UNIT – IV

FOURIER SERIES

Periodic Function :

Definition : **A function f(x) is said to be periodic with period T , if $\forall$ x , f(x+T) = f(x) where T is positive constant.**

The least value of T > 0 is called the periodic function of f(x).

Example: sin x = sin $(2\pi + x) = \sin(4\pi + x) = ----- -$

Here sinx is periodic function with period 2π. ***Def:***

Piecewise Continuous Function:

A function is said to be piece-wise continuous (or) Sectionally Continuous) over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided into a finite number of sub intervals, in each of which f(x) is continuous and both right and left hand limits at every end point if the sub intervals.

Dirichlet Conditions:

A function f(x) satisfies Dirichlet conditions if

(1) f(x) is well defined and single valued except at a finite no. of points in (-l,l)

(2) f(x) is periodic function with period 2l

(3) f(x) and f'(x) are piece wise continuous in (-l,l)

Fourier Series: If f(x) satisfies Dirichlet conditions , then it can be represented by an infinite series called Fourier Series in an interval (-l,l) as

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos\frac{n\pi x}{l} + + \sum_{n=1}^{\infty} bn \sin\frac{n\pi x}{l}$ ------------ (1) where

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$, $an = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}$ dx

bn $= \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}$ dx

Here a_0 , an and bn are called Fourier coefficients.

These are also calle Euler's formula.

Note (1): If x∈ $(-\pi, \pi)$ (i.e., inteval is $(-\pi, \pi)$)

Then f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty}(an\, cosnx + bn \sin nx)$

Where a_0 = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, an = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Note (2): In interval $(0,2\pi)$, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Note (3): The Fourier Series in (-l,l), $(-\pi,\pi)$, $(0,2\pi)$, $(c, c+2\pi)$ are called Full range expansion series

Note (4): The above series (1) converges to f(x) if x is a point of continuity

The above series (1) converges to $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Note (5): At $x = \pm\pi$, $f(x) = \frac{f(\pi-0)+f(-\pi+0)}{2}$ here $x \in (-\pi,\pi)$

Even and odd functions:

Case (1): If the function f(x) is an even function in the interval (-l,l)

i.e., f(-x) = f(x) then $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

an = $\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}$ dx (since f(x) & $\cos\frac{n\pi x}{l}$ are even functions)

bn = $\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \Rightarrow bn=0$ (since f(x) . $\sin\frac{n\pi x}{l}$ is odd function)

Therefore, in this case we get (only) Fourier cosine series only.

Case (2): If function f(x) is odd i.e., f(-x) = - f(x) then

an = 0 (since f(x) $\cos\frac{n\pi x}{l}$ is odd) (a_0=0 also)

And bn = $\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

In this case we get fourier sine series only.

[only for intervals (-l,l) , ($-\pi,\pi$)] **Problems**

:

1)Find Fourier series for the function f(x) = e^{ax} in (0,2π) Solution : Given function f(x) = e^{ax} in (0,2π)

$$\frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\,dx = \frac{1}{\pi}\left(\frac{e^{ax}}{a}\right)$$

a_0 =) apply limits 0 to 2π

$$= \frac{1}{a\pi}\left(e^{2\pi a} - 1\right)$$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} \cos nx \, dx$ an =

$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\right.$ (a cos nx + n sin nx)] apply limits 0 to 2π

$= \frac{1}{\pi}\left[\frac{e^{2\pi a}}{a^2+n^2}\right.$ (a cos 2nπ + 0) - $\frac{e^0}{a^2+n^2}$

$= \frac{1}{\pi}\frac{1}{a^2+n^2}$ [$e^{2\pi a}$ a – 1. a]

$= \frac{a}{\pi(a^2+n^2)}$ ($e^{2\pi a}$ - 1)

$= \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} \sin nx \, dx$

$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\right.$ (a sin nx + n cos nx)] apply limits 0 to 2π

$= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}\right.$ (0 – n cos 2nπ) - $\frac{e^0}{a^2+n^2}$ (0 - n)]

$= \frac{1}{\pi}\frac{n}{a^2+n^2}$ (1 - $e^{2\pi a}$) = $\frac{-n}{\pi(a^2+n^2)}$ ($e^{2\pi a}$ - 1)

$\frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$

(a + 0)] apply limits 0 to 2π

bn

Now the fourier series is f(x) =
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} an \ \cos nx + \sum_{n=1}^{\infty} bn \ \sin nx$$

$$= \frac{\frac{1}{a\pi}(e^{2\pi a} - 1)}{2} + \sum_{n=1}^{\infty} \frac{a}{\pi(a^2+n^2)} (e^{2\pi a} - 1)$$

$$\frac{-n}{\pi(a^2+n^2)} (e^{2\pi a} - 1) \ \sin nx$$

cosnx + $\sum_{n=1}^{\infty}$

(2): Find Fourier series for the function f(x) = e^x in $(0,2\pi)$

Solution : Given function f(x) = e^x in $(0,2\pi)$ a_o =

apply $\frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^x\, dx$ limits 0 to 2π

$= \frac{1}{\pi}(e^x)$ apply limits 0 to 2π

$= \frac{1}{\pi} (e^{2\pi} - 1)$

an = $\frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\, dx$

bn

$= \frac{1}{\pi} [\frac{e^x}{1+n^2} (1 \cos nx + n \sin nx)]$

$= \frac{1}{\pi} [\frac{e^{2\pi}}{1+n^2} (\cos 2n\pi + 0) - \frac{e^0}{1+n^2} (\cos 0 + 0)]$

