Unit – 2

Complex Integration

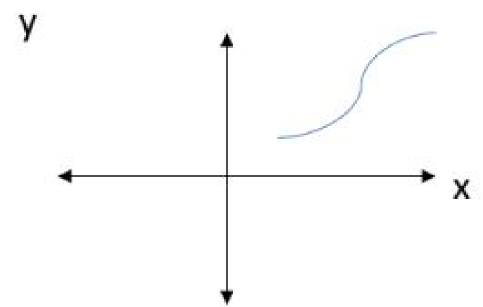
Line Integral:

suppose f(z) is a complex function in the region R, and C is a smooth curve in R. Consider an interval

$x_1 < x_2 \ldots < x_n < b$ are points in (a, b).

(a, b) and a <

$\Delta x_r = x_r - x_{r-1}$ are chord vectors, then



$$\lim_{n\to\infty} \sum_{r=1}^{n} f(z_r)\,\Delta x_r = \int_a^b f(z)\,dz$$

Where the summation tends to a limit and independent of the points choice. The limit exists if f(z) is continuous along the path.

Evaluation of the integrals: $\int f(z)\,dz = \int (u + iv)(dx + idy) = \int (udx - vdy) + i\int(udy + vdx)$ where u and v are functions of x.

Problems:

1) Evaluate $_c x^2 + ixydz$ from A(1, 1) to B(2, 8) along x = t and y = t^3.

Solution: Along x = t, y = t^3 , dx = dt, dy = 3 t^2 dt , The limits for t are 1 and 2 $_c($

( ))

$x^2 + ixy\ (dx+idy) = {}_c x^2dx - xydy) + i(xy\ dx + x^2dy$

$= {}_1 t\ {}^2_2$ dt - 3 t^6 dt + i4 t^4 dt = $\frac{t^3}{3}$-3 $\frac{t^7}{7}$+i4 $\frac{t^5}{5}$(apply the lower and upper limit)

$= -\frac{1094}{2} + \frac{124i}{5}$

2) Evaluate $_0^{1+i}\ z^2$ dz along y = x^2

Solution: $_0^{1+i}\ z^2$ dz along y = x^2 , dy = 2x dx

$= {}_0^{1+i} (x^2 - y^2+2ixy)(dx+ idy)$

$_1\ {}^2$- x^4) dx - 2 x^3 2x dx + i(x^2- $x^4$2xdx+2 x^3dx)

$= _0 (x$

2 2

$= - \quad +i$

3 3

3) Evaluate $\int_{1-i}^{2+i}(2x + 1 + iy)dz$ along (1-i) to (2+i).

Solution: Along (1-i) to (2+i) is the straight line AB joining (1,-1) to (2,1).

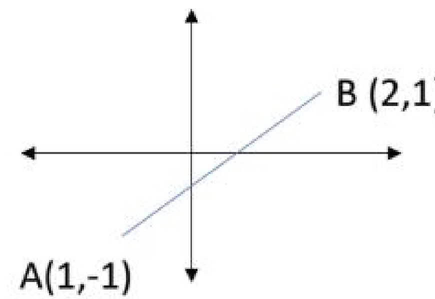
The equation of AB is y-1 = - $\frac{(-1-1)}{(1-2)}$ (x-2) y-2x = -3, y = 2x-3, dy = 2dx

X varies from 1 to 2

$\int_{1-i}^{2+i}(2x+1+iy)dz = \int_1^2 (2x+1\ dx) - (2x-3)2dx + i[2x-3]dx + (2x+1)2dx]$

$= \int_1^2 (-2x+7)dx + i(6x-1)dx$

$= -2\frac{x^2}{2} +7x+i(6\frac{x^2}{2} -x)|$ (apply the lower and upper limit)



$\int_{1-i}^{2+i}(2x+1+iy)dz = 4+8i$

4) Evaluate $\int_{(0,0)}^{(1,1)} [3x^2 + 5y + i(x^2 - y^2)]dz$ along $y^2 = x$.

Solution: Along y^2 =x, 2ydy = dx, y varies from 0 to 1.

$$_{0}(1,0),3)[3x^2+5y+i(x^2-y^2)][dx+idy] = {}_0^1 3y^4 2ydy+5y2y-(y^4-y^2)dy+i[(3y^4+5y)dy+(y^4-y^2)2ydy]$$

$$=5\frac{y6}{6}\ \frac{y5}{5}\quad -\frac{y3}{3}\quad +11\frac{y6}{6}\ \frac{y5}{5}+i(2\frac{y4}{4}+3\frac{y2}{2}-2\ +5\qquad)$$ (apply the lower

and upper limit)

$$=\frac{129}{30}+\frac{44i}{15}$$

$(1,3)$ $^2ydx+(x^2-y^2)dy$ along a) $y = 3x^2$ b) $y = 3x$.

5) Evaluate $_{(0,0)}$ x

Solution: a) $y = 3x^2$, dy = 6xdx, x varies from 0 to 1.

$$_{0}(1,0),3)\ x^2ydx+(x^2-y^2)dy = {}_0^1 3x^4dx+(x^2-9x^4)6xdx$$

(1,3) 5 4

(0,0) x2~~yd~~x +(~~x~~2 − y2)dy = 3 __

x5 + 6 x4 -54 x66

69

= - __

10

b) y = 3x , dy = 3dx, x varies from 0 to 1.

(1,3)

(0,0) $x^2ydx + (x^2 - y^2)dy$ =

$_0{}^1\, 3\, x^3dx + (x^2 - 9x^2)3dx$

x^4 x^3

= 3 __ - 24 __(apply the lower

4 3

and upper limit)

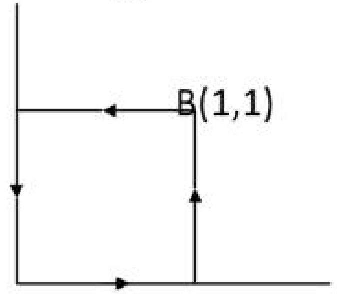
29

= - __

4

6) Evaluate $_c(3z + 1)dz$ where C is the boundary of the square with vertices at the points z = 0, z = 1, z =1+I, z = i and the orientation of C is anti-clockwise. **Solution: C is the square OABC**

$\int_c (3z+1)dz = \int_{c_1}(3z+1)dz + \int_{c_2}(3z+1)dz + \int_{c_3}(3z+1)dz + \int_{c_4}(3z+1)dz$



Along C_1= OA

y =0, dy =0 C(0,1)

X varies from 0 to 1 $\int_{c_1} 3z+1\, dz = \int_0^1 (3x+1)dx = 3\frac{x^2}{2} + x$ (apply the lower and upper limit)

Z=0 0 Z=1

A(1,0)

$= \frac{5}{2}$

Along C_2= AB

x =1, dx =0 **y**

varies from 0 to 1

1 3 c2(3z + 1)dz = i 0 [3 (1+iy)+1]dy = **4i** - 2

Along c_3**= BC** y =1, dy=0 x

varies from 1 to 0

0 3 c3(3z + 1)dz = 1 [3 (x +

i)+1]dx = - 2 -3i-1

Along c_4**= CO** x =0, dx=0 y

varies from 1 to 1

1 3) c4(3z + 1 dz = 1

[$3iy$ + 1]idx = 2 -i

5 3 5 3 c(3z+1)dz= =2 +$4i$− 2 − 2

−3i−i+ 2 =0

c(3z+1)dz=0

Evaluate $\int_{(0,0)}^{(1,1)} [3x^2 + 4xy + ix^2]dz$ along $y = x^2$ **7)**

Solution: $y = x^2$, $dy = 2xdx$,

$$\int_{(0,0)}^{(1,1)} [3x^2 + 4xy + ix^2] = \int_0^1 (3x^2 + 4x^3 + i x^2)(dx + i2xdx)$$

$$= \int_0^1 (3x^2 + 4x^3 - 2x^3)dx + i(6x^3 + 8x^4 + x^2)dx$$

$$= 1 + 1 - \frac{1}{2} + i\left(\frac{3}{2} + \frac{8}{5} + \frac{1}{3}\right) \text{ (apply the lower and upper limit)}$$

$$= \frac{3}{2} + \frac{103i}{30}$$

8) Evaluate $\int_c (y^2 + 2xy)dx + (x^2 - 2xy)dy$, where is the boundary of the region by $y = x^2$ and $x = y^2$

