LAQ

Numerical Methods & Probability Theory

(NMPT) (19A54401)

B-Tech - II - II Sem

R-19 Regulation

(Common to EEE & MECH)

Syllabus

Unit - II

Interpolation

Finite differences - Newton's Forward and Backward Interpolation formulae - Lagrange's formulae
Gauss Forward and Backward formula,
Stirling's formula, Bessel's formula

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Unit-II

Interpolation

Definition :-

Interpolation is the Process of finding out of the unknown value which lies in the given set of tabulated values

(or)

To add a value into a Series by Calculating it from surrounding known values is known as Interpolation

(or)

Interpolation is the method of Constructing new data points within the range for given set of points

(or)

The word Interpolation denotes the Process of Computing the value of a function $y = f(x)$, where x is the Independent Variable taking values $x_0, x_1, x_2 \ldots x_n$ and

y is the dependent Variable taking values $y_0, y_1, y_2 \ldots y_n$

x	$x_0, x_1, x_2, x_3 \ldots \ldots x_n$
y	$y_0, y_1, y_2, y_3 \ldots \ldots y_n$

Extp

(2) **Extrapolation** :- Extrapolation is the Process of finding out the unknown value which lies outside the given set of tabulated values

(or)

Extrapolation is the method of Constructing new data points outside the range for given set of Points

Finite Differences :-

In this Unit, we introduce what are Called forward, backward and Central differences of a function $y = f(x)$

Forward difference table :- (Δ)

x	y	1st diff Δy (or) $\Delta f(x)$	2nd diff $\Delta^2 y$ (or) $\Delta^2 f(x)$	3rd diff $\Delta^3 y$ (or) $\Delta^3 f(x)$	4th diff $\Delta^4 y$ (or) $\Delta^4 f(x)$	5th diff $\Delta^5 y$ (or) $\Delta^5 f(x)$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Let $y = f(x)$ be a function which takes values of Y as $y_0\ y_1\ y_2 \ldots y_n$ and Corresponding values of X as $x_0\ x_1\ x_2 \ldots x_n$

where x is an Independent Variable
y is an dependent Variable

Then the differences $(y_1 - y_0), (y_2 - y_1)\ (y_3 - y_1) \ldots$ are Called as First forward differences of Y and we denote them as $\Delta y_0, \Delta y_1\ \Delta y_2 \ldots$ respectively and defined as

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2 \ldots$$

The difference of the First forward differences is Called as "Second Forward differences" and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2 \ldots$ and define them as

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 \ldots$$

The differences of the second Forward differences is Called as "Third Forward differences" and they are denoted by $\Delta^3 y_0, \Delta^3 y_1, \Delta^3 y_2 \ldots$ and define them as

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$$
$$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2 \ldots$$

||ly we Can find 4th, 5th & 6th ----- nth forward differences.

→ The Symbol "Δ" is Called as Forward difference Operator

→ The above table is Called as Forward difference table (or) Diagonal difference table

→ Here x is Called as Argument and
y is Called as Function (or) Entry

→ "y_0" the 1st entry is Called as "Leading term" and Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ ---- are Called as Leading differences.

→ $\Delta f(x) = f(x+h) - f(x)$

Problems

(1) Construct the forward difference table for the following set of Values

x	1	2	3	4	5
y	4	13	34	73	136

Sol :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		9			
2	13		12		
		21		6	
3	34		18		0
		39		6	
4	73		24		
		63			
5	136				

(2) Construct the difference table for $y = x^3$ where x takes values 1, 2, 3, 4, 5, 6

x	1	2	3	4	5	6
$y = x^3$	1	8	27	64	125	216

Sol : The Forward difference table is.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		
5	125		30			
		91				
6	216					

(3) Construct the forward difference table for the following data and find the values of -

$\Delta f(3)$, $\Delta^2 f(2)$, $\Delta^3 f(4)$, $\Delta^4 f(2)$, $\Delta^5 f(1)$.

Sol :-

x	1	2	3	4	5	6
y	2	9	14	17	23	27

Forward difference table is

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1	2					
		7				
2	9		-2			
		5		0		
3	14		-2		5	
		3		5		-15
4	17		3		-10	
		6		-5		
5	23		-2			
		4				
6	27					

From the table we have,

$\Delta f(3) = 3$

$\Delta^2 f(2) = -2$

$\Delta^3 f(4) = 0$

$\Delta^4 f(2) = -10$

$\Delta^5 f(1) = -15$

(4) Construct the Forward difference table from the following data and find the values of

$\Delta f(15)$, $\Delta^2 f(20)$, $\Delta^3 f(25)$, $\Delta^4 f(30)$, $\Delta^5 f(35)$

$\Delta^4 f(10)$, $\Delta^3 f(20)$ $\Delta^2 f(25)$

x	10	15	20	25	30	35
y	110	220	202	309	108	100

Sol: The Forward difference table is

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
10	110					
		110				
15	220		-128			
		-18		253		
20	202		125		-686	
		107		-433		1620
25	309		-308		934	
		-201		501		
30	108		193			
		-8				
35	100					

From the above table we have

$\Delta f(15) = -18$

$\Delta^2 f(20) = -308$

$\Delta^3 f(25) = 0$

$\Delta^4 f(30) = 0$

$\Delta^5 f(35) = 0$

$\Delta^4 f(10) = -686$

$\Delta^3 f(20) = 501$

$\Delta^2 f(25) = 193$

(5) Construct the forward difference table for the following data and also find the values of $\Delta f(10)$, $\Delta^2 f(10)$, $\Delta^3 f(15)$, $\Delta^4 f(15)$, $\Delta f(15)$, $\Delta f(30)$, $\Delta^2 f(15)$, $\Delta^2 f(25)$, $\Delta^3 f(10)$, $\Delta^3 f(20)$, $\Delta^5 f(10)$, $\Delta^4 f(10)$

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Sol:-

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
10	19.97					
		1.54				
15	21.51		−0.58			
		0.96		0.67		
20	22.47		0.09		−0.68	
		1.05		−0.01		0.72
25	23.52		0.08		0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

From the above table we have

$\Delta f(10) = 1.54$

$\Delta^2 f(10) = -0.58$

$\Delta^3 f(15) = -0.01$

$\Delta^4 f(15) = 0.04$

$\Delta f(15) = 0.96$

$\Delta f(30) = 1.24$

$\Delta^2 f(15) = 0.09$

$\Delta^2 f(25) = 0.11$

$\Delta^3 f(10) = 0.67$

$\Delta^3 f(20) = 0.03$

$\Delta^5 f(10) = 0.72$

$\Delta^4 f(10) = -0.68$

(2) Backward difference Table :-

x	y	1st diff ∇y (or) $\nabla f(x)$	2nd diff $\nabla^2 f(x)$ (or) $\nabla^2 y$	3rd diff $\nabla^3 f(x)$ (or) $\nabla^3 y$	4th diff $\nabla^4 f(x)$ (or) $\nabla^4 y$	5th diff $\nabla^5 f(x)$ $\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