$= \frac{1}{\pi} \frac{1}{1+n^2} [e^{2\pi} - 1]$

$= \frac{1}{\pi(1+n^2)} (e^{2\pi} - 1)$

$= \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\, dx$

$= \frac{1}{\pi} [\frac{e^x}{1+n^2} (\sin nx + n \cos nx)]$ apply limits 0 to 2π

$= \frac{1}{\pi} [\frac{e^{2\pi}}{1+n^2} (0 - n \cos 2n\pi) - \frac{e^0}{1+n^2} (0 - n)]$

$= \frac{1}{\pi} \frac{n}{1+n^2} (1 - e^{2\pi}) = \frac{-n}{\pi(1+n^2)} (e^{2\pi} - 1)$

Now the fourier series is f(x) =

$\frac{a_0}{2} + \sum_{n=1}^{\infty} an\ \cos nx + \sum_{n=1}^{\infty} bn\ \sin nx$

$= \frac{\frac{1}{\pi}(e^{2\pi}-1)}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi(1+n^2)}\ (e^{2\pi} - 1)\ \cos \text{nx} + \sum_{n=1}^{\infty}$

$\frac{-n}{\pi(1+n^2)}\ (e^{2\pi} - 1)\ \sin nx$

Problem (3): H.W

Find Fourier series for the function f(x) = e^{-x} in $(0,2\pi)$

(Hint:- put a = - 1 in problem (1) we get the solution.)

(4) Express f(x) = x - π as Fourier Series in the interval - π < x < π Solution:

Given function f(x) = x - π a_o

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - \pi)\ dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\ \ dx - \frac{1}{\pi}\int_{-\pi}^{\pi} \pi\, dx$

= 0 – [x] with limits - π to π

= 0 – [π + π] = 2π an =

dx $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - \pi) \cos nx\, dx$ (since even)

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\, dx$ - $\frac{1}{\pi}$(- π) $\int_{-\pi}^{\pi} \cos nx\, dx$

$= \frac{1}{\pi}$(0) (since x cosnx is odd) + 2$\int_{0}^{\pi} cosnx$

= 0 + 2 [$\frac{\sin nx}{n}$] 0 to π limits apply we get an = 0+0 = 0

bn $= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - \pi) \sin nx\, dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$ - $\frac{1}{\pi}$(- π) $\int_{-\pi}^{\pi} \sin nx\, dx$

(even) (odd)

$= \frac{1}{\pi}$ 2$\int_{0}^{\pi} x\, sin\, nx$ dx – 0 (since sin nx is odd)

$= \frac{2}{\pi}$ [{ x(- $\frac{\cos nx}{n}$)} - $\int_{0}^{\pi} \frac{-\cos nx}{n}$ dx]

$= \frac{2}{\pi}$ [-$\pi \frac{\cos n\pi}{n}$ + 0 + $\frac{1}{n}$ ($\frac{\sin nx}{n}$)] apply limits 0 to π

$= \frac{2}{\pi}$ [-$\pi \frac{\cos n\pi}{n}$ + 0 + $\frac{1}{n}$ (0)] = - $\frac{2}{n}$ cos nπ = - $\frac{2}{n}(-1)^n$ = $\frac{2}{n}(-1)^{n+1}$, n=1,2,3...........

Now the Fourier Series of f(x) is f(x)

$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(an\ cosnx + bn\sin nx)$ f(x)

$= \frac{2\pi}{2} + \sum_{n=1}^{\infty}[(0)\ cosnx + \frac{2}{n}(-1)^{n+1}\sin nx]$

$= \pi + \sum_{n=1}^{\infty}[\frac{2}{n}(-1)^{n+1}\sin nx]$

(5)Obtain the Fourier series for f(x) = x - x² in the interval $[-\pi, \pi]$ $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{n}{1^2}$ 2

Hence show

that (or)

$\pi 2$

$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots =$

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Solution : Given function is f(x) = x - x² in $[-\pi, \pi]$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\ dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\ dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ dx$

$= 0\ (odd)\ - \frac{1}{\pi}\ [\frac{x^3}{3}] = -2\pi^2/3$

an = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

= $\frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$

= $\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$ - $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$

(odd) (even)

= 0 - $\frac{1}{\pi}2\int_{0}^{\pi} x^2\cos nx\,dx$

u = x^2 , dv = cos nx dx

= - $\frac{2}{\pi}$[($\frac{x^2\sin nx}{n}$) - $\frac{2}{n}\int_{0}^{\pi} x\sin nx\,dx$]

du = 2x dx , dv = ⍰ $\cos nx\,dx$

apply limits 0 to π

= - π2 [0 - π2 {($\overline{-x\cos n\ nx}$) + 0⍰π cos$n$ nx = $\frac{}{}$ n sin nx dx

apply limits 0 to π

= $\frac{4}{\pi n}$ [-$\pi\frac{(-1)^n}{n}$ + $\frac{1}{n^2}$

= $\frac{4}{n^2}(-1)^{n+1}$

4 (sin nx)] ⍰ $udv = \frac{4}{1^2} = 4$ $uv -$ ⍰ vdu

an = if n is odd a1 =

n^2

$-\frac{4}{n^2}$ if n is even

a2 = $\frac{4}{2^2}$ =1

a3 = $\frac{4}{3^2}$ = 4/9

bn = $\frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\, dx$

= $\frac{1}{\pi}$ [$\int_{-\pi}^{\pi} x \sin nx\, dx$ - $\int_{-\pi}^{\pi} x^2 \sin nx\, dx$

