferential stress

We know that circumferential stress,

$$\sigma_c = \frac{pd}{2t\eta} = \frac{2.4 \times (1.3 \times 10^3)}{2 \times 18 \times 0.7} = 124 \text{ N/mm}^2 = \mathbf{124 \text{ MPa}} \qquad \textbf{Ans.}$$

Longitudinal stress

We also know that longitudinal stress,

$$\sigma_l = \frac{pd}{4t\eta} = \frac{2.4 \times (1.3 \times 10^3)}{4 \times 18 \times 0.7} = 62 \text{ N/mm}^2 = \mathbf{62 \text{ MPa}} \qquad \textbf{Ans.}$$

EXAMPLE 3. *A gas cylinder of internal diameter 40 mm is 5 mm thick. If the tensile stress in the material is not to exceed 30 MPa, find the maximum pressure which can be allowed in the cylinder.*

SOLUTION. Given: Diameter of cylinder (d) = 40 mm ; Thickness of plates (t) = 5 mm and tensile stress (σ_c) = 30 MPa = 30 N/mm^2.

Let p = Maximum pressure which can be allowed in the cylinder.

We know that circumferential stress (σ_c),

$$30 = \frac{pd}{2t} = \frac{p \times 40}{2 \times 5} = 4p$$

$$\therefore \quad p = \frac{30}{4} = 7.5 \text{ N/mm}^2 = \mathbf{7.5 \text{ MPa}} \qquad \textbf{Ans.}$$

Change in Dimensions of a Thin Cylindrical Shell due to an Internal Pressure

Thin cylindrical shell subjected to an internal pressure, its walls will also be subjected to lateral strain. The effect of the lateral strains is to cause some change in the dimensions (i.e., length and diameter) of the shell. Now consider a thin cylindrical shell subjected to an internal pressure.

Let l = Length of the shell,

d = Diameter of the shell,

t = Thickness of the shell and p = Intensity of the internal pressure.

We know that the circumferential stress,

$$\sigma_c = \frac{pd}{2t}$$

and longitudinal stress, $$\sigma_l = \frac{pd}{4t}$$

Now let δd = Change in diameter of the shell,

δl = Change in the length of the shell and

$\frac{1}{m}$ = Poisson's ratio.

Now changes in diameter and length may be found out from the above equations, as usual (*i.e.*, by multiplying the strain and the corresponding linear dimension).

$$\therefore \quad \delta d = \varepsilon_1 \cdot d = \frac{pd}{2tE}\left(1 - \frac{1}{2m}\right) \times d = \frac{pd^2}{2tE}\left(1 - \frac{1}{2m}\right)$$

and $$\delta l = \varepsilon_2 \cdot l = \frac{pd}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right) \times l = \frac{pdl}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right)$$

EXAMPLE 4. *A cylindrical thin drum 800 mm in diameter and 4 m long is made of 10 mm thick plates. If the drum is subjected to an internal pressure of 2.5 MPa, determine its changes in diameter and length. Take E as 200 GPa and Poisson's ratio as 0.25.*

SOLUTION. Given: Diameter of drum (d) = 800 mm ; Length of drum (l) = 4 m = 4×10^3 mm ; Thickness of plates (t) = 10 mm ; Internal pressure (p) = 2.5 MPa = 2.5 N/mm^2 ; Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2 and poisson's ratio $\left(\frac{1}{m}\right) = 0.25$.

Change in diameter

We know that change in diameter,

$$\delta d = \frac{pd^2}{2tE}\left(1 - \frac{1}{2m}\right) = \frac{2.5 \times (800)^2}{2 \times 10 \times (200 \times 10^3)}\left(1 - \frac{0.25}{2}\right) \text{ mm}$$

$$= 0.35 \text{ mm} \quad \textbf{Ans.}$$

Change in length

We also know that change in length,

$$\delta l = \frac{pdl}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right) = \frac{2.5 \times 800 \times (4 \times 10^3)}{2 \times 10 \times (200 \times 10^3)}\left(\frac{1}{2} - 0.25\right) \text{ mm}$$

$$= 0.5 \text{ mm} \quad \textbf{Ans.}$$

Change in Volume of a Thin Cylindrical Shell due to an Internal Pressure

A little consideration will show that increase in the length and diameter of the shell will also increase its volume. Now consider a thin cylindrical shell subjected to an internal pressure.

Let l = Original length

d = Original diameter,

δl = Change in length due to pressure and

δd = Change in diameter due to pressure.

We know that original volume,

$$V = \frac{\pi}{4} \times d^2 \times l = \left[\frac{\pi}{4}(d + \delta d)^2 \times (l \times \delta l)\right] - \frac{\pi}{4} \times d^2 \times l$$

$$= \frac{\pi}{4}(d^2 \cdot \delta l + 2dl \cdot \delta d) \qquad \text{...(Neglecting small quantities)}$$

$$\therefore \quad \frac{\delta V}{V} = \frac{\frac{\pi}{4}(d^2 \cdot \delta l + 2dl \cdot \delta d)}{\frac{\pi}{4} \times d^2 \times l} = \frac{\delta l}{l} + \frac{2\delta d}{d} = \varepsilon_l + 2\varepsilon_c$$

or $\delta V = V(\varepsilon_l + 2\varepsilon_c)$

where ε_c = Circumferential strain and

ε_l = Longitudinal strain.

EXAMPLE 5. *A cylindrical vessel 2 m long and 500 mm in diameter with 10 mm thick plates is subjected to an internal pressure of 3 MPa. Calculate the change in volume of the vessel. Take E = 200 GPa and Poisson's ratio = 0.3 for the vessel material.*

SOLUTION. Given: Length of vessel (l) = 2 m = 2×10^3 mm ; Diameter of vessel (d) = 500 mm ; Thickness of plates (t) = 10 mm ; Internal pressure (p) = 3 MPa = 3 N/mm^2 ; Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2 and poisson's ratio $\left(\frac{1}{m}\right) = 0.3$.

We know that circumferential strain,

$$\varepsilon_c = \frac{pd}{2tE}\left(1 - \frac{1}{2m}\right) = \frac{3 \times 500}{2 \times 10 \times (200 \times 10^3)}\left(1 - \frac{0.3}{2}\right) = 0.32 \times 10^{-3} \qquad ...(i)$$

and logitudinal strain,

$$\varepsilon_l = \frac{pd}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right) = \frac{3 \times 500}{2 \times 10 \times (200 \times 10^3)}\left(\frac{1}{2} - 0.3\right) = 0.075 \times 10^{-3} \qquad ...(ii)$$

We also know that original volume of the vessel,

$$V = \frac{\pi}{4}(d)^2 \times l = \frac{\pi}{4}(500)^2 \times (2 \times 10^3) = 392.7 \times 10^6 \text{ mm}^3$$

$\therefore$ Change in volume,

$$\delta V = V(\varepsilon_c + 2\varepsilon_l) = 392.7 \times 10^6 [0.32 \times 10^{-3} + (2 \times 0.075 \times 10^{-3})] \text{ mm}^3$$

$$= \mathbf{185 \times 10^3 \text{ mm}^3} \quad \textbf{Ans.}$$