Let $y = f(x)$ be a function which takes values of y as $y_0\ y_1\ y_2 \cdots y_n$ and Corresponding values of x as $x_0\ x_1\ x_2 \cdots x_n$ then the differences $(y_1-y_0)\ (y_2-y_1)\ (y_3-y_2) \cdots$ are Called as First Backward difference of y and we denote them as $\nabla y_1, \nabla y_2, \nabla y_3 \cdots$ and we define them as

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\nabla y_3 = y_3 - y_2 \cdots$$

The differences of 1st backward differences are Called as second backward differences and they are defined as $(\nabla y_2 - \nabla y_1), (\nabla y_3 - \nabla y_2) \cdots$ and they are denoted by $\nabla^2 y_2, \nabla^2 y_3 \cdots$ and defined as

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 \quad \ldots$$

The differences of second backward differences are called as Third backward differences and they are defined as $(\nabla^2 y_4 - \nabla^2 y_3)$ .... and they are denoted by $\nabla^3 y_3$, $\nabla^3 y_4$ .... and defined as

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$$

$$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3 \quad \ldots$$

- ||ly, 4th, 5th, 6th ... nth backward differences.

→ The symbol '∇' is called as Backward difference Operator

→ The above table is called as Backword difference table (or) diagonal difference table.

Problems

1) Construct a Backward difference table for following set of values

x	1	3	5	7	9
y	8	12	21	36	62

Sol:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	8				
		4			
3	12		5		
		9		1	
5	21		6		4
		15		5	
7	36		11		
		26			
9	62				

(2) Construct a backward difference table for $y = \log x$

x	10	20	30	40	50
y	1.0000	1.3010	1.4771	1.6021	1.6990

Also find the values of $\nabla^3 \log(40)$, $\nabla^4 \log(50)$, $\nabla^4 \log(30)$, $\nabla^3 \log(50)$, $\nabla^2 \log(50)$, $\nabla^2 \log(40)$, $\nabla^2 \log(30)$, $\nabla \log(50)$, $\nabla \log(40)$, $\nabla \log(30)$

The Backward diff table is

x	y	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
10	1.0000				
		0.3010			
20	1.3010		−0.1249		
		0.1761		0.0738	
30	1.4771		−0.0511		−0.0508
		0.1250		0.0230	
40	1.6021		−0.0281		
		0.0969			
50	1.6990				

From the table we have

$\nabla^3 \log(40) = 0.0738$

$\nabla^4 \log(50) = -0.0508$

$\nabla^4 \log(30) = 0$

$\nabla^3 \log(50) = 0.0230$

$\nabla^2 \log(50) = -0.0281$

$\nabla^2 \log(40) = -0.0511$

$\nabla^2 \log(30) = -0.1249$

$\nabla \log(50) = 0.0969$

$\nabla \log(40) = 0.1250$

$\nabla \log(30) = 0.1761$

(3) Construct the Backward difference table from the data $\sin 30^\circ = 0.5000$

$\sin 35^\circ = 0.5736$

$\sin 40^\circ = 0.6428$

$\sin 45^\circ = 0.7071$

Assuming that the third difference to be Constant find the value of $\sin 25^\circ$

Sol :-

x	y	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
25°	0.4225			
		0.0775		
30°	0.5000		-0.0039	
		0.0736		-0.0005
35°	0.5736		-0.0044	
		0.0692		-0.0005
40°	0.6428		-0.0049	
		0.0643		
45°	0.7071			

Since the third difference to be Constant, the 1st value of 3rd difference is also -0.005

$\therefore \sin 25^\circ = 0.4225$

Shift Operator (E) :-

The shift operator of "f(x)" is denoted by $E\,f(x)$ and defined as

$$E\,f(x) = f(x+h)$$

$$E^2 f(x) = f(x+2h)$$

$$E^3 f(x) = f(x+3h) \ldots \quad \boxed{E^n f(x) = f(x+nh)}$$

The Inverse operator "E^{-1}" is defined by

$$\boxed{E^{-1} f(x) = f(x-h)}$$

Central difference Operator (δ) :-

The Central difference operator of f(x) is denoted by "$\delta f(x)$" and defined as

$$\boxed{\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\delta f(x) = \left(E^{1/2} - E^{-1/2}\right) f(x)$$

$$\therefore \boxed{\delta = E^{1/2} - E^{-1/2}}$$

Averaging Operator (μ) :-

The averaging operator of f(x) is denoted by "$\mu f(x)$" and defined as

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

$$\mu f(x) = \left(\frac{E^{1/2} + E^{-1/2}}{2} \right) f(x)$$

$$\boxed{\mu = \frac{E^{1/2} + E^{-1/2}}{2}}$$

→ Relation between the Operators Δ & E

(or)

P.T $\boxed{\Delta = E - 1}$ (or) $\boxed{E = 1 + \Delta}$

Proof:- we know that

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Rightarrow \Delta f(x) = E f(x) - f(x)$$

$$\Rightarrow \Delta \cancel{f(x)} = \cancel{f(x)} (E - 1)$$

$$\boxed{\Delta = E - 1} \text{ (or) } \boxed{E = 1 + \Delta}$$

→ $\Delta f(x) = f(x+h) - f(x)$

→ $f(x+h) = E f(x)$

→ $f(x-h) = E^{-1} f(x)$

→ $\nabla f(x) = f(x) - f(x-h)$

→ P.T $\boxed{\nabla = \Delta E^{-1}}$

Proof:- we know that $\nabla f(x) = f(x) - f(x-h)$

$$= \Delta f(x-h)$$

$$= \Delta E^{-1} f(x)$$

$$\nabla f(x) = \Delta E^{-1} f(x)$$

$$\boxed{\nabla = \Delta E^{-1}}$$

→ P.T $\boxed{\nabla = 1 - E^{-1}}$

<u>Sol</u> :- we know that

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x)$$
$$\nabla \cancel{f(x)} = \cancel{f(x)} (1 - E^{-1})$$
$$\boxed{\nabla = 1 - E^{-1}}$$