= $\frac{2}{\pi}$ [($\frac{-x\cos nx}{n}$) + $\frac{1}{n}\int_0^{\pi}$ $\cos nx\, dx$]

(even) (odd)

(= $\frac{2}{n}$ [$-\pi\frac{(-1)^n}{n}$ + $\frac{1}{n^2}$

sin nx)] b1 = 2/1 = 2 = $\frac{2}{n}(-1)^{n+1}$ = $\frac{2}{n}$ if n is

odd

b2

= -

2/2 = -1

= - $\frac{2}{n}$ if n is even

b3 = 2/3

= $\frac{a_0}{2} + \sum_{n=1}^{\infty}(an\, cosnx + bn \sin nx)$------(1)substitute

Now

, f(x)

$\Rightarrow f(x) = \frac{-\pi^2}{3}$ + 4 ($\frac{\cos x}{1^2}$ - $\frac{\cos 2x}{2^2}$ + $\frac{\cos 3x}{3^2}$ $-$ ... .)

+ 2 ($\frac{\sin x}{1}$ $-$ $\frac{\sin 2x}{2}$ + $\frac{\sin 3x}{3}$ +.......)--------(2)

in

(1)

put x = 0 in (2)

f(0) = 0 = $\frac{-\pi^2}{3}$+ 4($\frac{1}{1^2}$ - $\frac{1}{2^2}$ + $\frac{1}{3^2}$.......)

π^2

⇒ $\frac{1}{1^2}$ - $\frac{1}{2^2}$ + $\frac{1}{3^2}$ + ...=

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Half range series

(1) The half range cosine series in (0,l) is f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \ \cos\frac{n\pi x}{l}$

$a_0 = \frac{2}{l}\int_0^l f(x)dx$, $an = \frac{2}{l}\int_0^l f(x)\ \cos\frac{n\pi x}{l}$ dx

(2)The half range sine series in (0,l) is f(x) = $\sum_{n=1}^{\infty} b\ \sin\frac{n\pi x}{l}$

where bn = $\frac{2}{l}\int_0^l f(x)\ \sin\frac{n\pi x}{l}$ dx

Note :1) The half range cosine series in (0,π) is f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$, $an = \frac{2}{\pi}\int_0^{\pi} f(x)\ \cos n\pi$ dx

where

Note :2) The half range sine series in (0,π) is f(x) = $\sum_{n=1}^{\infty} bn \sin nx$ where

bn = $\frac{2}{\pi}\int_0^{\pi} f(x)\ \sin nx\, dx$

(1)Express $f(x) = \pi - x$ as Fourier cosine and sine series in $(0, \pi)$

Solution :

The half range cosine series for f(x) is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos nx$ ....(1)

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} \pi - x\, dx$

$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]$ apply limits o to π

$= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - (0-0)\right] = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi$

an = $\frac{2}{\pi}\int_0^{\pi} f(x) \cos n\pi\, dx$

$= \frac{2}{\pi}\int_0^{\pi} (\pi - x) \cos n\pi\, dx$

$= \frac{2}{\pi}\left[\left\{(\pi - x)\frac{\sin nx}{n}\right\} + \int_0^{\pi} \frac{\sin nx}{n}\, dx\right]$

(apply o to π)

$= \frac{2}{\pi}\left[(0-0) + \frac{1}{n}\left(-\frac{\cos nx}{n}\right)\right]$ apply o to π

$= -\frac{2}{\pi n^2}[\cos n\pi - \cos 0]$

$= -\frac{2}{\pi n^2}[\cos n\pi - \cos 0]$

$= -\frac{2}{\pi n^2}[[(-1)^n - 1] = \frac{2}{\pi n^2}[[1 - (-1)^n]$

Now (1) $\Rightarrow$ f(x) = $\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[[1 - (-1)^n] \cos nx$

H.W.) Express $f(x) = \pi - x$ as fourier sine series in (o, π Ans : $2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ $(bn = \frac{2}{n})$

2) Find the half range sine series of f(x) = x in the range $0 < x < \pi$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ .... =$

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Solution : The half range cosine series for f(x) is f(x)

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos nx$ ....(1)

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)$ dx = $\frac{2}{\pi}\int_0^{\pi}$ x dx = $\frac{2}{\pi}\,[\frac{x^2}{2}]$ apply limits o to π

$= \pi$

an = $\frac{2}{\pi}\int_0^{\pi} f(x)\ \cos nx\ \mathrm{dx}$

$= \frac{2}{\pi}\int_0^{\pi} (x) \cos nx\ \mathrm{dx}$

$= \frac{2}{\pi}\,[\{(x)\ \frac{\sin\ nx}{n}\}\ -\ \int_0^{\pi} \frac{\sin\ nx}{n}\ \mathrm{dx}]$

(apply o to π)