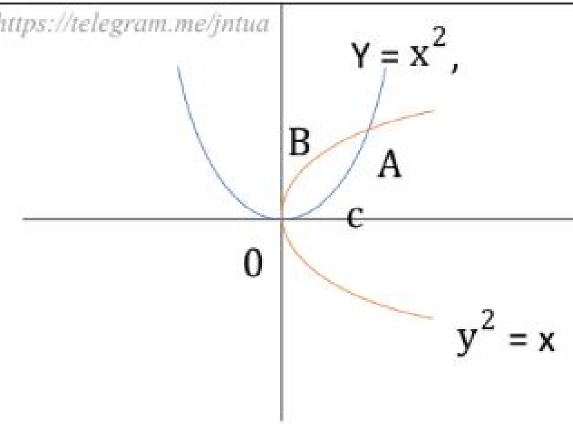
Solution:

C_1: Along OA, $y = x^2$, $dy = 2xdx$ X varies from 0 to 1 $\int_{c_1} (y^2 + 2xy)dx + (x^2 - 2xy)dy = \int_0^1 (x^4 + 2x^3)dx + (x^2 - 2x^3)2xdx = -\frac{2}{5}$

C_2: Along A**BO**, $x = y^2$, $dx = 2ydy$ y varies from 1 to 0 -

$\int_{c_2} (y^2 + 2xy)dx + (x^2 - 2xy)dy =$

$= \int_0^1 (y^2+2y^3)2ydy + (y^4 - 2y^3)dy = -1$

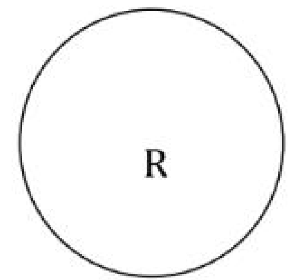


$\int_c (y^2 + 2xy)dx + (x^2 - 2xy)dy = -1 + \frac{2}{5} = -\frac{3}{5}$

Cauchy's theorem

If f(z) is analytical and $f^{I}(z)$ is continuous inside and c^{I} on a simple closed curve C, then $\int_c f(z)dz = 0$.

Proof: Suppose **R** is the region bounded by C f(z) = u+iv z = x+iy



Where C

$\int_c f(z)dz = \int_c (u + iv)(dx + idy)$ $= \int_c (udx - vdy) + i(udy + vdx)$

Since $f^{I}(z)$ is continuous, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous in R.

According to Green's theorem

$\frac{\partial v}{\partial} \frac{\partial u}{\partial}$

$$\int_c u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

$$\int_c f(z)\,dz = \iint_R \left(-\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dxdy + i \iint_R \left(\frac{\partial U}{\partial Y} - \frac{\partial u}{\partial x}\right) dxdy$$

Since f(z) is analytic $\int_c f(z)\,dz = \iint_R \left(\frac{\partial V}{\partial Y} - \frac{\partial V}{\partial y}\right) dxdy + i \iint_R \left(\frac{\partial U}{\partial Y} - \frac{\partial u}{\partial Y}\right) dxdy$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\int_c f(z)\,dz = 0$$

Cauchy's Integral Formula

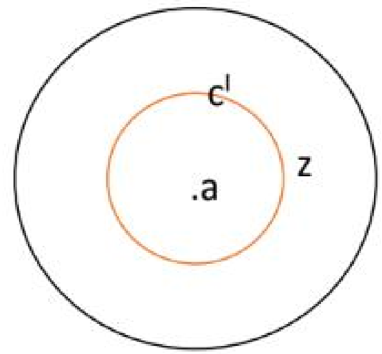
If f(z) is analytical within and on a simple closed curve and c' a is any point inside C, then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z-a)}$$

proof: C is a closed curve and a is any point inside C, Enclose a within a circle C whose radius is r and the centre is at a. Now C is inside C.

$\frac{f(z)}{(z-a)}$ is not analytical inside C.

By Cauchy's theorem for multiple connected region $\int_c g(z)\,dz = \int_{c^l} g(z)\,dz$



$$g(z) = \frac{f(z)}{(z-a)} \quad C$$

Where c^l is $|z-a| = r$

$z - a = re^{i\theta}$, $z = a + re^{i\theta}$

$dz = rie^{i\theta}d\theta$

θ varies from 0 to 2π in c^l

$$\int_c \frac{f(z)\,dz}{(z-a)} = \int_{c^l} \frac{f(z)\,dz}{(z-a)} = \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}}\, r i e^{i\theta}\, d\theta = i\int_0^{2\pi} f(a + re^{i\theta})\,d\theta$$

As $r \to 0, c^l \to 0$

$$\int_c \frac{f(z)\,dz}{(z-a)} = i\int_0^{2\pi} f(a)\,d\theta = f(a)\,2\pi i$$

$$f(a) = \frac{\int_c \frac{f(z)\,dz}{(z-a)}}{2\pi i}$$

Cauchy's integral formula for the derivatives

$\frac{f(z)dz}{} \; 1$

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}\,dz$$

Differentiating with respect to a successively

$$f^{I}(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^2}$$

$$f^{II}(a) = \frac{2}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^3}$$

$$f^{iii}(a) = \frac{2.3}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^4}$$

$$f^{iv}(a) = \frac{2.3.4}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^5}$$

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$$f^{n}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^{n+1}}$$

We can evaluate easily the integrals of complex functions using this formula.

Problems:

$ze^{z}dz$

1) Evaluate $\int_c \frac{ze^z dz}{(z+2)^3}$ where C is $|z| = 3$. **Solution:**

$z = -2$ lies inside $|z| = 3$

According to Cauchy's integral formula

$f^{II}(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^3}$, [$f(z) = z e^z$, $a = -2$]

$f^{I}(z) = z e^z + e^z$ $\quad$ $f^{II}(z) = z e^z + 2e^z$ $\quad$ $f^{II}(-2) = -2e^{-2} + 2e^{-2} = 0$

$$\int_c \frac{ze^z dz}{(z+2)^3} = \mathbf{0.}$$

2) Evaluate $\int_c \frac{dz}{z^3(z+4)}$ where C is $|z| = 2$ using Cauchy's integral formula.

Solution: z = 0 lies inside C and z = -4 lies outside.

According to Cauchy's integral formula

1

$f^{II}(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^3}$ [a=0 and $f(z) = \frac{1}{(z+4)}$] $f^{I}(z) = -\frac{1}{(z+4)^2}$ $f^{II}(z) = \frac{2}{(z+4)^3}$ and $f^{II}(0) = \frac{1}{32}$

$$\int_c \frac{dz}{z^3(z+4)} = \frac{i\pi}{32}$$

$(z^3-\sin 3z)dz$ | |

3) Evaluate $_c$ $\overline{(z-\frac{\pi}{2})^3}$ where C is $\mathbf{z}$ = 2 using Cauchy's integral formula.