<u>Properties</u> :-

→ $\Delta = E - 1$ (or) $E = 1 + \Delta$

→ $\nabla = 1 - E^{-1}$

→ $\delta = E^{1/2} - E^{-1/2}$

→ $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

→ $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

→ $(1+\Delta)(1-\Delta) = 1$

→ $\mu^{2} = 1 + \frac{1}{4}\delta^{2}$

→ $E = e^{hD}$

→ $\Delta f(x) = f(x+h) - f(x)$

→ $\nabla f(x+h) = f(x+h) - f(x)$

→ $\Delta f(x) = \nabla f(x+h)$

$\longrightarrow$ Evaluate Δe^{ax}

Sol: we know that

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

$$\Delta e^{ax} = e^{a(x+h)} - e^{ax}$$

$$= e^{ax} \cdot e^{ah} - e^{ax}$$

$$= e^{ax}(e^{ah} - 1)$$

$f(x) = e^{ax}$

$f(x+h) = e^{a(x+h)}$

$= e^{ax+ah}$

$$\therefore \boxed{\Delta e^{ax} = e^{ax}(e^{ah} - 1)}$$

$\longrightarrow$ Evaluate $\Delta \operatorname{Tan}^{-1} ax$

Sol: we know that

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

Let $f(x) = \operatorname{Tan}^{-1} ax$

$f(x+h) = \operatorname{Tan}^{-1} a(x+h)$

$= \operatorname{Tan}^{-1}(ax+ah)$

$\operatorname{Tan}^{-1} a - \operatorname{Tan}^{-1} b$

$= \operatorname{Tan}^{-1}\left(\frac{a-b}{1+ab}\right)$

$$\Delta \operatorname{Tan}^{-1} ax = \operatorname{Tan}^{-1}(ax+ah) - \operatorname{Tan}^{-1}(ax)$$

$$= \operatorname{Tan}^{-1}\left(\frac{\cancel{ax} + ah - \cancel{ax}}{1+(ax+ah)(ax)}\right)$$

$$\Delta \operatorname{Tan}^{-1} ax = \operatorname{Tan}^{-1}\left(\frac{ah}{1+a^2x(x+h)}\right)$$

$a^2x^2 + a^2xh$

$\Rightarrow a^2x(x+h)$

→ Evaluate $\Delta \log x$

Sol: We know that

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

$f(x) = \log x$

$f(x+h) = \log(x+h)$

$\log a - \log b = \log\left(\frac{a}{b}\right)$

$\Delta \log x = \log(x+h) - \log x$

$= \log\left(\frac{x+h}{x}\right)$

$\Delta \log x = \log\left(1 + \frac{h}{x}\right)$ //

→ Evaluate $\Delta \sin ax$

Sol: We know that $\boxed{\Delta f(x) = f(x+h) - f(x)}$

Let $f(x) = \sin ax$

$f(x+h) = \sin a(x+h)$

$= \sin(ax + ah)$

$\sin A - \sin B =$

$2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\Delta \sin ax = \sin(\overset{A}{ax+ah}) - \overset{B}{\sin ax}$

$= 2\cos\left(\frac{ax+ah+ax}{2}\right)\sin\left(\frac{\cancel{ax}+ah-\cancel{ax}}{2}\right)$

$\Delta \sin ax = 2\cos\left(\frac{2ax+ah}{2}\right)\sin\left(\frac{ah}{2}\right)$

→ Evaluate $\Delta^{2} e^{x}$, $h=1$

Sol :- we know that

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

Let $f(x) = e^{x}$

$f(x+h) = e^{x+h}$

$\Delta e^{x+h} = f(x) = e^{x+h}$

$e^{x+2h} - e^{x+h}$

$\Delta^{2} e^{x} = \Delta(\Delta e^{x})$

$= \Delta(e^{x+h} - e^{x})$

$= \Delta e^{x+h} - \Delta e^{x}$

$= e^{x+2h} - e^{x+h} - e^{x+h} + e^{x}$

$\Delta^{2} e^{x} = e^{x+2h} - 2e^{x+h} + e^{x}$

take $\boxed{h=1}$ we have

$\underset{h=1}{\Delta^{2} e^{x}} = e^{x+2} - 2e^{x+1} + e^{x}$

$= e^{x}\cdot e^{2} - 2\cdot e^{x} e^{1} + e^{x}$

$= e^{x}(e^{2} - 2e + 1)$

$\therefore \Delta^{2} e^{x}$ at $h=1$ $= e^{x}(e^{2} - 2e + 1)$

→ The Forward difference of a Constant is Zero

Sol :- Let $f(x) = a$ (Constant)

we know $\boxed{\Delta f(x) = f(x+h) - f(x)}$

$$\Delta a = a - a = 0$$

$$\Delta a = 0$$

∴ The difference of a Constant is Zero.

→ Prove $\Delta f(x) \cdot g(x) = f(x+h) \cdot \Delta g(x) + g(x) \cdot \Delta f(x)$

Sol :- $\Delta f(x) \cdot g(x) = f(x+h) g(x+h) - f(x) g(x)$

$= f(x+h) g(x+h) - f(x+h) g(x) +$

$f(x+h) g(x) - f(x) g(x)$

$= f(x+h) [g(x+h) - g(x)] + g(x) [f(x+h) - f(x)]$

$= f(x+h) \Delta g(x) + g(x) \Delta f(x)$

→ P.T $\Delta \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \cdot \Delta f(x) - f(x) \cdot \Delta g(x)}{g(x) \cdot g(x+h)}$

Sol :- Consider

$$\Delta \left(\frac{f(x)}{g(x)} \right) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{g(x) f(x+h) - g(x+h) f(x)}{g(x) \, g(x+h)}$$

$$= \frac{f(x+h) g(x) - f(x) g(x) + f(x) g(x) - g(x+h) f(x)}{g(x) \, g(x+h)}$$

$$\Rightarrow \frac{g(x)\left[f(x+h)-f(x)\right]+f(x)\left[g(x)-g(x+h)\right]}{g(x+h)\cdot g(x)}$$

$$\Rightarrow \frac{g(x)\,\Delta f(x)-f(x)\,\Delta g(x)}{g(x+h)\cdot g(x)}$$

$$\therefore \Delta\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\,\Delta f(x)-f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$$

$\longrightarrow$ P.T $\quad \Delta \log f(x) = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$

Sol: we know that $\boxed{\begin{array}{l}\Delta f(x) = f(x+h) - f(x) \\ f(x+h) = \Delta f(x) + f(x)\end{array}}$

now $\Delta \log f(x) = \log f(x+h) - \log f(x)$

$$= \log\left[\frac{f(x+h)}{f(x)}\right]$$

$$= \log\left[\frac{\Delta f(x) + f(x)}{f(x)}\right]$$

$$= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

1) Newton's Forward Interpolation formula

(or)

Newton Gregory Forward Interpolation formula

Let $y = f(x)$ be a function which takes values of y as $y_0, y_1, y_2 \ldots y_n$ and corresponding values of x as $x_0, x_1 x_2 \ldots x_n$ then the Newton's forward Interpolation formula is

$$y = f(x) = f(x_0+uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 +$$

$$\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots\cdots$$

where $x = x_0 + uh \implies \boxed{u = \frac{x-x_0}{h}}$

(or)

$$y = f(x) = f(x_0+uh) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) +$$

$$\frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(x_0) + \cdots\cdots$$

where $x = x_0 + uh \implies \boxed{u = \frac{x-x_0}{h}}$

Problems.