$= \frac{2}{\pi}\,[\ (0\text{-}0) - \frac{1}{n}\,(- \frac{\cos nx}{n}$

$= \frac{2}{\pi n^2}\,[\cos n\pi - \cos 0]$

$= \frac{2}{\pi n^2}\,[\ [(-1)^n - 1]\ )]$ apply o to π

an = 0 if n is even

$= -\dfrac{4}{\pi\, n^2}$ if n is odd

Now (1) $\Rightarrow$: f(x) = $\dfrac{\pi}{2} + \sum_{n=1}^{\infty} \dfrac{-4}{\pi n^2} \cos nx$ if n is odd

x = $\dfrac{\pi}{2} - \dfrac{4}{\pi} \left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} - \dots\dots.. \right)$

Put x=0 on both sides

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \dots\dots.. \right)$$

$$\Rightarrow \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \dots\dots.. \right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\dots = \frac{\pi^2}{8}$$

3) Express f(x) = $\cos x$, 0 < x < π in half range sine series

π $\qquad \sum_{n=1}^{\infty} bn \sin nx$ -------(1)

$= \frac{2}{\pi} \int_0^{\pi} f(x)\ \sin nx\ dx$

$= \frac{2}{\pi} \int_0^{\pi} \cos x\ \sin nx\ dx$

$= \frac{2}{\pi} \frac{1}{2} \int_0^{\pi} [\sin (n+1)x + \sin(n-1)x]\ dx$

$= \frac{1}{\pi} [\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}$] apply limits o to π

$= \frac{1}{\pi} [\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}$

$= \frac{1}{\pi} [\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}$

$= \frac{1}{\pi} [(\frac{(-1)^{2}(-1)^{n}}{n+1} + \frac{(-1)^{n}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}]$

$= \frac{1}{\pi}[(-1)^{n}\{ \frac{1}{n+1} + \frac{1}{n-1}\} + \{\frac{1}{n+1} + \frac{1}{n-1}\}]$

$= \frac{1}{\pi} [\{(-1)^{n}+1\}(\frac{1}{n+1} + \frac{1}{n-1})]$

$= \frac{2n}{\pi} [\frac{1+(-1)^{n}}{n^{2}-1}]$ (n not equal to 1)

Solution : The half range sine series in (0,) is f(x) = where

bn

]

] , n is not equal to 1

bn = 0 if n is odd.

$= \frac{4n}{\pi(n^2-1)}$ if n is even

b1 = b3 = b5 = -------- = 0

(1) $\Rightarrow$ f(x) = $\sum_{n=2}^{\infty} \frac{4n}{\pi(n^2-1)} \sin nx$, for n is even

4)Find half range sine series for f(x) = x(π −x) , in 0 < x < π

Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}$ **+…….=** $\frac{\pi^3}{32}$

Solution : Fourier series is f(x) = $\sum_{n=1}^{\infty} bn \sin nx$...(1)bn

$\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx\, dx$

$= \frac{2}{\pi}\pi\int_0^{\pi} x \sin nx\, dx - \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx$

$= \; = 2\left[\left(\frac{-x\, cosnx}{n}\right) - \int_0^{\pi}\frac{-\cos nx}{n}dx\right] - \frac{2}{\pi}\left[\left(\frac{-x^2\, cosnx}{n}\right) - \int_0^{\pi}\frac{-\cos nx}{n} 2x\, dx\right]$

(apply

0 to π) $= 2\left[\left(\frac{-\pi\, cosn\pi}{n}\right) + 0 + \frac{1}{n}\left(\frac{\sin\, nx}{n}\right) 0 \text{ to } \pi\right] - \frac{2}{\pi}\left[\left(\frac{-\pi^2\, cosn\pi}{n}\right) + \mathbf{0} + \frac{2}{n}\int_0^{\pi} x\cos nx\;\; dx\right]$

(apply

o to π) $= 2\left[-\pi\frac{(-1)^n}{n} + 0\right] + \frac{2}{\pi}.\pi^2\frac{(-1)^n}{n} - \frac{4}{\pi n}\left[\left(\frac{x\sin\, nx}{n}\right) 0 \text{ to } \pi - \int_0^{\pi}\frac{\sin\, nx}{n}\, dx\right.$

$= 2\left[-\pi\frac{(-1)^n}{n}\right] + 2\pi\frac{(-1)^n}{n} + \frac{4}{\pi n^2}\left(\frac{-\, cosnx}{n}\right) 0 \text{ to } \pi$

$= \frac{4}{\pi n^3}[-\cos n\pi + \cos 0]$

$= \frac{4}{\pi n^3}\left[\left[1-(-1)^n\right]\right.$ sub in (1)

bn

(1) ⇒ f(x) = $\sum_{n=1}^{\infty} \frac{4}{\pi n^3} [\,[1-(-1)^n\,]\sin nx$

(1) ⇒ f(x) = b1 sin x + b2 sin 2x + b3 sin 3x + ...

$= \frac{4}{\pi}\,(2)\sin x + 0 + \frac{4}{\pi.3^3}$

⇒ $x(\pi - x) = \frac{8}{\pi}\,[\,\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + ...]$(2) sin 3x + ... Put

x = π/2 on both sides

$\underline{\pi}\,\underline{\pi}$ [− ^{3} + ...] ⇒

$(2)^{=}$

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$\frac{8}{\pi}$ $\frac{1}{1}$ $\frac{1}{3}$

⇒ $\underline{\pi}4\,\underline{2}\,(\underline{\pi}\,8\,)$ = [^{3} − ^{3} + 5^3 .....]