Solution: **A**ccording to Cauchy's integral formula

$\frac{1}{}$ f z dz (π) = sin3z] $f^{II}(a)$ = c (z−a)3 [a= and f(z) = z

πi 2

$\frac{\pi}{2}$ <2, z = $\frac{\pi}{2}$ lies inside C: $\mathbf{z}$ = 2 | | $f^{I}(z)$= $3z^2$-

3cos3z $f^{II}(z)$ = 6z+9 sin3z $f^{II}(\frac{\pi}{2})$ = 3π-9

$\underline{f(z)dz}$

c (z−a)3 = **πi(**3π-9)

dz

4) Evaluate $_c$ ez(z−1)3 where C is | $\mathbf{z}$ | = 2 using Cauchy's integral formula.

dz $e^{-z}dz$

Solution: c ———— ez(z−1)3 = c ———— (z−1)3

z = 1 lies inside C i.e|z|=2

f(z) = e^{-z}

According to Cauchy's integral formula

1 $\underline{f(z)dz}$ c (z−a) =

f(a), [a =1] — $\underline{(\ )}$

2πi

1 f z dz $f^{II}(a) = \pi i$ c (z−a)3

$f^{I}(z)= - e^{-z}$ $f^{II}(z) = e^{-z}$, $f^{II}(1) = e^{-1}$

$$\int_c \frac{e^{-z}dz}{(z-1)^3} = \frac{i\pi}{e}$$

5) Using Cauchy's integral formula evaluate $\int_c \frac{z^4 dz}{(z+1)(z-i)^2}$ where C is ellipse and $9x^2+4y^2 =$ 36.

Solution: $\int_c \frac{z^4 dz}{(z+1)(z-i)^2}$

$= \int_c \frac{z^4 dz}{(z+1)(1+i)^2} - \int_c \frac{z^4 dz}{(z-i)(1+i)^2} + \frac{1}{(1+i)} \int_c \frac{z^4 dz}{(z-i)^2}$ Splitting into partial fractions z = -1 and z = i lie inside $9x^2+4y^2 = 36$

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)}$$

$$\frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^2} = f^{I}(a)$$

f(z) =z^4, a = -1, f(-1) = 1, a=I, f(i) = 1

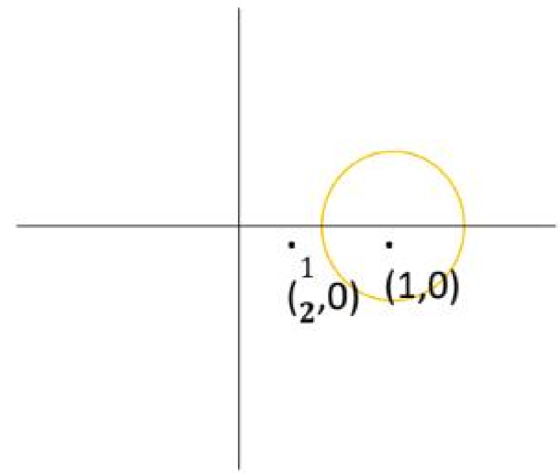
$f^{I}(z) = 4z^3$ and $f^{I}(i) = -4i$ ... $\frac{1}{(1+i)^2}$ ... $\frac{1}{(1+i)}$

$$\int_c \frac{z^4 dz}{(z+1)(z-i)^2} = \frac{1}{(1+i)^2}\ 2\pi i - 2\pi i + 2\pi i\ (-4i)$$

$$= \frac{8\pi}{(1+i)} = 4\pi(1\text{-}i)$$

6) Evaluate $\int_c \frac{\log z\, dz}{(z-1)^3}$ where C is $|z-1| = \frac{1}{2}$ using Cauchy's integral formula

Solution:



According to Cauchy's integral formula

$$\int_c \frac{f(z)\, dz}{(z-a)^3} = \frac{2\pi i}{2!} \left[f''(a) \right]_{a=1}$$

$|z-1| = \frac{1}{2}$ is a circle whose centre is (1,0)

1 and 1

radius is , a=1 lies inside C

2 $\overline{z^2}$

f(z)dz

1 f(z) =

logz, $f^{I}(z)$= , $f^{II}(z)$ = - , $f^{II}(1)$ = -1

z

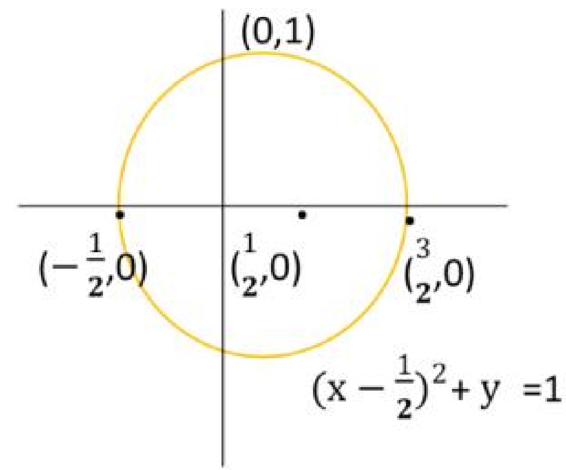
1

$f^{II}(a) = \pi i \int_c \frac{\log z\, dz}{(z-a)^3}$

$= -\pi i \int_c \frac{\log z\, dz}{(z-1)^3}$

7) Evaluate $\int_c \frac{(z^2-z-1)dz}{z(z-i)^2}$ where C is $\left|z - \frac{1}{2}\right| = 1$

Solution:



According to Cauchy's integral formula

$\int_c \frac{f(z)\,dz}{(z-a)} = 2\pi i f(a)$

z =0 inside C and z=i is outside C

f(z) = $\frac{(z^2-z-1)}{(z-i)^2}$, [a=0, f(0) =1]

$\int_c \frac{(z^2-z-1)dz}{z(z-i)^2}$ **= 2πi**

2

$(3z^2+7z+1)dz$

9) If F(a) = c ______ (z−a) using Cauchy's integral formula where C is |z| = 2, F(1), F(3), $F^{II}(1-i)$.

$(3z^2+7z+1)dz$

Solution: Suppose F(a) = c ______ (z−a)

$(3z^2+7z+1)dz$

F(1) = ______ , **[z=1 lies inside C]**

c (z−1)

f(z)dz

c (z−a) = 2πi f(a)

$[f(z) = 3z^2 + 7z+1$, f(1) = 3+7+1 =11]

(3z +7z+1)

2 c =

$2\pi i\ 11 = 22\ \pi i\ = F(1)$

(z−3)

$$F(z) = {}_c\frac{(3z^2+7z+1)}{(z-3)}dz,$$ [z=3 is outside C]

(3z +7z+1)

2 c

______ = 0 = F(3)

(z−3)

a = 1-i is inside C

$F(a) = 2\pi i(3\ a^2+7a+1)$

$F^I(a)\ = 2\pi i(6a+7)$

$F''(a) = 12\pi i$

$F''(1-i) = 12\pi i$

Complex Power Series

Taylor's Theorem:

If f(z) is analytic inside and a simple closed circle C with centre at a, then for z inside C f(z) = f(a) +

$f'(a)\,(z-a) + \underline{f''(a)}\,(z-a)^2 + \dfrac{f'''(a)}{}\,(z-a)^3 + \ldots$

2! 3!

Proof: Let Z be any point inside C, then enclose z with a circle c^{l}, with centre at a, let w be a point on c^{l}, then

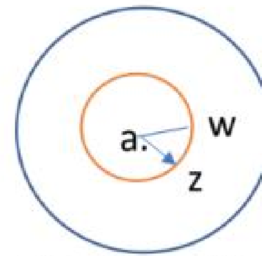
1 1 1 z−a −1

converges = = (1− ____) w−z w−a−(z−a) w−a w−a

$$= \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\frac{(z-a)^2}{(w-a)^2}+\frac{(z-a)^3}{(w-a)^3}+\ldots+\frac{\quad n}{\quad n}+\ldots\right]$$

(z−a)

(w−a)



uniformly

$$|z-a| < |w-a|$$

multiplying

$$\left|\frac{z-a}{w-a}\right| < 1$$

Therefore, this series

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both sides by f(w) and integrating with respect to w on c^{l} c^{l} f(w−zw dw) = c^{l} f(w−aw dw) +(z-a) c^{l} f(w−aw dw)2 + (z −

a)2 c^{l} f(w−aw dw)3 +...+ (z − a)n c^{l} (w−af w)dw$n$+1

f(w) is analytic on c^{l}

$1 \quad \frac{f(w)\,dw}{}$

c^{l} (w−z)

$f(z) = \frac{(f)^n}{(a)}$ 2πi $\frac{1}{}$ $\frac{(\ )}{f\ w\ dw}$

and n! = 2πi c^{l} $(w-a)_{n+1}$ Dividing by 2πi

$\frac{1}{}f$ w dw $\frac{1}{}f$ w dw $\frac{(z-a)}{}f$ w dw $\frac{(z-a)^2}{}f$ w dw $\frac{(z-a)^n}{}f$ w dw c^{l} (w−z) = 2πi c^{l} (w−a) + 2πi c^{l} $(w-a)_2$ + 2πi $(w-a)_3$ +...+ 2πi c^{l}

$(w-a)_{n+1}$+...