1) Find the value of $\sin 52^\circ$ from the values given below $\sin 45^\circ = 0.7071$

$\sin 50^\circ = 0.7660$

$\sin 55^\circ = 0.8192$

$\sin 60^\circ = 0.8660$ Using Newton Forward Interpolation formula.

Sol:- Given $\sin 45^\circ = 0.7071$ $\quad \sin 55^\circ = 0.8192$

$\sin 50^\circ = 0.7660$ $\quad \sin 60^\circ = 0.8660$

Forward difference table :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0 45°	0.7071 y_0			
		Δy_0 0.0589		
x_1 50°	0.7660 y_1		$\Delta^2 y_0$ -0.0057	
		0.0532		$\Delta^3 y_0$ -0.0007
x_2 55°	0.8192 y_2		-0.0064	
		0.0468		
x_3 60°	0.8660 y_3			

$x = x_0 + uh \implies u = \frac{x - x_0}{h} = \frac{52^\circ - 45^\circ}{5} = \frac{7}{5} = \underline{1.4}$.

h = Common diff b/w values $\quad \boxed{u = 1.4}$

we know that Newton Forward Interpolation formula is

$$y = f(x) = f(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$y = f(52^\circ) = f[(45^\circ + (1.4)(5)] = 0.7071 + (1.4)(0.0589) + \frac{(1.4)(1.4-1)}{2}(-0.0057) + \frac{(1.4)(1.4-1)(1.4-2)}{6}(-0.0007) + \cdots$$

$$\underline{f(52^\circ) = 0.7880}$$

$\therefore$ $\underline{\sin 52^\circ = 0.7880}$

(2) Find $f(1.6)$ Using Newton Forward Interpolation formula from the table given below

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

Sol:- Since the given value 1.6 is at the beginning of the table so we use Newton Forward Interpolation formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0 1	3.49 y_0			
		Δy_0 1.33		
1.4	4.82		$\Delta^2 y_0$ -0.19	
		1.14		$\Delta^3 y_0$ -0.41
1.8	5.96		-0.60	
		0.54		
2.2	6.5			

$x = 1.6$

$x_0 = 1$

$h = 1-1.4 = 0.4$

$$x = x_0 + uh \implies u = \frac{x - x_0}{h} = \frac{1.6 - 1}{0.4} = 1.5$$

Newton's Forward Interpolation formula is

$$y = f(x) = f(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y = f(1.6) = f[(1 + (1.5)(0.4)] = 3.49 + (1.5)(1.33) + \frac{(1.5)(1.5-1)}{2}(-0.19) + \frac{(1.5)(1.5-1)(1.5-2)}{6}(-0.41) + \cdots$$

$$\boxed{f(1.6) = 5.45}$$

(3) Find f(0.6) Using Newton Forward Interpolation formula from the table given below

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
f(x)	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Sol: Since the given Value 0.6 is at the beginning of the table so we use Newton Forward Interpolation formula

Forward difference table :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.1	0.003						
		0.064					
0.3	0.067		0.17				
		0.081		0.002			
0.5	0.148		0.019		0.001		
		0.100		0.003		0	
0.7	0.248		0.022		0.001		0
		0.122		0.004		0	
0.9	0.370		0.026		0.001		
		0.148		0.005			
1.1	0.518		0.031				
		0.179					
1.3	0.697						

$$x = x_0 + uh \Rightarrow u = \frac{x - x_0}{h} = \frac{0.6 - 0.1}{0.2} = 2.5$$

$$\boxed{f(0.6) = 0.196}$$

(4) For $x = 0, 1, 2, 3, 4$

$f(x) = 1, 14, 15, 5, 6$

find $f(3)$ Using Newton Forward Interpolation formula.

Sol :- since the given value 3 is at the beginning of the table so we used Newton Forward Interpolation formula.

Forward difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		13			
1	14		-12		
		1		1	
2	15		-11		21
		-10		22	
3	5		11		
		1			
4	6				

$x = x_0 + uh \implies u = \frac{x - x_0}{h} = \frac{3-0}{1} = 3 \quad \therefore \boxed{u=3}$

$$\boxed{f(3) = 5}$$

(5) Find $f(2.5)$ Using Newton Forward Interpolation formula from the following table.

x	0	1	2	3	4	5	6
y	0	1	16	81	256	625	1296

Sol :- $x = x_0 + uh \implies u = \frac{x - x_0}{h} = \frac{2.5 - 0}{1} = 2.5$

$$\boxed{f(2.5) = 39.0625}$$

(d) Newton Backward Interpolation formula

(or)

Newton Gregory Backward Interpolation formula

Let $y = f(x)$ be a function which takes values of y as $y_0, y_1, y_2 \ldots y_n$ and corresponding values of x as $x_0, x_1, x_2 \ldots x_n$ then the Newton Backward Interpolation formula is

$$y = f(x) = f(x_n + uh) = y_n + u \nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!} \nabla^4 y_n + \ldots$$

where $x = x_n + uh$

$$\Longrightarrow \boxed{u = \frac{x - x_n}{h}}$$

Difference b/w Forward & Backward : (Hint)

Forward	Backward
Δ	∇
x_0	x_n
y_0	y_n
$- +$	$+ + +$

Problems

(1) Calculate the value of $f(7.5)$ from the following data given below

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

Sol:- Since the given value 7.5 is at the end of the table, so we use Newton Backward Interpolation formula

Backward difference table :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$
1	1	7	12	6	0	0	0	0
2	8	19	18	6	0	0	0	
3	27	37	24	6	0	0		
4	64	61	30	6	0			
5	125	91	36	6				
6	216	127	42					
7	343	169						
8	512							

we know that $x = x_n + uh \Rightarrow u = \dfrac{x - x_n}{h} = \dfrac{7.5 - 8}{1} = -0.5$

$\therefore \boxed{u = -0.5}$

Newton Backward Interpolation formula is

$$y = f(x) = f(x_n + uh) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$$

$\boxed{f(7.5) = 421.875}$ $\therefore f(7.5) = 421.875$

(2) The Population of a town in a decimal census was given below now estimate the population for the year 1955

Year (x)	1921	1931	1941	1951	1961
Pop(y) in thousands	46	66	81	93	101

Sol :- The given year 1955 is at the end of the table so we use N.B.I.F

Backward diff table :-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1921	46				
		20			
1931	66		-5		
		15		2	
1941	81		-3		-3
		12		(-1)	
1951	93		-4		
		8			
1961	101				