$\frac{1}{1}$ $\frac{1}{3}$ $\frac{1}{}$

1 3

⇒ [$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3}$ .....] = $\frac{\pi^3}{32}$

- **<u>FOURIER SERIES IN AN ARBITRARY INTERVAL I,e in (-l,l</u>) & (0,2l)**

- **Problem : 1) Obtain the half range sine series for e^x in 0<x<1 Solution** : Given f(x) = e^x in (0,l)

The half range sine series for f(x) in (0,l) is f(x)= $\sum_{n=1}^{\infty} bn \ \sin\frac{n\pi x}{l}$ ......(1)

l=1 Where bn $= \frac{2}{l}\int_0^l f(x)\ \sin\frac{n\pi x}{l}\ dx$

$= \frac{2}{1}\int_0^1 f(x)\ \sin n\pi x\ dx$

bn

$= 2\int_0^1 e^x\, \sin(n\pi x)\, dx$

$=2\,\frac{e^x}{(1)^2+(n\pi)^2}$ (sin $n\pi x$ - $n\pi$. cos $n\pi x$) apply limits 0 to 1

$= \frac{2}{1+n^2\pi^2}[e^1(0 - n\pi . \cos n\pi) - e^0(0 - n\pi . \cos 0)]$

$= \frac{2}{1+n^2\pi^2}[-n\pi . e . \cos n\pi + n\pi]$

$= \frac{2}{1+n^2\pi^2}[-n\pi e(-1)^n + n\,\pi]$

$= \frac{2n\pi}{1+n^2\pi^2}[1 - e(-1)^n]$

bn

(1) $\Rightarrow$ f(x)= $\sum_{n=1}^{\infty}\frac{2n\pi}{1+n^2\pi^2}[1 - e(-1)^n]\ \sin n\pi x$

2) **Find the half range sine series of f(x) = 1 in (0,l)** **Solution** : The half range sine series in (0,l) is f(x) =

$$\sum_{n=1}^{\infty} bn \ \sin\frac{n\pi x}{l} \ ....(1)$$

$$= \frac{2}{l} \int_0^l f(x) \ \sin\frac{n\pi x}{l} \ dx$$

where bn

$$= \frac{2}{l} \int_0^l 1 \sin\frac{n\pi x}{l} \ dx$$

$$= \frac{2}{l} \left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right] \text{ apply limits o to l}$$

$$= -\frac{2}{l} \cdot \frac{l}{n\pi} [\cos n\pi - \cos 0]$$

$$= -\frac{2}{n\pi} [(-1)^n - 1]$$

bn = 0 if n is even

if n is odd

Now (1) $\rightarrow \sum_{n=1} \frac{}{n\pi} \sin \frac{}{l}$, if n is odd

3)Find the half range cosine series of f(x) = x(2-x) in the range $0 \leq x \leq 2$

Hence find sum of series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$

Solution : Given function f(x) = x(2-x) = 2x - x^2

The half range cosine series for f(x) is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos\frac{n\pi x}{l}$ ....(1)

where $a_0 = \frac{2}{2}\int_0^2 f(x)) \, dx = \frac{2}{2}\int_0 \int_0^2 f(x) \; 2x - x^2) \, dx$

$= \frac{2}{2}\left[\frac{2x^2}{2} - \frac{2x^3}{3}\right]$ apply 0 to 2 $= -\frac{4}{3}$

$an = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l} \, dx$

$= \frac{2}{2}\int_0^2 f(x) \cos\frac{n\pi x}{2} \, dx$ (l=2)

$= \int_0^2 (2x - x^2) \cos\frac{n\pi x}{2} \, dx$ (using integration by parts)

$= [(2x - x^2) \frac{2}{n\pi}\{\sin\frac{n\pi x}{2} + (2-2x)\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2} + (2)\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}]$

apply limits 0 to 2

$= \frac{-8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2} = \frac{-8}{n^2\pi^2}[1 - (-1)^n]$

$\frac{-16}{n^2\pi^2}$ when n is even an =

= 0 when n is odd

Substitute the values of a_o and an in (1) we get

$$(1) \Rightarrow 2x - x^2 = \frac{2}{3} - \frac{16}{\pi^2}\sum_{n=2,4,6}^{\infty}\left(\frac{1}{n^2}\cos\frac{n\pi x}{2}\right)$$

$$= \frac{2}{3} - \frac{16}{\pi^2}\left(\frac{1}{2^2}\cos \pi x + \frac{1}{4^2}\cos 2\pi x + \frac{1}{6^2}\cos 3\pi x + \text{------}\right)$$

$$= \frac{2}{3} - \frac{16}{\pi^2}.\frac{1}{2^2}\left(\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \text{-------}\right)$$

$$\Rightarrow 2x - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \text{-------}\right)\text{-------}(2)$$

Putting x = 1

in (2) we get

$$2 - 1 = \frac{2}{3} - \frac{4}{\pi^2}\left(\cos \pi + \frac{1}{2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \frac{1}{4^2}\cos 4\pi + \text{-------}\right)$$

$$\Rightarrow 1 - \frac{2}{3} = -\frac{4}{\pi^2}\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\ldots\right)$$

$$\Rightarrow \frac{1}{3} = \frac{4}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots\right)$$

$$+ \Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\ldots\ldots\ldots) = \frac{\pi 2}{12}$$

(4) Expand f(x) = e^{-x} *as Fourier series* in (-1,1)