2πi

(z−a) (z−a)

$2n$ f(z) =

f(a)+(z-a) f^{l}(a)+ ____ f^{ll}(a)+...+ $(f)^n$(a)+...

2! ! n

This is Taylor's series of f(z)

if z-a = h

h2 hn

f(a+h)=f(a) + h f^{l}(a)+ f^{ll}(a)+...+ n! (a)+... 2! — — $(f)^n$ if

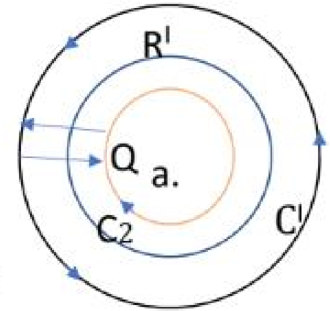
a=0, h=z

z2 zn n

—

f(z)=f(0) + z f^{l}(0)+ __2! f^{ll}(a)+...+ n!

(f) (a)+...



This is a Maclaurin's series of f(z)

Laurent series

If f(z) is analytic in a ring R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2, ($r_1 > r_2$) with centre at a then for all z in R P $f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots$ C_1

$$\text{Where } a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)\,dw}{(w-a)^{n+1}}$$

$$\text{and } \quad b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)\,dw}{(w-a)^{-n+1}}$$

Where c^{I} is any curve in R encircling C_2

Proof: Consider cross cut PQ and f(z) is analytic in the region R^{I} bounded by PQ, z is any point in R^{I}.

$$f(z) = \frac{1}{2\pi i}\left[\int_{PQ} \frac{f(w)\,dw}{(w-z)} - \int_{C_2} \frac{f(w)\,dw}{(w-z)} - \int_{QP} \frac{f(w)\,dw}{(w-z)} + \int_{C_1} \frac{f(w)\,dw}{(w-z)} \right]$$

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_1} \frac{f(w)\,dw}{(w-z)} - \int_{C_2} \frac{f(w)\,dw}{(w-z)} \right] \quad \textbf{Equation 1}$$

Where C_1 and C_2 are described anticlockwise

Consider

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)dw}{(w-z)} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)dw}{(w-a)} + \frac{(z-a)}{2\pi i}\oint_{C_1}\frac{f(w)dw}{(w-a)^2} + \dots + \frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)dw}{(w-a)^{n+1}} + \dots$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)dw}{(w-a)^{n+1}}$$

$$= \sum_{n=0}^{\infty}(z-a)^n a_n$$ ❼ **Equation 2**

Where $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)dw}{(w-a)^{n+1}}$

Consider $\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)dw}{(w-z)}$

For C_2, $w-a < z-a$

$\left|\frac{w-a}{z-a}\right| < 1$ $= + + \ldots]$

$\frac{(1)}{} \; \frac{1}{} \; \frac{()}{}$

$\frac{1}{(w-z)} = \frac{1}{w-a-(z-a)} = \frac{1}{(z-a)(1-\frac{w-a}{z-a})}$ f w dw $(z-a)^3$ 1 f w dw f w dw

C $2\pi i\, C_2 \; (z-a) +$ $2\pi i\; C_2 \,(w-a)_{-1}$ +

$c2\; 2\pi i\; (w-a)_{-3}$

$= \frac{1}{(z-a)}\left[1-\frac{w-a}{z-a}\right]^{-1}$

$\frac{1}{(z-a)}\left[1+\frac{w-a}{z-a} \quad \frac{(w-a)^2}{(z-a)^2} + \frac{(w-a)^3}{(z-a)^3}\right.$ $2\pi i$

$= \sum b_n (z-a)^{-n}$ ❼ **equation 3** Where

$-\frac{1}{} = \frac{f(w)dw}{z\;(w-z)} \quad \frac{1}{} \quad \frac{()}{}$ $(z-a)^2$ f w dw

$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{()}{(w-a)^{-n+1}}$

Substituting equations 2 & 3 in 1, we get $f(z) = \sum_{n=0}^{\infty}(z-a)^n a_n + \sum_{n=1}^{\infty}(z-a)^{-n} b_n$ This is called the Laurent series of f(z)

The first part $\sum_{n=0}^{\infty}(z-a)^n a_n$ is called the analytic part and the second part $\sum_{n=1}^{\infty}(z-a)^{-n} b_n$ is called the principal part. If the principal part is zero, the series reduces to the Taylor's series

Problems

1) Expand log z by Taylor's series about z = 1.

Solution: The given function is f(z) = log z

Taylor's series is

$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots$

a=1, f(1) = 0

$f^{I}(z) = \frac{1}{z}$, $f^{I}(1) = 1$,

$f^{II}(z) = -\frac{1}{z^2}$, $f^{II}(1) = -1$,

$f^{III}(z) = \frac{2}{z^3}$, $f^{III}(1) = 2$, $f^{iv}(z) = \frac{-3!}{z^4}$, $f^{iv}(1) = -3!$

$$\log z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \ldots + \frac{(-1)^{n-1}}{n}(z-1)^n + \ldots$$

2) Obtain all the Laurent series of the function $\frac{7z-2}{(z+1)z(z-2)}$ about z = -1

Solution: $f(z) = \frac{7z-2}{(z+1)z(z-2)}$

put z+1 = u , z = u-1 z-2 = u-3

$$\frac{7z-2}{(z+1)z(z-2)} = \frac{7(u-1)-2}{u(u-1)(u-3)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u-3}$$

7u−9

A = lim = -3

u→0 u−1 (u−3) 7u−9

B = lim = 1

u→1 u (u−3)

7u−9 C

= lim = 2 u→3 u−1 u

( )

- 3 + 1 + 2 = - 3 - 1 − u −1 (2) 1 − u −1 u u−1

u−3 u3 3

(−)

= - 3 - (1+u+ u2+ u3+...) - (1+ u + u2 +...) u 3 9

2

3

− —

= - u3 - 53 -(1 + 322)(z+1) -(1 + — 322) (z + 1)2−(1 + 324) (z + 1)3+...

3) Expand $\frac{1}{(z^2-3z+2)}$ is the region

(i) $0 < |z-1| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

Solution:

(i) $\frac{1}{(z^2-3z+2)}$ $\frac{1}{(z-2)}$ $\frac{1}{(z-1)}$ = -

$|z-1| < 1$

$\frac{1}{(z-2)} - \frac{1}{(z-1)} = \frac{1}{(z-1-1)} - \frac{1}{(z-1)}$

$= -\frac{1}{[1-(z-1)]} - \frac{1}{(z-1)} = -\left(1-(z-1)\right)^{-1} - \frac{1}{(z-1)}$

$= -(1+(z-1)+(z-1)^2+(z-1)^3+\cdots) - \frac{1}{(z-1)}$

(ii) $1<|z|<2$, $\frac{|z|}{2}<1$, $\frac{1}{|z|}<1$

$\frac{1}{(z-2)} - \frac{1}{(z-1)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$

$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$

(iii) $|z|>2$, $2<|z|$, $\frac{2}{|z|}<1$,

$\frac{1}{(z-2)} - \frac{1}{(z-1)} = \frac{1}{z(1-\frac{2}{z})} - \frac{1}{z(1-\frac{1}{z})}$

$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$

$\frac{1}{z}\left(1+\frac{2}{z}+\ldots\right) - \ldots$ z z z z z z

n−1

$= \frac{1}{z}$ $(2 = \sum_{n=1}^{\infty} \frac{}{n}$ $= \sum_{n=1}$ $\frac{2}{} + \frac{2^2}{2}$ $\frac{1}{}(1 + \frac{1}{} + \frac{1}{2} + \dots)$

1 $^{n-1}$-1)

∞ $\frac{2}{z^n}$ - ∞ $\frac{1}{}$ $n=1$ z^n

4) Find the Laurent series expansion of the function $\frac{(z^2-1)}{(z+2)(z+3)}$ if 2< z <3.