We know that NBIF is

$$y = f(x) = f(x_n + uh) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$$

$$y = f(1955) = f[(1961 + (-0.6)(10)] = 101 + (-0.6)(8) + \cdots$$

$$= 96.8368 \approx 97 \text{ Thousands}$$

$\therefore$ $f(1955) = 97$ thousands

$$u = \frac{x - x_n}{h} = \frac{1955 - 1961}{10} = -0.6 \quad \boxed{u = -0.6}$$

(3) The area A of a Circle of diameter is given below

d (x)	80	85	90	95	100
A (y)	5026	5674	6362	7088	7854

Find approximately the areas of Circles of diameter 82 and 91 Using Newton Forward & Backward Interpolation formula

Sol :- Since the given value 82 is at the beginning of the table so we use NGFI formula.

x	y	1st diff	2nd diff	3rd diff	4th diff
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Case-1 : when $\boxed{x = 82}$ $\Rightarrow x = x_0 + uh \Rightarrow$

$$u = \frac{x - x_0}{h} = \frac{82 - 80}{5} = 0.4 \qquad \boxed{u = 0.4}$$

$f(82) = 5280.1056$

Case-2 : when $\boxed{x = 91}$

$$u = \frac{x - x_n}{h} = \frac{91 - 100}{5} = -1.8 \quad \therefore \boxed{u = -1.8}$$

$f(91) = 6504.1248$

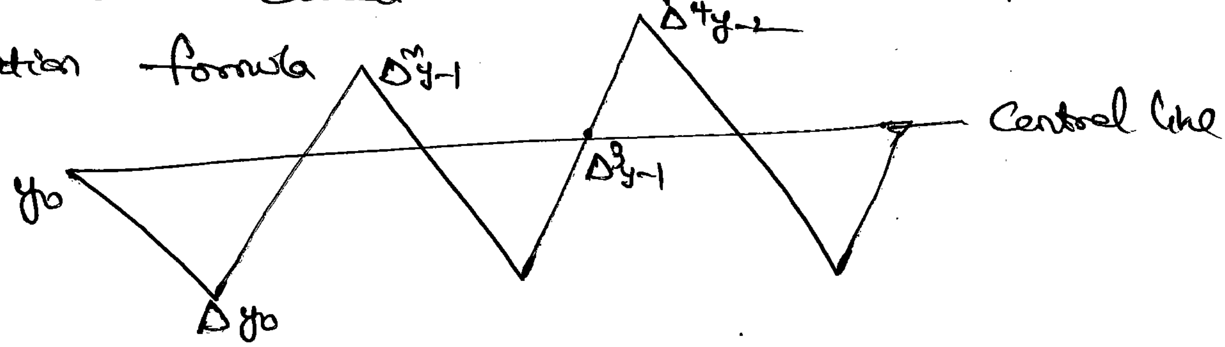
Central difference Interpolation

(3) Gauss Forward Central difference Interpolation Formula

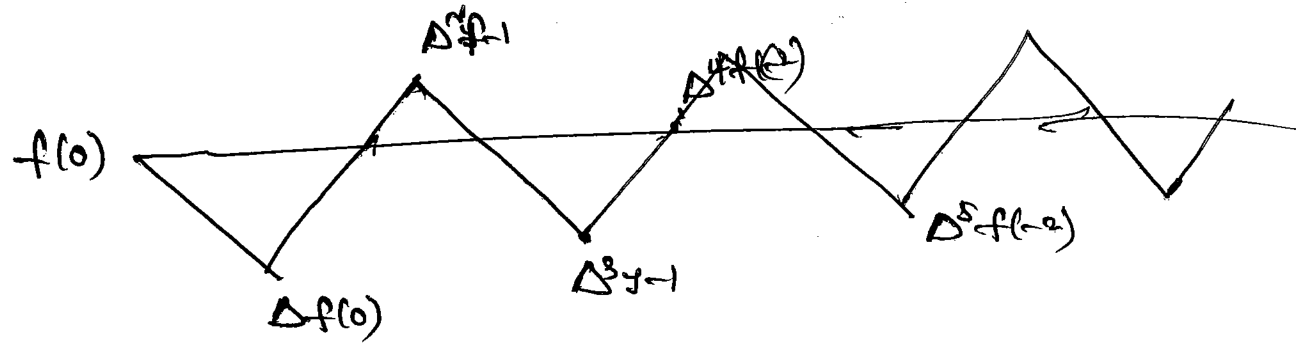
Let $y = f(x)$ be a function which takes values of Y as $y_0\, y_1\, y_2 \ldots y_n$ and corresponding values of X as $x_0\, x_1\, x_2 \ldots x_n$ then the ~~gauss~~ Gauss Forward Central difference Interpolation formula is

$$Y = f(x) = f(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

This is called Gauss Forward Central difference Interpolation formula



where $x = x_0 + uh \implies u = \frac{x - x_0}{h}$



(1) Apply Gauss Forward Interpolation formula and find $f(32)$ from the following table

x	25	30	35	40
y	0.2707	0.3027	0.3386	0.3794

Sol :- Given $f(25) = 0.2707$ $f(35) = 0.3386$

$f(30) = 0.3027$ $f(40) = 0.3794$

Here we have to find $f(32)$ which is nearest to $f(30)$ so we use gauss forward Central difference formula

now taking Origin as 30

$x_0 = 30$ $x = x_0 + uh \Rightarrow u = \frac{x - x_0}{h}$

$h = 5$

$x = 32$ $u = \frac{32-30}{5} = 0.4$ $\therefore \boxed{u = 0.4}$

Central difference table is given by

	x	y	Δy $\Delta f(x)$	$\Delta^2 y$ $\Delta^2 f(x)$	$\Delta^3 y$ $\Delta^3 f(x)$
-1	$25_{x_{-1}}$	$0.2707_{y_{-1}}$		$\Delta^2 y_{-1}$	
			0.0320		$\Delta^3 y_{-1}$
0	30_{x_0}	0.3027_{y_0}		0.0039	
			0.0359 Δy_0		0.0010
1	35_{x_1}	0.3386_{y_1}		0.0049	
			0.0408		
2	40_{x_2}	0.3794_{y_2}			

By Gauss Forward Interpolation formula

$$Y = f(x) = f(x_0+uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$x = x_0 + uh$