Solution : Here l = 1

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$= \frac{1}{1}\int_{-1}^{1} e^{-x}dx$ $=(\frac{e^{-x}}{-1})$apply limits -1 to 1

$= - e^{-1} + e^{1} = e - \frac{1}{e} = 2 \sinh 1$

an = $\frac{1}{l}\int_{-l}^{l} f(x)\ \cos\frac{n\pi x}{l}\ dx$

$= 1\int_{-1}^{1} e^{-x}\cos(n\pi x)$ dx

$= \frac{e^{-x}}{(-1)^2+(n\pi)^2}$ (- cos $n\pi x$ + nπ . sin $n\pi x$) apply limits -1 to 1

$= \frac{1}{1+n^2\pi^2} [e^{-1}\{-(-1)^n + 0\} - e^{1}\{-(-1)^n + 0\}]$

$= \frac{1}{1+n^2\pi^2} (-1)^n (\mathrm{e} - e^{-1})$

$= \frac{1}{1+n^2\pi^2} (-1)^n$ 2sinh1

bn $= \frac{1}{l} \int_{-l}^{l} f(x)\ \sin\frac{n\pi x}{l}\ \mathrm{dx}$ bn

$= \frac{1}{l} \int_{-l}^{l} f(x)\ \sin\frac{n\pi x}{l}\ \mathrm{dx}$ bn

$= \frac{1}{l} \int_{-l}^{l} f(x)\ \sin\frac{n\pi x}{l}\ \mathrm{dx}$ (l=1)

$= \int_{-1}^{1} e^{-x} \sin(n\pi x)$ dx

$= \frac{e^{-x}}{(-1)^2+(n\pi)^2}$ (- sin $n\pi x$ - $n\pi$. cos $n\pi x$) apply limits -1 to 1

$= \frac{1}{1+n^2\pi^2} [e^{-1}(0 - \mathrm{n}\pi$. cos nπ) - e^{1}(0 - nπ . cos nπ)]

$= \frac{1}{1+n^2\pi^2}$ nπ . cos nπ $(\mathrm{e} - e^{-1})$

$= \frac{1}{1+n^2\pi^2}$ n$\pi(-1)^n$ 2sinh1

Now Fourier series of f(x) in (-l,l) is

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos n\,\pi x + + \sum_{n=1}^{\infty} bn \sin n\pi x$ ....(1)

f(x) = $\frac{2\sinh 1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}(-1)^n\, 2\sinh 1 \cos n\pi x + \sum_{n=1}^{\infty}$ $\frac{1}{1+n^2\pi^2}$ $n\pi(-1)^n\, 2\sinh 1 \sin n\pi x$

$\Rightarrow$ f(x) = 2 sinh1 + [$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}(-1)^n \{\cos n\pi x + n\pi \sin n\pi x\}$]

- Functions having points of discontinuity : Problems:

(1) If f(x) is a function with period 2π is defined by f(x) =

0 , for - π < x ≤ 0

$= x$, for $0 \leq$ **x < π then write the fourier series for f(x)**

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$

Solution : The Fourier series in (- π, π) is **f(x) =**

$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ---------(1)

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$

$= \frac{1}{\pi}[\,0 + \int_0^{\pi} x\,dx\,] = \frac{1}{\pi}(\frac{x^2}{2})$ 0 to π = $\frac{\pi}{2}$

an = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}[\,0 + \int_0^{\pi} x\cos nx\,dx\,]$ ⍰ $u\,dv$ = uv - ⍰ $v\,du$

$= \frac{1}{\pi n^2}[(-1)^n - 1]$ u = x , dv = cos nx dx = 0 , if n is even

$= -\frac{2}{\pi n^2}$, if n is odd

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

bn $= \frac{1}{\pi}[\,0 + \int_0^{\pi} x\sin nx\,dx\,]$

$= \frac{1}{\pi}[\,(\frac{-x\cos nx}{n}) - \int_0^{\pi}\frac{-\cos nx}{n}\,dx]$ (apply 0 to π)

$= \frac{1}{\pi}[\,(\frac{-\pi\cos n\pi}{n}) + 0 + \frac{1}{n}(\frac{\sin nx}{n})$) 0 to π]

$= \frac{1}{\pi}[\frac{-\pi(-1)^n}{n} + 0 + 0 = -\frac{(-1)^n}{n}$

bn $= \frac{1}{n}$, if n is odd

$= -\frac{1}{n}$, if n is even

(1) $\Rightarrow$ f(x) = $\frac{1}{2}\,\frac{\pi}{2} - \frac{2}{\pi}[(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \text{-----}) + (\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\ \text{------})$------(2)

Put x = 0 on both sides f(0) = 0

$(2) \Rightarrow 0 = \frac{\pi}{4}\ - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{------}\right.$

$\frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{------}\right) = \frac{\pi}{4}$

$\Rightarrow \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{------}\right) = \frac{\pi^2}{8}$) + 0

Problem (2) : Find Fourier series to represent the function f(x) given by

f(x) = - k , for $-\pi < x < 0$

k , for $0 < x < \pi$ hence show

that $1\ \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{------} = \frac{\pi}{4}$ Solution : In

$-\pi < x < 0$

i.e., $x \in (-\pi, 0)$, f(x) = - k

f(-x) = - f(x) in $(0, \pi)$

In $0 < x < \pi$ i.e., $x \in (0, \pi)$ f(x)