Solution:

$$f(z) = \frac{(z^2-1)}{(z+2)(z+3)} = 1 - \frac{(5z+7)}{(z^2+5z+6)}$$

$$\frac{3}{(z+2)} \; \frac{8}{(z+3)} = 1+$$

-

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \dots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \dots\right)$$

$$= 1+3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{n-1}}{z^n} + 8\sum_{n=1}^{\infty}\frac{(-1)^n z^{n-1}}{3^n}$$

$$= 1+\sum_{n=1}^{\infty}(-1)^n\left(-\frac{2^{n-1}}{z^n}+\frac{8z^{n-1}}{3^n}\right)$$

5) Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about z=1 as Laurent series. Also indicate the region of convergence of the series.

Solution: $f(z) = \frac{e^{2z}}{(z-1)^3}$

put z-1 =u, z= 1+u

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+u)}}{(u)^3} = \frac{e^2e^{2u}}{(u)^3} = \frac{e^2}{u^3}\left(1+2u+\frac{(2u)^2}{2!}+\dots\right)$$

$$= \frac{e^2}{(z-1)^3}\left(1+2(z-1)+\frac{(2(z-1))^2}{2!}+\dots\right)$$

$$= e^2\left(\frac{1}{(z-1)^3}+\frac{1}{(z-1)^2}+\frac{2}{z-1}+\cdots\right)$$

6) Express $f(z) = \frac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of z-1.

Solution: $f(z) = \frac{z}{(z-1)(z-3)}$

z A B

$\frac{}{(z-1)(z-3)} = \frac{}{(z-1)} + \frac{}{(z-3)}$

$A = \lim_{z \to 1} \frac{z}{(z-3)} = -\frac{1}{2}$

$B = \lim_{z \to 3} \frac{z}{(z-1)} = \frac{3}{2}$

$$f(z) = \frac{3}{2(z-3)} - \frac{1}{2(z-1)} = \frac{3}{2(z-1-2)} - \frac{1}{2(z-1)}$$

$$= \frac{3}{-4\left(1-\frac{z-1}{2}\right)} - \frac{1}{2(z-1)}$$

$$= -\frac{3}{4}\left(1-\frac{z-1}{2}\right)^{-1} - \frac{1}{2(z-1)} = -\frac{3}{4}\left(1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\ldots\right) - \frac{1}{2(z-1)}$$

$$= -\frac{1}{2(z-1)} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

Contour Integration

Singular points

Singular point: A point at which f(z) ceases to be analytic is called a singular point.

Isolated singular point: Suppose z=a is a singular point of a function f(z) and no other singular point of f(z) exists in a circle with centre at a, then z=a is said to be an isolated singular point.

In such a case f(z) can be expanded by Laurent series around z=a

Pole: If the principal part of f(z) consists of a finite number of terms $b_1, b_2 \ldots b_n$ $b_n \neq$

0 then (z-a) is said to be a pole of order n.

if n=1, z=a is said to be a simple pole.(note: if f(z) has a pole at z=a, then $\lim_{z \to a} f(z) = \infty$)

Removable singularity: If a single valued function f(z) is not defined at z=a $\lim_{z \to \infty} (\ )$ and f z exists, then z=a is said to be a $\sin z$ removable singularity f(z) = ___ , z=0 is a removable singularity. z

Essential singularity: If the principal part of f(z) consists of an infinite number of terms, then z=a is said to be an essential singularity

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$ z=0 is an essential singularity.

Singularity at infinity: Suppose we substitute $z = \frac{1}{w}$, $f(\frac{1}{w}) = F(w)$ (say), then the singularity at w=0 of F(w) is called the singularity at infinity. e^z has an essential singularity at z =∞, since $e^{\frac{1}{z}}$ has an essential singularity at z=0.

Entire function: A function which is analytic everywhere in the finite plane is called an entire function or integral function.

Examples: e^z , sin z, cos z are entire functions.

Note: An entire function can be represented by a Taylor series which has an infinite radius of convergence. Conversely, if a power series has an infinite radius of convergence, it represents an entire function.

Liouville's theorem: If f(z) is analytic and bounded, i.e $|f(z)| < m$ for some constant m in the entire complex plane, then f(z) is a constant.

Residue: We know that $\int_c \frac{dz}{(z-a)} = 2\pi i$ where C is $|z-a| = R$ and $\int_c \frac{dz}{(z-a)^n} = 0$, if $n \neq -1$.

$\int_c f(z)\,dz = 2\pi i\, b_1$ where C is the circle with centre at a and f(z) is expanded in Laurent series. b_1 is said to be the residue of f(z) at z=a [the coefficient of $\frac{1}{(z-a)}$ in the principal part of the Laurent series of f(z)].

Cauchy's Residue Theorem:

Statement: If f(z) is an analytic function inside and on a closed curve 'C' except at a finite number of points, inside C, then

$\int_c f(z)\,dz = 2\pi i$ (sum of the residues at the points where f(z) is not analytic and which lie inside C).

If the poles of order one and n then the residues are

$\lim_{z\to a} \frac{d}{dz}[(z-a) f(z)]$, $\lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$

1) Find the poles of the function and the corresponding residues at each pole, $f(z) = \frac{e^{iz}}{(z^2+1)}$

Solution: The given function is $f(z) = \frac{e^{iz}}{(z^2+1)}$, f(z) is not analytic at z = i and z = -i

Therefore, the poles of f(z) are i and -i, both are simple poles

If z=a is a simple pole, then the residue at z= a is $\lim_{z\to a}(z-a)f(z)$

Res z=i= $\lim(z-i)f(z) = \lim(z-i) \frac{e^{iz}}{(z^2+1)}$ $= \frac{-i}{2}e^{-1}$

Res z= -i= $\lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}(z+i)\frac{e^{iz}}{(z-i)(z+i)} = \frac{e \cdot i}{2}$

2) Find the poles of the function and the corresponding residues at each pole, f(z) = $\frac{\sin^2 z}{(z-\frac{\pi}{6})^2}$.

Solution: The given function is f(z) = $\frac{\sin^2 z}{(z-\frac{\pi}{6})^2}$, z- $\frac{\pi}{6}$ is a double pole

$$\text{Res at } z = \frac{\pi}{6} = \lim_{z\to\frac{\pi}{6}} \frac{d}{dz}\left[(z-\frac{\pi}{6})^2 \frac{\sin^2 z}{(z-\frac{\pi}{6})^2}\right]$$

$$= \lim_{z\to\frac{\pi}{6}} 2\sin z\cos z = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

3) Find the residue of $\frac{z\sin z}{(z-\pi)^3}$ at z = π.

Solution: The given function is f(z) = $\frac{z\sin z}{(z-\pi)^3}$, z = π is a pole of order 3

If z = a is a pole of order 3, then residue at z = a is

1

[(z − a)

$\lim_{z\to a} \frac{1}{(n-1)} \frac{d^{n-1}}{dz^{n-1}}$ $^n f(z)]$ (a = π)

1 d2

Res at z = π = $\lim_{z\to\pi} \frac{1}{2} \frac{d^2}{dz^2}$ (z sinz)

2

1 d

= lim __ (z

2

=

cosz + sinz) z→π dz lim (cosz

1

z→π 2

− z sinz + cosz) = **-1**.