$u = \frac{x - x_0}{h} = \frac{32-30}{5} = 0.4$

$\boxed{u = 0.4}$

$$Y = f(32) = f[30 + (0.4)5]$$

$$= 0.3027 + (0.4)(0.0359) + \frac{(0.4)(-0.6)}{2}(0.0039)$$

$$+ \frac{(1.4)(0.4)(-0.6)}{6}(0.0010)$$

$$Y = f(32) = 0.3165$$

$$\therefore \boxed{f(32) = 0.3165}$$

(2) Apply Gauss Forward Central formula and find $f(3.5)$ using the following table given below

x	2	3	4	5
y	2.626	3.454	4.784	6.986

Sol :- given

$f(2) = 2.626$ $\quad f(4) = 4.784$

$f(3) = 3.454$ $\quad f(5) = 6.986$

Here we have to find $f(3.5)$ which is nearer to $f(3)$

now taking Origin as 3

$x_0 = 3$

$h = 1$ $\quad u = \frac{x - x_0}{h} = \frac{3.5 - 3}{1} = 0.5$

$x = 3.5$

$\therefore \boxed{u = 0.5}$

Central difference table :-

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	2	2.626			
			0.828	$\Delta^2 y_{-1}$	
0	3 x_0	3.454		0.502	
		y_0	1.330		0.370
1	4	4.784	Δy_0	0.872	$\Delta^3 y_{-1}$
			2.202		
2	5	6.986			

Since the given value 3.5 is nearer to value 3

So we can consider $\boxed{x_0 = 3}$ & $\boxed{y_0 = 3.454}$

The Gauss Forward formula is

$$y = f(x) = f(x_0+uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1}$$

$$+ \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$y = f(3.5) = f[3+(0.5)1]$$

$$= 3.454 + (0.5)(1.330) + \frac{(0.5)(0.5-1)}{2}(0.502)$$

$$+ \frac{(0.5+1)(0.5)(0.5-1)}{6}(0.370) + \cdots$$

$$= 3.454 + 0.665 - 0.06275 - 0.023125$$

$$= \underline{4.033}$$

$$\therefore f(3.5) = \underline{4.033125}$$

(3) Apply Gauss Forward Central difference interpolation formula and find the value of U_9 if $U_0 = 14$, $U_4 = 24$, $U_8 = 32$, $U_{12} = 35$, $U_{16} = 40$.

x	0	4	8	12	16
U(x)	14	24	32	35	40

$x = x_0 + uh$

$$\Rightarrow u = \frac{x - x_0}{h} = \frac{9-8}{4} = \frac{1}{4} = 0.25$$

$$\therefore \boxed{u = 0.25}$$

The Central difference table is :

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	0	14				
			10			
-1	4	24		-2		
			8		-3	
0	8 (x_0)	32 (y_0)		-5 ($\Delta^2 y_{-1}$)		10 ($\Delta^4 y_{-2}$)
			3 (Δy_0)		7 ($\Delta^3 y_{-1}$)	
1	12	35		2		
			5			
2	16	40				

Since the given value 9 is nearer to value 8, So we can consider $x_0 = 8$, $y_0 = 32$

$$x = x_0 + uh \Rightarrow u = \frac{x - x_0}{h} = \frac{9-8}{4} = \frac{1}{4} = 0.25$$

$$y = f(x) = f(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y = f(9) = f[8 + (0.25)4]$$

$$= 33.117$$

$$\boxed{f(9) = 33}$$

(4) Apply Gauss Forward formula find y when $x = 30$ given that

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154

Sol: Taking 29 as origin which is nearer to 30

$x_0 = 29$

$h = 4$

$x = 30$

$$u = \frac{x - x_0}{h} = \frac{30-29}{4} = 0.25$$

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	21	18.4708				
			-0.6564			
-1	25	17.8144		-0.0510		
			-0.7074		-0.0054	
0	29	17.1070		-0.0564		-0.002
			-0.7638		-0.0076	$\Delta^4 y_{-1}$
1	33	16.3432		-0.0640		
			-0.8278			
2	37	15.5154				

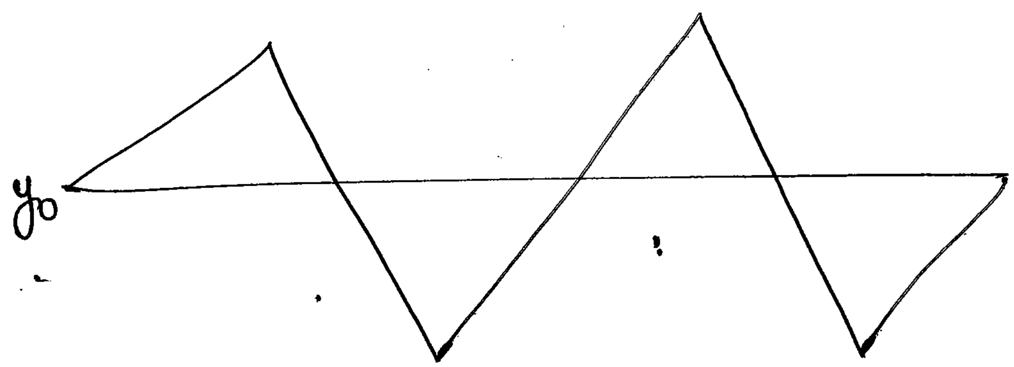
$$f(30) = 16.9303$$

(4) Gauss Backward Central difference formula

Let $y = f(x)$ be a function which takes values of y as $y_0\, y_1\, y_2 \ldots y_n$ and corresponding values of x as $x_0\, x_1\, x_2 \ldots x_n$ then the Gauss Backward difference formula is

$$y = f(x) = f(x_n + uh) = y_0 + u\,\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \cdots$$

This is called Gauss Backward Central difference formula.



$f(0)$ — $\Delta f(-1)$ — $\Delta^2 f(-1)$ — $\Delta^3 f(-2)$ — $\Delta^4 f(-2)$

(1) Apply Gauss Backward diff formula. find y when $x = 25$ from the following table given below

x	20	24	28	32
y	2854	3162	3544	3992

Sol :- Taking Origin as 28

$x_0 = 28$ $\quad u = \frac{x - x_0}{h} = \frac{25 - 28}{4} = -0.75$

$x = 25$

$h = 4$

The Central difference table is

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	20	2854			
			308		
-1	24	3162		74	
			382 (Δy_{-1})		-8 ($\Delta^3 y_{-2}$)
0	28	3544		66 ($\Delta^2 y_{-1}$)	
			448		
1	32	3992			

Gauss Backward Central diff formula is

$$y = f(x) = f(x_0 + uh) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \cdots$$

$$y = f(25) = f[28 + (-0.75)4]$$

$$= 3544 + (-0.75)(382) + \frac{(-0.75)(0.25)}{2}(66)$$

$$+ \frac{(0.25)(-0.75)(-1.75)}{6}(-8)$$

$$= 3544 - 286.5 - 6.1875 - 0.4375$$

$$\boxed{f(25) = 3251}$$

(2) Apply Gauss Backward Central difference formula to find the population for the year at 1936 for the data given below