= k f(-x) = k = -

(-k)

= - f(x) in $(-$

π,0) There fore f(x) is odd function in $(-\pi, \pi)$

so $a_0 = 0$, an = 0

bn $= \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi}\int_0^{\pi} k \sin nx\, dx$

$= \frac{2k}{\pi} \left(\frac{-\cos nx}{n}\right.$

$= \frac{2k}{\pi n} [(-1)^n - 1]$

bn

) apply limits 0 to π

= 0 , if n is even

$= \frac{4k}{\pi n}$, if n is odd

Now f(x) = $\sum_{n=1}^{\infty} bn \sin nx$

= $b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x$ --------f(x)

$= \frac{4k}{\pi}\sin x + 0 + \frac{4k}{\pi}\frac{\sin 3x}{3} + 0 +$ ---------(1)

Deduction : put x = $\frac{\pi}{2}$ on both sides in (1)

(1) $\Rightarrow$ $k = \frac{4k}{\pi}(1) + \frac{4k}{\pi}(-\frac{1}{3}) + \frac{4k}{\pi}(\frac{1}{5}) +$ --------

$\Rightarrow k = \frac{4k}{\pi}[1 - \frac{1}{3} + \frac{1}{5}$ ---------

$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5}$ ---------

<u>Parseval's Formula</u> :-

Prove That $\int_{-l}^{l}[f(x)]^2$ dx = l [$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(an^2 + bn^2)$

Proof :- We know that the Fourier series of f(x) in (-l,l) is **f(x)**

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} bn \sin\frac{n\pi x}{l}$ -----------**(1)**

Multiplying on both sides of (1) by f(x) and integrate term by term from -l to l we get $\int_{-l}^{l}[f(x)]^2$ dx =

$$\frac{a_0}{2}\int_{-l}^{l} f(x)dx + \sum_{n=1}^{\infty} an \int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx$$

$$+ \sum_{n=1}^{\infty} bn \int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx \text{ --------(2)}$$

Now $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \Rightarrow \int_{-l}^{l} f(x)dx = l\, a_0$

an = $\frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx \Rightarrow \int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx$ = l an

and bn = $\frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx \Rightarrow \int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx$ = l bn

Substitute these in (2)

$$\frac{a_0}{2}.\mathrm{l}\ a_0 + \sum_{n=1}^{\infty} an\ .\ \mathrm{l}\ an + \sum_{n=1}^{\infty} bn\ .\ \mathrm{l}\ bn$$

$$(2) \Rightarrow \int_{-l}^{l}[f(x)]^2\,dx = \mathrm{l}\left[\frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty}(an^2 + bn^2)\right]$$

This is called parseval's formula.

Note 1): In (0,2l) the parseval's formula is

$$\int_0^{2l}[f(x)]^2\,dx = \mathrm{l}\left[\frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty}(an^2 + bn^2)\right]$$

Note :2) If 0 < x < l (for half range cosine series of f(x)) parsevel's formula is

$$\int_0^{l}[f(x)]^2\,dx = \frac{l}{2}\left[\frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty} an^2\right]$$

Note :3) If 0 < x < l (for half range sine series of f(x)) parsevel's formula is

$$\int_0^{l}[f(x)]^2\,dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} bn^2\right]$$

Problem : prove that in 0 < x < l, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\frac{\cos\frac{\pi x}{l}}{1^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} + \text{--------}\right)$ and hence deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \text{------} = \frac{\pi}{96}$$

Solution : Let f(X) = x , 0 < X < l

The Fourier cosine series for f(x) in (0,l) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \text{ ------- (1)}$$

Here $a_0 = \frac{2}{l}\int_0^l f(x)\, dx$

$= \frac{2}{l}\int_0^l x \, dx$

$= \frac{2}{l}\left[\frac{x^2}{2}\right]$ apply limits 0 to l

$= \frac{2}{l}\left[\frac{l^2}{2}\right] = l$

an = $\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\, dx$

$= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\, dx$

u = x, dv = $\cos\frac{n\pi x}{l}$ dx

$= \frac{2}{l}\left[\left\{\frac{x\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_{0 \text{ to } l} - \int_0^l \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\, dx\right]$

$= \frac{2}{l}\cdot\frac{l}{n\pi}\left[(0-0) - \left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_{0 \text{ to } l}\right]$

$= \frac{2}{n\pi}\cdot\frac{l}{n\pi}\left[\cos n\pi - \cos 0\right]$

$= \frac{2l}{n^2\pi^2}\left[\left[(-1)^n - 1\right]\right]$

$-4l$ $-4l$ an = 0 ,

n is even $a_1 = \pi^2.1^2$, $a_3 = \pi^2.3^2$

$= \frac{-4l}{n^2\pi^2}$, n is odd $a_2 = 0$, $a_4 = 0$ ....