4) Evaluate $\int_c \frac{(\cos\pi z^2+\sin\pi z^2)dz}{(z-1)^2(z-2)}$ where C is |z| = 3.

$(\cos\pi z^2 +\sin\pi z^2)$

$(\quad)^2(z-2)$

Solution: The given function is f(z) = ~~, z = 1 is a do~~uble pole and z = 2 is a simple pole, both lie inside C. z−1

Res at z = 1 = lim $\frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to1} \frac{d}{dz}\left(\frac{\cos\pi z^2+\sin\pi z^2}{(z-2)}\right)$

$= \lim_{z\to1} \frac{(z-2)(-2z\sin\pi z^2+2z\cos\pi z^2)-\cos\pi z^2-\sin\pi z^2}{(z-2)^2}$ =

3 $\lim_{z\to2} \frac{(\cos\pi z^2+\sin\pi z^2)}{(z-1)^2}$ = **1**

Res at z = 2 = lim z →

2 f(z) = lim $_{z\to 2}$

According to residue theorem

$(\cos\pi z^2+\sin\pi z^2)dz$

$\int($) = 2 πi(sum of the residues) = **2 πi(3+1) =8 πi** $_c$ $z-1$

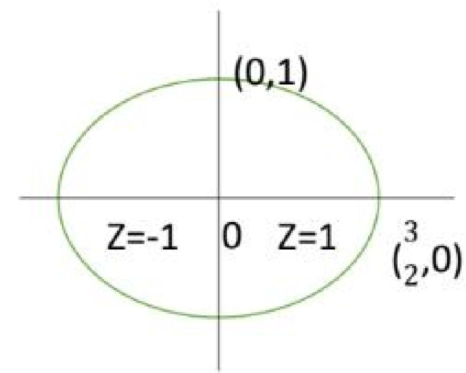
$2(z-2)$

z secz dz ^{2}+9 y^2=9

5) Evaluate $\int_c \frac{z\sec z\,dz}{1-z^2}$ where C is $4x^2+9y^2=9$

Solution:

The given function is f(z) = $\frac{z\sec z}{1-z^2}$ z=1 and -1 are simple poles and 4 x^2+9 y^2=9 is a ellipse whose semi minor and major axes are 1 and $\frac{3}{2}$.1 and -1 both lie inside C.



Res at z=1 = $\lim_{z\to 1}$ (z-1)f(z) = $\lim_{z\to 1}$ - $\frac{z\sec z}{z+1}$ = - $\frac{\sec 1}{2}$

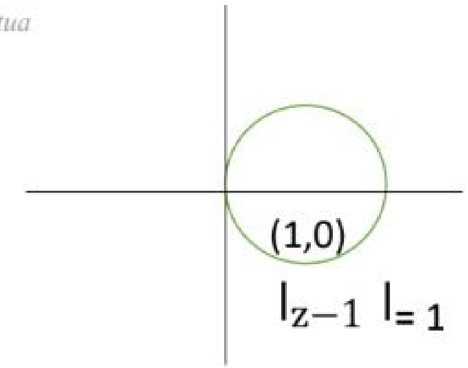
Res at z= - 1 = $\lim_{z\to -1}$ (z+1)f(z) = $\lim_{z\to -1}$ - $\frac{z\sec z}{z-1}$ = - $\frac{\sec 1}{2}$

$\int_c \frac{z\sec z\,dz}{1-z^2}$ = 2 πi (sum of the residues, by residue theorem) x

= 2 πi (-sec 1) = **- 2 πi (sec 1)**

$e^z dz$

6) Evaluate $\int_c \frac{e^z dz}{(z+2)(z-1)}$ Where C is the circle $|z-1| = 1$.



Solution: The given function is f(z) = $\frac{e^z}{(z+2)(z-1)}$, z =-2 and 1 are simple poles , z=1 lies inside C and z =-2 lies outside C.

Res at z=1 = lim

$(z-1)f(z) = \lim_{z\to 1} = \lim_{z\to 1} \frac{e^z}{z+2} = \frac{e}{3}$

$\int_c f(z)\, dz = 2\pi i$ (sum of residues at the poles which lie inside C)

$$\int_c \frac{e^z dz}{(z+2)(z-1)} = \frac{2\pi i e}{3}$$

Evaluation of real integrals in unit circle

We can evaluate the integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function, using residue theorem.

we know that if $z = e^{i\theta}$, we can write $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} =$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$\frac{1}{}$ 1

sin θ = (z-) 2i z

i

$e^{i\theta}$

dθ =

dz

and

dθ =

—

dz

iz

By this substitution we can change the integral into a function of z.

We know that $\int_c f(z)dz = 2\pi i$ (sum of the integrals) We

| |

take C is z =1, then θ varies from 0 to 2π

2π

0 f(cosθ, sinθ)dθ = $\int_c g(z)dz$ where C is |z| =1

1 1 1 dz g(z) = f [2 (z+) , (z-)] — z 2i z iz

We can evaluate using residue theorem

Problems

2π dθ 2π

1) Show that $_0$ $\frac{}{a+b\sin\theta} = \frac{}{a^2-b^2}$, a>b>0 using residue theorem.

Solution: Consider C = |z|=1, z = $e^{i\theta}$

1 1 1 $\cos\theta = \frac{1}{2}(z+\frac{}{z})$, $\sin\theta = \frac{}{2i}(z-\frac{}{z})$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_c \frac{dz}{iz[a+\frac{b}{2i}(z-\frac{1}{z})]}$$

$$f(z) = [\frac{2}{bz^2+2aiz-b}]$$

2 dz

$$\int_c f(z)dz = \int_c \frac{}{bz^2+2aiz-b}dz$$

$bz^2 + 2aiz - b = b(z-\alpha)(z-\beta)$

where $(\alpha+\beta) = -\frac{2ai}{b}$, $\alpha\beta = -1$

$$\alpha = \frac{-ai+i\sqrt{a^2-b^2}}{b} \text{ and } \beta = \frac{-ai-i\sqrt{a^2-b^2}}{b}$$

$|\alpha| <1$ and $|\beta| >1$ α lies in C $\int_c f(z)\, dz = 2\pi i \operatorname{Res} Z = \alpha$

$$\operatorname{Res} Z = \alpha = \lim_{z\to\alpha} (Z - \alpha)\, f(z) = \lim_{z\to\alpha} \frac{2}{b(z-\beta)}$$

$$= \frac{2}{b(\alpha-\beta)}$$

$$= \frac{2}{b\left[\frac{-ai+i\sqrt{a^2-b^2}}{b} + \frac{ai+i\sqrt{a^2-b^2}}{b}\right]}$$

$$= \frac{1}{i\sqrt{a^2-b^2}}$$

$$\int_c f(z)dz = \frac{1}{i}\int \frac{2\,dz}{bz^2+2aiz-b} = \frac{2\pi i}{i\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

2) Evaluate $\int_0^{2\pi} \frac{d\theta}{(6-3\cos\theta)^2}$ using residue theorem

Solution: $\int_0^{2\pi} \frac{d\theta}{(6-3\cos\theta)^2}$

Substitute $z = e^{i\theta}$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right),\ \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$dz = i\,e^{i\theta}d\theta \ \text{ and } d\theta = \frac{dz}{iz}$$

2π dθ dz 4zdz

$$\int_0^{2\pi} \frac{d\theta}{(6-3\cos\theta)^2} = \int_c \frac{dz}{iz[6-\frac{3}{2}(z+\frac{1}{z})]^2} = \int_c \frac{4z\,dz}{9i(z^2-4z+1)^2}$$