Year (x)	1901	1911	1921	1931	1941	1951
Pop (Thos)	12	15	20	27	39	52

<u>Sol</u> :- Taking Origin as 1941

$x_0 = 1941$

$x = 1936$ $\quad U = \frac{x - x_0}{h} = \frac{1936 - 1941}{10} = -0.5$

$h = 10$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	12					
		3				
1911	15		2			
		5		0		
1921	20		2		3	
		7		3		$\Delta^5 y_{-2}$ −10
1931	27		5		−7 $\Delta^4 y_{-2}$	
		12 Δy_{-1}		−4 $\Delta^3 y_{-2}$		
1941 x_0	39 y_0		1 $\Delta^2 y_{-1}$			
		13				
1951	52					

The Gauss Backward formula is

$$y = f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y = f(1936) = 39 + (-0.5)(12) + \frac{(-0.5)(-0.5+1)}{2}(1) + \frac{(-0.5+1)(-0.5)(-0.5-1)}{6}(-4) + \cdots$$

$$= \underline{\underline{32.625}} \text{ Thousands}$$

(3) Apply Gauss Backward Central difference formula to find $f(32)$, given

$f(25) = 0.2707$

$f(30) = 0.3027$

$f(35) = 0.3386$

$f(40) = 0.3794$

Sol: Given that

x	25	30	35	40
y	0.2707	0.3027	0.3386	0.3794

Central difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	0.2707			
		0.032		
30	0.3027		0.0039	
		Δy_{-1} 0.0359		0.0010
35 x_0	0.3386 y_0		0.0049	
		0.0408	$\Delta^2 y_{-1}$	
40	0.3794			

Since the given value 32 nearer to 35 from Backward

So we Consider $x_0 = 35$ $y_0 = 0.3386$

$$x = x_0 + uh \Rightarrow u = \frac{x - x_0}{h} = \frac{32-35}{5} = \frac{-3}{5} = -0.6$$

$$\boxed{u = -0.6}$$

The Gauss Backward Central diff formula is

$$y = f(x) = f(x_0 + uh) =$$

$$y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{6}\Delta^3 y_{-2} + \cdots$$

$$y = f(32) = f(35 + (-0.6)5)$$

$$\boxed{f(32) = 0.3166}$$

(5) Legrange Interpolation

Let $y = f(x)$ be a function which takes values of x as $x_0\ x_1\ x_2 \ldots x_n$ and corresponding values of y as $y_0\ y_1\ y_2 \ldots y_n$ then the Legrange Interpolation formula for the given data is

x	x_0	x_1	x_2	x_3
y	y_0	y_1	y_2	y_3

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

replace $(x_0 = x_1)$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

replace $(x_1 = x_2)$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 +$$

replace $(x_2 = x_3)$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 + \cdots$$

Problems

1) Find $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively

Using Lagrange Interpolation formula

Sol :- given data

x	1 (x_0)	7 (x_1)	15 (x_2)
y	168 (y_0)	192 (y_1)	336 (y_2)

The Lagrange Interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 + \cdots$$

$$y = f(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)}(168) + \frac{(10-1)(10-15)}{(7-1)(7-15)}(192) + \frac{(10-1)(10-7)}{(15-1)(15-7)}(336) + \cdots$$

$$\boxed{f(10) = 231}$$

(Q) Using Lagrange Interpolation formula Find the value of $f(10)$ from the following table given below

x	5 x_0	6 x_1	9 x_2	11 x_3
y	12 y_0	13 y_1	14 y_2	16 y_3

Sol :- Lagrange Interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 + \cdots$$

$$y = f(10) = 14.666 = 15$$

$$\therefore \boxed{f(10) = 15}$$

(3) Using Lagrange Interpolation formula find the value of u_3 from the following data given below $u_0 = 580$, $u_1 = 556$, $u_2 = 520$, $u_4 = 385$

Sol: given data

x	0 x_0	1 x_1	2 x_2	4 x_3
y	580 y_0	556 y_1	520 y_2	385 y_3

Lagrange Interpolation formula :-

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 +$$

$(x_0 = x_1)$ $$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$(x_1 = x_2)$ $$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 +$$

$(x_2 = x_3)$ $$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 + \cdots$$

$$y = f(3) = 465$$

$$\boxed{u_3 = 465}$$

(4) Using Legrange Interpolation formula. find f(3) from the following data

x	0 x_0	1 x_1	2 x_2	4 x_3	5 x_4	6 x_5
y	1 y_0	14 y_1	15 y_2	5 y_3	6 y_4	19 y_5

Sol: $\boxed{f(3) = 10}$

(5) Using Legrange Interpolation formula find f(2) from the following table

x	0 x_0	1 x_1	3 x_2	4 x_3
y	5 y_0	6 y_1	50 y_2	100 y_3

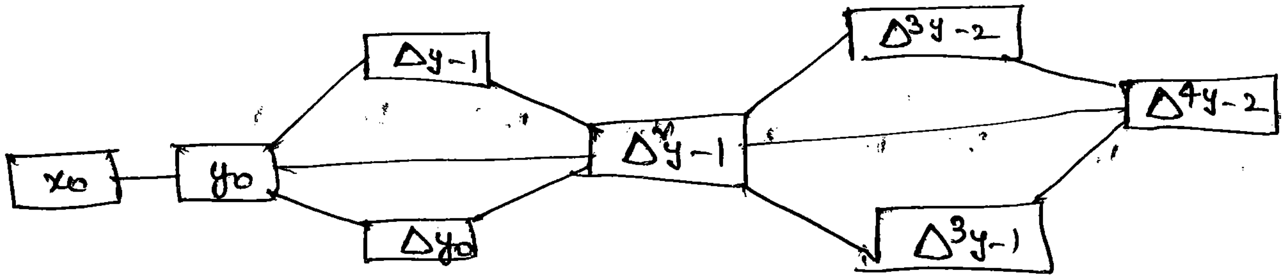
Sol: $\boxed{f(2) = 19.833}$

(6) Using Legrange formula find f(4) from the table given below

x	0 x_0	2 x_1	3 x_2	6 x_3
y	-4 y_0	2 y_1	14 y_2	158 y_3

Sol: $\boxed{f(4) = 40}$

6. Stirling's formula



Let $y = f(x)$ be a function which takes values of y as $y_0\ y_1\ y_2 \dots y_n$ and corresponding values of x as $x_0, x_1\ x_2 \dots x_n$ then the Stirling's formula is given by

$$y = f(x) = f(x_0 + uh) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \dots$$

This is called Stirling's formula

→ Stirlings formula can be obtained by taking average of Gauss Forward & Gauss Backward

→ This formula involves means of odd differences just above and below the central line and even differences on this line as shown below

$$y_0 - \begin{pmatrix}\Delta y_{-1} \\ \Delta y_0\end{pmatrix} - \Delta^2 y_{-1} - \begin{pmatrix}\Delta^3 y_{-2} \\ \Delta^3 y_{-1}\end{pmatrix} - \Delta^4 y_{-2} - \begin{pmatrix}\Delta^5 y_{-3} \\ \Delta^5 y_{-2}\end{pmatrix}$$