Substitute a_o,an in (1)

$$(1) \Rightarrow \frac{l}{2} . \frac{-4l}{\pi^2} \left(\frac{\cos\frac{\pi x}{l}}{1^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} + \text{--------}\right)$$

Now $a_o = l$, $a_1 = \frac{-4l}{\pi^2.1^2}$, $a_3 = \frac{-4l}{\pi^2.3^2}$ -------

From parseval's formula , we have

$$\int_0^l [f(x)]^2 \, dx = \frac{l}{2} \left[\frac{a_0^2}{2} \right.$$

$$\Rightarrow \int_0^l x^2 \qquad \frac{l}{2} \left[\frac{l}{2} + \frac{16l}{\pi^4.1^4} + a_1^2 + \frac{16l}{\pi^4.3^4} + \text{---------}\right] \quad a_2^2 + a_3^2 + \text{-------}]$$

2 2 2

dx = + 0^2 +

$$\Rightarrow \left(\frac{x^3}{3} \quad l \qquad \frac{1}{2} + \frac{16}{\pi^4.1^4} + \frac{16}{\pi^4.3^4} + \text{-------}\right]$$

) 0 to l = .

l^2 [2

$\Rightarrow \frac{1}{3}(2l^3).\ \frac{2}{l^3} = \frac{1}{2} + \frac{16}{\pi^4.1^4} + \frac{16}{\pi^4.3^4} + \text{-------}$

$\Rightarrow \frac{2}{3} - \frac{1}{2} = \frac{16}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \text{------}\right)$

$\Rightarrow \frac{1}{6}\ .\frac{\pi^4}{16} = \frac{1}{1^4} + \frac{1}{3^4} + \text{------}$

There fore $\frac{1}{1^4} + \frac{1}{3^4} + \text{------} = \frac{\pi^4}{96}$

COMPLEX FOURIER SERIES in (-l,l) or (0,2l):-

The complex form of Fourier series of a periodic function f(x) of period 2l is defined by

f(x) = $\sum_{n=-\infty}^{\infty} cn\ e^{\frac{in\pi x}{l}}$ --- (1) where $= \frac{1}{2l}\int_{-l}^{l} f(x)\ e^{\frac{-in\pi x}{l}}$ cndx , n=0,-1,1,2....

Note (1) : If period of function is 2π, i.e., in ($-\pi$, π) or (0,2 π) then complex fourier series is f(x) = $\sum_{n=-\infty}^{\infty} cn\ e^{inx}$ -----(2)

Where cn $= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}$ dx , n = 0,-1,1,-2,2 ....

Problem : Find complex fourier series of f(x) = e^x if $-\pi < x < \pi$ and f(x) = f(x + 2 π)

Solution : Complex fourier series of f(x) = e^x is f(x) = $\sum_{n=-\infty}^{\infty} cn\ e^{inx}$ ----(1)

When cn = $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}$ dx

cn = $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}\ e^{-inx}$ dx = $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}$

= $\frac{1}{2\pi}[\frac{e^{(1-in)x}}{1-in}]$ limits ($-\pi, \pi$) = $\frac{1}{2\pi(1-in)}[e^{(1-in)\pi} - e^{(1-in)(-\pi)}]$

= $\frac{1}{2\pi(1-in)}[e^{\pi}.\ e^{-in\pi} - e^{-\pi}.\ e^{in\pi}]$

= $\frac{1}{2\pi}.\ \frac{1}{(1-in)}[e^{\pi}.\ (-1)^{n} - e^{-\pi}.\ (-1)^{n}$

= $\frac{1}{2\pi(1-in)}(-1)^{n}(e^{\pi} - e^{-\pi})$

$\pm in\pi$ = $(-1)^{n}$+-0

1 $1+in$

] e = cos n π + I

sin n π

$(1-in$) $1+in$ *

$= \frac{(-1)^n}{2\pi} . \frac{1+in}{(1+n^2)}$. (2 sin h π)

Therefore cn $= (-1)^n \cdot \frac{1+in}{\pi(1+n^2)}$ (sin h π) sub in (1)

(1) $\Rightarrow$ f(x) $= \sum_{n=-\infty}^{\infty} (-1)^n . \frac{1+in}{\pi(1+n^2)}$ (sin h π) e^{inx}

Problem : Find the complex form of the fourier series of f(x) = e^{-x} ,-1 $\le$ x $\le$ 1

here(l=1)

Solution : The complex fourier series of f(x) in (-1,1) is

f(x) = $\sum_{n=-\infty}^{\infty} cn\, e^{\frac{in\pi x}{l}}$ -----(1)

Where cn = $\frac{1}{2}\int_{-1}^{1} e^{-x}\ e^{-in\pi x}$ dx = $\frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}$ dx

$= \frac{1}{2} [\frac{e^{-(1+in\pi)x}}{-(1+in\pi)x}$] limits(-1,1)

$= -\frac{1}{2} . \frac{1}{1+in\pi} [e^{-(1+in\pi)} - e^{(1+in\pi)}$

$= \frac{1}{2} [\frac{1-in\pi}{1+\pi^2 n^2}] [e^{(1+in\pi)} - e^{-(1+in\pi)}]$

$= \frac{1}{2} [\frac{1-in\pi}{1+\pi^2 n^2}]$ [e. $e^{in\pi} - e^{-1} . e^{-in\pi}$]

$= \frac{1}{2} [\frac{1-in\pi}{1+\pi^2 n^2}] [(-1)^n$ (e - e^{-1})]

$= \frac{1}{2} (-1)^n ([\frac{1-in\pi}{1+\pi^2 n^2}]$ 2 sin h)

(1) $\Rightarrow$ f(x) $= \sum_{n=-\infty}^{\infty} (-1)^n [\frac{1-in\pi}{1+\pi^2 n^2}]$ sin h . $e^{-in\pi x}$