The poles are α and β where α = 2 - √3 and β = 2 + √3 and both are double poles, among which α lies inside C.

$$\text{Res at } z = \alpha = \lim_{z\to\alpha} \frac{d}{dz}[(Z-\alpha)^2 f(z)]$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{z}{(Z-\beta)^2}\right] = \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$$

$(\alpha + \beta) = 4$, $\alpha - \beta = -2\sqrt{3}$ Res at $z=\alpha = \frac{4}{24\sqrt{3}} = \frac{1}{6\sqrt{3}}$

$$\int_c \frac{4z\,dz}{9i(z^2-4z+1)^2} = \frac{4}{9i}\, 2\pi i \frac{1}{6\sqrt{3}} = \frac{4\pi}{27\sqrt{3}}$$

3) Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$, a>b>0 using residue theorem

Solution: $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ put $z = e^{i\theta}$, $dz = e^{i\theta}$ $\frac{dz}{d\theta}$ $=d\theta$ $d\theta = \frac{dz}{iz}$ $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \int_c \frac{4z\,dz}{i(2az+bz^2+b)^2}$$

The poles are α and β, both are double poles

$$\frac{-a+\sqrt{a^2-b^2}}{b} \qquad \frac{-a-\sqrt{a^2-b^2}}{b}$$

Where $\alpha =$ and β

$= b\ b$

a lies inside C

$$\text{Residue at } z = \alpha = \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{z}{b^2(Z-\beta)^2}\right]$$

$$= -\left(\frac{1}{b^2}\right)\frac{(\alpha+\beta)}{(\alpha-\beta)^2}$$

$$= -\frac{2}{b^2}\left(\frac{-2ab^3}{8(a^2-b^2)^{\frac{3}{2}}}\right) = \frac{a}{4(a^2-b^2)^{\frac{3}{2}}}$$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = 2\pi i\ (\text{Res } z = \alpha \text{ by residue theorem})$$

$$= \frac{2\pi i a^4}{4i(a^2-b^2)(a^2-b^2)^{\frac{3}{2}}} = \frac{2\pi a}{(a^2-b^2)^{\frac{3}{2}}}$$

Contour integration when the poles lie on imaginary axis

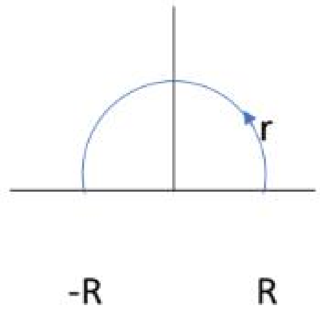
We can evaluate integrals of the type $\frac{f(x)}{g(x)} = h(x)$, using residue theorem.

Consider $\int_c h(z)\,dz$ when the poles of h(z) lie on imaginary axis. We take positive imaginary axis. Integration is taken over the semicircle and the line – R to R. The poles lie on upper half plane. If the poles lie on real axis

$$\int_c h(z)\,dz = \int_{-R}^{R} h(z)\,dz + \int_\Gamma h(z)\,dz$$

We know that by residue theorem $\int_c h(z)\,dz = 2\pi i$ (sum of the residues of h(z) at its poles which lie on upper half plane)

$$\int_{-R}^{R} h(z)\,dz + \int_\Gamma h(z)\,dz = 2\pi i \text{ (sum of the residues)}$$



In the limiting case $R \to \infty$ we get

$$\int_{-\infty}^{\infty} h(x)\,dx \text{ (if } \int_\Gamma h(z)\,dz = 0)$$

Problems:

1) Evaluate by contour integration $\int_0^\infty \frac{dx}{1+x^2}$

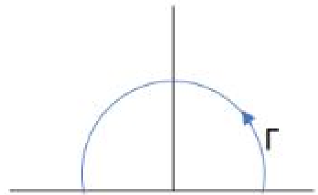
Solution: Consider $\int_c \frac{dz}{1+z^2}$ where C is the contour consisting of semicircle Γ and the line (diameter) from –R to R.

dz dz dz R c

$\frac{}{1+z^2} = {}_{-R} \frac{}{1+z^2} + {}_r \frac{}{1+z^2}$

r $\frac{dz}{1+z^2} = 0$ ∞ dx dz

$_{-\infty} \frac{}{1+x^2} = {}_c \frac{}{1+z^2}$



The poles of f(z) are ± i lie on upper half plane.

1 1

Res at z=i= $\lim_{z\to i}$ (z-i) f(z) = $\lim_{z \to i} \frac{}{(z+i)}$ = $\frac{}{(2i)}$ —

dz

$\int_c \frac{}{1+z^2} = 2\pi i$ (residue at z=i)

$= 2\pi i \cdot \frac{1}{(2i)} = \pi$

$2\int_0^\infty \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ [f(x) is even]

$= \int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_c \frac{dz}{1+z^2} = \frac{\pi}{2}$

2) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}$ using residue theorem.

Solution: $\int_{-\infty}^{\infty} f(x)\,dx$

$\int_{-R}^{R} f(z)\,dz = \int f(z) + \int_\Gamma f(z)\,dz$ [$\int_\Gamma f(z)\,dz = 0$]

$= \int_c f(Z)\,dz$ $\int_{-\infty}^{\infty} f(z)\,dz = \int_c f(Z)\,dz$

The poles of $f(z) = \frac{z^2}{(1+z^2)(4+z^2)}$ are i, -i, 2i,-2i.

All are simple poles i and 2i lie on upper half plane.

Res at z=i= lim (z-i)f(z) $z \to i$

$$= \lim_{z \to i} \frac{z^2}{(i+z)(4+z^2)} = -\frac{1}{6i}$$

Res at z=2i = lim (z-2i)f(z) $z \to 2i$

$$= \lim_{z \to 2i} \frac{z^2}{(z+2i)(1+z^2)} = -\frac{4}{4i(-3)} = \frac{1}{3i}$$

According to residue theorem $\int_c f(Z)\,dz$ = $2\pi i$ (sum of residues)

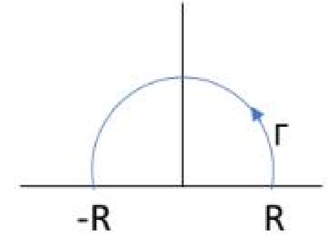
$$= 2\pi i\left(-\frac{1}{6i} + \frac{1}{3i}\right) = \frac{\pi}{3}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\,dx = \frac{\pi}{3}$$

3) Evaluate $\int_0^{\infty} \frac{x^2}{1+x^6}$ using residue theorem.

Solution: $\int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz \qquad \left[\int_{\Gamma} f(z)\,dz = 0\right]$$



$$= \int_{c} f(z)\,dz$$

$$\int_{-R}^{R} f(z)\,dz = \int_{c} f(z)\,dz$$

The poles are , $e^{(2n+1)\pi i/6}$ where n=0,1,2,3,4,5

[-1=cosπ+isinπ= $e^{\pi i}$ =cos(2n+1)π+isin$(2n+1)\pi$

$(-1)^{\frac{1}{6}} = \frac{\cos(2n+1)\pi}{6} + i\frac{\sin(2n+1)\pi}{6} = e^{(2n+1)\pi i/6}$

When n = 0, 1, 2 i.e , $e^{\frac{\pi i}{6}}$, $e^{\frac{3\pi i}{6}}$, $e^{\frac{5\pi i}{6}}$ lie on upper half plane.

$$\text{Res at } z \to e^{\frac{\pi i}{6}} = \lim_{z \to e^{\frac{\pi i}{6}}} \left(z - e^{\frac{\pi i}{6}}\right) f(z) \qquad \text{form } \frac{0}{0}$$

$$= \lim_{z \to e^{\frac{\pi i}{6}}} \frac{z^2\left(z - e^{\frac{\pi i}{6}}\right)}{(1+z^6)}$$

$$= \lim_{z \to e^{\frac{\pi i}{6}}} \frac{\left(3z^2 - 2z\, e^{\frac{\pi i}{6}}\right)}{6z^5}$$