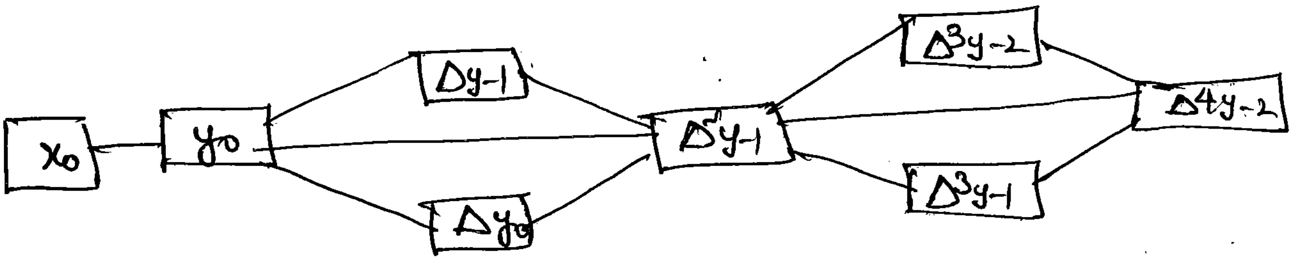
$- \Delta^6 y_{-3} - \dots$ Central line

① Using Stirling's formula find $f(1.22)$ from the table given below

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749	0.99957	0.99385	0.97385

Sol: Since we require y at $\boxed{x = 1.22}$ take the Origin at $\boxed{x = 1.2}$ & $\boxed{h = 0.1}$

$$U = \frac{x - x_0}{h} = \frac{1.22 - 1.2}{0.1} = \frac{0.02}{0.1} = 0.2$$



	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	1.0	0.84147				
			0.04974			
-1	1.1	0.89121	Δy_{-1}	-0.00891	$\Delta^3 y_{-2}$	
			0.04083		-0.00040	
0	1.2 x_0	0.93204 y_0		-0.00931		0.0008
			0.03152 Δy_0	$\Delta^2 y_{-1}$	-0.00032	$\Delta^4 y_{-2}$
1	1.3	0.96356		-0.00963	$\Delta^3 y_{-1}$	
			0.02189			
2	1.4	0.98545				

Stirlings Formula

$$y = f(x) = f(x_0+uh) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) +$$

$$\frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$

$$\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y = f(1.2) = 0.93204 + (0.2)\left(\frac{0.03152 + 0.04083}{2}\right) +$$

$$\frac{(0.2)^2}{2}(-0.00931) + \frac{(0.2)\left[(0.2)^2-1\right]}{6}\left(\frac{-0.00040 - 0.00032}{2}\right) +$$

$$\frac{(0.2)^2\left[(0.2)^2-1\right]}{24}(0.0008)$$

$$= \underline{\underline{0.93910}}$$

$$\underline{\underline{f(1.22) = 0.93910}}$$

(2) Using Stirlings formula find $f(35)$ from the table given below

x	20	30	40	50
y	512	439	346	243

<u>Sol</u>: $U = \frac{x - x_0}{h} = \frac{35-30}{10} = \underline{0.5}$

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	20	512			
			Δy_{-1} -73		
0	30 x_0	439 y_0		$\Delta^2 y_{-1}$ -20	
			-93 Δy_0		10
1	40	346		-10	
			-103		
2	50	243			

Stirling's formula is

$$y = f(x) = f(x_0 + uh) = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots$$

$$= 439 + \frac{0.5}{2}(-93-73) + \frac{0.25}{2}(-20)$$

$$= \underline{395}$$

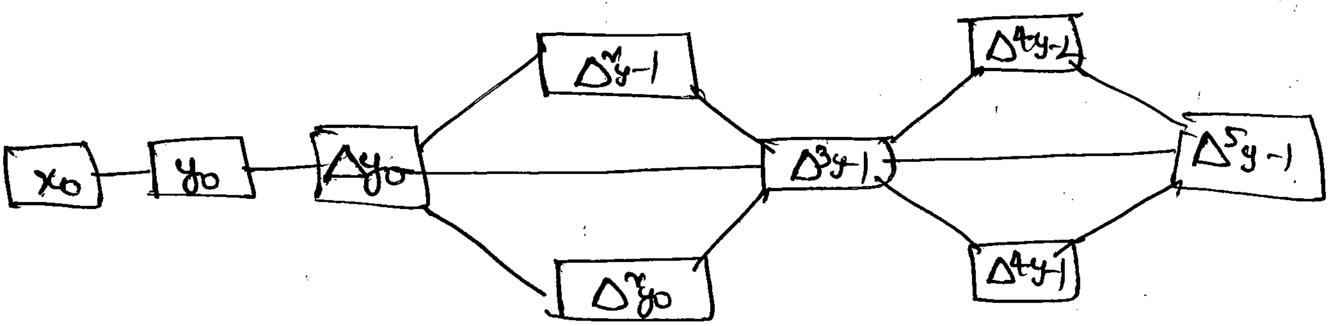
$$\therefore \boxed{y(35) = 395}$$

(7) Bessel's Interpolation

$$y = f(x) = f(x_0+uh) = \left(\frac{y_0+y_1}{2}\right) + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right] + \frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}\left[\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right] + \cdots$$

$$x = x_0+uh \Longrightarrow u = \frac{x-x_0}{h}$$

This is called Bessel's Interpolation formula



Problems

(1) Find the value of f(2.73) Using Bessels formula

x	2.5	2.6	2.7	2.8	2.9	3.0
y	0.4938	0.4953	0.4965	0.4974	0.4981	0.4987

Sol : $U = \frac{x - x_0}{h} = \frac{2.73 - 2.7}{0.1} = 0.3$ $\boxed{U = 0.3}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.5	0.4938					
		0.0015				
2.6	0.4953		-0.0003			
		0.0012		0.000		
2.7 (x_0)	0.4965 (y_0)		-0.0003		0.0001 ($\Delta^4 y_{-2}$)	
		0.0009 (Δy_0)		-0.0001 ($\Delta^3 y_{-1}$)		-0.0001 ($\Delta^5 y_{-1}$)
2.8	0.4974		-0.0002 ($\Delta^2 y_0$)		0.000 ($\Delta^4 y_{-1}$)	
		0.0007		0.0001		
2.9	0.4981		-0.0001			
		0.0006				
3.0	0.4987					

$U = \frac{x - x_0}{h} = \frac{2.73 - 2.7}{0.1} = 0.3$ $\boxed{U = 0.3}$

$$f(2.73) = 0.4968$$

(2) Using Bessels Interpolation formula find $f(1.22)$ from the table given below

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749	0.99937	0.99385	0.97385

$$f(1.22) = 0.93910$$