$\frac{(3z-2e^{\frac{\pi i}{6}})}{6z^4} = \lim$

$\underline{\pi i}\, z \to e^{\underline{\pi i}}\, 6$

$-3\pi i$

$\frac{e^{6}}{e^{\frac{2\pi i}{3}}} \frac{1}{6}$ $= \frac{1}{6} = \frac{1}{2} e^{\frac{\pi i}{6}} = (\cos\frac{\pi}{2} - i\sin\frac{\pi}{6}) = -\frac{i}{6}$

$6e$

Res at $z \to e^{\frac{3\pi i}{6}}$ $\lim$ $= (z - e^{\frac{\pi i}{2}}) f(z)$ $\frac{0}{0}$ form

$\underline{3\pi i}$

$z \to e^{\frac{\pi i}{6}}$

$= \lim \frac{z^2(z - e^{\frac{\pi i}{2}})}{(1+z^6)}$

$z \to e^{\frac{\pi i}{2}}$

$= \lim \frac{(3z^2 - 2z e^{\frac{\pi i}{2}})}{6z^5}$

$\underline{\pi i}$

$z \to e^{\frac{\pi i}{2}}$

$= \lim \frac{(3z - 2e^{\frac{\pi i}{2}})}{6z^4}$

$\underline{\pi i}$

$z \to e^{\frac{\pi i}{2}}$

$= \frac{1}{6} e^{-\frac{3\pi i}{2}} (\cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2}) = \frac{i}{6}$

$5\pi i$ $5\pi i$

Res at z → e = (z− $\frac{\ }{6}$ lim $\frac{\ }{6}$ e)f(z) ^{0} form

$\underline{5\pi i}$

z →e 6

$\underline{5\pi i}$

z^2(z− lim e 6)

= $\frac{\qquad}{(1+z^6)}$

$\underline{5\pi i}$ z →e 6

$\underline{5\pi i}$

^{2}−2z e 6) lim (3z

$\frac{5\pi i}{6}$ $6z^5$

lim $\frac{1}{6}$ e$^{\frac{-15\pi i}{6}}$ $\frac{1}{6}$ $\frac{15\pi}{6}$

= z

→e

15π i

= = (cos

$\underline{5\pi i}$

6

z →e

According to residue theorem

2πi (sum of residues) ( ) $_c$f Z dz =

2

= 2 (- -) = π

6 6 6 3 —— = $\frac{\ }{1+x^6}$

π 2

πi i i

π

– —

∞ xdx

∞ = 3

∞ xdx

=

6

-i sin) = -

6

6

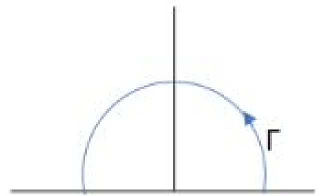
4) Evaluate $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^3}$ using residue theorem.

Solution: $\int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz \qquad \left[\int_{\Gamma} f(z)\,dz = 0\right]$$

$$= \int_{c} f(z)\,dz$$

$$\int_{-R}^{R} f(z)\,dz = \int_{c} f(z)\,dz$$



R

The function is $f(z) = \frac{1}{(z^2+1)^3}$

The poles are i and –i of order 3, z=i lies on upper half plan and inside the semicircle

Res at z=i = $\lim_{z\to i} \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-i)^3 f(z)\right]$

$$= \lim_{z\to i} \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right)$$

$$= \frac{1}{2}\lim_{z\to i}\frac{12}{(z+i)^5}$$

$$= \frac{6}{} \frac{3}{} = \mathbf{i}$$

$(2i)^5$ 16

According to residue theorem $\int_c f(Z)\, dz = 2\pi i$ (residue at z = i)

$$= 2\pi i \frac{3}{16i} = \frac{3\pi}{8}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$$

Evaluation of the integrals of the type

$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx$

Jordan's Lemma

If f(z) is a function of z satisfying the following properties:

(i) f(z) is analytic in upper half plane except at a finite number of poles

(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ with $0 \leq \arg z \leq \pi$

(iii) a is a positive integer, then

$$\lim_{r \to \infty} \int_c f(z)\, e^{iaz} dz = 0$$

Where C is a semicircle with radius r and centre at the origin

∞ ^{imx}f(x) dx = $_c$ e^{imx}f z dz = 2πi

∞ e

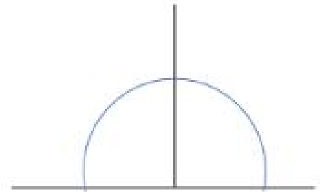
(sum of the residues which lie on upper half plane)

Problems

$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+16)(x^2+9)}$

1) Evaluate ( ) using residue theorem.

Solution: $_c$ f z e^{imz}dz ( ) = $_r$ e^{imx}f z dz ( ) R lim $_{\to\infty}$ $\int_{-+}$ R R e^{imx}f(z) dz



=> $_r$ e^{imx}f z dz = 0 (Jordan's Lemma)

∞ ^{imx}f(x) dx = $_c$ e^{imx}f z dz = 2πi

-R R

∞ e

(sum of the residues which lie on upper half plane)

$\frac{e^{iz}dz}{}$

c $(z2+16)(z2+9)$ z=3i , -3i, 4i and -4i are simple poles. 3i and 4i lie on upper half plane.

Res at z = 3i = lim (z-3i)f(z) z → 3i

$$= \lim_{z \to 3i} \frac{e^{iz}}{(z^2+16)(z+3i)}$$

$$= \frac{e^{-3}}{(-9+16)(6i)} = \frac{-ie^{-3}}{42}$$

Res at z = 4i = lim (z-4i)f(z) z → 4i

$$= \lim_{z \to 4i} \frac{e^{iz}}{(z+4i)(z^2+9)}$$

$$= \frac{e^{-4}}{(9-16)(8i)} = \frac{ie^{-4}}{56}$$

$$\int_c \frac{e^{iz}dz}{(z^2+16)(z^2+9)} = 2\pi i\left(\frac{-i}{42e^3} + \frac{i}{56e^4}\right) = \frac{\pi(4e^{-3}-3e^{-4})}{84}$$

$$\text{R.P} \int_c \frac{e^{iz}dz}{(z^2+16)(z^2+9)} = \int_c \frac{\cos z\, dz}{(z^2+16)(z^2+9)}$$

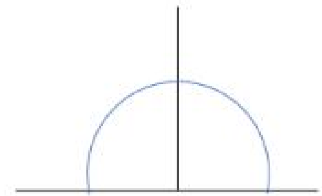
$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+16)(x^2+9)} = \frac{\pi(4e^{-3}-3e^{-4})}{84}$$

∞ $\frac{x\sin x\,dx}{(a^2+x^2)}$
0

2) Evaluate ( )

Solution: : $_c f\,z\,e^{imz}dz =\ _r e^{imx}f$ ( ) $_R \lim_{\to\infty} \int_{-}$ $z\,dz$ +
( )

$^R{}_R\ e^{imx}f(z)\ dz$



=> $_r e^{imx}f\,z\,dz = 0$

z

f(z) = ______(a2+z2)

-R R

z = ai and –ai are simple poles.

Res at z =

ai = lim (zai)f(z)

z → ai

zeiz

lim

→

$\overline{\qquad}$ z ai

(z+ai)

aie^{-a} **e**

c $\qquad$ $-a$ $\qquad$ e−a **−a**

$\frac{\quad}{(2ai)}$ $\frac{\quad}{2}$ =

−a

i e

0 $\overline{\qquad}$ (a^2+z^2) zsinx dz **=** $2\pi i \frac{\quad}{2}$ **= πi e** ∞ $\frac{x\sin x\,dx}{}$ **=** $\frac{\pi}{}$ **i e2** $-a$

(a^2+x^2)

$$\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{(a^2+x^2)} = \pi$$